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Abstract

Let (X_k) be a strictly stationary sequence of random variables with values in some Polish space E and common marginal μ, and $(A_k)_{k>0}$ be a sequence of Borel sets in E. In this paper, we give some conditions on (X_k) and (A_k) under which the events $\{X_k \in A_k\}$ satisfy the Borel-Cantelli (or strong Borel-Cantelli) property. In particular we prove that, if $\mu(\limsup_n A_n) > 0$, the Borel-Cantelli property holds for any absolutely regular sequence. In case where the A_k's are nested, we show, on some examples, that a rate of convergence of the mixing coefficients is needed. Finally we give extensions of these results to weaker notions of dependence, yielding applications to non-irreducible Markov chains and dynamical systems.

1 Introduction

Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space. Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of random variables defined on $(\Omega, \mathcal{T}, \mathbb{P})$ and with values in some Polish space E, and $(A_k)_{k>0}$ be a sequence of Borel sets in E. Assume that

$$\mathbb{P}(B_1) > 0 \text{ and } \sum_{k>0} \mathbb{P}(B_k) = \infty, \text{ where } B_k = \{X_k \in A_k\}. \tag{1.1}$$

Our aim in this paper is to find nice sufficient conditions implying the so-called Borel-Cantelli property

$$\sum_{k>0} \mathbf{1}_{B_k} = \infty \text{ almost surely (a.s.)} \tag{1.2}$$

or the stronger one

$$\lim_{n\to\infty} (S_n/E_n) = 1 \ \text{ a.s., where } S_n = \sum_{k=1}^{n} \mathbf{1}_{B_k} \ \text{ and } \ E_n = \mathbb{E}(S_n)\,, \tag{1.3}$$

usually called strong Borel-Cantelli property. The focus will be mainly on irreducible or non-irreducible Markov chains. Nevertheless we will apply some of our general criteria to dynamical systems and compare them with the results of Kim (2007) and Gouëzel (2007) concerning the transformation defined by Liverani-Saussol-Vaienti (1999).

Let us now recall some known results on this subject. On one hand, if the sequence $(X_i)_{i\in\mathbb{Z}}$ is strictly stationary, ergodic, and if $A_k = A_1$ for any positive k, then $\lim_n n^{-1} S_n = \mu(A_1)$ a.s., where μ denotes the law of X_1. Hence (1.2) holds. However, as pointed out for instance by Chernov and Kleinbock (2001), the ergodic theorem cannot be used to handle sequences of sets $(A_k)_k$ such that $\lim_k \mu(A_k) = 0$. On the other hand, if the random variables X_k are independent, then (1.2) holds for any sequence $(A_k)_{k>0}$ of Borel sets in E satisfying (1.1) (see Borel (1909), page 252). Extending this result to non necessarilly independent random variables has been the object of intensive researches. Let $\mathcal{F}_k = \sigma(X_i : i \le k)$ and recall that $B_k = \{X_k \in A_k\}$. Lévy (1937, p. 249) proved that, with probability 1,

$$\sum_{k>0} \mathbf{1}_{B_k} = \infty \ \text{ if and only if } \ \sum_{k>1} \mathbb{P}(X_k \in A_k \mid \mathcal{F}_{k-1}) = \infty. \tag{1.4}$$

However the second assertion is still difficult to check in the case of sequences of dependent random variables. As far as we know, the first tractable criterion for (1.2) to hold is due to Erdős and Rényi (1959) and reads as follows:

$$\lim_{n\to\infty} E_n = \infty \quad \text{and} \quad \lim_{n\to\infty} E_n^{-2}\mathrm{Var}(S_n) = 0\,. \tag{1.5}$$

Suppose now that the sequence $B_k = \{X_k \in A_k\}$ satisfies the following uniform mixing condition:

$$|\mathbb{P}(B_k \cap B_{k+n}) - \mathbb{P}(B_k)\mathbb{P}(B_{k+n})| \le \varphi_n\big(\mathbb{P}(B_k) + \mathbb{P}(B_{k+n})\big)\,. \tag{1.6}$$

Then, if

$$\lim_{n\to\infty} E_n = \infty \quad \text{and} \quad \sum_{n\ge1} \varphi_n < \infty\,, \tag{1.7}$$

the criterion (1.5) is satisfied and consequently (1.2) holds. Furthermore, if (1.7) holds, then the strong Borel-Cantelli property (1.3) also holds, according to Theorem 8 and Remark 7 in Chandra and Ghosal (1998). This result has applications to dynamical systems. For example, Philipp (1967) considered the Gauss map $T(x) = 1/x \pmod 1$ and the β-transforms $T(x) = \beta x \pmod 1$ with $\beta > 1$, with $(X_k)_{k\ge0} = (T^k)_{k\ge0}$ viewed as a random sequence on the probability space $([0,1], \mu)$, where μ is the unique T-invariant probability measure absolutely continuous w.r.t. the Lebesgue measure. For such maps and sequences (A_k) of intervals satisfying

$$\sum_{k>0} \mu(A_k) = \infty, \tag{1.8}$$

he proved that (1.7) is satisfied. More recently, Chernov and Kleinbock (2001) proved that (1.7) is satisfied when $(X_k)_{k\geq 0}$ are the iterates of Anosov diffeomorphisms preserving Gibbs measures and (A_k) belongs to a particular class of rectangles (called EQR rectangles). We also refer to Conze and Raugi (2003) for non-irreducible Markov chains satisfying (1.7).

However some dynamical systems do not satisfy (1.7). We refer to Haydn *et al.* (2013) and Luzia (2014) for examples of such dynamical systems and Borel-Cantelli type results, including the strong Borel-Cantelli property. In particular, estimates as in (1.7) are not available for non uniformly expanding maps such as the Liverani-Saussol-Vaienti map (1999) with parameter $\gamma \in]0,1[$. Actually, for such maps, Kim (2007) proved in his Proposition 4.2 that for any $\gamma \in]0,1[$, the sequence of intervals $A_k = [0, k^{1/(\gamma-1)}]$ satisfies (1.8) but (B_k) does not satisfy (1.2). Moreover, there are many irreducible, positively recurrent and aperiodic Markov chains which do not satisfy (1.6) with $\varphi_n \to 0$ even for regular sets A_k, such as the Markov chain considered in Remark 5.1 in the case where $A_k = [0, 1/k]$ (see Chapter 9 in Rio (2017) for more about irreducible Markov chains). However, these Markov chains are β-mixing in the sense of Volkonskiĭ and Rozanov (1959), and therefore strongly mixing in the sense of Rosenblatt (1956).

The case where the sequence of events $(B_k)_{k>0}$ satisfies a strong mixing condition has been considered first by Tasche (1997). For $n > 0$, let

$$\bar{\alpha}_n = \frac{1}{2} \sup \big\{ \mathbb{E}\big(|\mathbb{P}(B_{k+n} \mid \mathcal{F}_k) - \mathbb{P}(B_{k+n})| \big) : k > 0 \big\}. \tag{1.9}$$

Tasche (1997) obtained sufficient conditions for (1.2) to hold. However these conditions are more restrictive than (1.1): even in the case where the sequence $(\bar{\alpha}_n)_n$ decreases at a geometric rate and $(\mathbb{P}(B_k))_k$ is non-increasing, Theorem 2.2 in Tasche (1997) requires the stronger condition $\sum_{k>1} \mathbb{P}(B_k)/\log(k) = \infty$. Under slower rates of mixing, as a consequence of our Theorem 3.2 (see Remark 3.4), we obtain that if $(\mathbb{P}(B_k))_k$ is non-increasing and $\bar{\alpha}_n \leq Cn^{-a}$ for some $a > 0$, $(B_k)_k$ satisfies the Borel-Cantelli property (1.2) provided that

$$\sum_{n\geq 1} (\mathbb{P}(B_n))^{(a+1)/a} = \infty \ \text{ and } \ \lim_{n\to+\infty} n^a \mathbb{P}(B_n) = \infty \,,$$

which improves Item (i) of Theorem 2.2 in Tasche (1997). Furthermore, we will prove that this result cannot be improved in the specific case of irreducible, positive recurrent and aperiodic Markov chains for some particular sequence $(A_k)_{k>0}$ of nested sets (see Remark 3.5 and Section 5). Consequently, for this class of Markov chains, the size property (1.1) is not enough for $(B_k)_{k>0}$ to satisfy (1.2).

In the stationary case, denoting by μ the common marginal distribution, a natural question is then: for sequences of sets $(A_k)_{k>0}$ satisfying the size property (1.8), what conditions could be added to get the Borel-Cantelli property? Our main result in this direction is Theorem 3.1 (i) stating that if

$$\mu(\limsup_n A_n) > 0 \ \text{ and } \ \lim_{n\to\infty} \beta_{\infty,1}(n) = 0 \,, \tag{1.10}$$

then $(B_k)_{k>0}$ satisfies the Borel-Cantelli property (1.2) without additional conditions on the sizes of the sets A_k (see (3.3) for the definition of the coefficients $\beta_{\infty,1}(n)$). Notice that the first part of (1.10) implies the size property (1.8) : this follows from the direct part of the Borel-Cantelli lemma. For the weaker coefficients $\tilde\beta_{1,1}(n)$ defined in (4.2) (resp. $\tilde\beta_{1,1}^{\mathrm{rev}}(n)$ defined in Remark 4.2) and when the A_k's are intervals, Item (i) of our Theorem 4.1 implies the Borel-Cantelli property under the conditions

$$\mu(\limsup_n A_n) > 0 \ \text{ and } \sum_{n>0} \tilde\beta_{1,1}(n) < \infty \ \Big(\text{resp. } \sum_{n>0} \tilde\beta_{1,1}^{\mathrm{rev}}(n) < \infty\Big). \tag{1.11}$$

The proof of this result is based on the following characterization of sequences (A_k) of intervals satisfying the above condition: For a sequence (A_k) of intervals, $\mu(\limsup_n A_n) > 0$ if and only if there exists a sequence of intervals (J_k) such that $J_k \subset A_k$ for any positive k, $\sum_{k>0} \mu(J_k) = \infty$ and (J_k) fulfills the asymptotic equirepartition property

$$\limsup_n \Big\| \frac{\sum_{k=1}^n \mathbf{1}_{J_k}}{\sum_{k=1}^n \mu(J_k)} \Big\|_{\infty,\mu} < \infty\,, \tag{1.12}$$

where $\|\cdot\|_{\infty,\mu}$ denotes the supremum norm with respect to μ. Up to our knowledge, this elementary result is new. We then prove that, under the mixing condition given in (1.11), the sequence $(\{X_k \in J_k\})$ has the strong Borel-Cantelli property (see Item (ii) of Theorem 4.1). In the case of the Liverani-Saussol-Vaienti map (1999) with parameter $\gamma \in]0,1[$, the mixing condition in (1.11) holds for $\tilde\beta_{1,1}^{\mathrm{rev}}(n)$ and any γ in $]0,1/2[$. For γ in $]0,1/2[$, our result can be applied to prove that $(B_k)_{k>0}$ satisfies the Borel-Cantelli property (1.2) for any sequence (A_k) of intervals satisfying $\mu(\limsup_n A_n) > 0$, and the strong Borel-Cantelli property (1.3) under the additional condition (1.12) with $J_k = A_k$. However, for the LSV map, Gouëzel (2007) obtains the Borel-Cantelli property (1.2) under the condition

$$0 < \gamma < 1 \ \text{ and } \sum_{k>0} \lambda(A_k) = \infty \tag{1.13}$$

(but not the strong Borel-Cantelli property). Now

$$\mu(\limsup_n A_n) > 0 \Rightarrow \lambda(\limsup_n A_n) > 0 \Rightarrow \sum_{k>0} \lambda(A_k) = \infty\,,$$

by the direct part of the Borel-Cantelli lemma. Hence, for the LSV map, (1.13) is weaker than (1.11). Actually the condition (1.13) is the minimal one to get the Borel-Cantelli property in the case $A_n = [0, a_n]$ (see Example 4.1 of Section 4.3).

A question is then to know if a similar condition to (1.13) can be obtained in the setting of irreducible Markov chains. In this direction, we prove that, for aperiodic, irreducible and positively recurrent Markov chains, the renewal measure plays the same role as the Lebesgue measure for the LSV map. More precisely, if $(X_k)_{k\in\mathbb{N}}$ and ν are respectively the stationary Markov chain and the renewal measure defined in Section 5, we obtain the Borel-Cantelli property in Theorem 5.2 (but not the strong Borel-Cantelli property) for sequences of Borel sets such that

$$\sum_{k>0} \nu(A_k) = \infty \text{ and } A_{k+1} \subset A_k \ \text{ for any } k > 0\,, \tag{1.14}$$

without additional condition on the rate of mixing. Furthermore we prove in Theorem 5.4 that this condition cannot be improved in the nested case.

The paper is organized as follows. In Section 2, we give some general conditions on a sequence of events $(B_k)_{k>0}$ to satisfy the Borel-Cantelli property (1.2), or some stronger properties (such as the strong Borel-Cantelli property (1.3)). The results of this section, including a more general criterion than (1.5) stated in Proposition 2.3, will be applied all along the paper to obtain new results in the case where $B_k = \{X_k \in A_k\}$, under various mixing conditions on the sequence $(X_k)_{k>0}$. In Section 3, we state our main results for β-mixing and α-mixing sequences; in Section 4, we consider weaker type of mixing for real-valued random variables, and we give three examples (LSV map, auto-regressive processes with heavy tails and discrete innnovations, symmetric random walk on the circle) to which our results apply; in Section 5, we consider the case where $(X_k)_{k>0}$ is an irreducible, positively recurrent and aperiodic Markov chain: we obtain very precise results, which show in particular that some criteria of Section 3 are optimal in some sense. Section 6 is devoted to the proofs, and some complementary results are given in Appendix (including Borel-Cantelli criteria under pairwise correlation conditions).

2 Criteria for the Borel-Cantelli properties

In this section, we give some criteria implying Borel-Cantelli type results. Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space and $(B_k)_{k>0}$ be a sequence of events.

Definition 2.1. The sequence $(B_k)_{k>0}$ is said to be a Borel-Cantelli sequence in $(\Omega, \mathcal{T}, \mathbb{P})$ if $\mathbb{P}(\limsup_k B_k) = 1$, or equivalently, $\sum_{k>0} \mathbf{1}_{B_k} = \infty$ almost surely.

From the first part of the classical Borel-Cantelli lemma, if $(B_k)_{k>0}$ is a Borel-Cantelli sequence, then $\sum_{k>0} \mathbb{P}(B_k) = \infty$.

We now define stronger properties. The first one is the convergence in L^1.

Definition 2.2. We say that the sequence $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence in $(\Omega, \mathcal{T}, \mathbb{P})$ if $\sum_{k>0} \mathbb{P}(B_k) = \infty$ and $\lim_{n\to\infty} \|(S_n/E_n) - 1\|_1 = 0$, where $S_n = \sum_{k=1}^n \mathbf{1}_{B_k}$ and $E_n = \mathbb{E}(S_n)$.

Notice that, if $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence, then S_n converges to ∞ in probability as n tends to ∞. Since $(S_n)_n$ is a non-decreasing sequence, it implies that $\lim_n S_n = \infty$ almost surely. Therefrom $(B_k)_{k>0}$ is a Borel-Cantelli sequence.

The second one is the so-called strong Borel-Cantelli property.

Definition 2.3. With the notations of Definition 2.2, the sequence $(B_k)_{k>0}$ is said to be a strongly Borel-Cantelli sequence if $\sum_{k>0} \mathbb{P}(B_k) = \infty$ and $\lim_{n\to\infty}(S_n/E_n) = 1$ almost surely.

Notice that $\mathbb{E}(S_n/E_n) = 1$. Since the random variables S_n/E_n are nonnegative, by Theorem 3.6, page 32 in Billingsley [1], if $(B_n)_{n>0}$ is a strongly Borel-Cantelli sequence, then $(S_n/E_n)_{n>0}$ is a uniformly integrable sequence and consequenly $(S_n/E_n)_{n>0}$ converges in L^1 to 1. Hence any strongly Borel-Cantelli sequence is a L^1 Borel-Cantelli sequence.

We start with the following characterizations of the Borel-Cantelli property.

Proposition 2.1. *Let $(A_k)_{k>0}$ be a sequence of events in $(\Omega, \mathcal{T}, \mathbb{P})$ and $\delta \in]0,1]$ be a real number. The two following statements are equivalent:*

1. *$\mathbb{P}(\limsup_k A_k) \geq \delta$.*
2. *There exists a sequence $(\Gamma_k)_{k>0}$ of events such that $\Gamma_k \subset A_k$, $\sum_{k>0} \mathbb{P}(\Gamma_k) = \infty$ and*

$$\limsup_n \left\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} \right\|_\infty \leq 1/\delta\,. \tag{2.1}$$

Furthermore, if there exists a triangular sequence of events $(A_{k,n})_{1\leq k\leq n}$ with $A_{k,n} \subset A_k$, such that $\tilde{E}_n := \sum_{k=1}^n \mathbb{P}(A_{k,n}) > 0$, $\lim_n \tilde{E}_n = \infty$ and $\left(\tilde{E}_n^{-1} \sum_{k=1}^n \mathbf{1}_{A_{k,n}}\right)_{n\geq 1}$ is uniformly integrable, then $\mathbb{P}(\limsup_k A_k) > 0$.

Before going further on, we give an immediate application of this proposition which shows that a Borel-Cantelli sequence is characterized by the fact that it contains a subsequence which is a L^1 Borel-Cantelli sequence.

Corollary 2.1. *Let $(A_k)_{k>0}$ be a sequence of events in $(\Omega, \mathcal{T}, \mathbb{P})$ and $\delta \in]0,1]$ be a real number. Then the following statements are equivalent:*

1. *$\mathbb{P}(\limsup_k A_k) = 1$.*
2. *There exists a L^1 Borel-Cantelli sequence $(\Gamma_k)_{k>0}$ of events such that $\Gamma_k \subset A_k$.*

Now, if the sets A_k are intervals of the real line, then one can construct intervals Γ_k satisfying the conditions of Proposition 2.1, as shown by the proposition below, which will be applied in Section 4 to the LSV map.

Proposition 2.2. *Let J be an interval of the real line and let μ be a probability measure on its Borel σ-field. Let $(I_k)_{k>0}$ be a sequence of subintervals of J and $\delta \in]0,1]$ be a real number. The two following statements are equivalent:*

1. *$\mu(\limsup_k I_k) \geq \delta$.*
2. *There exists a sequence $(\Gamma_k)_{k>0}$ of intervals such that $\Gamma_k \subset I_k$, $\sum_{k>0} \mu(\Gamma_k) = \infty$ and (2.1) holds true.*

Let us now state some new criteria, which differ from the usual criteria based on pairwise correlation conditions. Here it will be necessary to introduce a function f with bounded derivatives up to order 2.

Definition 2.4. Let f be the application from $\mathbb{R}$ in $\mathbb{R}^+$ defined by $f(x) = x^2/2$ for x in $[-1, 1]$ and $f(x) = |x| - 1/2$ for x in $]-\infty, -1[\cup]1, +\infty[$.

We now give criteria involving the so defined function f.

Proposition 2.3. *Let f be the real-valued function defined in Definition 2.4 and $(B_k)_{k>0}$ be a sequence of events in $(\Omega, \mathcal{T}, \mathbb{P})$ such that $\mathbb{P}(B_1) > 0$ and $\sum_{k>0} \mathbb{P}(B_k) = \infty$.*

(i) *Suppose that there exists a triangular sequence $(g_{j,n})_{1\le j\le n}$ of non-negative Borel functions such that $g_{j,n} \le \mathbf{1}_{B_j}$ for any j in $[1, n]$, and that this sequence satisfies the criterion below: if $\tilde{S}_n = \sum_{k=1}^n g_{k,n}$ and $\tilde{E}_n = \mathbb{E}(\tilde{S}_n)$, there exists some increasing sequence $(n_k)_k$ of positive integers such that*

$$\lim_{k\to\infty} \tilde{E}_{n_k} = \infty \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}\big(f\big((\tilde{S}_{n_k} - \tilde{E}_{n_k})/\tilde{E}_{n_k}\big)\big) = 0. \tag{2.2}$$

Then $(B_k)_{k>0}$ is a Borel-Cantelli sequence.

(ii) *Let $S_n = \sum_{k=1}^n \mathbf{1}_{B_k}$ and $E_n = \mathbb{E}(S_n)$. If*

$$\lim_{n\to\infty} \mathbb{E}\big(f\big((S_n - E_n)/E_n\big)\big) = 0, \tag{2.3}$$

then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence.

(iii) *If*

$$\sum_{n>0} \frac{\mathbb{P}(B_n)}{E_n} \sup_{k\in[1,n]} \mathbb{E}\big(f\big((S_k - E_k)/E_n\big)\big) < \infty, \tag{2.4}$$

then $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence.

Remark 2.1. Since $f(x) \le x^2/2$ for any real x, (2.3) is implied by the usual L^2 criterion (1.5), which is the sufficient condition given in Erdős and Rényi (1959) to prove that $(B_k)_{k>0}$ is a Borel-Cantelli sequence. Moreover, (2.4) is implied by the more elementary criterion

$$\sum_{n>0} E_n^{-3} \mathbb{P}(B_n) \sup_{k\in[1,n]} \operatorname{Var}(S_k) < \infty, \tag{2.5}$$

which is a refinement of Corollary 1 in Etemadi (1983) (see also Chandra and Ghosal (1998) for a review).

3 β-mixing and α-mixing sequences

In order to state our results, we need to recall the definitions of the α-mixing, β-mixing and φ-mixing coefficients between two σ-fields of $(\Omega, \mathcal{T}, \mathbb{P})$.

Definition 3.1. The α-mixing coefficient $\alpha(\mathcal{A}, \mathcal{B})$ between two σ-fields $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{T}$ is defined by

$$2\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{E}(|\mathbb{P}(B|\mathcal{A}) - \mathbb{P}(B)|) : B \in \mathcal{B}\}.$$

One also has $\alpha(\mathcal{A},\mathcal{B}) = \sup\{\, |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| : (A,B) \in \mathcal{A}\times\mathcal{B}\}$, which is the usual definition. Now, if X and Y are random variables with values in some Polish space and $\mathcal{A}$ and $\mathcal{B}$ are the σ-fields generated respectively by X and Y, one can define the β-mixing coefficient $\beta(\mathcal{A},\mathcal{B})$ and the φ-mixing coefficient $\varphi(\mathcal{A},\mathcal{B})$ between the σ-fields $\mathcal{A}$ and $\mathcal{B}$ by

$$\beta(\mathcal{A},\mathcal{B}) = \mathbb{E}\Big(\sup_{B\in\mathcal{B}} |\mathbb{P}(B|\mathcal{A}) - \mathbb{P}(B)|\Big) \text{ and } \varphi(\mathcal{A},\mathcal{B}) = \Big\| \sup_{B\in\mathcal{B}} |\mathbb{P}(B|\mathcal{A}) - \mathbb{P}(B)| \Big\|_\infty ,$$

where $\mathbb{P}(\cdot|\mathcal{A})$ is a regular version of the conditional probability given $\mathcal{A}$. In contrast to the other coefficients $\varphi(\mathcal{A},\mathcal{B}) \neq \varphi(\mathcal{B},\mathcal{A})$ in the general case.

From these definitions $2\alpha(\mathcal{A},\mathcal{B}) \le \beta(\mathcal{A},\mathcal{B}) \le \varphi(\mathcal{A},\mathcal{B}) \le 1$. According to Bradley (2007), Theorem 4.4, Item (a2), one also has

$$4\alpha(\mathcal{A},\mathcal{B}) = \sup\{\|\mathbb{E}(Y|\mathcal{A})\|_1 : \ Y \ \mathcal{B}\text{-measurable}, \ \|Y\|_\infty = 1 \text{ and } \mathbb{E}(Y) = 0\}\,. \tag{3.1}$$

Let us now define the the β-mixing an α-mixing coefficients of the sequence $(X_i)_{i\in\mathbb{Z}}$. Throughout the sequel

$$\mathcal{F}_m = \sigma(X_k : k \le m) \text{ and } \mathcal{G}_m = \sigma(X_i : i \ge m). \tag{3.2}$$

Define the β-mixing coefficients $\beta_{\infty,1}(n)$ of $(X_i)_{i\in\mathbb{Z}}$ by

$$\beta_{\infty,1}(n) = \beta(\mathcal{F}_{-n}, \sigma(X_0)) \text{ for any } n > 0\,, \tag{3.3}$$

and note that the sequence $(\beta_{\infty,1}(n))_{n\ge 0}$ is non-increasing. $(X_i)_{i\in\mathbb{Z}}$ is said to be absolutely regular or β-mixing if $\lim_{n\uparrow\infty} \beta_{\infty,1}(n) = 0$. Similarly, define the α-mixing coefficients $\alpha_{\infty,1}(n)$ by

$$\alpha_{\infty,1}(n) = \alpha(\mathcal{F}_{-n}, \sigma(X_0))\,, \tag{3.4}$$

and note that the sequence $(\alpha_{\infty,1}(n))_{n\ge 0}$ is non-increasing. $(X_i)_{i\in\mathbb{Z}}$ is said to be strongly mixing or α-mixing if $\lim_{n\uparrow\infty} \alpha_{\infty,1}(n) = 0$.

3.1 Mixing criteria for the Borel-Cantelli properties

We start with some criteria when the underlying sequence is β-mixing and $\mu(\limsup_n A_n) > 0$ (see Remark 3.1).

Theorem 3.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables with values in some Polish space E. Denote by μ the common marginal law of the random variables X_i. Assume that $\lim_{n\uparrow\infty} \beta_{\infty,1}(n) = 0$. Let $(A_k)_{k>0}$ be a sequence of Borel sets in E satisfying $\sum_{k>0} \mu(A_k) = +\infty$. Set $B_k = \{X_k \in A_k\}$ for any positive k.*

(i) *If $\mu(\limsup_n A_n) > 0$, then $(B_k)_{k>0}$ is a Borel-Cantelli sequence.*

(ii) *Set $E_n = \sum_{k=1}^n \mu(A_k)$ and $H_n = E_n^{-1} \sum_{k=1}^n \mathbf{1}_{A_k}$. If $(H_n)_{n>0}$ is a uniformly integrable sequence in $(E, \mathcal{B}(E), \mu)$, then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence.*

(iii) *Let Q_{H_n} be the cadlag inverse of the tail function $t \mapsto \mu(H_n > t)$. Set*

$$Q^*(0) = 0 \quad \textit{and} \quad Q^*(u) = u^{-1} \sup_{n>0} \int_0^u Q_{H_n}(s)ds \quad \textit{for any } u \in]0,1]. \tag{3.5}$$

If

$$\sum_{j>0} j^{-1} \beta_{\infty,1}(j) Q^*(\beta_{\infty,1}(j)) < \infty, \tag{3.6}$$

then $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence in $(\Omega, \mathcal{T}, \mathbb{P})$.

Remark 3.1. By the second part of Proposition 2.1 applied with $A_{k,n} = A_k$, if $(H_n)_{n>0}$ is uniformly integrable, then $\mu(\limsup_n A_n) > 0$. Hence (ii) does not apply if $\mu(\limsup_n A_n) = 0$. On another hand, the map $u \mapsto uQ^*(u)$ is non-decreasing. Thus, if $\beta_{\infty,1}(j) > 0$ for any j, (3.6) implies that $\lim_{u \downarrow 0} uQ^*(u) = 0$. Then, by Proposition A.1, $(H_n)_{n>0}$ is uniformly integrable and therefrom $\mu(\limsup_n A_n) > 0$. Consequently, if $\mu(\limsup_n A_n) = 0$, (iii) cannot be applied if $\beta_{\infty,1}(j) > 0$ for any j.

Remark 3.2. If the sequence $(H_n)_{n>0}$ is bounded in $L^p(\mu)$ for some p in $]1, \infty]$, $Q^*(u) = \mathcal{O}(u^{-1/p})$ as u tends to 0. Then, by Proposition A.1, this sequence is uniformly integrable and consequently, by (ii), $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence as soon as $\lim_{n\uparrow\infty} \beta_{\infty,1}(n) = 0$. If furthermore $\sum_{j>0} j^{-1} \beta_{\infty,1}^{1-1/p}(j) < \infty$, then, by (iii), $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence. In particular, if $\mu(A_i \cap A_j) \le C\mu(A_i)\mu(A_j)$ for any (i,j) with $i \ne j$, for some constant C, $(H_n)_{n>0}$ is bounded in $L^2(\mu)$, and consequently $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence as soon as $\sum_{j>0} j^{-1} \sqrt{\beta_{\infty,1}(j)} < \infty$.

Remark 3.3. Let $S_n = \sum_{k=1}^n \mathbf{1}_{A_k}(X_k)$ and $E_n = \mathbb{E}(S_n)$. Inequality (6.31) in the proof of the above theorem applied with $\Gamma_{k,n} = A_k$ gives

$$\limsup_n \mathbb{E}\big(f_n(S_n - E_n)\big) \le 2 \limsup_n \int_E G_n \psi_m d\mu \,,$$

for any $m > 0$, where ψ_m is defined in (6.22), $G_n = S_n/E_n$ and $f_n(x) = f(x/E_n)$. It follows that

$$\limsup_n \mathbb{E}\big(f_n(S_n - E_n)\big) \le 2\|\psi_m\|_\infty$$

for any positive integer m. Now, from inequality (6.22) in the proof of Theorem 3.1, we have $\|\psi_m\|_\infty \le \varphi(\sigma(X_0), \mathcal{F}_{-m})$. Hence, if $\varphi(\sigma(X_0), \mathcal{F}_{-m})$ converges to 0 as m tends to ∞, then $\lim_n \mathbb{E}\big(f_n(S_n - E_n)\big) = 0$ and consequently $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence (see Item (ii) of Proposition 2.3). Similarly, one can prove that, if $\varphi(\sigma(X_0), \mathcal{G}_m)$ converges to 0 as m tends to ∞, then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence. For other results in the φ-mixing setting, see Chapter 1 in Iosifescu and Theodorescu (1969).

Let us now turn to the general case where $\mu(\limsup_n A_n)$ is not necessarily positive. In this case, assuming absolute regularity does not yield any improvement compared to the strong mixing case (see Remark 3.5 after Corollary 3.1). Below, we shall use the following definition of the inverse function associated with some non-increasing sequence of reals.

Definition 3.2. For any non-increasing sequence $(v_n)_{n\in\mathbb{N}}$ of reals, the function v^{-1} is defined by $v^{-1}(u) = \inf\{n \in \mathbb{N} : v_n \le u\} = \sum_{n\ge 0} \mathbf{1}_{\{u<v_n\}}$.

Theorem 3.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables with values in some Polish space E. Let $(\alpha_{\infty,1}(n))_{n\ge 0}$ be its associated sequence of strong-mixing coefficients defined by (3.4). Denote by μ the law of X_0. Let $(A_k)_{k>0}$ be a sequence of Borel sets in E satisfying $\sum_{k>0} \mu(A_k) = +\infty$. Set $B_k = \{X_k \in A_k\}$ for any positive k. Assume that there exist $n_0 > 0$, $C > 0$, $\delta > 0$ and a non-increasing sequence $(\alpha_*(n))_{n\ge 0}$ such that for all $n \ge n_0$,*

$$\alpha_{\infty,1}(n) \le C\alpha_*(n) \ \text{ and } \ \alpha_*(2n) \le (1-\delta)\alpha_*(n)\,. \tag{3.7}$$

Suppose in addition that $(\mu(A_n))_{n\ge 1}$ is a non-increasing sequence,

$$\frac{\mu(A_n)}{\alpha_*(n)} \to \infty \ \text{ as } n \to \infty, \text{ and } \sum_{n\ge 1} \frac{\mu(A_n)}{\alpha_*^{-1}(\mu(A_n))} = \infty\,. \tag{3.8}$$

Then $(B_k)_{k>0}$ is a Borel-Cantelli sequence.

Remark 3.4. Let us first notice that Theorem 3.2 still holds with $\bar\alpha_n$ defined in (1.9) instead of $\alpha_{\infty,1}(n)$ (the proof is unchanged). To compare Theorem 3.2 with Theorem 2.2 (i) in Tasche (1997), let us consider

$$\mu(A_n) \sim C_1 n^{-(r+1)/(r+2)}(\log n)^{-b} \text{ and } \bar\alpha_n \sim C_2 n^{-(r+1)}(\log n)^{-a}$$

with $r \ge -1$. Theorem 2.2 (i) in Tasche (1997) requires $a > 1$ and $b \le 1$ whereas an application of Theorem 3.2 gives the weaker conditions: $(r+2)b \le a + r + 1$ if $r > -1$ and $a > b$ if $r = -1$.

Theorem 3.3. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables with values in some Polish space E. Let $(\alpha_{\infty,1}(n))_{n\ge 0}$ be its associated sequence of strong-mixing coefficients defined by (3.4). Denote by μ the law of X_0. Let $(A_k)_{k>0}$ be a sequence of Borel sets in E satisfying $\sum_{k>0} \mu(A_k) = +\infty$. Set $B_k = \{X_k \in A_k\}$ for any positive k. Let $E_n = \sum_{k=1}^n \mu(A_k)$.*

1. *Let $\eta(x) = x^{-1}\alpha_{\infty,1}([x])$. Assume that $\lim_n E_n^{-1}\eta^{-1}(1/n) = 0$. Then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence.*

2. *Assume that there exist a sequence $(u_n)_{n>0}$ of positive reals such that*

$$\sum_{n>0} \frac{\mu(A_n)}{E_n} u_n < \infty \ \text{ and } \ \sum_{n>0} \frac{\mu(A_n)}{E_n^2} \alpha_{\infty,1}^{-1}(E_n u_n/n) < \infty\,. \tag{3.9}$$

   Then $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence.

We now apply these results to rates of mixing $\mathcal{O}(n^{-a})$ for some positive constant a.

Corollary 3.1. *Let $(A_k)_{k>0}$ be a sequence of Borel sets in E satisfying $\sum_{k>0}\mu(A_k) = +\infty$. For any $k > 0$, let $B_k = \{X_k \in A_k\}$. Assume that there exists $a > 0$ such that $\alpha_{\infty,1}(n) \le Cn^{-a}$, for $n \ge 1$.*

1. *If $\sum_{n\ge1}(\mu(A_n))^{(a+1)/a} = \infty$, $\lim_n n^a\mu(A_n) = \infty$ and $(\mu(A_n))_{n\ge1}$ is non-increasing, then $(B_k)_{k>0}$ is a Borel-Cantelli sequence.*
2. *If $\lim_n n^{-1/(a+1)}E_n = \infty$ then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence.*
3. *If $\sum_{n>0} n^{1/(a+1)}\mu(A_n)E_n^{-2} < \infty$ then $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence.*

Remark 3.5. According to the second item of Remark 5.1, Item 1. of Corollary 3.1 cannot be improved, even in the β-mixing case.

Remark 3.6. Theorems 3.2 and 3.3 (and therefore Corollary 3.1) also hold if the coefficients $\alpha_{\infty,1}(n)$ are replaced by the reversed ones $\alpha_{1,\infty}(n) = \alpha(\sigma(X_0), \mathcal{G}_n)$ (see Section 6.2.3 for a short proof of this remark).

Remark 3.7. Let $\alpha_{1,1}(n) = \alpha(\sigma(X_0), \sigma(X_n))$. From the criteria based on pairwise correlation conditions stated in Annex B, if $\alpha_{1,1}(n) = \mathcal{O}(n^{-a})$ with $a > 1$ then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence if $\lim_n n^{-1/(a+1)}E_n = \infty$ (see Remark B.1), which is the same condition as in Corollary 3.1. Now if $\alpha_{1,1}(n) = \mathcal{O}(n^{-a})$ with $a \in]0,1[$, $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence when $\lim_n n^{-1+a/2}E_n = \infty$ (see Remark B.1), which is more restrictive. Recall that, for Markov chains $\alpha_{\infty,1}(n) = \alpha_{1,1}(n)$. Hence criteria based on pairwise correlation conditions are less efficient in the context of α-mixing Markov chains and slow rates of α-mixing.

4 Weakening the type of dependence

In this section, we consider stationary sequences of real-valued random variables. In order to get more examples than α-mixing or β-mixing sequences, we shall use less restrictive coefficients, where the test functions are indicators of half lines instead of indicators of Borel sets. Some exemples of slowly mixing dynamical systems and non-irreducible Markov chains to which our results apply will be given in Subsection 4.3.

4.1 Definition of the coefficients

Definition 4.1. The coefficients $\tilde\alpha(\mathcal{A}, X)$ and $\tilde\beta(\mathcal{A}, X)$ between a σ-field $\mathcal{A}$ and a real-valued random variable X are defined by

$$\tilde\alpha(\mathcal{A}, X) = \sup_{t\in\mathbb{R}} \|\mathbb{E}(\mathbf{1}_{X\le t}|\mathcal{A}) - \mathbb{P}(X \le t)\|_1 \quad\text{and}\quad \tilde\beta(\mathcal{A}, X) = \left\| \sup_{t\in\mathbb{R}} |\mathbb{E}(\mathbf{1}_{X\le t}|\mathcal{A}) - \mathbb{P}(X \le t)| \right\|_1 .$$

The coefficient $\tilde\varphi(\mathcal{A}, X)$ between $\mathcal{A}$ and X is defined by

$$\tilde\varphi(\mathcal{A}, X) = \sup_{t\in\mathbb{R}} \|\mathbb{E}(\mathbf{1}_{X\le t}|\mathcal{A}) - \mathbb{P}(X \le t)\|_\infty .$$

From this definition it is clear that $\tilde{\alpha}(\mathcal{A},X) \le \tilde{\beta}(\mathcal{A},X) \le \tilde{\varphi}(\mathcal{A},X) \le 1$.

Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real-valued random variables. We now define the dependence coefficients of $(X_i)_{i\in\mathbb{Z}}$ used in this section. The coefficients $\tilde{\alpha}_{\infty,1}(n)$ are defined by

$$\tilde{\alpha}_{\infty,1}(n) = \tilde{\alpha}(\mathcal{F}_0, X_n) \quad \text{for any } n > 0. \tag{4.1}$$

Here $\mathcal{F}_0 = \sigma(X_k : k \le 0)$ (see (3.2)). The coefficients $\tilde{\beta}_{1,1}(n)$ and $\tilde{\varphi}_{1,1}(n)$ are defined by

$$\tilde{\beta}_{1,1}(n) = \tilde{\beta}(\sigma(X_0), X_n) \quad \text{and} \quad \tilde{\varphi}_{1,1}(n) = \tilde{\varphi}(\sigma(X_0), X_n) \text{ for any } n > 0. \tag{4.2}$$

4.2 Results

Theorem 4.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables. Denote by μ the common marginal law of the random variables X_i. Let $(I_k)_{k>0}$ be a sequence of intervals such that $\mu(I_1) > 0$ and $\sum_{k>0} \mu(I_k) = \infty$. Set $B_k = \{X_k \in I_k\}$ for any positive k, and $E_n = \sum_{k=1}^n \mu(I_k)$.*

(i) *If $\mu(\limsup_n I_n) > 0$ and $\sum_{k>0} \tilde{\beta}_{1,1}(k) < \infty$, then $(B_k)_{k>0}$ is a Borel-Cantelli sequence.*

(ii) *Let $p \in [1,\infty)$ and q be the conjugate exponent of p. If*

$$\lim_{n\to\infty} \frac{1}{E_n^p} \sum_{k=1}^{n-1} k^{p-1} \tilde{\beta}_{1,1}(k) = 0 \quad \text{and} \quad \sup_{n>0} \frac{1}{E_n} \Big\| \sum_{k=1}^{n} \mathbf{1}_{I_k}(X_0) \Big\|_q < \infty ,$$

then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence.

(iii) *Let $p \in [1,\infty)$ and q be the conjugate exponent of p. If*

$$\sum_{n>0} \frac{\mu(I_n)}{E_n^2} \left(\sum_{k=1}^{n-1} k^{p-1} \tilde{\beta}_{1,1}(k) \right)^{1/p} < \infty \quad \text{and} \quad \sup_{n>0} \frac{1}{E_n} \Big\| \sum_{k=1}^{n} \mathbf{1}_{I_k}(X_0) \Big\|_q < \infty ,$$

then $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence.

(iv) *If $\lim_{n\to\infty} E_n^{-1} \sum_{k=1}^{n-1} \tilde{\varphi}_{1,1}(k) = 0$, then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence.*

(v) *If $\sum_{n>0} E_n^{-1} \tilde{\varphi}_{1,1}(n) < \infty$, then $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence.*

Remark 4.1. Item (v) on the uniform mixing case can be derived from Theorem 8 and Remark 7 in Chandra and Ghosal (1998). Note that, if $p = 1$, the condition in Item (iii) becomes

$$\sum_{n>0} \frac{\tilde{\beta}_{1,1}(n)}{E_n} < \infty \quad \text{and} \quad \sup_{n>0} \frac{1}{E_n} \left\| \sum_{k=1}^{n} \mathbf{1}_{I_k}(X_0) \right\|_\infty < \infty .$$

Note that, for intervals $(I_k)_{k>0}$ satisfying the condition on right hand, we get the same condition as in (v), but for $\tilde{\beta}_{1,1}(n)$ instead of $\tilde{\varphi}_{1,1}(n)$.

Remark 4.2. Theorem 4.1 remains true if we replace the coefficients $\tilde{\beta}_{1,1}(n)$ (resp. $\tilde{\varphi}_{1,1}(n)$) by $\tilde{\beta}_{1,1}^{\text{rev}}(n) = \tilde{\beta}(\sigma(X_n), X_0)$ (resp. $\varphi_{1,1}^{\text{rev}}(n) = \tilde{\varphi}(\sigma(X_n), X_0)$).

Remark 4.3. *Comparison with usual pairwise correlation criteria.* Let us compare Theorem 4.1 wit the results stated in Annex B in the case $\mu(\limsup_n I_n) > 0$. From the definition of the coefficients $\tilde\beta_{1,1}(n)$,

$$|\mathbb{P}(B_k \cap B_{k+n}) - \mathbb{P}(B_k)\mathbb{P}(B_{k+n})| \le \tilde\beta_{1,1}(n).$$

Hence the assumptions of Proposition B.1 hold true with $\gamma_n = \varphi_n = 0$ and $\alpha_n = \tilde\beta_{1,1}(n)$. In particular, from Proposition B.1(i), if

$$\lim_n E_n^{-2} \sum_{k=1}^n \sum_{j=1}^k \min(\tilde\beta_{1,1}(j), \mu(I_k)) = 0, \tag{4.3}$$

$(B_k)_{k>0}$ is a Borel-Cantelli sequence. For example, if $\tilde\beta_{1,1}(n) = \mathcal{O}(n^{-a})$ for some constant $a > 1$, then, from Remark B.1, (4.3) holds if $\lim_n n^{-1/(a+1)} E_n = \infty$. In contrast Theorem 4.1(i) ensures that $(B_k)_{k>0}$ is Borel-Cantelli sequence as soon as $\sum_{k>0} \tilde\beta_{1,1}(k) < \infty$, without conditions on the sizes of the intervals I_k. Next, if $\tilde\beta_{1,1}(n) = \mathcal{O}(n^{-a})$ for some $a < 1$, then, according to Remark B.1, (4.3) is fulfilled if $\lim_n n^{-1+(a/2)} E_n = \infty$. Under the same condition, Theorem 4.1(ii) ensures that $(B_k)_{k>0}$ is a Borel-Cantelli sequence if, for some real q in $(1, \infty]$,

$$\lim_n n^{-1+(a/p)} E_n = \infty \quad \text{and} \quad \sup_{n>0} \frac{1}{E_n} \Big\| \sum_{k=1}^n \mathbf{1}_{I_k}(X_0) \Big\|_q < \infty\,, \tag{4.4}$$

where $p = q/(q-1)$. Consequently Theorem 4.1(ii) provides a weaker condition on the sizes of the intervals I_k if the sequence $(\sum_{k=1}^n \mathbf{1}_{I_k}(X_0)/E_n)_{n>0}$ is bounded in L^q for some $q > 2$.

As quoted in Remark 3.1, if $\mu(\limsup_n I_n) = 0$ then (i), (ii), (iii) of Theorem 4.1 cannot be applied. Instead, the analogue of Theorems 3.2 and 3.3 and of Corollary 3.1 hold (the proofs are unchanged).

Theorem 4.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables. Denote by μ the common marginal law of the random variables X_i. Let $(I_k)_{k>0}$ be a sequence of intervals such that $\mu(I_1) > 0$ and $\sum_{k>0} \mu(I_k) = \infty$. Set $B_k = \{X_k \in I_k\}$ for any positive k, and $E_n = \sum_{k=1}^n \mu(I_k)$. Then the conclusion of Theorem 3.2 (resp. Theorem 3.3, Corollary 3.1) holds by replacing the conditions on $(\alpha_{\infty,1}(n))_{n>0}$ and $(A_k)_{k>0}$ in Theorem 3.2 (resp. Theorem 3.3, Corollary 3.1) by the same conditions on $(\tilde\alpha_{\infty,1}(n))_{n>0}$ and $(I_k)_{k>0}$.*

Remark 4.4. Theorem 4.2 remains true if we replace the coefficients $\tilde\alpha_{\infty,1}(n)$ by $\tilde\alpha_{1,\infty}(n) = \tilde\alpha(\mathcal{G}_n, X_0)$ where $\mathcal{G}_n = \sigma(X_i, i \ge n)$ (see the arguments given in the proof of Remark 3.6).

4.3 Examples

Example 4.1. Let us consider the so-called LSV map (Liverani, Saussol and Vaienti (1999)) defined as follows:

$$\text{for } 0 < \gamma < 1, \quad \theta(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1]\,. \end{cases} \tag{4.5}$$

Recall that if $\gamma \in]0,1[$, there is only one absolutely continuous invariant probability μ whose density h satisfies $0 < c \leq h(x)/x^{-\gamma} \leq C < \infty$. Moreover, it has been proved in [7], that the $\tilde{\beta}^{\mathrm{rev}}_{1,1}(n)$ coefficients of weak dependence associated with $(\theta^n)_{n\geq 0}$, viewed as a random sequence defined on $([0,1],\mu)$, satisfy $\tilde{\beta}^{\mathrm{rev}}_{1,1}(n) \leq \kappa n^{-(1-\gamma)/\gamma}$ for any $n \geq 1$ and some $\kappa > 0$.

Let us first recall Theorem 1.1 of Gouëzel (2007): let λ be the Lebesgue measure over $[0,1]$ and let $(I_k)_{k>0}$ be a sequence of intervals such that

$$\sum_{k>0} \lambda(I_k) = \infty\,. \tag{4.6}$$

Then $B_n = \{\theta^n \in I_n\}$ is a Borel-Cantelli sequence. If furthermore the intervals I_k are included in $[1/2,1]$ then $B_n = \{\theta^n \in I_n\}$ is a strongly Borel-Cantelli sequence (this follows from inequality (1.3) in [15], and Item (ii) of Proposition B.1.) If (I_n) is a decreasing sequence of intervals included in $(d,1]$ with $d > 0$ satisfying (4.6), then $B_n = \{\theta^n \in I_n\}$ is strongly Borel-Cantelli as shown in Kim (2007, Prop. 4.1).

We consider here two particular cases:

- Consider $I_n = [0, a_n]$ with $(a_n)_{n>0}$ a decreasing sequence of real numbers in $]0,1]$ converging to 0. Set $B_n = \{\theta^n \in I_n\}$. Using the same arguments as in Proposition 4.2 in Kim (2007), one can prove that, if $\sum_{n>0} a_n < \infty$, then $\mu(\limsup_{n\to\infty} B_n) = 0$. Conversely, if $\sum_{n>0} a_n = \infty$, which is exactly condition (4.6), then $(B_n)_{n\geq 1}$ is a Borel-Cantelli sequence.

  Now, to apply Theorem 4.2 (and its Remark 4.4), we first note that it has been proved in [8], that the $\tilde{\alpha}_{1,\infty}(n)$ coefficients of weak dependence associated with $(\theta^n)_{n\geq 0}$, viewed as a random sequence defined on $([0,1],\mu)$, satisfy $\kappa_1 n^{-(1-\gamma)/\gamma} \leq \tilde{\alpha}_{1,\infty}(n) \leq \kappa_2 n^{-(1-\gamma)/\gamma}$ for any $n \geq 1$ and some positive constants κ_1 and κ_2. Hence, in that case, Theorem 4.2 gives the same condition (4.6) for the Borel-Cantelli property, up to the mild additional assumption $n^{1/\gamma} a_n \to \infty$. This shows that the approach based on the $\tilde{\alpha}_{1,\infty}(n)$ dependence coefficients provides optimal results in this case. Now, if $n a_n \to \infty$, then $(B_n)_{n\geq 1}$ is a L^1 Borel-Cantelli sequence. Finally, if $\sum_{n\geq 1} n^{-1}(n a_n)^{\gamma-1} < \infty$, then $(B_n)_{n\geq 1}$ is a strongly Borel-Cantelli sequence.

- Let now $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers in $[0,1]$ such that $a_0 > 0$ and $b_{n+1} = b_n + a_n \mod 1$. Define, for any $n \in \mathbb{N}$, $I_{n+1} = [b_n, b_{n+1}]$ if $b_n < b_{n+1}$ and $I_{n+1} = [b_n, 1] \cup [0, b_{n+1}]$ if $b_{n+1} < b_n$. It follows that $(I_n)_{n\geq 1}$ is a sequence of consecutive intervals on the torus $\mathbb{R}/\mathbb{Z}$. Assume that $\sum_{n\in\mathbb{N}} a_n = \infty$ (which is exactly (4.6)). Since $\mu(I_{n+1}) \geq C a_n$, the divergence of the series implies that $\sum_{n>0} \mu(I_n) = \infty$. Applying Theorem 4.1 (iii), it follows that for any $\gamma < 1/2$, $(B_n)_{n\geq 1}$ is a strongly Borel-Cantelli sequence. Now if $\gamma = 1/2$, applying Theorem 4.1 (ii) and (iii) with $p = 1$, we get that $(B_n)_{n\geq 1}$ is a L^1 Borel-Cantelli sequence as soon as $(\sum_{k=1}^n a_k)/\log(n) \to \infty$, and a strongly Borel-Cantelli sequence as soon as $(\sum_{k=1}^n a_k)/(\log(n))^{2+\varepsilon} \to \infty$ for some $\varepsilon > 0$. If $\gamma > 1/2$, we get that $(B_n)_{n\geq 1}$ is a L^1 Borel-Cantelli sequence as soon as $(\sum_{k=1}^n a_k)/n^{(2\gamma-1)/\gamma} \to \infty$, and a strongly Borel-Cantelli sequence as soon as $(\sum_{k=1}^n a_k)/(n^{(2\gamma-1)/\gamma}(\log(n))^{1+\varepsilon}) \to \infty$ for some $\varepsilon > 0$.

Example 4.2. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of iid random variables with values in $\mathbb{R}$, such that $\mathbb{E}(\log(1+|\varepsilon_0|)) < \infty$. We consider here the stationary process

$$X_k = \sum_{i\geq 0} 2^{-i}\varepsilon_{k-i}\,, \tag{4.7}$$

which is defined almost surely (this is a consequence of the three series theorem). The process $(X_k)_{k\geq 0}$ is a Markov chain, since $X_{n+1} = \frac{1}{2}X_n + \varepsilon_{n+1}$. However this chain fails to be irreducible when the innovations are with values in $\mathbb{Z}$. Hence the results of Sections 3 and 5 cannot be applied in general. Nevertheless, under some mild additional conditions, the coefficients $\tilde\beta_{1,1}(n)$ of this chain converge to 0 as shown by the lemma below.

Lemma 4.1. *Let μ be the law of X_0. Assume that μ has a bounded density. If*

$$\sup_{t>0} t^p\mathbb{P}(\log(1+|\varepsilon_0|) > t) < \infty \quad \textit{for some } p > 1\,, \tag{4.8}$$

then $\tilde\beta_{1,1}(n) = \mathcal{O}(n^{-(p-1)/2})$.

Remark 4.5. The assumption that μ has a bounded density can be verified in many cases. For instance, it is satisfied if $\varepsilon_i = \xi_i + \eta_i$ where (ξ_i) and (η_i) are two independent sequences of iid random variables, and ξ_0 has the Bernoulli(1/2) distribution. Indeed, in that case, $X_0 = U_0 + Z_0$ with $U_0 = \sum_{i=0}^\infty 2^{-i}\xi_{-i}$ and $Z_0 = \sum_{i=0}^\infty 2^{-i}\eta_{-i}$. Since U_0 is uniformly distributed over $[0,2]$, it follows that the density of μ is uniformly bounded by 1/2.

Since $(X_k)_{k\in\mathbb{Z}}$ is a stationary Markov chain, $\tilde\alpha_{\infty,1}(n) \leq \tilde\beta_{1,1}(n)$. Hence, under the assumptions of Lemma 4.1, we also have that $\tilde\alpha_{\infty,1}(n) = \mathcal{O}(n^{-(p-1)/2})$. Let then $B_n = \{X_n \in I_n\}$. As a consequence, we infer from Lemma 4.1, Theorems 4.1 and 4.2 that

- If $\mu(\limsup_n I_n) > 0$, μ has a bounded density and (4.8) holds for some $p > 3$, then $(B_n)_{n\geq 1}$ is a Borel-Cantelli sequence.
- If μ has a bounded density, (4.8) holds, $\sum_{n\geq 1}\big(\mu(I_n)\big)^{(p+1)/(p-1)} = \infty$, $(\mu(I_n))_{n\geq 1}$ is non-increasing, and $\lim_n n^{(p-1)/2}\mu(I_n) = \infty$, then $(B_n)_{n\geq 1}$ is a Borel-Cantelli sequence.

Example 4.3. We consider the symmetric random walk on the circle, whose Markov kernel is defined by

$$Kf(x) = \frac{1}{2}\big(f(x+a) + f(x-a)\big) \tag{4.9}$$

on the torus $\mathbb{R}/\mathbb{Z}$ with a irrational in $[0,1]$. The Lebesgue-Haar measure λ is the unique probability which is invariant by K. Let $(X_i)_{i\in\mathbb{N}}$ be the stationary Markov chain with transition kernel K and invariant distribution λ. We assume that a is badly approximable in the weak sense meaning that, for any positive ϵ, there exists some positive constant c such that

$$d(ka,\mathbb{Z}) \geq c|k|^{-1-\epsilon} \quad \text{for any } k > 0. \tag{4.10}$$

From Roth's theorem the algebraic numbers are badly approximable in the weak sense (see for instance Schmidt [26]). Note also that the set of numbers in $[0,1]$ satisfying (4.10) has Lebesgue measure 1. For this chain, we will obtain the bound below on the coefficients $\tilde{\beta}_{1,1}(n)$.

Lemma 4.2. *Let a be badly approximable in the weak sense, and let $(X_i)_{i\in\mathbb{N}}$ be the stationary Markov chain with transition kernel K and invariant distribution λ. Then, for any b in $(0,1/2)$, $\tilde{\beta}_{1,1}(n) = \mathcal{O}(n^{-b})$.*

Since $(X_k)_{k\in\mathbb{Z}}$ is a stationary Markov chain, $\tilde{\alpha}_{\infty,1}(n) \leq \tilde{\beta}_{1,1}(n)$. Hence, under the assumptions of Lemma 4.2, $\tilde{\alpha}_{\infty,1}(n) = \mathcal{O}(n^{-b})$ for any b in $(0,1/2)$. As a consequence, we infer from Lemma 4.2, Theorems 4.1 and 4.2 the corollary below on the symmetric random walk on the circle with linear drift.

Corollary 4.1. *Let t be a real in $[0,1[$. Set $Y_k = X_k - kt$. For any positive integer n, let $I_n = [0, n^{-\delta}]$. Set $B_n = \{Y_n \in I_n\}$. If $\delta < 1/3$, $(B_n)_{n\geq 1}$ is a strongly Borel-Cantelli sequence for any t in $[0,1[$. Now, if t is badly approximable in the strong sense, which means that (4.10) holds with $\epsilon = 0$, $(B_n)_{n\geq 1}$ is a strongly Borel-Cantelli sequence for any $\delta < 1/2$.*

5 Harris recurrent Markov chains

In this section, we are interested in the Borel-Cantelli lemma for irreducible and positively recurrent Markov chains. Let E be a Polish space and $\mathcal{B}$ be its Borel σ-field. Let P be a stochastic kernel. We assume that there exists a measurable function s with values in $[0,1]$ and a probability measure ν such that $\nu(s) > 0$ and

$$P(x,A) \geq s(x)\nu(A) \text{ for any } (x,A) \in E \times \mathcal{B}. \tag{5.1}$$

Then the chain is aperiodic and irreducible. Let us then define the sub-stochastic kernel Q by

$$Q(x,A) = P(x,A) - s(x)\nu(A) \text{ for any } (x,A) \in E \times \mathcal{B}. \tag{5.2}$$

Throughout this section, we assume furthermore that

$$\sum_{n\geq 0} \nu Q^n(1) < \infty. \tag{5.3}$$

Then the probability measure

$$\mu = \Big(\sum_{n\geq 0} \nu Q^n(1)\Big)^{-1} \sum_{n\geq 0} \nu Q^n \tag{5.4}$$

is the unique invariant probability measure under P. Furthermore the stationary Markov chain $(X_i)_{i\in\mathbb{N}}$ with kernel P is positively recurrent (see Rio (2017), Chapter 9 for more details) and β-mixing according to Corollary 6.7 (ii) in Nummelin (1984). Thus a direct application of Theorem 3.1 (i) gives the following result.

Theorem 5.1. *Let $(A_k)_{k>0}$ be a sequence of Borel subsets of E such that $\mu(\limsup_n A_n) > 0$. Then $\sum_{k>0} \mathbf{1}_{A_k}(X_k) = \infty$ a.s.*

Obviously the result above does not apply in the case where the events are nested and $\lim_n \mu(A_n) = 0$. However in this case, the regeneration technique can be applied to prove the following result.

Theorem 5.2. *Let $(A_k)_{k>0}$ be a sequence of Borel subsets of E such that $\sum_{k>0} \nu(A_k) = \infty$ and $A_{k+1} \subset A_k$ for any positive k. Then $\sum_{k>0} \mathbf{1}_{A_k}(X_k) = \infty$ a.s.*

Suppose now that $\mu(\limsup_n A_n) = 0$ and that the events $(A_n)_{n\geq 1}$ are not necessarily nested. Then applying Corollary 3.1 and using Proposition 9.7 in Rio (2017) applied to arithmetic rates of mixing (see Rio (2017) page 164 and page 165 lines 8-11), we derive the following result:

Theorem 5.3. *Let T_0 be the first renewal time of the extended Markov chain (see (6.75) for the exact definition). Assume that there exists $a > 1$ such that $\mathbb{P}_\mu(T_0 > n) \leq Cn^{-a}$ for $n \geq 1$. Suppose furthermore that $(A_k)_{k>0}$ is a sequence of Borel subsets of E such that $\sum_{n\geq 1}(\mu(A_n))^{(a+1)/a} = \infty$, $\lim_n n^a \mu(A_n) = \infty$ and $(\mu(A_n))_{n\geq 1}$ is non-increasing. Then $\sum_{k>0} \mathbf{1}_{A_k}(X_k) = \infty$ a.s.*

If the stochastic kernel $Q_1(x, .)$ defined in (6.72) is equal to δ_x, then Theorem 5.2 cannot be further improved, as shown in Theorem 5.4 below

Theorem 5.4. *Let E be a Polish space. Let ν be a probability measure on E and s be a measurable function with values in $]0, 1]$ such that $\nu(s) > 0$. Suppose furthermore that*

$$\int_E \frac{1}{s(x)} d\nu(x) < \infty. \tag{5.5}$$

Let

$$P(x, .) = s(x)\nu + (1 - s(x))\delta_x. \tag{5.6}$$

Then P is irreducible, aperiodic and positively recurrent. Let $(X_i)_{i\in\mathbb{N}}$ denote the strictly stationary Markov chain with kernel P and $(A_k)_{k>0}$ be a sequence of Borel subsets of E such that $\sum_{k>0} \nu(A_k) < \infty$ and $A_{k+1} \subset A_k$ for any positive k. Then $\sum_{k>0} \mathbf{1}_{A_k}(X_k) < \infty$ a.s.

Remark 5.1. Let us compare Theorems 5.2 and 5.3 when P is the Markov kernel defined by (5.6) with $E = [0, 1]$, $s(x) = x$ and $\nu = (a + 1)x^a\lambda$ with $a > 0$ (here λ is the Lebesgue measure on $[0, 1]$). For this example, $\mu = ax^{a-1}\lambda$ and $\mathbb{P}_\mu(T_0 > n) \sim a\Gamma(a)n^{-a}$. Furthermore, from Lemma 2, page 75 in Doukhan, Massart and Rio (1994), if $(\beta_n)_{n>0}$ denotes the sequence of β-mixing coefficients of the stationary Markov chain with kernel P, then

$$a\Gamma(a) \leq \liminf_n n^a \beta_n \leq \limsup_n n^a \beta_n \leq 3a\Gamma(a)2^a.$$

Now, for any $k \geq 1$, let $A_k = I_k =]a_k^{1/a}, b_k^{1/a}]$.

- Assume that $I_{k+1} \subset I_k$, which means that (a_k) is non-decreasing and (b_k) is non-increasing. Then Theorem 5.2 applies if $\sum_{k>0}(b_k^{(a+1)/a} - a_k^{(a+1)/a}) = \infty$ whereas Theorem 5.3 applies if $\lim_n n^a(b_n - a_n) = \infty$ and $\sum_{k>0}(b_k - a_k)^{(a+1)/a} = \infty$. Note that the first condition is always weaker than the second one. Note also that, if $\lim_k a_k > 0$, the first condition is equivalent to $\sum_{k>0}(b_k - a_k) = \infty$, which is then strictly weaker than $\sum_{k>0}(b_k - a_k)^{(a+1)/a} = \infty$. Since $(b_k - a_k) = \mu(I_k) = \mathbb{P}(X_k \in I_k)$, the condition $\sum_{k>0}(b_k - a_k) = \infty$ is the best possible for the Borel-Cantelli property (this is due to the direct part of the Borel-Cantelli lemma).

- Assume now that $a_k \equiv 0$ and (b_k) is non-increasing. In that case, $\nu(I_k) = (\mu(I_k))^{(a+1)/a}$, for any $k \geq 1$. According to Theorem 5.4, it follows that $\sum_{n\geq 1}(\mu(I_n))^{(a+1)/a} = \infty$ is a necessary condition to get the Borel-Cantelli property.

- Assume now that $I_k =]a_k^{1/a}, (2a_k)^{1/a}] \subset [0,1]$ with $(a_k)_k \downarrow 0$. Since $I_{k+1} \not\subset I_k$ in this case, Theorem 5.2 does not apply whereas the conditions of Theorem 5.3 hold provided that $\lim_n n^a a_n = \infty$ and $\sum_{k>0} a_k^{(a+1)/a} = \infty$.

6 Proofs

6.1 Proofs of the results of Section 2

6.1.1 Proof of Proposition 2.1.

We start by showing that 2. $\Rightarrow$ 1. Let $\Gamma = \limsup_k \Gamma_k$. It suffices to prove that $\mathbb{P}(\Gamma) \geq \delta$. Note first that

$$\left\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} \right\|_1 = 1 \quad \text{and} \quad \limsup_n \left\| \frac{\mathbf{1}_\Gamma \sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} \right\|_1 \leq \delta^{-1}\mathbb{P}(\Gamma)\,,$$

by (2.1). Hence it is enough to prove that

$$\lim_n \left\| \mathbf{1}_{\Gamma^c} \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} \right\|_1 = 0\,.$$

This follows directly from (2.1) and the fact that, by definition of the lim sup and since $\sum_{k>0} \mathbb{P}(\Gamma_k) = +\infty$,

$$\lim_n \mathbf{1}_{\Gamma^c} \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} = 0 \ \mathbb{P}\text{-a.s.}$$

We prove now that 1. $\Rightarrow$ 2. Proceeding by induction on k one can construct an increasing sequence $(n_k)_{k\geq 0}$ of integers such that $n_0 = 1$ and

$$\mathbb{P}\Big(\bigcup_{j=n_{k-1}}^{n_k - 1} A_j \Big) \geq \delta(1 - 2^{-k}) \quad \text{for any } k > 0. \tag{6.1}$$

Define now the sequence $(\Gamma_j)_{j>0}$ of Borel sets by

$$\Gamma_{n_k} = A_{n_k} \ \text{ and } \Gamma_j = A_j \setminus \Big(\bigcup_{i=n_k}^{j-1} A_i\Big) \ \text{ for any } j \in]n_k, n_{k+1}[, \ \text{ for any } k \geq 0 \,.$$

From the definition of $(\Gamma_j)_{j>0}$

$$\sum_{i=n_k}^{n_{k+1}-1} \mathbf{1}_{\Gamma_i} = \mathbf{1}_{\left(\bigcup_{i\in[n_k,n_{k+1}[} \Gamma_i\right)} = \mathbf{1}_{\left(\bigcup_{i\in[n_k,n_{k+1}[} A_i\right)} \leq 1 \ \text{ for any } k \geq 0 \,.$$

Consequently, for any $j \geq 0$ and any n in $[n_j, n_{j+1}[$,

$$\sum_{i=1}^{n} \mathbf{1}_{\Gamma_i} \leq \sum_{k=0}^{j}\Big(\sum_{i=n_k}^{n_{k+1}-1} \mathbf{1}_{\Gamma_i}\Big) \leq j+1 \,.$$

Furthermore, from (6.1),

$$\sum_{i=1}^{n} \mathbb{P}(\Gamma_i) \geq \sum_{k=1}^{j} \mathbb{P}\Big(\bigcup_{i=n_{k-1}}^{n_k-1} A_i\Big) \geq (j-1)\delta$$

for any $j \geq 1$ and any n in $[n_j, n_{j+1}[$. Hence, if $G_n = \big(\sum_{i=1}^n \mathbb{P}(\Gamma_i)\big)^{-1} \sum_{i=1}^n \mathbf{1}_{\Gamma_i}$, then $G_n \leq (j+1)/((j-1)\delta)$ for n in $[n_j, n_{j+1}[$, which ensures that $\limsup_n G_n \leq 1/\delta$.

We now prove the second part of Proposition 2.1. Suppose that there exists a triangular sequence of events $(A_{k,n})_{1\leq k\leq n}$ with $A_{k,n} \subset A_k$, such that $\tilde{E}_n = \sum_{k=1}^n \mathbb{P}(A_{k,n}) \to \infty$ and that the sequence $(Z_n)_{n\geq 1}$ defined by $Z_n = \tilde{E}_n^{-1} \sum_{k=1}^n \mathbf{1}_{A_{k,n}}$ is uniformly integrable. Set $C_N = \bigcup_{k>N} A_k$. For any $n > N$,

$$\mathbb{E}(Z_n) = \mathbb{E}\big(Z_n \mathbf{1}_{C_N^c}\big) + \mathbb{E}\big(Z_n \mathbf{1}_{C_N}\big) \leq (N/\tilde{E}_n) + \mathbb{E}\big(Z_n \mathbf{1}_{C_N}\big),$$

since $\sum_{k=1}^n \mathbf{1}_{A_{k,n}} \leq N$ on C_N^c. Using Lemma 2.1 (a) in Rio (2017), it follows that

$$1 = \mathbb{E}(Z_n) \leq (N/\tilde{E}_n) + \int_0^1 Q_{Z_n}(u) Q_{\mathbf{1}_{C_N}}(u) du \leq (N/\tilde{E}_n) + \sup_{n>0} \int_0^{\mathbb{P}(C_N)} Q_{Z_n}(u) du,$$

where Q_Z denotes the cadlag inverse of the tail function $t \mapsto \mathbb{P}(Z > t)$. Hence,

$$1 = \lim_n \mathbb{E}(Z_n) \leq \sup_{n>0} \int_0^{\mathbb{P}(C_N)} Q_{Z_n}(u) du.$$

Now, if $\mathbb{P}(\limsup_k A_k) = 0$, then $\lim_N \mathbb{P}(C_N) = 0$. If furthermore $(Z_n)_{n>0}$ is uniformly integrable, then, by Proposition A.1, the term on right hand in the above inequality tends to 0 as N tends to ∞, which is a contradiction. The proof of Proposition 2.1 is complete. $\diamond$

6.1.2 Proof of Corollary 2.1.

The fact that 2. implies 1. is immediate. Now, if 1. holds true, then, by Proposition 2.1, there exists a sequence $(\Gamma_k)_{k>0}$ of events such that $\Gamma_k \subset A_k$, $\sum_{k>0} \mathbb{P}(\Gamma_k) = +\infty$ and (2.1) holds with $\delta = 1$. Since $\| \sum_{k=1}^n \mathbf{1}_{\Gamma_k} / \sum_{k=1}^n \mathbb{P}(\Gamma_k) \|_1 = 1$, it follows that

$$\lim_n \left\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} \right\|_\infty = 1\,. \tag{6.2}$$

Now

$$\left\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} - 1 \right\|_1 = 2 \left\| \left(\frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} - 1 \right)_+ \right\|_1 \le 2 \left(\left\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mathbb{P}(\Gamma_k)} \right\|_\infty - 1 \right)_+ ,$$

which, together with (6.2), implies that the above sequence $(\Gamma_k)_{k>0}$ is a L^1 Borel-Cantelli sequence. Hence Corollary 2.1 holds. $\diamond$

6.1.3 Proof of Proposition 2.2.

The fact that 2. $\Rightarrow$ 1. follows immediately from Proposition 2.1. We now prove the direct part. Proceeding by induction on k one can construct an increasing sequence $(n_k)_{k\ge 0}$ of integers such that $n_0 = 1$ and

$$\mu\Big(\bigcup_{j=n_{k-1}}^{n_k - 1} I_j \Big) \ge \delta(1 - 2^{-k}) \ \text{ for any } k > 0. \tag{6.3}$$

Now, for any $k \ge 0$, we construct the intervals Γ_j for j in $[n_k, n_{k+1}[$. This will be done by using the lemma below.

Lemma 6.1. *Let $(J_k)_{k\in[1,m]}$ be a sequence of intervals of $\mathbb{R}$. Then there exists a sequence $(\Gamma_k)_{k\in[1,m]}$ of disjoint intervals such that $\bigcup_{k=1}^m \Gamma_k = \bigcup_{k=1}^m J_k$ and $\Gamma_k \subset J_k$ for any k in $[1, m]$.*

Proof of Lemma 6.1. We prove the Lemma by induction on m. Clearly the result holds true for $m = 1$. Assume now that Lemma 6.1 holds true at range m. Let then $(J_k)_{k\in[1,m+1]}$ be a sequence of intervals. By the induction hypothesis, there exists a sequence $(\Gamma_{k,m})_{1\le k\le m}$ of disjoint intervals such that $\bigcup_{k=1}^m \Gamma_{k,m} = \bigcup_{k=1}^m J_k$ and $\Gamma_{k,m} \subset J_k$ for any k in $[1, m]$. Now, at the range $m + 1$, define now the intervals Γ_k for k in $[1, m]$ by $\Gamma_k = \emptyset$ if $\Gamma_{k,m} \subset J_{m+1}$ and $\Gamma_k = \Gamma_{k,m}$ if $\Gamma_{k,m} \not\subset J_{m+1}$. Clearly these intervals are disjoint. Set

$$\Gamma_{m+1} = \bigcap_{k=1}^m \Big(\Gamma_k^c \cap J_{m+1} \Big). \tag{6.4}$$

If $\Gamma_k = \emptyset$, then $\Gamma_k^c \cap J_{m+1} = J_{m+1}$. Otherwise, from the definition of Γ_k, Γ_k is a nonempty interval and $\Gamma_k \not\subset J_{m+1}$, which implies that $\Gamma_k^c \cap J_{m+1}$ is an interval. Hence Γ_{m+1} is a finite intersection of intervals, which ensures that Γ_{m+1} is an interval. By 6.4, Γ_{m+1} does not intersect Γ_k for any k in $[1, m]$. Hence the so defined intervals Γ_k are disjoint, $\Gamma_k \subset J_k$ for any k in $[1, m+1]$. Finally

$$\bigcup_{k=1}^{m+1} \Gamma_k = J_{m+1} \bigcup \Big(\bigcup_{k=1}^{m} \Gamma_k \Big) = J_{m+1} \bigcup \Big(\bigcup_{k=1}^{m} \Gamma_{k,m} \Big) = J_{m+1} \bigcup \Big(\bigcup_{k=1}^{m} J_k \Big) \tag{6.5}$$

Hence, if Lemma 6.1 holds true at range m, then Lemma 6.1 holds true at range $m+1$, which ends the proof of the lemma. $\diamond$

End of the proof of Proposition 2.2. For any $k \geq 0$, by Lemma 6.1 applied to $(I_j)_{j \in [n_k, n_{k+1}[}$, there exists a sequence $(\Gamma_j)_{j \in [n_k, n_{k+1}[}$ of disjoint intervals such that

$$\bigcup_{j \in [n_k, n_{k+1}[} \Gamma_j = \bigcup_{j \in [n_k, n_{k+1}[} I_j \quad \text{and } \Gamma_j \subset I_j \ \text{ for any } j \in [n_k, n_{k+1}[. \tag{6.6}$$

From now on the end of the proof is exactly the same as the end of the proof of the first part of Proposition 2.1. $\diamond$

6.1.4 Proof of Proposition 2.3.

We start by proving Item (ii). Let f be the function defined in Definition 2.4 and X be any integrable real-valued random variable. Then

$$\|X\|_1 \leq \|X \mathbf{1}_{|X| \leq 1}\|_2 + \|X \mathbf{1}_{|X|>1}\|_1 \leq \sqrt{2\mathbb{E}(f(X))} + 2\mathbb{E}(f(X)). \tag{6.7}$$

Consequently, if (2.3) holds, then $\lim_{n\to\infty} \|(S_n - E_n)/E_n\|_1 = 0$, which proves Item (ii).

Proof of Item (i). Applying (6.7), we get that $\lim_{k\to\infty} \|(\tilde{S}_{n_k}/\tilde{E}_{n_k}) - 1\|_1 = 0$. Hence, by the Markov inequality, $\lim_{k\to\infty} \mathbb{P}(\tilde{S}_{n_k} \leq \tilde{E}_{n_k}/2) = 0$, which proves that $\tilde{S}_{n_k}$ converges to ∞ in probability as k tends to ∞. Now $g_{j,n_k} \leq \mathbf{1}_{B_j}$ any j in $[1, n_k]$. Therefrom $\tilde{S}_{n_k} \leq S_{n_k}$ and consequently S_{n_k} converges to ∞ in probability as k tends to ∞. Since $(S_n)_n$ is a non-decreasing sequence of random variables, it implies immediately that $\lim_{n\to\infty} S_n = +\infty$ almost surely, which completes the proof of Item (i).

Proof of Item (iii). For any non-negative real x, define $E : x \mapsto E(x) = \mathbb{E}(S_{[x]})$. E is a non-decreasing and cadlag function defined on $\mathbb{R}^+$ with values in $\mathbb{R}^+$. Let E^{-1} be its generalized inverse on $\mathbb{R}^+$ defined by $E^{-1}(u) = \inf\{x \in \mathbb{R}^+ : E(x) \geq u\}$. Hence

$$x \geq E^{-1}(u) \iff E(x) \geq u. \tag{6.8}$$

Note that $E([x]) = E_{[x]}$. Let $\tau_n = \alpha^n$ for a fixed $\alpha > 1$ and define

$$m_n = E^{-1}(\tau_n) = \inf\{k \geq 1 : E(k) \geq \tau_n\}.$$

Hence $(m_n)_{n\geq 1}$ is a non-decreasing sequence of integers. Note also that there exists a positive integer n_0 depending on α such that, for any $n \geq n_0$, $m_n < m_{n+1}$. Indeed, let assume that there exists $n \geq n_0$ such that $m_n = m_{n+1}$. By definition $E(m_n - 1) < \alpha^n$ and $E(m_n) = E(m_{n+1}) \geq \alpha^{n+1}$. This implies that

$$\alpha^{n+1} \leq E(m_n - 1) + \mathbb{P}(B_{m_n}) < \alpha^n + 1\,.$$

Since $\alpha > 1$, there exists an integer n_0 such that the above inequality fails to hold for any $n \geq n_0$. This contradicts the fact that there exists $n \geq n_0$ such that $m_n = m_{n+1}$. Let us then show that

$$(S_{m_n}/E_{m_n}) \to 1 \text{ almost surely, as } n \to \infty\,. \tag{6.9}$$

By the first part of the Borel-Cantelli lemma, (6.9) will hold provided that

$$\sum_{n \geq n_0} \mathbb{E}\big(f\big((S_{m_n} - E_{m_n})/E_{m_n}\big)\big) < \infty\,. \tag{6.10}$$

Hence, setting, for any real $b > 0$,

$$f^*(x/b) := \sup_{1 \leq k \leq [x]} \mathbb{E}\big(f\big((S_k - E_k)/b\big)\big)\,,$$

to prove (6.10), it suffices to show that

$$\sum_{n \geq n_0} f^*(m_n/E_{m_n}) < \infty\,. \tag{6.11}$$

Write

$$\begin{aligned}
\sum_{n \geq n_0} f^*(m_n/E_{m_n}) &= \sum_{n \geq n_0} \sum_{k=m_n+1}^{m_{n+1}} \mathbb{P}(B_k) f^*(m_n/E_{m_n}) \Big(\sum_{k=m_n+1}^{m_{n+1}} \mathbb{P}(B_k) \Big)^{-1} \\
&\leq \sum_{n \geq n_0} \sum_{k=m_n+1}^{m_{n+1}} \mathbb{P}(B_k) f^*(k/E_{m_n}) \Big(\sum_{k=m_n+1}^{m_{n+1}} \mathbb{P}(B_k) \Big)^{-1}.
\end{aligned}$$

Note now that, for any real $a \geq 1$, $f(ax) \leq a^2 f(x)$. Therefore

$$\sum_{n \geq n_0} f^*(m_n/E_{m_n}) \leq \sum_{n \geq n_0} \sum_{k=m_n+1}^{m_{n+1}} \mathbb{P}(B_k)(E_k/E_{m_n})^2 f^*(k/E_k) \Big(\sum_{\ell=m_n+1}^{m_{n+1}} \mathbb{P}(B_\ell) \Big)^{-1}.$$

Next, for any $k \leq m_{n+1}$, $E_k \leq E_{m_{n+1}} < \tau_{n+1} + \mathbb{P}(B_{m_{n+1}}) \leq \alpha^{n+1} + 1$, $E_{m_n} \geq \alpha^n$ and

$$\sum_{\ell=m_n+1}^{m_{n+1}} \mathbb{P}(B_\ell) = E_{m_{n+1}} - E_{m_n} \geq \alpha^{n+1} - (\alpha^n + \mathbb{P}(B_{m_n})) \geq \alpha^n(\alpha - 1) - 1 \geq \alpha^n(\alpha - 1)/2\,,$$

for any $n \geq n_1$. Hence, for any $n \geq n_1$ and any $1 \leq k \leq m_{n+1}$,

$$\sum_{\ell=m_n+1}^{m_{n+1}} \mathbb{P}(B_\ell) \geq \frac{\alpha^{n+1}(\alpha - 1)}{2\alpha} \geq \frac{(\alpha^{n+1} + 1)(\alpha - 1)}{4\alpha} \geq \frac{E_k(\alpha - 1)}{4\alpha}\,.$$

So, overall, setting $n_2 = \max(n_0, n_1)$,

$$\sum_{n\geq n_2} f^*(m_n/E_{m_n}) \leq \frac{4\alpha(\alpha+1)^2}{\alpha-1} \sum_{n\geq n_2} \sum_{k=m_n+1}^{m_{n+1}} \frac{\mathbb{P}(B_k)}{E_k} f^*(k/E_k) =: C_\alpha \sum_{k\geq m_{n_2}} \frac{\mathbb{P}(B_k)}{E_k} f^*(k/E_k)\,,$$

proving (6.11) (and subsequently (6.9)) under (2.4). The rest of the proof is quite usual but we give it for completeness. Since $(S_n)_{n\geq 1}$ is a non-decreasing sequence as well as the normalizing sequence $(E_n)_{n\geq 1}$, if $1 < m_n \leq k \leq m_{n+1}$,

$$\frac{E_{m_n}}{E_{m_{n+1}}} \frac{S_{m_n}}{E_{m_n}} \leq \frac{S_k}{E_k} \leq \frac{E_{m_{n+1}}}{E_{m_n}} \frac{S_{m_{n+1}}}{E_{m_{n+1}}}\,.$$

But, for any positive integer k, $\alpha^k \leq E_{m_k} < \alpha^k + \mathbb{P}(B_k)$. Therefore $E_{m_{n+1}}/E_{m_n} \to \alpha$, as $n \to \infty$. Hence, by using (6.9), almost surely,

$$(1/\alpha) \leq \liminf_{k\to\infty} (S_k/E_k) \leq \limsup_{k\to\infty} (S_k/E_k) \leq \alpha\,.$$

Taking the intersection of all such events for rationals $\alpha > 1$, Item (iii) follows. $\diamond$

6.2 Proofs of the results of Section 3

6.2.1 Proof of Theorem 3.1 (β-mixing case)

Throughout this section, $\beta_j = \beta_{\infty,1}(j)$. Items (i) and (ii) will be derived from the proposition below.

Proposition 6.1. *With the notations of Theorem 3.1, let $(\Gamma_{k,n})_{1\leq k\leq n}$ be a double array of Borel sets in E. Set $\tilde{E}_n = \sum_{k=1}^n \mu(\Gamma_{k,n})$ and $G_n = \tilde{E}_n^{-1} \sum_{k=1}^n \mathbf{1}_{\Gamma_{k,n}}$. Suppose that $\tilde{E}_n > 0$ for any positive n, $\lim_{n\uparrow\infty} \tilde{E}_n = \infty$ and $(G_n)_{n>0}$ is a uniformly integrable sequence in $(E, \mathcal{B}(E), \mu)$. Let $B_{k,n} = \{X_k \in \Gamma_{k,n}\}$ and $\tilde{S}_n = \sum_{k=1}^n \mathbf{1}_{B_{k,n}}$. If $\lim_{n\uparrow\infty} \beta_n = 0$, then*

$$\lim_{n\to\infty} \|(\tilde{S}_n - \tilde{E}_n)/\tilde{E}_n\|_1 = 0. \tag{6.12}$$

Proof of Proposition 6.1. From (6.7), it is enough to prove that

$$\lim_{n\to\infty} \mathbb{E}\big(f_n(\tilde{S}_n - \tilde{E}_n)\big) = 0, \text{ where } f_n(x) = f(x/\tilde{E}_n). \tag{6.13}$$

Now, by setting $\tilde{S}_0 = \tilde{E}_0 = 0$, we first write

$$f_n(\tilde{S}_n - \tilde{E}_n) = \sum_{k=1}^n \big(f_n(\tilde{S}_k - \tilde{E}_k)) - f_n(\tilde{S}_{k-1} - \tilde{E}_{k-1})\big). \tag{6.14}$$

Let then $T_0 = 0$ and, for $k > 0$,

$$T_k = \tilde{S}_k - \tilde{E}_k,\ \xi_k = T_k - T_{k-1} = \mathbf{1}_{\Gamma_{k,n}}(X_k) - \mu(\Gamma_{k,n}). \tag{6.15}$$

With these notations, by the Taylor integral formula at order 1,

$$\begin{aligned} f_n(\tilde{S}_k - \tilde{E}_k)) - f_n(\tilde{S}_{k-1} - \tilde{E}_{k-1}) &= f_n(T_k) - f_n(T_{k-1}) \\ &= f_n'(T_{k-1})\xi_k + \int_0^1 \big(f_n'(T_{k-1} + t\xi_k) - f_n'(T_{k-1})\big)\xi_k \, dt. \end{aligned}$$

Now $f_n'(x) = \tilde{E}_n^{-1} f'(x/\tilde{E}_n)$. Moreover, from the definition of f, f' is 1-Lipschitzian. Hence

$$\big(f_n'(T_{k-1} + t\xi_k) - f_n'(T_{k-1})\big)\xi_k \leq \tilde{E}_n^{-2}\xi_k^2$$

for any t in $[0, 1]$, which implies that

$$f_n(T_k) - f_n(T_{k-1}) \leq f_n'(T_{k-1})\xi_k + \tilde{E}_n^{-2}\xi_k^2. \tag{6.16}$$

Now, using (6.14), (6.16), taking the expectation and noticing that $f_n'(T_0) = f_n'(0) = 0$, we get that

$$\mathbb{E}\big(f_n(\tilde{S}_n - \tilde{E}_n)\big) \leq \sum_{k=2}^{n} \mathbb{E}\big(f_n'(T_{k-1})\xi_k\big) + \tilde{E}_n^{-2} \sum_{k=1}^{n} \mu(\Gamma_{k,n}). \tag{6.17}$$

Next, let $m \geq 2$ be a fixed integer. For $n \geq m$,

$$f_n'(T_{k-1})\xi_k = f_n'(T_{(k-m)_+})\xi_k + \sum_{j=1}^{m-1} \big(f_n'(T_{k-j}) - f_n'(T_{(k-j-1)_+})\big)\xi_k.$$

Taking the expectation in the above equality, we then get that

$$\mathbb{E}(f_n'(T_{k-1})\xi_k) = \operatorname{Cov}(f_n'(T_{(k-m)_+}), \mathbf{1}_{B_{k,n}}) + \sum_{j=1}^{m-1} \operatorname{Cov}\big(f_n'(T_{(k-j)_+}) - f_n'(T_{(k-j-1)_+}), \mathbf{1}_{B_{k,n}}\big). \tag{6.18}$$

In order to bound up the terms appearing in (6.18), we will use Delyon's covariance inequality, which we now recall. We refer to Rio (2017, Theorem 1.4) for an available reference with a proof.

Lemma 6.2. - *Delyon (1990) - Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-fields of $(\Omega, \mathcal{T}, \mathbb{P})$. Then there exist random variables $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$ respectively $\mathcal{A}$-measurable with values in $[0, \varphi(\mathcal{A}, \mathcal{B})]$ and $\mathcal{B}$-measurable with values in $[0, \varphi(\mathcal{B}, \mathcal{A})]$, satisfying $\mathbb{E}(d_{\mathcal{A}}) = \mathbb{E}(d_{\mathcal{B}}) = \beta(\mathcal{A}, \mathcal{B})$ and such that, for any (p, q) in $[1, \infty]^2$ with $(1/p)+(1/q) = 1$ and any random vector (X, Y) in $L^p(\mathcal{A})\times L^q(\mathcal{B})$,*

$$|\operatorname{Cov}(X, Y)| \leq 2\big(\mathbb{E}(d_{\mathcal{A}}|X|^p)\big)^{1/p}\big(\mathbb{E}(d_{\mathcal{B}}|Y|^q)\big)^{1/q}, \tag{6.19}$$

where $\big(\mathbb{E}(d_{\mathcal{A}}|X|^p)\big)^{1/p} = \|X\|_\infty$ if $p = \infty$ and $\big(\mathbb{E}(d_{\mathcal{B}}|Y|^q)\big)^{1/q} = \|Y\|_\infty$ if $q = \infty$.

We now bound up the first term in the right-hand side of equality (6.18). If $k \leq m$, then $T_{(k-m)_+} = 0$, whence

$$\operatorname{Cov}\big(f_n'(T_{(k-m)_+}), \mathbf{1}_{B_{k,n}}\big) = 0.$$

Set

$$W_{k,l} = \sum_{-k<i\leq -l} \big(\mathbf{1}_{\Gamma_{i+k,n}}(X_i) - \mu(\Gamma_{i+k,n})\big) \ \text{ for any } l \leq k. \tag{6.20}$$

If $k > m$, using the stationarity of $(X_i)_{i\in\mathbb{Z}}$, we obtain that

$$\mathbb{E}\big(f_n'(T_{(k-m)_+})\xi_k\big) = \mathrm{Cov}(f_n'(W_{k,m}), \mathbf{1}_{\Gamma_{k,n}}(X_0)). \tag{6.21}$$

Let us now apply Lemma 6.2 with $\mathcal{A} = \mathcal{F}_{-m}$, $\mathcal{B} = \sigma(X_0)$, $p = \infty$, $q = 1$, $X = f_n'(W_{k,m})$ and $Y = \mathbf{1}_{\Gamma_{k,n}}(X_0)$: there exists some measurable function ψ_m satisfying

$$0 \leq \psi_m \leq \varphi(\sigma(X_0), \mathcal{F}_{-m}) \ \text{ and } \ \int_E \psi_m d\mu = \beta_m, \tag{6.22}$$

such that, for any $k > m$,

$$\mathrm{Cov}(f_n'(T_{(k-m)_+}), \mathbf{1}_{B_{k,n}}) \leq 2\|f_n'\|_\infty \int_E \mathbf{1}_{\Gamma_{k,n}}\psi_m d\mu. \tag{6.23}$$

Next $f_n'(x) = \tilde{E}_n^{-1} f'(x/\tilde{E}_n)$. Since $\|f'\|_\infty \leq 1$, it follows that $\|f_n'\|_\infty \leq \tilde{E}_n^{-1}$. Summing (6.23) on k and using this bound, we finally get that

$$\sum_{k=2}^{n} \mathrm{Cov}(f_n'(T_{(k-m)_+}), \mathbf{1}_{B_{k,n}}) \leq 2 \int_E G_n \psi_m d\mu, \tag{6.24}$$

where G_n is defined in Proposition 6.1.

We now bound up the other terms in the right-hand side of equality (6.18). If $j \geq k$, then $T_{(k-j)_+} = T_{(k-j-1)_+} = 0$, which implies that

$$\mathrm{Cov}\big(f_n'(T_{(k-j)_+}) - f_n'(T_{(k-j-1)_+}), \mathbf{1}_{B_{k,n}}\big) = 0.$$

If $j < k$, using the stationarity of $(X_i)_{i\in\mathbb{Z}}$, we obtain that

$$\mathrm{Cov}\big(f_n'(T_{(k-j)_+}) - f_n'(T_{(k-j-1)_+}), \mathbf{1}_{B_{k,n}}\big) = \mathrm{Cov}(f_n'(W_{k,j}) - f_n'(W_{k,j+1}), \mathbf{1}_{\Gamma_{k,n}}(X_0)), \tag{6.25}$$

where $W_{k,j}$ and $W_{k,j+1}$ are defined in (6.20). Applying Lemma 6.2 with $\mathcal{A} = \mathcal{F}_{-j}$, $\mathcal{B} = \sigma(X_0)$, $p = q = 2$, $X = f_n'(W_{k,j}) - f_n'(W_{k,j+1})$ and $Y = \mathbf{1}_{\Gamma_{k,n}}(X_0)$, we obtain that there exist some $\sigma(X_0)$-measurable random variable b_j and some $\mathcal{F}_{-j}$-measurable random variable η_j with values in $[0, 1]$, satisfying

$$\mathbb{E}(b_j) = \mathbb{E}(\eta_j) = \beta_j \tag{6.26}$$

and such that

$$\mathrm{Cov}(f_n'(W_{k,j}) - f_n'(W_{k,j+1}), \mathbf{1}_{\Gamma_{k,n}}(X_0)) \leq 2\sqrt{\mathbb{E}(\eta_j|f_n'(W_{k,j}) - f_n'(W_{k,j+1})|^2)\mathbb{E}(b_j\mathbf{1}_{\Gamma_{k,n}}(X_0))}\,. \tag{6.27}$$

Next, from the definitions of f_n and f, $f_n'(x) = \tilde{E}_n^{-1} f'(x/\tilde{E}_n)$ and f' is 1-Lipschitzian. Consequently

$$|f_n'(W_{k,j}) - f_n'(W_{k,j+1})| \leq \tilde{E}_n^{-2}|W_{k,j} - W_{k,j+1}| = \tilde{E}_n^{-2}|\mathbf{1}_{\Gamma_{k-j,n}}(X_{-j}) - \mu(\Gamma_{k-j,n})|,$$

which implies that

$$\mathbb{E}(\eta_j|f_n'(W_{k,j}) - f_n'(W_{k,j+1})|^2) \le \tilde{E}_n^{-4}\mathbb{E}(b_j'|\mathbf{1}_{\Gamma_{k-j,n}}(X_{-j}) - \mu(\Gamma_{k-j,n})|^2),$$

with $b_j' = \mathbb{E}(\eta_j \mid \sigma(X_{-j}))$. Combining the above inequality, (6.27) and the elementary inequality $2\sqrt{ab} \le a+b$, we infer that

$$\tilde{E}_n^2 \operatorname{Cov}(f_n'(W_{k,j}) - f_n'(W_{k,j+1}), \mathbf{1}_{\Gamma_{k,n}}(X_0)) \le \mathbb{E}(b_j\mathbf{1}_{\Gamma_{k,n}}(X_0) + b_j'|\mathbf{1}_{\Gamma_{k-j,n}}(X_{-j}) - \mu(\Gamma_{k-j,n})|^2). \tag{6.28}$$

Recall now that b_j is $\sigma(X_0)$-measurable and b_j' is $\sigma(X_{-j})$-measurable. Hence there exists Borelian functions $\varphi_{j,0}$ and $\varphi_{j,1}$ with values in $[0,1]$ such that $b_j = \varphi_{j,0}(X_0)$ and $b_j' = \varphi_{j,1}(X_{-j})$. Using now the stationarity of $(X_i)_{i\in\mathbb{Z}}$, we get

$$\mathbb{E}(b_j\mathbf{1}_{\Gamma_{k,n}}(X_0) + b_j'|\mathbf{1}_{\Gamma_{k-j,n}}(X_{-j}) - \mu(\Gamma_{k-j,n})|^2) = \int_E \big(\varphi_{j,0}\mathbf{1}_{\Gamma_{k,n}} + \varphi_{j,1}|\mathbf{1}_{\Gamma_{k-j,n}} - \mu(\Gamma_{k-j,n})|^2\big)d\mu.$$

Next, applying the elementary inequality

$$|\mathbf{1}_{\Gamma_{k-j,n}} - \mu(\Gamma_{k-j,n})|^2 \le \mathbf{1}_{\Gamma_{k-j,n}} + \mu(\Gamma_{k-j,n}),$$

noticing that $\int_E \varphi_{j,1}d\mu = \beta_j$ and putting together (6.25), (6.28) and the above inequalities, we get

$$\tilde{E}_n^2 \operatorname{Cov}\big(f_n'(T_{(k-j)_+}) - f_n'(T_{(k-j-1)_+}), \mathbf{1}_{B_{k,n}}\big) \le \beta_j\mu(\Gamma_{k-j,n}) + \int_E \big(\varphi_{j,0}\mathbf{1}_{\Gamma_{k,n}} + \varphi_{j,1}\mathbf{1}_{\Gamma_{k-j,n}}\big)d\mu, \tag{6.29}$$

for some Borelian functions $\varphi_{j,0}$ and $\varphi_{j,1}$ with values in $[0,1]$ satisfying

$$\int_E \varphi_{j,0}d\mu = \int_E \varphi_{j,1}d\mu = \beta_j. \tag{6.30}$$

Finally, summing (6.29) on j and k, using (6.17), (6.18) and (6.24), we obtain

$$\mathbb{E}\big(f_n(\tilde{S}_n - \tilde{E}_n)\big) \le \tilde{E}_n^{-1}\Big(1 + \sum_{j=1}^{m-1}\beta_j\Big) + 2\int_E G_n\Big(\psi_m + \tilde{E}_n^{-1}\sum_{j=1}^{m-1}\psi_j'\Big)d\mu, \tag{6.31}$$

where

$$\psi_j' = (\varphi_{j,0} + \varphi_{j,1})/2. \tag{6.32}$$

Let λ denote the Lebesgue measure on $[0,1]$ and let Q_{G_n} be the cadlag inverse function of the the tail function of G_n. Then, by Lemma 2.1 (a) in Rio (2017) applied to the functions G_n and $\mathbf{1}_{u\le\psi_m}$,

$$\int_E G_n\psi_m d\mu = \iint_{E\times[0,1]} G_n\mathbf{1}_{u\le\psi_m}d\mu\otimes\lambda \le \int_0^{\beta_m} Q_{G_n}(s)ds. \tag{6.33}$$

In a similar way

$$\int_E G_n\psi_j' d\mu \le \int_0^{\beta_j} Q_{G_n}(s)ds. \tag{6.34}$$

Putting the two above inequalities in (6.31), we get:

$$\mathbb{E}\big(f_n(\tilde{S}_n - \tilde{E}_n)\big) \le \tilde{E}_n^{-1}\Big(1 + \sum_{j=1}^{m-1} \int_0^{\beta_j} (1 + 2Q_{G_n}(s))ds\Big) + 2\int_0^{\beta_m} Q_{G_n}(s)ds. \tag{6.35}$$

We now complete the proof of Proposition 6.1. Since $\int_0^1 Q_{G_n}(s)ds = \int_E G_n d\mu = 1$, the above inequality ensures that

$$\mathbb{E}\big(f_n(\tilde{S}_n - \tilde{E}_n)\big) \le \tilde{E}_n^{-1}(3m-2) + 2\int_0^{\beta_m} Q_{G_n}(s)ds. \tag{6.36}$$

It follows that

$$\limsup_{n\to\infty} \mathbb{E}\big(f\big((\tilde{S}_n - \tilde{E}_n)/\tilde{E}_n\big)\big) \le 2\limsup_{n\to\infty} \int_0^{\beta_m} Q_{G_n}(s)ds \tag{6.37}$$

for any integer $m \ge 2$. Now $\lim_{m\uparrow\infty} \beta_m = 0$. Consequently, if the sequence $(G_n)_{n>0}$ is uniformly integrable, then, by Proposition A.1, the term on right hand in the above inequality tends to 0 as m tends to ∞, which ends the proof of Proposition 6.1. $\diamond$

End of the proof of Theorem 3.1. Item (ii) follows immediately from Proposition 6.1 applied with $\Gamma_{k,n} = A_k$. To prove Item (i), we note that applying Proposition 2.1 with $(\Omega, \mathcal{T}, \mathbb{P}) = (X, \mathcal{B}(X), \mu)$, there exists a sequence of events $(\Gamma_k)_{k>0}$ such that $(\Gamma_{k,n})_{k>0} \equiv (\Gamma_k)_{k>0}$ satisfies the assumptions of Proposition 6.1. Item (i) then follows by applying Proposition 6.1.

It remains to prove Item (iii). Here we will apply Proposition 2.3 (iii). Thoughout the proof of Item (iii), $\beta_0 = 1$ by convention. For any positive integer k, let $S_k = \sum_{j=1}^k \mathbf{1}_{B_j}$ and $E_k = \mathbb{E}(S_k)$. Since f is convex and $f(0) = 0$,

$$f((S_k - E_k)/E_n) \le (E_k/E_n) f((S_k - E_k)/E_k)$$

for any k in $[1, n]$. Applying now Inequality (6.35) in the case $\Gamma_{j,n} = A_j$, we get that

$$\mathbb{E}\big(f((S_k - E_k)/E_k)\big) \le E_k^{-1} \sum_{j=0}^{m-1} \int_0^{\beta_j} (1 + 2Q_{H_k}(s))ds + 2\int_0^{\beta_m} Q_{H_k}(s)ds.$$

Now, from the definition of Q^*,

$$\int_0^u Q_{H_k}(s))ds \le uQ^*(u) \ \text{ for any } u \in]0,1] \ \text{ and any } k > 0.$$

The three above inequalities ensure that

$$\sup_{k\le n} \mathbb{E}\big(f_n(S_k - E_k)\big) \le E_n^{-1} \sum_{j\in[0,m-1]} \beta_j(2Q^*(\beta_j) + 1) + 2\beta_m Q^*(\beta_m). \tag{6.38}$$

Let n_0 be the smallest integer such that $E_{n_0} \geq 2$. For $n \geq n_0$, choose $m := m_n = 1 + [E_n]$ in the above inequality. For this choice of m_n, noticing that $Q^*(\beta_j) \geq Q^*(1) = 1$, we get

$$\sum_{n \geq n_0} \frac{\mathbb{P}(B_n)}{3E_n} \sup_{k \in [1,n]} \mathbb{E}\big(f_n\big(S_k - E_k\big)\big) \leq \sum_{n \geq n_0} \Big[\sum_{0 \leq j \leq [E_n]} \beta_j Q^*(\beta_j) \frac{\mathbb{P}(B_n)}{E_n^2} + \beta_{m_n} Q^*(\beta_{m_n}) \frac{\mathbb{P}(B_n)}{E_n} \Big]. \tag{6.39}$$

We now bound up the first term on the right-hand side. Clearly

$$\sum_{n \geq n_0} \sum_{0 \leq j \leq [E_n]} \beta_j Q^*(\beta_j) E_n^{-2} \mathbb{P}(B_n) = \sum_{j \geq 0} \beta_j Q^*(\beta_j) \sum_{n : E_n \geq j \vee 2} E_n^{-2} \mathbb{P}(B_n).$$

Next, noticing that $E_n - E_{n-1} = \mathbb{P}(B_n)$, we get that $\mathbb{P}(B_n)/E_n^2 \leq 1/E_{n-1} - 1/E_n$. It follows that

$$\sum_{n : E_n \geq j \vee 2} E_n^{-2} \mathbb{P}(B_n) \leq 1/E_{n_j - 1}\,,$$

where n_j is the smallest integer such that $E_{n_j} \geq j \vee 2$. Since $E_{n_j - 1} \geq E_{n_j} - 1$, $1/E_{n_j-1} \leq 2/(j \vee 2)$. Hence

$$\sum_{n \geq n_0} \sum_{0 \leq j \leq [E_n]} \beta_j Q^*(\beta_j) E_n^{-2} \mathbb{P}(B_n) \leq 1 + 2 \sum_{j > 0} j^{-1} \beta_j Q^*(\beta_j) < \infty \tag{6.40}$$

under condition (3.6). To complete the proof of (iii), it remains to prove that

$$\sum_{n \geq n_0} \beta_{m_n} Q^*(\beta_{m_n}) E_n^{-1} \mathbb{P}(B_n) < \infty \tag{6.41}$$

under condition (3.6), where $m_n = 1 + [E_n]$. For any integer $k \geq 2$, let I_k be the set of integers n such that $[E_n] = k$. By definition, I_k is an interval of $\mathbb{N}$. Furthermore, from the fact that $\mu(A_n) \leq 1$, $I_k \neq \emptyset$. Since $\lim_n E_n = \infty$, this interval is finite. Consequently

$$\sum_{n \in I_k} \mathbb{P}(B_n) = E_{\sup I_k} - E_{\inf I_k - 1} \leq E_{\sup I_k} - E_{\inf I_k} + 1 \leq 2.$$

Now, recall that n_0 is the first integer such that $E_{n_0} \geq 2$. Consequently $n_0 = \inf I_2$ and

$$\sum_{n \geq n_0} \beta_{m_n} Q^*(\beta_{m_n}) \frac{\mathbb{P}(B_n)}{E_n} = \sum_{k \geq 2} \beta_{k+1} Q^*(\beta_{k+1}) \sum_{n \in I_k} \frac{\mathbb{P}(B_n)}{E_n} \leq 2 \sum_{k \geq 2} k^{-1} \beta_{k+1} Q^*(\beta_{k+1}) < \infty\,,$$

under condition (3.6). This ends the proof of Item (iii). Theorem 3.1 is proved. $\diamond$

6.2.2 Proofs of Theorems 3.2 and 3.3 (α-mixing case)

Proof of Theorem 3.2. To apply Item (i) of Proposition 2.3, we shall prove that under (3.7) and (3.8), there exists a sequence $(\psi_n)_{n>0}$ of positive integers such that setting $m_n = \inf\{k \in \mathbb{N}^* \,:\, \psi_k \geq n\}$, $\tilde{S}_n = \sum_{k=1}^{m_n - 1} \mathbf{1}_{A_{\psi_k}}(X_{\psi_k})$ and $\tilde{E}_n = \mathbb{E}(\tilde{S}_n) = \sum_{k=1}^{m_n-1} \mu(A_{\psi_k})$ (so here $g_{j,n} = g_j = \mathbf{1}_{A_j}(X_j)$ if $j \in \psi(\mathbb{N}^*)$ and 0 otherwise), we have

$$\lim_{N \to \infty} \tilde{E}_{2^N} = \infty \ \text{ and } \lim_{N \to \infty} \big(f\big((\tilde{S}_{2^N} - \tilde{E}_{2^N})/\tilde{E}_{2^N}\big) \big) = 0\,. \tag{6.42}$$

To construct the sequence $\psi = (\psi_n)_{n\geq 1}$, let us make the following considerations. By the second part of (3.8), there exists a positive decreasing sequence $(\delta_n)_{n\geq 1}$ such that $\delta_n \to 0$, as $n \to \infty$, and

$$\sum_{n\geq 1} \delta_n \frac{\mu(A_n)}{\alpha_*^{-1}(\mu(A_n))} = \infty \,. \tag{6.43}$$

Now, note that, by the second part of (3.7), there exist $u_0 > 0$ and $\kappa > 1$ such that for any $u \in]0, u_0[$, $\alpha_*^{-1}(u/2) \leq \kappa\alpha_*^{-1}(u)$. Hence setting $j_n = \sup\{j \geq 0 \, : \, \kappa^{-j} \geq \delta_n\}$ and $\varepsilon_n = 2^{-j_n}$, it follows that $\alpha_*^{-1}(\mu(A_n)) \geq \delta_n\alpha_*^{-1}(\varepsilon_n\mu(A_n))$, which combined with (6.43) implies that

$$\sum_{n\geq 1} \frac{\mu(A_n)}{\alpha_*^{-1}(\varepsilon_n\mu(A_n))} = \infty \,. \tag{6.44}$$

Definition 6.1. Let $(k_L)_{L\geq 0}$ be the sequence of integers defined by

$$k_L = L \wedge \lceil \log_2 \alpha_*^{-1}(\varepsilon_{2^L}\mu(A_{2^L}))\rceil \,, \quad \text{where } \log_2 x = \log(x \vee 1)/\log 2$$

and $\lceil x\rceil = \inf \mathbb{Z} \cap [x, \infty[$. Set $j_0 = 0$ and $j_{L+1} = j_L + 2^{L-k_L}$ for any $L \geq 0$, Finally, for any $L \geq 0$, we set $\psi_{j_L} = 2^L$ and for any $i = j_L + \ell$ with $\ell \in [1, j_{L+1} - j_L - 1] \cap \mathbb{N}^*$, $\psi_i = 2^L + \ell 2^{k_L}$.

Recall the notation, $f_{2^N}(x) = f\big(x/\tilde{E}_{2^N}\big)$. Noticing that $\tilde{S}_{2^N} = \sum_{k=1}^{j_N-1} \mathbf{1}_{A_{\psi_k}}(X_{\psi_k})$ and recalling that $f(0) = 0$, we have

$$\begin{aligned}
&\mathbb{E}\big(f\big((\tilde{S}_{2^N} - \tilde{E}_{2^N})/\tilde{E}_{2^N}\big)\big) = \mathbb{E}f_{2^N}(\tilde{S}_{2^N} - \tilde{E}_{2^N}) \\
&\quad = \sum_{L=2}^{N}\sum_{\ell=j_{L-1}}^{j_L-1} \Big\{\mathbb{E}f_{2^N}\Big(\sum_{i=1}^{\ell}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big) - \mathbb{E}f_{2^N}\Big(\sum_{i=1}^{\ell-1}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big)\Big\} \,.
\end{aligned} \tag{6.45}$$

Using Taylor's formula (as to get (6.16)) and taking the expectation, we derive

$$\begin{aligned}
&\mathbb{E}f_{2^N}\Big(\sum_{i=1}^{\ell}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big) - \mathbb{E}f_{2^N}\Big(\sum_{i=1}^{\ell-1}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big) \\
&\qquad\qquad \leq \operatorname{Cov}\Big(f'_{2^N}\Big(\sum_{i=1}^{\ell-1}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big), \mathbf{1}_{A_{\psi_\ell}}(X_{\psi_\ell})\Big) + \frac{\mu(A_{\psi_\ell})}{(\tilde{E}_{2^N})^2} \,.
\end{aligned}$$

Since $\|f'_{2^N}\|_\infty \leq 1/\tilde{E}_{2^N}$, it follows from (3.1) that

$$\mathbb{E}\big(f\big((\tilde{S}_{2^N} - \tilde{E}_{2^N})/\tilde{E}_{2^N}\big)\big) \leq \sum_{L=2}^{N}\sum_{\ell=j_{L-1}}^{j_L-1} \Big\{\frac{\mu(A_{\psi_\ell})}{(\tilde{E}_{2^N})^2} + \frac{4\alpha_{\infty,1}(2^{k_{L-2}})}{\tilde{E}_{2^N}}\Big\} \,.$$

Now, since $j_L - j_{L-1} = 2^{L-k_L}$ and $\tilde{E}_{2^N} = \sum_{L=2}^{N}\sum_{\ell=j_{L-1}}^{j_L-1} \mu(A_{\psi_\ell})$, we get

$$\mathbb{E}\big(f\big((\tilde{S}_{2^N} - \tilde{E}_{2^N})/\tilde{E}_{2^N}\big)\big) \leq \frac{1}{\tilde{E}_{2^N}} + 4\sum_{L=2}^{N} \alpha_{\infty,1}(2^{k_{L-2}})\frac{2^{L-k_L}}{\tilde{E}_{2^N}} \,.$$

Note then that, since $(\mu(A_n))_{n\geq 1}$ is a non-increasing sequence,

$$\tilde{E}_{2^N} = \sum_{L=2}^{N}\sum_{\ell=j_{L-1}}^{j_L-1} \mu(A_{\psi_\ell}) \geq \sum_{L=2}^{N}(j_L - j_{L-1})\mu(A_{\psi_{j_L}}) = \sum_{L=2}^{N} 2^{L-k_L}\mu(A_{2^L})\,. \tag{6.46}$$

Thus

$$\mathbb{E}\big(f\big((\tilde{S}_{2^N} - \tilde{E}_{2^N})/\tilde{E}_{2^N}\big)\big) \leq \frac{1}{\tilde{E}_{2^N}} + \frac{4\sum_{L=2}^{N}\alpha_{\infty,1}(2^{k_{L-2}})2^{L-k_L}}{\sum_{L=2}^{N}2^{L-k_L}\mu(A_{2^L})}\,.$$

This shows that (6.42) will be satisfied if

$$\lim_{N\to\infty}\tilde{E}_{2^N} = \infty \quad\text{and}\quad \lim_{L\to\infty}\big(\alpha_{\infty,1}(2^{k_L})/\mu(A_{2^L})\big) = 0\,. \tag{6.47}$$

Since $(\mu(A_n))_{n\geq 1}$ is a non-increasing sequence, condition (6.44) is equivalent to

$$\sum_{k\geq 0} 2^k \frac{\mu(A_{2^k})}{\alpha_*^{-1}(\varepsilon_{2^k}\mu(A_{2^k}))} = \infty\,. \tag{6.48}$$

Together with (6.46) and the definition of 2^{k_L}, (6.48) implies the first part of (6.47). Next, taking into account the definition of 2^{k_L},

$$\alpha_{\infty,1}(2^{k_L})/\mu(A_{2^L}) \leq \max\big(C\varepsilon_{2^L}, \alpha_{\infty,1}(2^L)/\mu(A_{2^L})\big) \to 0\,, \text{ as } L\to\infty\,,$$

by the first parts of conditions (3.7) and (3.8). This ends the proof. $\diamond$

Proof of Theorem 3.3. Starting from (6.17), taking into account (6.18) and the facts that

$$\big|\mathrm{Cov}\big(f_n'(T_{(k-m)_+}), \mathbf{1}_{A_k}(X_k)\big)\big| \leq 4\alpha_{\infty,1}(m)/E_n$$

and

$$\big|\mathrm{Cov}\big(f_n'(T_{(k-j)_+}) - f_n'(T_{(k-j-1)_+}), \mathbf{1}_{A_k}(X_k)\big)\big| \leq E_n^{-2}\mu(A_k)\,,$$

we infer that, for any positive integer m and any integer k in $[1,n]$,

$$\mathbb{E}\big(f\big((S_k - E_k)/E_n\big)\big) \leq 4n\alpha_{\infty,1}(m)/E_n + m/E_n\,.$$

Item 1. follows by choosing $m = m_n = \eta^{-1}(1/n)$ and by taking into account Item (ii) of Proposition 2.3. To prove Item 2., we choose $m = m_n = \alpha_{\infty,1}^{-1}(u_nE_n/n)$. Item 2. then follows by taking into account Item (iii) of Proposition 2.3. $\diamond$

6.2.3 Proof of Remark 3.6

To prove that Theorem 3.2 still holds with $\alpha_{1,\infty}(n)$ replacing $\alpha_{\infty,1}(n)$, it suffices to modify the decomposition (6.45) as follows:

$$\mathbb{E}\big(f\big((\tilde{S}_{2^N} - \tilde{E}_{2^N})/\tilde{E}_{2^N}\big)\big) = \mathbb{E}f_{2^N}(\tilde{S}_{2^N} - \tilde{E}_{2^N})$$
$$= \sum_{L=2}^{N}\sum_{\ell=j_{L-1}}^{j_L-1}\Big\{\mathbb{E}f_{2^N}\Big(\sum_{i=\ell}^{j_N-1}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big) - \mathbb{E}f_{2^N}\Big(\sum_{i=\ell+1}^{j_N-1}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big)\Big\}\,.$$

Next, as in the proof of Theorem 3.2, we use Taylor's formula and the fact that, by (3.1), for any $\ell \in \{j_{L-1}, \dots, j_L - 1\}$,

$$\mathrm{Cov}\Big(f'_{2^N}\Big(\sum_{i=\ell+1}^{j_N-1}(\mathbf{1}_{A_{\psi_i}}(X_{\psi_i}) - \mu(A_{\psi_i}))\Big), \mathbf{1}_{A_{\psi_\ell}}(X_{\psi_\ell})\Big) \le \frac{4\alpha_{1,\infty}(2^{k_{L-2}})}{\tilde{E}_{2^N}}.$$

The rest of the proof is unchanged.

To prove that Theorem 3.3 still holds with $\alpha_{1,\infty}(n)$ replacing $\alpha_{\infty,1}(n)$, we start by setting

$$S^*_k = \sum_{i=k}^{n} \mathbf{1}_{B_{i,n}}, \; E^*_k = \sum_{i=k}^{n} \mu(\Gamma_{i,n}), \; T^*_k = S^*_k - E^*_k \text{ and } \xi_k = T^*_k - T^*_{k+1}.$$

Then, setting $S^*_{n+1} = E^*_{n+1} = 0$, instead of (6.14), we write

$$f_n(\tilde{S}_n - \tilde{E}_n) = \sum_{k=1}^{n} \big(f_n(S^*_k - E^*_k)) - f_n(S^*_{k+1} - E^*_{k+1})\big).$$

By the Taylor integral formula at order 1, it follows that

$$f_n(\tilde{S}_n - \tilde{E}_n) = \sum_{k=1}^{n} \Big(f'_n(T^*_{k+1})\xi_k + \int_0^1 \big(f'_n(T^*_{k+1} + t\xi_k) - f'_n(T^*_{k+1})\big)\xi_k \, dt\Big).$$

Then, instead of (6.18), we use the following decomposition:

$$\mathbb{E}(f'_n(T^*_{k+1})\xi_k) = \mathrm{Cov}(f'_n(T^*_{(k+m)_+}), \mathbf{1}_{B_{k,n}}) + \sum_{j=1}^{m-1} \mathrm{Cov}\big(f'_n(T^*_{(k+j)_+}) - f'_n(T^*_{(k+j+1)_+}), \mathbf{1}_{B_{k,n}}\big).$$

Hence, the only difference with the proof of Theorem 3.3 is the following estimate:

$$\big|\mathrm{Cov}\big(\mathbf{1}_{A_k}(X_k), f'_n(T^*_{(k+m)_+})\big)\big| \le 4\alpha_{1,\infty}(m)/E_n.$$

This ends the proof of the remark. $\diamond$

6.3 Proofs of the results of Section 4

6.3.1 Proof of Theorem 4.1.

To prove Item (i), we first apply Proposition 2.2. Since $\mu(\limsup_n I_n) > 0$, it follows from that proposition that there exists a sequence $(\Gamma_k)_k$ of intervals such that $\Gamma_k \subset I_k$, $\sum_{k>0} \mu(\Gamma_k) = \infty$ and

$$\sup_{n>0} \Big\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mu(\Gamma_k)} \Big\|_{\infty,\mu} < \infty, \tag{6.49}$$

where $\|\cdot\|_{\infty,\mu}$ is the essential supremum norm with respect to μ.

Let us prove now that $\tilde{B}_k = \{X_k \in \Gamma_k\}$ is a L^1-Borel-Cantelli sequence. Since $\tilde{B}_k \subset B_k$, this will imply that $(B_k)_{k>0}$ is a Borel-Cantelli sequence. From (1.5) applied to $\tilde{S}_n = \sum_{k=1}^n \mathbf{1}_{\tilde{B}_k}$, it is enough to prove that

$$\lim_{n\to\infty} \big(\mathbb{E}(\tilde{S}_n)\big)^{-2}\mathrm{Var}(\tilde{S}_n) = 0. \tag{6.50}$$

By stationarity,

$$\mathrm{Var}(\tilde{S}_n) = \sum_{k=1}^{n} \mathrm{Var}(\mathbf{1}_{\Gamma_k}(X_0)) + 2\sum_{k=1}^{n-1}\sum_{j=1}^{n-k} \mathrm{Cov}(\mathbf{1}_{\Gamma_k}(X_0), \mathbf{1}_{\Gamma_{k+j}}(X_j)) . \tag{6.51}$$

Let $b_j = \sup_{t\in\mathbb{R}} |\mathbb{E}(\mathbf{1}_{X_j\le t}|X_0) - \mathbb{P}(X_j \le t)|$. Clearly, since Γ_ℓ is an interval,

$$|\mathrm{Cov}(\mathbf{1}_{\Gamma_k}(X_0), \mathbf{1}_{\Gamma_{k+j}}(X_j))| \le 2\mathbb{E}(\mathbf{1}_{\Gamma_k}(X_0)b_j) . \tag{6.52}$$

Setting $\bar{B}_n = b_1 + \cdots + b_{n-1}$, we infer from (6.51) and (6.52) that

$$\mathrm{Var}(\tilde{S}_n) \le \mathbb{E}\Big((1+4\bar{B}_n)\sum_{k=1}^{n} \mathbf{1}_{\Gamma_k}(X_0)\Big) . \tag{6.53}$$

Since $\mathbb{E}(\tilde{S}_n) = \mu(\Gamma_1) + \cdots + \mu(\Gamma_n)$, we infer from (6.53) that

$$\frac{\mathrm{Var}(\tilde{S}_n)}{\big(\mathbb{E}(\tilde{S}_n)\big)^2} \le \frac{\mathbb{E}(1+4\bar{B}_n)}{\sum_{k=1}^n \mu(\Gamma_k)} \Big\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mu(\Gamma_k)} \Big\|_{\infty,\mu} \le \frac{\big(1+4\sum_{k=1}^{n-1} \tilde{\beta}_{1,1}(k)\big)}{\sum_{k=1}^n \mu(\Gamma_k)} \Big\| \frac{\sum_{k=1}^n \mathbf{1}_{\Gamma_k}}{\sum_{k=1}^n \mu(\Gamma_k)} \Big\|_{\infty,\mu} , \tag{6.54}$$

the last inequality being true because $\mathbb{E}(b_k) = \tilde{\beta}_{1,1}(k)$. Hence (6.50) follows from (6.49), (6.54), and the fact that $\sum_{k>0} \tilde{\beta}_{1,1}(k) < \infty$ and $\sum_{k\ge 1} \mu(\Gamma_k) = +\infty$. The proof of Item (i) is complete.

We now prove Item (ii). Let $S_n = \sum_{k=1}^n \mathbf{1}_{B_k}$. Arguing as for (i), it is enough to prove (6.50) with S_n instead of $\tilde{S}_n$. Since the I_k are intervals, the same computations as for (i) lead to

$$\mathrm{Var}(S_j) \le \mathbb{E}\Big((1+4\bar{B}_j)\sum_{k=1}^{j} \mathbf{1}_{I_k}(X_0)\Big) \le \mathbb{E}\Big((1+4\bar{B}_n)\sum_{k=1}^{n} \mathbf{1}_{I_k}(X_0)\Big) \tag{6.55}$$

for any $j \le n$. Set $\tilde{\beta}_{1,1}(0) = 1$. Applying Hölder's inequality, we get that, for any $j \le n$,

$$\mathrm{Var}(S_j) \le \Big\|1{+}4\bar{B}_n\Big\|_p \Big\|\sum_{k=1}^{n} \mathbf{1}_{I_k}(X_0)\Big\|_q \le 4\Big(p\sum_{k=0}^{n-1}(1{+}k)^{p-1}\tilde{\beta}_{1,1}(k)\Big)^{1/p} \Big\|\sum_{k=1}^{n} \mathbf{1}_{I_k}(X_0)\Big\|_q , \tag{6.56}$$

(the last inequality follows from Remark 1.6 and Inequality (C.5) in [24]). Consequently

$$\frac{\mathrm{Var}(S_n)}{\big(\mathbb{E}(S_n)\big)^2} \le \frac{4\big(p\sum_{k=0}^{n-1}(1+k)^{p-1}\tilde{\beta}_{1,1}(k)\big)^{1/p}}{E_n} \Big\| \frac{\sum_{k=1}^n \mathbf{1}_{I_k}(X_0)}{E_n} \Big\|_q .$$

Hence Item (ii) follows via (1.5). In addition, Item (iii) follows from (6.56) by applying (2.5).

To prove (iv) and (v), we start from (6.55), and we get that, for any $j \le n$,

$$\frac{\mathrm{Var}(S_j)}{\big(\mathbb{E}(S_n)\big)^2} \le \frac{\|1+4\bar{B}_n\|_\infty}{E_n^2} \sum_{k=1}^{n} \mu(I_k) \le \frac{1}{E_n}\Big(1+4\sum_{k=1}^{n-1} \tilde{\varphi}_{1,1}(k)\Big) . \tag{6.57}$$

Then (iv) follows from (6.57) with $j = n$ and (1.5) and (v) from (6.57) and (2.5). $\diamond$

6.3.2 Proof of Lemma 4.1

We consider the natural coupling

$$X_k^* = \sum_{i=0}^{k-1} 2^{-i}\varepsilon_{k-i} + \sum_{i\geq k} 2^{-i}\varepsilon'_{k-i}\,,$$

where $(\varepsilon'_i)_{i\in\mathbb{Z}}$ is an independent copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$. Note that X_k^* distributed as X_k and independent of X_0. Let then

$$\delta(k) = \mathbb{E}\left(\min(|X_k - X_k^*|, 1)\right)\,.$$

We first give a bound on $\delta(k)$. By definition

$$\delta(k) \leq \mathbb{E}\Big(\min\Big(\Big|\sum_{i\geq k} 2^{-i}(\varepsilon_{k-i} - \varepsilon'_{k-i})\Big|, 1\Big)\Big)\,.$$

By sub-additivity and stationarity,

$$\delta(k) \leq \sum_{i\geq k} \mathbb{E}\left(\min(2^{-i}|\varepsilon_0 - \varepsilon'_0|, 1)\right)\,.$$

Hence

$$\delta(k) \leq \sum_{i\geq k} 2^{-i}\mathbb{E}\Big(|\varepsilon_0 - \varepsilon'_0|\mathbf{1}_{|\varepsilon_0-\varepsilon'_0|\leq 2^{i/2}}\Big) + \sum_{i\geq k}\mathbb{P}\left(|\varepsilon_0 - \varepsilon'_0| > 2^{i/2}\right)$$

and, consequently,

$$\delta(k) \leq \kappa 2^{-k/2} + \mathbb{E}\Big(\big(2(\log 2)^{-1}\log|\varepsilon_0 - \varepsilon'_0| - k\big)_+\Big) \tag{6.58}$$

with $\kappa = 1/(1-2^{-1/2})$. This gives the upper bound

$$\delta(k) \leq K2^{-k/2} + K\mathbb{E}\big((\log|\varepsilon_0 - \varepsilon'_0|\mathbf{1}_{\log|\varepsilon_0-\varepsilon'_0|>k\log(\sqrt{2})}\big)\,.$$

Now, if (4.8) holds,

$$\sup_{t>1} t^{p-1}\mathbb{E}\big(\log|\varepsilon_0 - \varepsilon'_0|\mathbf{1}_{\log|\varepsilon_0-\varepsilon'_0|>t}\big) < \infty\,,$$

and it follows then easily from (6.58) that there exists some positive constant B such that

$$\delta(k) \leq Bk^{1-p} \ \text{ for any } k\geq 1. \tag{6.59}$$

Now let F_μ be the distribution function of μ. By Lemma 2, Item 2. in [9], for any $y\in[0,1]$

$$\tilde\beta_{1,1}(k) \leq y + \mathbb{P}(|F_\mu(X_k) - F_\mu(X_k^*)| > y) \tag{6.60}$$

Since μ has a bounded density, F_μ is Lipshitz. Moreover $|F_\mu(X_k) - F_\mu(X_k^*)| \leq 1$. Hence

$$|F_\mu(X_k) - F_\mu(X_k^*)| \leq A\min(1, |X_k - X_k^*|) \ \text{ for some constant } A\geq 1. \tag{6.61}$$

Now, by (6.60), (6.61) and the Markov inequality, $\tilde\beta_{1,1}(k) \leq y + A\delta(k)/y$ for any positive y. Consequently $\tilde\beta_{1,1}(k) \leq 2\sqrt{A\delta(k)}$. The conclusion of Lemma 4.1 follows then from (6.59). ⋄

6.3.3 Proof of Lemma 4.2

We first note that, for any function g in $L^2(\lambda)$, one has

$$K^n(g)(x) - \lambda(g) = \sum_{k\in\mathbb{Z}^*} (\cos(2\pi k a))^n \hat{g}(k) \exp(2i\pi k x)\,, \tag{6.62}$$

where $(\hat{g}(k))_{k\in\mathbb{Z}}$ are the Fourier coefficients of g.

Next, we need to approximate the function $\mathbf{1}_{[0,t]}$ by smooth functions. To do this, we start from an infinitely differentiable density ℓ supported in $[0,1]$, and we define

$$g_1(x) = \left(\int_0^x \ell(t)dt\right) \mathbf{1}_{[0,1]}(x) \quad \text{and} \quad g_2(x) = (1 - g_1(x))\mathbf{1}_{[0,1]}(x)\,.$$

Now, for $0 < h < 1/4$, $t \in [2h, 1-h]$ and $x \in [0,1]$, we have

$$f^-_{t,h}(x) \le \mathbf{1}_{[0,t]}(x) \le f^+_{t,h}(x)\,,$$

where

$$\begin{aligned} f^+_{t,h}(x) &= \mathbf{1}_{[0,t]}(x) + g_2((x-t)/h) + g_1((x+h-1)/h) \\ f^-_{t,h}(x) &= \mathbf{1}_{[h,t-h]}(x) + g_2((x+h-t)/h) + g_1(x/h)\,. \end{aligned}$$

Hence, for $t \in [2h, 1-h]$

$$K^n(f^-_{t,h}) - \lambda(f^-_{t,h}) - 2h \le K^n(\mathbf{1}_{[0,t]}) - t \le K^n(f^+_{t,h}) - \lambda(f^+_{t,h}) + 2h\,. \tag{6.63}$$

On the other hand

$$\Big\| \sup_{t\in[0,2h]} |K^n(\mathbf{1}_{[0,t]}) - t| \Big\|_1 \le 4h \quad \text{and} \quad \Big\| \sup_{t\in[1-h,1]} |K^n(\mathbf{1}_{[0,t]}) - t| \Big\|_1 \le 2h\,. \tag{6.64}$$

From (6.63) and (6.64), we get

$$\Big\| \sup_{t\in[0,1]} |K^n(\mathbf{1}_{[0,t]}) - t| \Big\|_1 \le 10h + \Big\| \sup_{f\in\mathcal{F}_h} |K^n(f) - \lambda(f)| \Big\|_1 \tag{6.65}$$

where $\mathcal{F}_h = \{f^+_{t,h}, f^-_{t,h}, t \in [2h, 1-h]\}$.

Note that the functions belonging to $\mathcal{F}_h$ are infinitely differentiable, so that one can easily find some upper bounds on their Fourier coefficients. More precisely, by two elementary integrations by parts, we obtain that there exist a positive constant C such that, for any $f \in \mathcal{F}_h$,

$$|\hat{f}(k)| \le \frac{C}{h(|k|+1)^2}\,. \tag{6.66}$$

From (6.62) and (6.66), we get that

$$\sup_{f\in\mathcal{F}_h} \|K^n(f) - \lambda(f)\|_{\infty,\lambda} \le \frac{C}{h} \sum_{k\in\mathbb{Z}^*} |k|^{-2} |\cos(2\pi k a)|^n\,. \tag{6.67}$$

Take $\beta \in (0, 1/2)$. By the properties of the Gamma function there exists a positive constant K such that,

$$\sum_{n \geq 1} \frac{1}{n^\beta} |\cos(2\pi ka)|^n \leq \frac{K}{(1 - |\cos(2\pi ka)|)^{1-\beta}}\,.$$

Since $(1 - |\cos(\pi u)|) \geq \pi(d(u, \mathbf{Z}))^2$, we derive that

$$\sum_{n \geq 1} \frac{1}{n^\beta} \sum_{k \in \mathbf{Z}^*} |k|^{-2} |\cos(2\pi ka)|^n \leq \frac{K}{\pi^{1-\beta}} \sum_{k \in \mathbf{Z}^*} \frac{|k|^{-2}}{(d(2ka, \mathbf{Z}))^{2-2\beta}}\,.$$

Note that, if a is badly approximable by rationals in the weak sense, then so is $2a$. Therefore if a satisfies (4.10), proceeding as in the proof of Lemma 5.1 in [10], we get that, for any $\eta > 0$,

$$\sum_{k=2^N}^{2^{N+1}-1} \frac{1}{(d(2ka, \mathbf{Z}))^{2-2\beta}} = \mathcal{O}(2^{(2-2\beta)N(1+\eta)})\,.$$

Therefore, since $\beta \in (0, 1/2)$, taking η close enough to 0, we get

$$\sum_{n \geq 1} \frac{1}{n^\beta} \sum_{k \in \mathbf{Z}^*} |k|^{-2} |\cos(2\pi ka)|^n \ll \sum_{N \geq 0} 2^{(2-2\beta)N(1+\eta)} \max_{2^N \leq k \leq 2^{N+1}} |k|^{-2} < \infty\,. \tag{6.68}$$

From (6.67) and (6.68), for any c in $(0, 1)$ there exists a constant B such that

$$\sup_{f \in \mathcal{F}_h} \|K^n(f) - \lambda(f)\|_{\infty,\lambda} \leq Bn^{-c}h^{-1}\,. \tag{6.69}$$

From (6.65) and (6.69), we infer that, for any c in $(0, 1)$ there exists a constant κ such that

$$\Big\| \sup_{t \in [0,1]} |K^n(\mathbf{1}_{[0,t]}) - t| \Big\|_1 \leq \kappa\big(h + n^{-c}h^{-1}\big)\,.$$

Taking $h = n^{c/2}$ in the above inequality, we then get Lemma 4.2. $\diamond$

6.3.4 Proof of Corollary 4.1

The first part of Corollary 4.1 follows immediately from Lemma 4.2 and Theorem 4.2 applied to $(X_i)_{i \in \mathbb{Z}}$ and the sequence (J_n) of intervals on the circle defined by $J_n = [nt, nt + n^{-\delta}]$. In order to prove the second part, we will apply Theorem 4.1(iii) to the sequence $(X_i)_{i \in \mathbb{Z}}$. The main step is to prove that

$$\sup_{n > 0} \frac{1}{E_n} \Big\| \sum_{k=1}^{n} \mathbf{1}_{J_k}(X_0) \Big\|_\infty < \infty. \tag{6.70}$$

Now $E_n \sim n^{1-\delta}/(1 - \delta)$ as $n \to \infty$. Therefrom one can easily see that (6.70) follows from the inequality below: for some positive constant c_0,

$$\sum_{k=m+1}^{2m} \mathbf{1}_{J_k} \leq c_0 m^{1-\delta} \text{ for any integer } m > 0. \tag{6.71}$$

Now $\sum_{k=m+1}^{2m} \mathbf{1}_{J_k}(x) \leq \sum_{k=m+1}^{2m} \mathbf{1}_{x-m^{-\delta}\leq kt\leq x}$. Furthermore, if t is badly approximable, then, from (4.10) with $\varepsilon = 0$, $d(kt, lt) = d(t(l-k), \mathbb{Z}) \geq c(l-k)^{-1} \geq c/m$ for any (k, l) such that $m < k < l \leq 2m$, which ensures that $\sum_{k=m+1}^{2m} \mathbf{1}_{x-m^{-\delta}\leq kt\leq x} \leq 1 + c^{-1}m^{1-\delta}$ for any x. This inequality and the above facts imply (6.71) and, consequently, (6.70). Now Corollary 4.1 follows easily from Lemma 4.2, (6.70) and Theorem 4.1(iii) $\diamond$

6.4 Proofs of the results of Section 5

6.4.1 Proof of Theorem 5.2.

Recall that, for any Polish space E, there exists a one to one bimeasurable mapping from E onto a Borel subset of $[0, 1]$. Consequently we may assume without loss of generality that $E = [0, 1]$. We define the Markov chain and the renewal process in the same way as in Subsection 9.3 in Rio (2017). Let $(U_i, \varepsilon_i)_{i\geq 0}$ be a sequence of independent random variables with the uniform law over $[0, 1]^2$ and ζ_0 be a random variable with law μ independent of $(U_i, \varepsilon_i)_{i\geq 0}$. Let $(\xi_k)_{k>0}$ be a sequence of independent random variables with law ν. Suppose furthermore that this sequence $(\xi_k)_{k>0}$ is independent of the σ-field generated by ζ_0 and $(U_i, \varepsilon_i)_{i\geq 0}$. Define the stochastic kernel Q_1 by

$$Q_1(x, A) = (1 - s(x))^{-1}(P(x.A) - s(x)\nu(A)) \text{ if } s(x) < 1 \text{ and } Q_1(x, A) = \nu(A) \text{ if } s(x) = 1 \tag{6.72}$$

and the conditional distribution function G_x by

$$G_x(t) = Q_1(x, \,]-\infty, t]\,) \text{ for any } (x, t) \in [0, 1] \times [0, 1]. \tag{6.73}$$

Define the sequence $(X_n)_{n>0}$ by induction in the following way: $X_0 = \zeta_0$ and

$$X_{n+1} = \xi_{n+1} \text{ if } s(X_n) \geq U_n \text{ and } X_{n+1} = G^{-1}_{X_n}(\varepsilon_n) \text{ if } s(X_n) < U_n. \tag{6.74}$$

Then the sequence $(X_n)_{n\geq 0}$ is a Markov chain with kernel P and initial law μ. The incidence process $(\eta_n)_{n\geq 0}$ is defined by $\eta_n = \mathbf{1}_{U_n\leq s(X_n)}$ and the renewal times $(T_k)_{k\geq 0}$ by

$$T_k = 1 + \inf\{j \geq 0 : \eta_0 + \cdots + \eta_j = k + 1\}. \tag{6.75}$$

We also set $\tau_j = T_{j+1} - T_j$ for any $j \geq 0$. Under the assumptions of Theorem 5.2, $(\tau_j)_{j\geq 0}$ is a sequence of integrable, independent and indentically distributed random variables. Note also that (5.3) implies that $T_0 < \infty$ almost surely (see Rio (2017), Subsection 9.3). Hence, by the strong law of large numbers,

$$\lim_{k\to\infty}(T_k/k) = \mathbb{E}(\tau_1) \text{ a.s.} \tag{6.76}$$

Let m be a positive integer such that $m > \mathbb{E}(\tau_1)$. Then there exists some random integer k_0 such that $T_k \leq km$ for any $k \geq k_0$. Since the sequence of sets $(A_j)_{j>0}$ is non-increasing, it follows that $\mathbf{1}_{A_{T_k}} \geq \mathbf{1}_{A_{km}}$ for any $k \geq k_0$. Furthermore

$$\sum_{k>0} \mathbf{1}_{A_k}(X_k) \geq \sum_{k>0} \mathbf{1}_{A_{T_k}}(X_{T_k}). \tag{6.77}$$

Consequently, if $\sum_{k>0} \mathbf{1}_{A_{km}}(X_{T_k}) = \infty$ a.s., then a.s. $\sum_{k>0} \mathbf{1}_{A_k}(X_k) = \infty$. Now, from the construction of the Markov chain, the random variables $(X_{T_k})_{k>0}$ are iid with law ν. Next, since the sequence of sets $(A_j)_{j>0}$ is non-increasing and $\sum_k \nu(A_k) = \infty$, the series $\sum_k \nu(A_{km})$ is divergent. Hence, by the second Borel-Cantelli lemma for sequences of independent events, $\sum_{k>0} \mathbf{1}_{A_{km}}(X_{T_k}) = \infty$ a.s., which completes the proof of Theorem 5.2. $\diamond$

6.4.2 Proof of Theorem 5.4.

From Lemma 9.3 in Rio (2017), the stochastic kernel P is irreducible, aperiodic and positively recurrent. Furthermore

$$\mu = \left(\int_E \frac{1}{s(x)} d\nu(x) \right)^{-1} \frac{1}{s(x)} \, \nu$$

is the unique invariant law under P. Now, let $(X_i)_{i \in \mathbb{N}}$ denote the strictly stationary Markov chain with kernel P. Define the renewal times T_k as in (6.75). Then the random variables $(X_{T_k})_{k>0}$ are iid with law ν. Since $\sum_{k>0} \nu(A_k) < \infty$, it follows that $\sum_{k>0} \mathbf{1}_{A_k}(X_{T_k}) < \infty$ almost surely. Now $T_k \geq k$, from which $A_{T_k} \subset A_k$. Hence

$$\mathbb{P}(X_{T_k} \in A_{T_k} \text{ infinitely often }) = 0. \tag{6.78}$$

Since $Q_1(x, .) = \delta_x$, $X_m = X_{T_k}$ for any m in $[T_k, T_{k+1}[$. Furthermore $A_m \subset A_{T_k}$ for any $m \geq T_k$. Consequently, if X_{T_k} does not belong to A_{T_k}, then, for any m in $[T_k, T_{k+1}[$, X_m does not belong A_m. Now (6.78) and the above fact imply Theorem 5.4. $\diamond$

A Uniform integrability

In this section, we recall the definition of the uniform integrability and we give a criterion for the uniform integrability of a family $(Z_i)_{i \in I}$ of nonnegative random variables. We first recall the usual definition of uniform integrability, as given in Billingsley (1999).

Definition A.1. A family $(Z_i)_{i \in I}$ of nonnegative random variable is said to be uniformly integrable if $\lim_{M \to +\infty} \sup_{i \in I} \mathbb{E}\big(Z_i \mathbf{1}_{Z_i > M}\big) = 0$.

Below we give a proposition, which provides a more convenient criterion. In order to state this proposition, we need to introduce some quantile function.

Notation A.1. Let Z be a real-valued random variable and H_Z be the tail function of Z, defined by $H_Z(t) = \mathbb{P}(Z > t)$ for any real t. We denote by Q_Z the cadlag inverse of H_Z.

Proposition A.1. *A family $(Z_i)_{i \in I}$ of nonnegative random variables is uniformly integrable if and only if*

$$\lim_{\varepsilon \searrow 0} \sup_{i \in I} \int_0^\varepsilon Q_{Z_i}(u) du = 0. \tag{A.1}$$

Proof. Assume that the family $(Z_i)_{i\in I}$ is uniformly integrable. Let U be a random variable with uniform distribution over $[0,1]$. Since $Q_{Z_i}(U)$ has the same distribution as Z_i,

$$\sup_{i\in I}\int_0^\varepsilon Q_{Z_i}(u)du \le M\varepsilon + \sup_{i\in I}\mathbb{E}\big(Z_i\mathbf{1}_{Z_i>M}\big).$$

Choosing $M = \varepsilon^{-1/2}$ in the above inequality, we then get (A.1). Conversely, assume that condition (A.1) holds true. Then one can easily prove that $A := \sup_{i\in I}\mathbb{E}(Z_i) < \infty$. It follows that $\mathbb{P}(Z_i > A/\varepsilon) \le \varepsilon$, which ensures that $Q_{Z_i}(\varepsilon) \le A/\varepsilon$. Consequently, for any $i \in I$,

$$\mathbb{E}\big(Z_i\mathbf{1}_{Z_i>A/\varepsilon}\big) = \int_0^\varepsilon Q_{Z_i}(u)\mathbf{1}_{Q_{Z_i}(u)>A/\varepsilon}\,du \le \int_0^\varepsilon Q_{Z_i}(u)du,$$

which implies the uniform integrability of $(Z_i)_{i\in I}$. $\diamond$

B Criteria under pairwise correlation conditions

Proposition B.1. *Let $(B_k)_{k>0}$ be a sequence of events in $(\Omega,\mathcal{T},\mathbb{P})$ such that $\mathbb{P}(B_1) > 0$ and $\sum_{k>0}\mathbb{P}(B_k) = \infty$. Set $E_n = \sum_{k=1}^n \mathbb{P}(B_k)$. Assume that there exist a non-increasing sequence $(\gamma_n)_n$ of reals in $[0,1]$ and sequences $(\alpha_n)_n$ and $(\varphi_n)_n$ of reals in $[0,1]$ such that for any integers k and n,*

$$\big|\mathbb{P}(B_k\cap B_{k+n}) - \mathbb{P}(B_k)\mathbb{P}(B_{k+n})\big| \le \gamma_n\mathbb{P}(B_k)\mathbb{P}(B_{k+n}) + \varphi_n\big(\mathbb{P}(B_k)+\mathbb{P}(B_{k+n})\big) + \alpha_n\,.$$

(i) *Assume that*

$$\gamma_n \to 0\ ,\ E_n^{-1}\sum_{k=1}^n \varphi_k \to 0 \ \text{ and } \ E_n^{-2}\sum_{k=1}^n\sum_{j=1}^k \min(\alpha_j,\mathbb{P}(B_k)) \to 0\,,\ \text{ as } n\to\infty\,. \tag{B.2}$$

Then $(B_k)_{k>0}$ is a L^1 Borel-Cantelli sequence.

(ii) *Assume that*

$$\sum_{k\ge1}\frac{\gamma_k}{k} < \infty\ ,\ \sum_{k\ge1}\frac{\varphi_k}{E_k} < \infty \ \text{ and } \ \sum_{k\ge1}E_k^{-2}\sum_{j\in[1,k]}\min(\alpha_j,\mathbb{P}(B_k)) < \infty\,. \tag{B.3}$$

Then $(B_k)_{k>0}$ is a strongly Borel-Cantelli sequence.

Remark B.1. If $\alpha_n = \mathcal{O}(n^{-a})$ with $a \in]0,1[$, then $\sum_{j=1}^k \alpha_j = \mathcal{O}(k^{1-a})$. Hence the third condition in (B.2) holds as soon as $n^{-1+(a/2)}E_n \to \infty$. On the other hand, the third condition in (B.3) holds as soon as $\sum_{n\ge1} n^{1-a}E_n^{-2} < \infty$ (note that this latter condition is satisfied when $n^{-1+a/2}(\log n)^{-(1/2+\varepsilon)}E_n \to \infty$ for some $\varepsilon > 0$).

If $\alpha_n = \mathcal{O}(n^{-1})$ then $\sum_{j=1}^k \alpha_j = \mathcal{O}(\log k)$. Hence the third condition in (B.2) holds as soon as $E_n(n\log n)^{-1/2} \to \infty$. On the other hand, the third condition in (B.3) holds as soon as $\sum_{n\ge1}(n\log n)/E_n^2 < \infty$ (note that this latter condition is satisfied when $n^{-1/2}(\log n)^{-(1+\varepsilon)}E_n \to \infty$ for some $\varepsilon > 0$).

If $\alpha_n = \mathcal{O}(n^{-a})$ with $a > 1$, then $\sum_{j=1}^{\infty} \min(\alpha_j, \mathbb{P}(B_k)) = \mathcal{O}\big(\mathbb{P}(B_k)^{1-1/a}\big)$. Hence the third condition in (B.2) holds if $n^{-1/(a+1)} E_n \to \infty$ (use the fact that $\sum_{k=1}^{n} \mathbb{P}(B_k)^{1-1/a} \leq n(E_n/n)^{1-1/a}$). Next, the third condition in (B.3) holds as soon as $\sum_{n\geq 1} E_n^{-2}\mathbb{P}(B_n)^{1-1/a} < \infty$ (note that this latter condition is satisfied when $\mathbb{P}(B_n) \geq n^{-a/(a+1)}(\log n)^{a/(a+1)+\varepsilon}$ for some $\varepsilon > 0$).

If $\alpha_n = \mathcal{O}(a^n)$ with $a \in]0,1[$ then $\sum_{j=1}^{\infty} \min(\alpha_j, \mathbb{P}(B_k)) = O\big(\mathbb{P}(B_k) \log\big(e/\mathbb{P}(B_k)\big)\big)$. Hence the third condition in (B.2) holds as soon as $n\,\mathbb{P}(B_n) \to \infty$. On the other hand, the third condition in (B.3) holds as soon as $\mathbb{P}(B_n) \geq n^{-1}(\log n)^{\varepsilon}$ for some $\varepsilon > 0$.

Proof of Proposition B.1. Note that

$$\begin{aligned}\max_{k\leq n} \mathrm{Var} S_k &\leq E_n + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \Big(\gamma_{j-i}\mathbb{P}(B_i)\mathbb{P}(B_j) + \varphi_{j-i}\big(\mathbb{P}(B_i)+\mathbb{P}(B_j)\big) + (\mathbb{P}(B_j)\wedge\alpha_{j-i})\Big) \\ &\leq E_n\big(1 + 4\sum_{k=1}^{n-1}\varphi_k\big) + 2\sum_{j=2}^{n}\sum_{k=1}^{j-1}(\mathbb{P}(B_j)\wedge\alpha_k) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\gamma_{j-i}\mathbb{P}(B_i)\mathbb{P}(B_j)\,. \qquad \text{(B.4)}\end{aligned}$$

Moreover, for any positive integer m,

$$\begin{aligned}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\gamma_{j-i}\mathbb{P}(B_i)\mathbb{P}(B_j) &\leq \sum_{i=1}^{n-1}\sum_{j=i+1}^{(i+m-1)\wedge n}\mathbb{P}(B_i) + \gamma_m\sum_{i=1}^{n-1}\sum_{j=i+m}^{n}\mathbb{P}(B_i)\mathbb{P}(B_j) \\ &\leq mE_n + \gamma_m E_{n-1}E_n\,. \qquad \text{(B.5)}\end{aligned}$$

Now, from (B.4) and (B.5), one easily infers that criteria (1.5) holds true under (B.2), which proves Item (i) of Proposition B.1.

To prove Item (ii), we shall apply criteria (2.5). Starting from (B.4) and using the facts that $\sum_{j\geq k} E_j^{-2}\mathbb{P}(B_j) \leq E_{k-1}^{-1}$ and $\sum_{n\geq j} E_n^{-3}\mathbb{P}(B_n) \leq E_{j-1}^{-2}$, we get that

$$\sum_{n\geq 2}\frac{\mathbb{P}(B_n)}{4E_n^3}\max_{k\leq n}\mathrm{Var}S_k \leq \frac{2}{E_1} + \sum_{k\geq 2}\frac{\varphi_k}{E_{k-1}} + \sum_{j\geq 2}\sum_{k=1}^{j-1}\frac{\mathbb{P}(B_j)\wedge\alpha_k}{E_{j-1}^2} + \sum_{n\geq 2}\frac{\mathbb{P}(B_n)}{E_n^2}\min_{m\geq 1}\big(m + \gamma_m E_{n-1}\big).$$

By the second and the third conditions in (B.3), it follows that (2.5) will be satisfied if

$$\sum_{n\geq 2}\frac{\mathbb{P}(B_n)}{E_n^2}\min_{m\geq 1}\big(m + \gamma_m E_{n-1}\big) < \infty\,. \qquad \text{(B.6)}$$

Define the function $\psi : [1,\infty[\mapsto [0,\infty[$ by $\psi(x) = (\gamma_{[x]}/[x])$ and let ψ^{-1} denote the cadlag generalized inverse function of ψ. Let $m_n = \psi^{-1}(E_{n-1}^{-1})$. Then $m_n \geq 1$ and

$$\min_{m\geq 1}\big(m + \gamma_m E_{n-1}\big) \leq m_n + m_n\psi(m_n)E_{n-1} \leq 2m_n,$$

since $\psi(\psi^{-1}(x)) \leq x$. Using the fact that $x \mapsto \psi^{-1}(1/x)$ is non-decreasing, it follows that

$$\begin{aligned}\sum_{n\geq 2}\frac{\mathbb{P}(B_n)}{E_n^2}\min_{m\geq 1}\big(m + \gamma_m E_{n-1}\big) &\leq 2\sum_{n\geq 2}\frac{\mathbb{P}(B_n)}{E_n^2}\psi^{-1}(E_{n-1}^{-1}) \\ &\leq 2\int_0^{1/E_1}\psi^{-1}(x)dx = 2\int_{\psi^{-1}(1/E_1)}^{\infty}\psi(y)dy + \frac{2}{E_1}\psi^{-1}(1/E_1)\,,\end{aligned}$$

which is finite under the first part of condition (B.3). This ends the proof of Item (ii). $\diamond$

References

[1] Billingsley, P. Convergence of Probability Measures, Second Edition *John Wiley & Sons, New York*, 1999.

[2] Borel, E. Les probabilités dénombrables et leurs applications arithmtiques. *Palermo Rend.* **27**, (1909), 247-271 .

[3] Bradley, R. C. Introduction to strong mixing conditions. Vol. 1. *Kendrick Press, Heber City, UT*, 2007.

[4] Chandra, T. K. and Ghosal, S. Some elementary strong laws of large numbers: a review. *Frontiers in probability and statistics* (Calcutta, 1994/1995), 61–81, Narosa, New Delhi, 1998.

[5] Chernov, N. and Kleinbock, D. Dynamical Borel-Cantelli lemmas for Gibbs measures. *Israel J. Math.* **22** (2001), 1–27.

[6] Conze, J. and Raugi, A. Convergence of iterates of a transfer operator, application to dynamical systems and to Markov chains. *ESAIM Probab. Stat.* **7**, (2003), 115-146.

[7] Dedecker, J., Dehling, H.G. and Taqqu, M.S. Weak convergence of the empirical process of intermittent maps in L^2 under long-range dependence. *Stochastics and Dynamics* **15**, (2015), 29 pages.

[8] Dedecker, J., Gouëzel, S. and Merlevède, F. Some almost sure results for unbounded functions of intermittent maps and their associated Markov chains. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010), no. 4, 796–821.

[9] Dedecker, J. and Prieur, C. New dependence coefficients. Examples and applications to statistics. *Probability Theory and Related Fields* **132** (2005) 203-236.

[10] Dedecker, J. and Rio, E. On mean central limit theorems for stationary sequences. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** (2008), no. 3, 693–726.

[11] Delyon, B. Limit theorem for mixing processes. *Tech. Report IRISA, Rennes 1*, 546 (1990).

[12] Doukhan, P., Massart, P. and Rio, E. The functional central limit theorem for strongly mixing processes. Ann. Inst. H. Poincaré Probab. Statist. **30** (1994), no. 1, 63–82.

[13] Erdős, P. and Rényi, A., On Cantor's series with convergent $\sum 1/q_n$, *Ann. Univ. Sci. Budapest. Eötvös. Sect. Math.* **2**, (1959), 93-109. .

[14] Etemadi, N., Stability of sums of weighted nonnegative random variables. *J. Multivariate Anal.* **13** (1983), no. 2, 361-365.

[15] Gouëzel, S. A Borel-Cantelli lemma for intermittent interval maps. *Nonlinearity* **20** (2007), no. 6, 1491–1497.

[16] Haydn, N., Nicol, M., Persson, T. and Vaienti, S. A note on Borel-Cantelli lemmas for non-uniformly hyperbolic dynamical systems. *Ergodic Theory Dynam. Systems* **33** (2013), no. 2, 475–498.

[17] Iosifescu, M. and Theodorescu, R. Random processes and learning. Die Grundlehren der mathematischen Wissenschaften, Band 150. *Springer-Verlag, New York*, 1969.

[18] Kim, D.H. The dynamical Borel-Cantelli lemma for interval maps. *Discrete Contin. Dyn. Syst.* **17** (2007), no. 4, 891–900.

[19] Lévy, P. Théorie de l'addition des variables aléatoires. XVII+ 328 p. *Paris, Gauthier-Villars*, 1937.

[20] Liverani, C., Saussol, B. and Vaienti, S. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems* **19** (1999), no. 3, 671–685.

[21] Luzia, N. A Borel-Cantelli lemma and its applications. *Trans. Amer. Math. Soc.* **366** (2014), no. 1, 547–560.

[22] Nummelin, E. General irreducible Markov chains and nonnegative operators. *Cambridge Tracts in Mathematics*, **83**. *Cambridge University Press, Cambridge*, 1984.

[23] Philipp, W. Some metrical theorems in number theory. *Pacific Journal of Mathematics* **20** (1967), 109-127.

[24] Rio, E. Asymptotic theory of weakly dependent random processes. Translated from the 2000 French edition. Probability Theory and Stochastic Modelling, 80. *Springer, Berlin*, 2017.

[25] Rosenblatt, M. A central limt theorem ans a strong mixing condition. *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 43-47.

[26] Schmidt, W. M. Diophantine approximation. Lectures Notes in Math. ,**785**, Springer, Berlin.

[27] Tasche, D. On the second Borel-Cantelli lemma for strongly mixing sequences of events. *J. Appl. Probab.* **34** (1997), no. 2, 381–394.

[28] Volkonskiĭ, V. A. and Rozanov, Y. A., Some limit theorems for random functions I. *Theory Probab. Appl.* **4** (1959), 178-197.