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Consensus and Flocking in a Class of Cucker-Smale Systems under Communication Failures

Benoît Bonnet*and Émilien Flayac†

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Abstract

We study sufficient conditions for the emergence of consensus and flocking in a class of strongly cooperative non-linear multi-agent systems subject to communication failures. Our approach is based on a combination of Lyapunov analysis along with the formulation of a novel persistence condition for cooperative systems. The latter can be interpreted as a lower bound on the algebraic connectivity of the time-average of the interaction graph generated by the communication weights, and provides quantitative convergence rates towards consensus and flocking.

Keywords : Multi-agent systems, Persistence of excitation, Asymptotic Flocking, Lyapunov Methods.

1 Introduction

The study of emerging patterns in dynamical systems describing collective behaviour has been the object of an increasing attention in the past decades. There is by now a large literature devoted to the analysis of *cluster formation* in the class of so-called *cooperative systems*, see e.g. [31]. These systems are widely used, for example, to study crowd motion [11], robot swarms [3, 15] and animal groups such as bird flocks or fish schools [1, 4].

Since the seminal papers [12, 13], a great deal of interest has been manifested towards the analysis of the so-called *flocking behaviour* (see Definition 4 below), which describes the appearance of alignment patterns in second-order cooperative multi-agent systems. In [18], the authors proposed a simpler proof of the emergence of asymptotic flocking based on Lyapunov-type methods. One of the main strength of the latter approach is that it could be applied to both finite and infinite dimensional multi-agent systems, while providing a strong unifying framework for consensus and flocking analysis with very diverse interaction topologies (see e.g. [6, 25]). It also allowed to design efficient control laws for key models, see [7, 8, 29].

When communications between agents are subject to possibly severe failures, it is then of high interest to identify conditions under which convergence can still be guaranteed. For discrete-time first and second order systems, opinion formation models have been thoroughly investigated in a graph-theoretic framework, see for instance the seminal paper [27] and subsequent developments in [23, 33]. These contributions include in particular asymptotic flocking analysis in some specific classes of discrete and continuous time systems. Further results allowed to incorporate asymmetric communication rates and random communication failures e.g. in [14, 30], as well as stochastic perturbations described by Brownian motions [16, 17]. However, to the best of our knowledge, the problem of studying the convergence towards flocking for general non-linear time-continuous systems subject to *multiplicative* communication failures has not been fully addressed in the literature.

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In this paper, we investigate the formation of asymptotic flocking for general second-order multi-agent systems of the form

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} \xi_{ij}(t)\phi(|x_i(t) - x_j(t)|)(v_j(t) - v_i(t)), \end{cases}$$

where we assume that the interaction kernel $\phi \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ is **strictly positive**. The functions $\xi_{ij}(\cdot) \in L^\infty(\mathbb{R}_+, [0,1])$ represent **communication weights**, taking into account potential communication failures that can occur in the system (e.g. when $\xi_{ij}(t) = 0$, see Section 4 below). We require them to be *symmetric*, i.e. $\xi_{ij}(\cdot) = \xi_{ji}(\cdot)$, which means that the interaction graph of the system is *undirected* at almost every time. This simplifying assumption is fairly standard in consensus analysis, as it encodes the fact that the interactions between agents are fully reciprocal. One of the main motivations for this choice of communication rates is to propose a formalism allowing to encompass *random interaction failures*. This article is aimed at being the first step towards a more general theory for such systems, in which the $\xi_{ij}(\cdot)$ will be realisations of stochastic processes.

The main novelty of the approach developed in this work is the introduction of an adequate *persistence* condition on the time-varying weights. We shall say that the collection of weights $(\xi_{ij}(\cdot))_{1\le i,j\le N}$ (written $\xi_{ij}(\cdot)$ for conciseness) satisfy the persistence condition (PE) (see Definition 3 below) if there exist $(\tau, \mu) \in \mathbb{R}_+^* \times (0,1]$ such that

$$B\left(\left(\tfrac{1}{\tau}\textstyle\int_t^{t+\tau} \boldsymbol{L}_\xi(s)\mathrm{d}s\right)\boldsymbol{v}, \boldsymbol{v}\right) \ge \mu B(\boldsymbol{v}, \boldsymbol{v}),$$

for all $\boldsymbol{v} \in (\mathbb{R}^d)^N$. Here, $B(\cdot,\cdot)$ denotes the *variance bilinear form* over $(\mathbb{R}^d)^N$ (see Definition 2 below), and $\boldsymbol{L}_\xi(\cdot)$ is the time-dependent graph-Laplacian associated to the interaction weights $\xi_{ij}(\cdot)$ of the system, see (2) below. Similar integral conditions were already investigated in the literature (see e.g. [2, 22, 32]), but to the best of our knowledge this article is the first one to formulate a persistence condition in terms of positive-definiteness of the averaged graph-Laplacian with respect to the variance bilinear form.

Persistence conditions are quite standard in classical control theory (see e.g. [9, 10, 28]), and have proven their adaptability in stability theory, in particular in allowing to build *strict Lyapunov functions* for perturbed systems (see e.g. [20, 21, 24]). Besides their practical interest (e.g. study input-to-state stability), strict Lyapunov functions allow to recover quantitative convergence rates towards the equilibrium, which are crucial to ensure the formation of flocking behaviours in multi-agent systems.

In the context of cooperative dynamics, our notion of persistence of excitation has both a deep and simple meaning in terms of interaction topology. Indeed, it transcribes the fact that on average on a sliding time window, the interaction graph describing the multi-agent system is connected. It also imposes a uniform lower-bound on the *algebraic connectivity* of the time-averaged graph corresponding to the weights $(\xi_{ij}(\cdot))$, which is represented by the first non-zero eigenvalue of the graph-Laplacian (see e.g. [23]). This type of average connectedness assumption is standard when studying first-order time-varying interaction topologies (see e.g. [2, 5, 27]), and is even proven to be necessary for consensus to arise in a large number of cases in [27]. In the way we formulate it, this condition further encodes the idea that one only requires the system to be persistently exciting with respect to the agents which have *not reached flocking*.

Our approach can be related more precisely to other existing contributions as follows. In [27, 19] and other works in the literature, it is proven that asymptotic consensus can be recovered in oriented first-order systems under mere average connectivity assumptions on the interacting topology, together with other assumptions ensuring that the interactions are somewhat reciprocal. However in general, it is known that quantitative exponential convergence rates can only be recovered with strong average connectedness assumptions such as (PE), see also [22] for similar conditions. Moreover, notice that under the persistence assumption (PE), the interaction graph of the system need not be connected at all times, nor does it need to exhibit dwell times. This is stark in contract with other contributions on the topic of multi-agent flocking, such as [33]. In the subsequent work [23], the authors relaxed this connectivity assumption, at the price of restricting their attention to discrete-time systems in which the maximal spreading in position of the agents is *a priori bounded*. In this article, the difficulty to be handled is precisely that the agents may scatter too quickly and arbitrarily far away in the absence of communications. Indeed, our persistence assumption (PE) only imposes a lower-bound on the algebraic connectivity of the graph generated by the weights $\xi_{ij}(\cdot)$. Whence, the true interaction graph incorporating the magnitude function $\phi(\cdot)$ may a priori have an averaged connectivity which vanishes. We also stress that the model of communication failures considered in the present paper is substantially different from several other contributions of the literature, such as [16, 17]. In the latter, the communications of the agents are supposed to remain unaltered as the system evolves, and the random disturbances due to the environment are modelled by additive white noises.

The structure of the article is the following. In Section 2, we introduce our Lyapunov approach by recovering a known result of consensus formation for persistently excited first-order dynamics. We then extend this result in Section 3 to prove the formation of flocking in a class of Cucker-Smale type systems with strongly interacting kernels in the sense of Hypothesis **(K)**, which is the main result of this paper. In Section 4, we illustrate our persistence condition on a general class of communication weights, and we conclude with some remarks and open perspectives in Section 5.

2 Exponential consensus under persistent excitation for first-order dynamics

In this section, we introduce the main tools used throughout this article in the particular case of consensus formation. We study first-order cooperative systems of the form

$$\begin{cases} \dot{x}_i(t) = \frac{1}{N}\sum\limits_{j=1}^{N} \xi_{ij}(t)\phi(|x_i(t) - x_j(t)|)(x_j(t) - x_i(t)), \\ x_i(0) = x_i^0, \end{cases} \tag{CS$_1$}$$

where $(x_1^0, \ldots, x_N^0) \in (\mathbb{R}^d)^N$ is an initial datum.

From now on, we use the notation $\boldsymbol{x} = (x_1, \ldots, x_N)$ for the state in $(\mathbb{R}^d)^N$ and $\bar{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^N x_i$ for its mean value. For systems of the form (CS$_1$), we aim at studying the formation of *asymptotic consensus*, defined as follows.

Definition 1. *A solution $\boldsymbol{x}(\cdot)$ of* (CS$_1$) converges to consensus *if for any $i \in \{1, \ldots, N\}$ it holds*

$$\lim_{t \to +\infty} |x_i(t) - \bar{\boldsymbol{x}}(t)| = 0.$$

By the symmetry of the rates $\xi_{ij}(\cdot)$, the system (CS$_1$) can be rewritten in matrix form as

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}(t, \boldsymbol{x}(t))\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{CSM$_1$}$$

where $\boldsymbol{L} : \mathbb{R}_+ \times (\mathbb{R}^d)^N \to \mathcal{L}((\mathbb{R}^d)^N)$ is the so-called *graph-Laplacian*, defined by

$$(\boldsymbol{L}(t, \boldsymbol{x})\boldsymbol{y})_i := \frac{1}{N}\sum_{j=1}^{N} \xi_{ij}(t)\phi(|x_i - x_j|)(y_i - y_j). \tag{1}$$

In the sequel, we will also use the graph-Laplacian $\boldsymbol{L}_\xi(\cdot)$ associated to the weights $\xi_{ij}(\cdot)$, defined by

$$(\boldsymbol{L}_\xi(t)\boldsymbol{y})_i := \frac{1}{N}\sum_{j=1}^{N} \xi_{ij}(t)(y_i - y_j). \tag{2}$$

Observe that since the communication rates $\xi_{ij}(\cdot)$ are L^∞-functions, both $\boldsymbol{L}(\cdot, \cdot)$ and $\boldsymbol{L}_\xi(\cdot)$ are defined only for almost every $t \geq 0$.

The structure displayed in (1) is fairly general and allows for a comprehensive study of both consensus and flocking problems via Lyapunov methods. With this goal in mind, we introduce the following bilinear form in the spirit of [6, 7].

Definition 2. *The* variance bilinear form $B(\cdot,\cdot)$ *is*

$$B(\boldsymbol{x},\boldsymbol{y}) := \tfrac{1}{N}\textstyle\sum_{i=1}^{N}\langle x_i, y_i\rangle - \langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}\rangle. \tag{3}$$

It is symmetric and positive semi-definite.

The evaluation $B(\boldsymbol{x},\boldsymbol{x})$ of this bilinear form is the distance of a given $\boldsymbol{x} \in (\mathbb{R}^d)^N$ from the so-called *consensus manifold*

$$\mathscr{C} = \{\boldsymbol{x} \in (\mathbb{R}^d)^N \text{ s.t. } x_1 = \cdots = x_N\}.$$

It then follows that $B(\boldsymbol{x},\boldsymbol{x}) = 0$ if and only if $x_i = \bar{\boldsymbol{x}}$ for any index $i \in \{1,\ldots,N\}$, i.e. if $\boldsymbol{x}$ is a consensus. We now list some properties of $B(\cdot,\cdot)$ and $\boldsymbol{L}(\cdot,\cdot)$.

Proposition 1. *The graph-Laplacian $\boldsymbol{L}(t,\boldsymbol{x})$ is positive-semi definite with respect to $B(\cdot,\cdot)$. Moreover, vectors of the form $\boldsymbol{L}(t,\boldsymbol{x})\boldsymbol{y}$ have zero mean.*

Proof. By summing over $i \in \{1,\ldots,N\}$ the components in (1), one can check that $\boldsymbol{L}(t,\boldsymbol{x})\boldsymbol{y}$ has zero mean. This together with the symmetry of the communication rates $\xi_{ij}(\cdot)$ implies that

$$\begin{aligned} B(\boldsymbol{L}(t,\boldsymbol{x})\boldsymbol{y},\boldsymbol{y}) &= \tfrac{1}{N^2}\sum_{i,j=1}^{N} \xi_{ij}(t)\phi(|x_i - x_j|)\langle y_i, y_i - y_j\rangle \\ &= \tfrac{1}{2N^2}\sum_{i,j=1}^{N} \xi_{ij}(t)\phi(|x_i - x_j|)|y_i - y_j|^2 \geq 0, \end{aligned}$$

for all $\boldsymbol{x},\boldsymbol{y} \in (\mathbb{R}^d)^N$. □

We now introduce the concept of *persistence of excitation* for multi-agent systems.

Definition 3. *The weights $\xi_{ij}(\cdot)$ satisfy the* **persistence of excitation** *condition* (PE) *if there exist $(\tau,\mu) \in \mathbb{R}_+^* \times (0,1]$ such that*

$$B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau} \boldsymbol{L}_\xi(s)\mathrm{d}s\right)\boldsymbol{x},\boldsymbol{x}\right) \geq \mu B(\boldsymbol{x},\boldsymbol{x}), \tag{PE}$$

for all $\boldsymbol{x} \in (\mathbb{R}^d)^N$ and almost every $t \geq 0$.

Remark 1. *Condition* (PE) *only involves the communication weights $\xi_{ij}(\cdot)$ through $\boldsymbol{L}_\xi(\cdot)$ and not the state of the system. Moreover, it is formulated using the bilinear form $B(\cdot,\cdot)$, illustrating the fact that one only needs the persistence to hold along directions which are orthogonal to the consensus manifold $\mathscr{C}$. Finally,* (PE) *can be interpreted as a lower bound on the so-called* algebraic connectivity *(see e.g. [23]) of the time-average of the graphs with weights $\xi_{ij}(\cdot)$, as illustrated in Section 4 below.*

In the following theorem, we prove that solutions of (CS_1) exponentially converge to consensus when the persistence assumption (PE) holds. This result is not new, and can be derived from earlier works dealing with consensus in undirected graphs, see e.g. in [27, 5]. However, we believe that this exposition allows for a progressive introduction of some of the concepts that shall be necessary for the establishment of our main result Theorem 2.

Theorem 1 (Consensus). *Let $\phi(\cdot)$ be positive and $\xi_{ij}(\cdot)$ be weights such that* (PE) *holds with parameters $(\tau,\mu) \in \mathbb{R}_+^* \times (0,1]$. Then, any solution $\boldsymbol{x}(\cdot)$ of (CS_1) exponentially converges to consensus.*

Proof. Let $\|\boldsymbol{L}(t,\boldsymbol{x})\|_B$ be the operator norm of $\boldsymbol{L}(t,\boldsymbol{x})$ with respect to $B(\cdot,\cdot)$, defined by

$$\|\boldsymbol{L}(t,\boldsymbol{x})\|_B = \sup_{\boldsymbol{y}\in(\mathbb{R}^d)^N} \sqrt{\frac{B(\boldsymbol{L}(t,\boldsymbol{x})\boldsymbol{y},\boldsymbol{L}(t,\boldsymbol{x})\boldsymbol{y})}{B(\boldsymbol{y},\boldsymbol{y})}}.$$

Define the constant $c := \sup_{(t,\boldsymbol{x})} \|\boldsymbol{L}(t,\boldsymbol{x})\|_B^{1/2}$ – which is finite since $\phi(\cdot)$ is bounded –, and denote by X the *standard deviation* of $\boldsymbol{x} \in (\mathbb{R}^d)^N$, defined by

$$X := \sqrt{B(\boldsymbol{x},\boldsymbol{x})}.$$

By definition of $B(\cdot,\cdot)$, a solution $\boldsymbol{x}(\cdot)$ of (CS$_1$) asymptotically converges to consensus if and only if $X(\cdot)$ vanishes at infinity.

We introduce the time-dependent family of linear operators $\psi_\tau : \mathbb{R}_+ \to \mathcal{L}((\mathbb{R}^d)^N)$ along $\boldsymbol{x}(\cdot)$, given by

$$\psi_\tau(t) := (1+c^2)\tau \,\mathrm{Id} - \tfrac{1}{\tau}\int_t^{t+\tau}\int_t^s \boldsymbol{L}(\sigma,\boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s. \tag{4}$$

Then, $\psi_\tau(\cdot)$ is Lipschitz with pointwise derivative

$$\dot{\psi}_\tau(t) = \boldsymbol{L}(t,\boldsymbol{x}(t)) - \tfrac{1}{\tau}\int_t^{t+\tau} \boldsymbol{L}(s,\boldsymbol{x}(s))\mathrm{d}s. \tag{5}$$

By construction, it further holds that

$$\sqrt{\tau}X \le \sqrt{B(\psi_\tau(t)\boldsymbol{x},\boldsymbol{x})} \le \sqrt{(1+c^2)\tau}X. \tag{6}$$

We thus define the candidate Lyapunov function

$$\mathscr{X}_\tau(t) := \lambda X(t) + \sqrt{B(\psi_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}, \tag{7}$$

where $\lambda > 0$ is a tuning parameter and $\boldsymbol{x}(\cdot)$ solves (CSM$_1$). Notice that by (6), it holds that

$$(\lambda + \sqrt{\tau})X(t) \le \mathscr{X}_\tau(t) \le (\lambda + \sqrt{(1+c^2)\tau})X(t). \tag{8}$$

This type of construction is inspired from [24] and appears frequently in strict Lyapunov design for persistent systems.

By Proposition 1, any solution $\boldsymbol{x}(\cdot)$ of (CSM$_1$) has constant mean, i.e. $\bar{\boldsymbol{x}}(\cdot) \equiv \bar{\boldsymbol{x}}_0$. By invariance with respect to translation of (CSM$_1$), we can assume without loss of generality that $\bar{\boldsymbol{x}}(\cdot) \equiv 0$. We now want to prove that $\mathscr{X}_\tau(\cdot)$ satisfies a strict-dissipation inequality of the form

$$\dot{\mathscr{X}}_\tau(t) \le -\alpha \mathscr{X}_\tau(t), \tag{9}$$

for some $\alpha > 0$. With this goal, we first compute

$$\begin{aligned}\dot{\mathscr{X}}_\tau(t) = &-\tfrac{\lambda}{X(t)}B(\boldsymbol{L}(t,\boldsymbol{x}(t)),\boldsymbol{x}(t)) \\ &+ \tfrac{B(\dot{\psi}_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}{2\sqrt{B(\psi_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}} - \tfrac{B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\psi_\tau(t)\boldsymbol{x}(t))}{\sqrt{B(\psi_\tau(t)\boldsymbol{x}(t),\boldsymbol{x}(t))}}.\end{aligned}$$

By (5)-(6), it holds that

$$\begin{aligned}&\dot{\mathscr{X}}_\tau(t) \\ &\le -\tfrac{1}{2\sqrt{(1+c^2)\tau}X(t)}B\big((\tfrac{1}{\tau}\int_t^{t+\tau}\boldsymbol{L}(s,\boldsymbol{x}(s))\mathrm{d}s)\boldsymbol{y},\boldsymbol{y}\big) \\ &+ \tfrac{1}{\sqrt{\tau}X(t)}B\big((\tfrac{1}{\tau}\int_t^{t+\tau}\int_t^s \boldsymbol{L}(\sigma,\boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s)\boldsymbol{y},\boldsymbol{L}(t,\boldsymbol{y})\boldsymbol{y}\big) \\ &+ \tfrac{1}{\sqrt{\tau}X(t)}(\tfrac{1}{2} - \sqrt{(1+c^2)\tau} - \sqrt{\tau}\lambda)B(\boldsymbol{L}(t,\boldsymbol{y})\boldsymbol{y},\boldsymbol{y}),\end{aligned} \tag{10}$$

where we wrote $\boldsymbol{y} := \boldsymbol{x}(t)$ for conciseness.

To estimate the first line of (10), recall that first-order cooperative systems have compactly supported trajectories, see e.g. [29, Lemma 1] . Since $\phi(\cdot)$ is positive and continuous, there exists a positive constant C_0 depending only on $\boldsymbol{x}_0$, such that

$$\min_{1\le i,j\le N} \phi(|x_i(t) - x_j(t)|) \ge C_0,$$

for all times $t \geq 0$. By (1), this further implies that

$$
\begin{aligned}
B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{y},\boldsymbol{y}) &\geq \tfrac{1}{2N^2}\sum_{i,j=1}^{N} C_0\xi_{ij}(t)|y_i-y_j|^2 \\
&= C_0\, B(\boldsymbol{L}_\xi(t)\boldsymbol{y},\boldsymbol{y}),
\end{aligned}
$$

for any $\boldsymbol{y} \in (\mathbb{R}^d)^N$. By (PE), this further yields

$$
B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau} \boldsymbol{L}(s,\boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{x}(t),\boldsymbol{x}(t)\right) \geq C_0\mu X^2(t). \tag{11}
$$

For the second line of (10), one has that

$$
\begin{aligned}
&B\Big(\tfrac{1}{\tau}\big(\int_t^{t+\tau}\int_t^{s} \boldsymbol{L}(\sigma,\boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s\big)\boldsymbol{x}(t), \boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t)\Big) \\
&\leq \tau c^2 X(t)\sqrt{B\big(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t)\big)} \\
&\leq \tau c^2 X(t)\ \|\boldsymbol{L}(t,\boldsymbol{x}(t))^{1/2}\|_B\ \sqrt{B\big(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{x}(t)\big)} \\
&\leq \tau c^3\Big(\tfrac{\epsilon}{2}X(t)^2 + \tfrac{1}{2\epsilon}B\big(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{x}(t)\big)\Big),
\end{aligned} \tag{12}
$$

for any $\epsilon > 0$, by definition of $\|\cdot\|_B$ and by applying the Cauchy-Schwartz and Young inequalities. Merging (10)-(11)-(12) and recalling that $\boldsymbol{L}(\cdot,\cdot)$ is positive semi-definite, we obtain that

$$
\begin{aligned}
\dot{\mathscr{X}}_\tau(t) \leq& -\left(\tfrac{C_0\mu}{2\sqrt{(1+c^2)\tau}} - \tfrac{c^3\sqrt{\tau}}{2}\epsilon\right)X(t) \\
&+ \tfrac{1}{X(t)}\left(\tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon} - \lambda\right)B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{x}(t),\boldsymbol{x}(t)).
\end{aligned}
$$

Choosing furthermore the parameters

$$
\epsilon = \tfrac{C_0\mu}{2c^3\tau\sqrt{(1+c^2)}}, \qquad \lambda = \tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon}
$$

and using (8), we recover that

$$
\begin{aligned}
\dot{\mathscr{X}}_\tau(t) &\leq -\tfrac{C_0\,\mu}{4\sqrt{(1+c^2)\tau}}X(t) \\
&\leq -\tfrac{C_0\,\mu}{4\sqrt{(1+c^2)\tau}(\lambda+\sqrt{(1+c^2)\tau})}\mathscr{X}_\tau(t)
\end{aligned}
$$

so that (9) holds with a given constant $\alpha > 0$. By an application of Grönwall's Lemma, we obtain that $\lim_{t\to+\infty}\mathscr{X}_\tau(t) = 0$, and thus $\lim_{t\to+\infty}X(t) = 0$ by (8).

By definition of $X(\cdot)$, this implies that $\boldsymbol{x}(\cdot)$ exponentially converges to consensus with rate $\alpha > 0$. □

3 Flocking for Cucker-Smale systems with strong interactions

In this section, we prove the main result of this article, which is the formation of asymptotic flocking for a class of Cucker-Smale type subject to communication failures. These systems are of the form

$$
\begin{cases}
\dot{x}_i(t) = v_i(t), \\
\dot{v}_i(t) = \tfrac{1}{N}\sum_{j=1}^{N}\xi_{ij}(t)\phi(|x_i(t)-x_j(t)|)(v_j(t)-v_i(t)).
\end{cases} \tag{CS$_2$}
$$

Similarly to Section 2, (CS$_2$) can be rewritten in matrix form as follows using the graph-Laplacian

$$
\begin{cases}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{v}(t), \qquad\qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \\
\dot{\boldsymbol{v}}(t) &= -\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{v}(t), \quad \boldsymbol{v}(0) = \boldsymbol{v}_0.
\end{cases} \tag{CSM$_2$}
$$

We now recall the definition of *asymptotic flocking*.

Definition 4. *A solution* $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ *of* (CS$_2$) converges to flocking *if for any* $i \in \{1, \dots, N\}$ *it holds*

$$\sup_{t \geq 0} |x_i(t) - \bar{\boldsymbol{x}}(t)| < +\infty, \quad \lim_{t \to +\infty} |v_i(t) - \bar{\boldsymbol{v}}(t)| = 0.$$

For this problem, we assume that the interaction kernel $\phi(\cdot) \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ satisfies the following *strong interaction* assumption.

Hypotheses (K). *There exist positive constants* K, σ *along with a parameter* $\beta \in (0, \frac{1}{2})$ *such that*

$$\phi(r) \geq \frac{K}{(\sigma + r)^\beta}. \tag{13}$$

In particular, $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$*. Up to replacing* $\phi(\cdot)$ *by this lower estimate, we can assume without of generality that* $\phi(\cdot)$ *is* ***non-increasing****.*

Remark 2. *Hypothesis* **(K)** *is a strengthened version of the usual strong interaction condition, which requires that* $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$*, see e.g. [18]. Remark that here, we require that the Cucker-Smale exponent* β *be less that* $\frac{1}{2}$*, whereas in the literature the expected critical exponent beyond which unconditional flocking may fail to occur is* $\beta = 1$*.*

Remark 3. *When* $\phi(\cdot)$ *is bounded from below by a positive constant, flocking always occurs in the full-communication setting, see e.g. [12, 18, 29]. For systems with communications failures satisfying* (PE)*, this result is a simple consequence of Theorem 1. For positive kernels* $\phi(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}_+^*)$*, one can construct examples of initial conditions* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *for which flocking does not occur [7].*

One can check that solutions of (CSM$_2$) satisfy

$$\dot{\bar{\boldsymbol{x}}}(t) = \bar{\boldsymbol{v}}(t), \qquad \dot{\bar{\boldsymbol{v}}}(t) = 0.$$

By invariance properties of multi-agent systems, we can assume without loss generality that $\bar{\boldsymbol{x}}(\cdot) = \bar{\boldsymbol{v}}(\cdot) \equiv 0$. We define the standard deviation maps

$$X(t) := \sqrt{B(\boldsymbol{x}(t), \boldsymbol{x}(t))}, \quad V(t) := \sqrt{B(\boldsymbol{v}(t), \boldsymbol{v}(t))}.$$

As a consequence of the symmetry of $\xi_{ij}(\cdot)$, the system (CSM$_2$) is *weakly dissipative* in the sense that

$$\dot{X}(t) \leq V(t), \qquad \dot{V}(t) \leq 0. \tag{14}$$

In the seminal paper [18], the authors introduced a concise proof of the Cucker-Smale flocking based on the analysis of a system of *strictly dissipative inequalities.* More precisely, if it holds that

$$\dot{X}(t) \leq V(t), \quad \dot{V}(t) \leq -\phi(2\sqrt{N} X(t)) V(t), \tag{15}$$

with an interaction kernel $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$, then the system converges to flocking. Our aim is to adapt their strategy while taking into account possible communication failures. We prove the following main result of this paper.

Theorem 2 (Main result - Flocking)**.** *Let* $\phi(\cdot)$ *be positive, non-increasing and satisfying* **(K)***. Assume that the weights* $\xi_{ij}(\cdot)$ *are such that* (PE) *holds with parameters* $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$*. Then any solution* $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ *of* (CS$_2$) *converges to flocking.*

The proof of this result relies on the construction of a locally-strict and time-dependent Lyapunov function for (CSM$_2$), for which a system of inequalities akin to (15) holds **only on a bounded time interval**. This local-in-time strict dissipation allows us to recover the asymptotic flocking of the system by a reparametrisation of the time variable. To the best of our knowledge, this combination of strict Lyapunov design and flocking analysis via locally dissipative inequalities is fully new in the context of multi-agent systems.

Notation 1. *Define the* rescaled interaction kernel

$$\boldsymbol{\phi}_\tau(r) := \phi(2\sqrt{N}(r + \tau V(0))) \tag{16}$$

for any $r \geq 0$, and denote by $\boldsymbol{\Phi}_\tau(\cdot)$ its uniquely determined primitive which vanishes at $X(0)$, i.e.

$$\boldsymbol{\Phi}_\tau(X) := \int_{X(0)}^{X} \boldsymbol{\phi}_\tau(r)\mathrm{d}r. \tag{17}$$

We start the proof of Theorem 2 by a series of lemmas which will progressively highlight the role of the different assumptions made on the system.

Lemma 1. *Let $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ solve* (CSM$_2$)*. If* (PE) *holds with $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$, then*

$$B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau} \boldsymbol{L}(s, \boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \geq \mu\boldsymbol{\phi}_\tau(X(t))B(\boldsymbol{w}, \boldsymbol{w}), \tag{18}$$

for any $\boldsymbol{w} \in (\mathbb{R}^d)^N$, with $\boldsymbol{\phi}_\tau(\cdot)$ defined as in (16)*.*

Proof. By definition of $\boldsymbol{L}(\cdot, \cdot)$, it holds that

$$\begin{aligned} & B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau} \boldsymbol{L}(s, \boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \\ & \geq \tfrac{1}{2N^2}\sum_{i,j=1}^{N}\left(\tfrac{1}{\tau}\int_t^{t+\tau} \xi_{ij}(s)\phi(|x_i(s) - x_j(s)|)\mathrm{d}s\right)|w_i - w_j|^2 \\ & \geq \tfrac{1}{2N^2}\sum_{i,j=1}^{N}\left(\tfrac{1}{\tau}\int_t^{t+\tau} \xi_{ij}(s)\phi(2\sqrt{N}X(s))\mathrm{d}s\right)|w_i - w_j|^2, \end{aligned} \tag{19}$$

since $\phi(\cdot)$ is non-increasing. As a consequence of the weak dissipation (14), it further holds that

$$X(s) = X(t) + \int_t^s \dot{X}(\sigma)\mathrm{d}\sigma \leq X(t) + \tau V(0),$$

for all $s \in [t, t + \tau]$. By (19), and using again that $\phi(\cdot)$ is non-increasing, it holds

$$\begin{aligned} & B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau} \boldsymbol{L}(s, \boldsymbol{x}(s))\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \\ \geq \;& \frac{\phi\big(2\sqrt{N}(X(t)+\tau V(0))\big)}{2N^2}\sum_{i,j=1}^{N}\left(\tfrac{1}{\tau}\int_t^{t+\tau} \xi_{ij}(s)\mathrm{d}s\right)|w_i - w_j|^2 \\ = \;& \boldsymbol{\phi}_\tau(X(t))B\left(\left(\tfrac{1}{\tau}\int_t^{t+\tau} \boldsymbol{L}_\xi(s)\mathrm{d}s\right)\boldsymbol{w}, \boldsymbol{w}\right) \\ \geq \;& \boldsymbol{\phi}_\tau(X(t))\mu B(\boldsymbol{w}, \boldsymbol{w}), \end{aligned}$$

where we used (PE) in the last inequality. □

We now define the candidate Lyapunov function

$$\mathscr{V}_\tau(t) := \lambda(t)V(t) + \sqrt{B(\psi_\tau(t)\boldsymbol{v}(t), \boldsymbol{v}(t))}, \tag{20}$$

where $\psi_\tau(\cdot)$ is chosen in (4) and $\lambda(\cdot)$ is a smooth tuning curve. We have the following lemma.

Lemma 2. *For any $\epsilon_0 > 0$, there exists a time horizon $T_{\epsilon_0} > 0$ such that*

$$\dot{\mathscr{V}}_\tau(t) \leq -\tfrac{\mu\,\boldsymbol{\phi}_\tau(X(t))}{2\sqrt{(1+c^2)\tau}}V(t). \tag{21}$$

for all times $t \in [0, 2T_{\epsilon_0})$.

Proof. Following the proof of Theorem 1, we can estimate the time-derivative of $\mathscr{V}_\tau(\cdot)$ as

$$\begin{aligned}\dot{\mathscr{V}}_\tau(t) \leq &-\left(\tfrac{\mu\,\phi_\tau(X(t))}{2\sqrt{(1+c^2)\tau}} - \tfrac{c^3\sqrt{\tau}}{2}\epsilon(t) - \dot{\lambda}(t)\right)V(t)\\ &+\tfrac{1}{V(t)}\left(\tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon(t)} - \lambda(t)\right)B(\boldsymbol{L}(t,\boldsymbol{x}(t))\boldsymbol{v}(t),\boldsymbol{v}(t)).\end{aligned} \tag{22}$$

The two main differences with respect to the proof of Theorem 1 are the choice of *time-dependent families of parameters* $(\lambda(\cdot),\epsilon(\cdot))$ and the use of refined estimate (18) instead of directly using (PE).

We start by fixing for all times $t \geq 0$

$$\lambda(t) := \tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon(t)}. \tag{23}$$

This implies in particular that $\dot{\lambda}(t) = -\frac{c^3\sqrt{\tau}}{2\epsilon^2(t)}\dot{\epsilon}(t)$. We choose now $\epsilon(\cdot)$ as the solution of

$$\dot{\epsilon}(t) = \epsilon^3(t), \qquad \epsilon(0) = \epsilon_0,$$

for a given constant $\epsilon_0 > 0$, i.e.

$$\epsilon(t) = \tfrac{\epsilon_0}{\sqrt{1-2\epsilon_0^2 t}}, \tag{24}$$

for $t \in [0, 1/2\epsilon_0^2)$. Then, (22) reads as

$$\dot{\mathscr{V}}_\tau(t) \leq -\tfrac{\mu\,\phi_\tau(X(t))}{2\sqrt{(1+c^2)\tau}}V(t),$$

and (21) holds with $T_{\epsilon_0} = 1/4\epsilon_0^2$. □

Observe that (21) involves both $V(\cdot)$ and $\mathscr{V}_\tau(\cdot)$. We now derive an estimate involving solely $V(\cdot)$.

Lemma 3. *There exists a function $\epsilon_0 \in \mathbb{R}_+^* \mapsto X_M(\epsilon_0)$ such that $X(t) \leq X_M(\epsilon_0)$ for all $t \in [0, T_{\epsilon_0}]$. Moreover, for any $\epsilon_0 > 0$ one has that*

$$V(T_{\epsilon_0}) \leq \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right)V(0)\exp\left(-\tfrac{\mu\,\phi_\tau(X_M(\epsilon_0))}{4(\alpha_3+\beta_3\epsilon_0)\epsilon_0}\right), \tag{25}$$

where $\{\alpha_k,\beta_k\}_{k=1}^3$ depend on (c,τ).

Proof. Choose $\epsilon_0 > 0$ and denote by $(\lambda(\cdot),\epsilon(\cdot))$ the corresponding functions given respectively by (23)-(24). Similarly to (6), it holds that

$$\sqrt{\tau}V \leq \sqrt{B(\psi_\tau(t)\boldsymbol{v},\boldsymbol{v})} \leq \sqrt{(1+c^2)\tau}V.$$

By definition of $\mathscr{V}_\tau(\cdot)$ in (20), we then have that

$$\begin{cases}\mathscr{V}_\tau(t) \leq \left(\sqrt{(1+c^2)\tau} + \tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{\tau}}{2\epsilon_0}\right)V(t),\\ \mathscr{V}_\tau(t) \geq \left(\sqrt{\tau} + \tfrac{1}{2\sqrt{\tau}} + \tfrac{c^3\sqrt{2\tau}}{4\epsilon_0}\right)V(t),\end{cases}$$

for any $t \in [0, T_{\epsilon_0}]$, where we used the fact that $\epsilon(t) \in [\epsilon_0, \sqrt{2}\epsilon_0]$ on this time interval. By simple identification of the coefficients, these estimates can be rewritten as

$$\left(\tfrac{\alpha_2}{\epsilon_0} + \beta_2\right)V(t) \leq \mathscr{V}_\tau(t) \leq \left(\tfrac{\alpha_1}{\epsilon_0} + \beta_1\right)V(t), \tag{26}$$

for constants $\{\alpha_k,\beta_k\}_{k=1}^2$ depending on (c,τ).

We can further integrate (21) on $[0,t]$ to recover

$$\mathscr{V}_\tau(t) \leq \mathscr{V}_\tau(0) - \tfrac{\mu}{2\sqrt{(1+c^2)\tau}}\int_0^t \phi_\tau(X(s))V(s)\mathrm{d}s.$$

which in turn implies that

$$\begin{aligned} V(t) \leq & \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right) V(0) \\ & - \tfrac{\mu\epsilon_0}{\alpha_2'+\beta_2'\epsilon_0} \int_0^t \boldsymbol{\phi}_\tau(X(s))V(s)\mathrm{d}s, \end{aligned} \tag{27}$$

where $(\alpha_2', \beta_2') = 2\sqrt{(1+c^2)\tau}(\alpha_2, \beta_2)$. Recall now that $\dot{X}(s) \leq V(s)$ by (14) and apply the change of variable $r = X(s)$ in (27) to obtain that

$$\begin{aligned} V(t) & \leq \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right) V(0) - \tfrac{\mu\epsilon_0}{\alpha_2'+\beta_2'\epsilon_0} \int_{X(0)}^{X(t)} \boldsymbol{\phi}_\tau(r)\mathrm{d}r \\ & = \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right) V(0) - \tfrac{\mu\epsilon_0}{\alpha_2'+\beta_2'\epsilon_0} \boldsymbol{\Phi}_\tau(X(t)). \end{aligned} \tag{28}$$

Since $\boldsymbol{\phi}_\tau \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$, its primitive $\boldsymbol{\Phi}_\tau(\cdot)$ is a strictly increasing map which image continuously spans $\mathbb{R}_+$. It is therefore invertible, and for any $\epsilon_0 > 0$ there exists a radius $X_M(\epsilon_0)$ such that

$$X_M(\epsilon_0) = \boldsymbol{\Phi}_\tau^{-1}\left(\tfrac{2\sqrt{(1+c^2)\tau}(\alpha_1+\beta_1\epsilon_0)}{\mu\epsilon_0} V(0)\right). \tag{29}$$

Since $V(\cdot)$ is a non-negative quantity by definition, it necessarily follows by plugging (29) into (28) that $X(t) \leq X_M(\epsilon_0)$ on $[0, T_{\epsilon_0}]$.

Going back to the dissipative differential inequality (21) combined with (26), we can again use the fact that $\boldsymbol{\phi}_\tau(\cdot)$ is non-increasing to obtain

$$\dot{\mathscr{V}}_\tau(t) \leq -\tfrac{\mu\epsilon_0\, \boldsymbol{\phi}_\tau(X_M(\epsilon_0))}{(\alpha_3+\beta_3\epsilon_0)} \mathscr{V}_\tau(t)$$

for almost every $t \in [0, T_{\epsilon_0}]$, where we denoted $(\alpha_3, \beta_3) = 2\sqrt{(1+c^2)\tau}(\alpha_1, \beta_1)$. By an application of Grönwall Lemma to $\mathscr{V}_\tau(\cdot)$ along with (26), we conclude that

$$V(T_{\epsilon_0}) \leq \left(\tfrac{\alpha_1+\beta_1\epsilon_0}{\alpha_2+\beta_2\epsilon_0}\right) V(0) \exp\left(-\tfrac{\mu\, \boldsymbol{\phi}_\tau(X_M(\epsilon_0))}{4(\alpha_3+\beta_3\epsilon_0)\epsilon_0}\right)$$

where we used the fact that $T_{\epsilon_0} = 1/4\epsilon_0^2$. □

Building on the estimate (25) obtained in Lemma 3, we now conclude the proof of our main result Theorem 2. To lighten the computations, most of the argument will be carried out in terms of asymptotic estimates.

Notation 2. *We will use the notations*

$$f(x) \underset{x \to a}{\gtrsim} g(x) \quad \textit{and} \quad f(x) \underset{x \to a}{\lesssim} g(x),$$

to mean that a map $f(\cdot)$ is bounded from below (respectively from above) by a map which is equivalent to $g(\cdot)$ as $x \to a$.

Proof (Theorem 2). In order to recover the emergence of flocking in (CS_2), we look into the asymptotic behaviour of our estimates as $\epsilon_0 \to 0^+$, or equivalently as $T_{\epsilon_0} \to +\infty$. Using the analytical expression (29) of $X_M(\epsilon_0)$, we have that

$$\boldsymbol{\phi}_\tau(X_M(\epsilon_0)) = \boldsymbol{\phi}_\tau \circ \boldsymbol{\Phi}_\tau^{-1}\left(C_1 + \frac{C_2}{\epsilon_0}\right),$$

where C_1, C_2 are positive constants depending on the data of the problem. Moreover by integrating (13), we obtain that

$$\Phi(X) \geq \tfrac{K}{1-\beta}\Big((\sigma + X)^{1-\beta} - (\sigma + X(0))^{1-\beta}\Big),$$

which along with standard monotonicity properties of inverse functions and the fact that $\boldsymbol{\phi}_\tau(\cdot)$ is non-increasing, yields the existence of a positive constants C such that

$$\boldsymbol{\phi}_\tau \circ \boldsymbol{\Phi}_\tau^{-1}\left(C_1 + \frac{C_2}{\epsilon_0}\right) \underset{\epsilon_0 \to 0^+}{\gtrsim} C\epsilon_0^{\frac{\beta}{1-\beta}}. \tag{30}$$

Combining the expression (30) with (25) and recalling that $T_{\epsilon_0} = 1/4\epsilon_0^2$, we recover that

$$V(T_{\epsilon_0}) \underset{T_{\epsilon_0} \to +\infty}{\lesssim} \tfrac{\alpha_1}{\alpha_2} V(0) \exp\left(-\tfrac{C\mu}{8\alpha_3} T_{\epsilon_0}^{\frac{1-2\beta}{2(1-\beta)}} \right). \tag{31}$$

Since $\epsilon_0 \in \mathbb{R}_+^* \mapsto T_{\epsilon_0}$ continuously spans the whole of $\mathbb{R}_+$, we can reparametrise time using $T := T_{\epsilon_0}$. As we assumed in **(K)** that $\beta \in (0, \frac{1}{2})$, the estimate of (31) implies that

$$V(T) \underset{T \to +\infty}{\longrightarrow} 0.$$

We now turn our attention to the uniform boundedness of the position radius $X(\cdot)$. The weak-dissipativity (14) of (CSM_1) expressed in terms of the new time variable $T := T_{\epsilon_0}$ writes

$$\sup_{T \geq 0} X(T) \leq X(0) + \int_0^{+\infty} V(T) \mathrm{d}T.$$

This implies that $\sup_{T \geq 0} X(T) < +\infty$ as a consequence of (31) and of the fact that $\beta \in (0, \frac{1}{2})$, which concludes the proof of Theorem 2. $\square$

4 Illustration of the persistence condition

In this section, we exhibit a general situation in which (PE) holds. We start by fixing a constant $\mu \in (0, 1]$ and by recalling known facts about interaction graphs and their Laplacians, for which we refer the reader e.g. to [26].

Definition 5. *The* algebraic connectivity *of a graph with weights ξ_{ij} is the smallest non-zero eigenvalue of $\boldsymbol{L}_\xi$, denoted by $\lambda_2(\boldsymbol{L}_\xi)$.*

Lemma 4. *If an interaction graph with weights ξ_{ij} is such that $\lambda_2(\boldsymbol{L}_\xi) \geq \mu$, then*

$$B(\boldsymbol{L}_\xi \boldsymbol{v}, \boldsymbol{v}) \geq \mu B(\boldsymbol{v}, \boldsymbol{v}),$$

for any $\boldsymbol{v} \in (\mathbb{R}^d)^N$.

Proof. This follows from the definition of algebraic connectivity, along with the fact that

$$B(\boldsymbol{L}_\xi \, \boldsymbol{v}, \boldsymbol{v}) = \tfrac{1}{2N^2} \sum_{i,j=1}^{N} \xi_{ij} |v_i - v_j|^2,$$

for any $\boldsymbol{v} \in (\mathbb{R}^d)^N$. $\square$

Lemma 5. *Let $\boldsymbol{L}_{\xi_1}, \boldsymbol{L}_{\xi_2}$ be the graph-Laplacian associated to two interaction graphs with weights ξ_{ij}^1 and ξ_{ij}^2 respectively. Then*

$$\boldsymbol{L}_{\boldsymbol{\xi}} := \boldsymbol{L}_{\xi_1 + \xi_2} = \boldsymbol{L}_{\xi_1} + \boldsymbol{L}_{\xi_2},$$

is the graph-Laplacian of the union *of the two graphs, which weights are $\boldsymbol{\xi}_{ij} = \xi_{ij}^1 + \xi_{ij}^2$.*

From now on, we fix $\tau \in \mathbb{R}_+^*$, an integer $n \geq 1$, and time-dependent communication rates $\xi_{ij}(\cdot)$ which are constant on all the intervals of the form $[\frac{m\tau}{n}, \frac{(m+1)\tau}{n})$ for some $m \geq 0$.

Proposition 2. *Suppose that for all $m \geq 0$, the time-average of the graphs $(\xi_{ij}(\frac{m+k}{n}\tau))_{k=0}^{n-1}$, which weights are given by*

$$\boldsymbol{\xi}_{ij}^m := \tfrac{1}{n} \sum_{k=0}^{n-1} \xi_{ij} \left(\tfrac{m+k}{n} \tau \right), \tag{32}$$

is connected with $\lambda_2(\boldsymbol{L}_{\boldsymbol{\xi}^m}) \geq \mu$. Then (PE) *holds.*

Proof. For $m \geq 0$ and $t \in [\frac{m\tau}{n}, \frac{(m+1)\tau}{n})$, we have

$$\begin{aligned} \tfrac{1}{\tau}\textstyle\int_t^{t+\tau} \boldsymbol{L}_\xi(s)\mathrm{d}s = {} & \big(\tfrac{(m+1)}{n} - \tfrac{t}{\tau}\big)\boldsymbol{L}_\xi\big(\tfrac{m\tau}{n}\big) \\ & + \tfrac{1}{n}\sum_{k=1}^{n-1}\boldsymbol{L}_\xi\big(\tfrac{m+k}{n}\tau\big) \\ & + \big(\tfrac{t}{\tau} - \tfrac{m\tau}{n}\big)\boldsymbol{L}_\xi\big(\tfrac{(m+n)\tau}{n}\big). \end{aligned} \tag{33}$$

Now, remark that $\max\{\frac{(m+1)}{n} - \frac{t}{\tau}, \frac{t}{\tau} - \frac{m\tau}{n}\} \geq \frac{1}{2n}$. Without loss of generality, assume that $\frac{(m+1)}{n} - \frac{t}{\tau} \geq \frac{1}{2n}$, so that by (33) it holds that

$$\begin{aligned} & B\Big(\Big(\tfrac{1}{\tau}\textstyle\int_t^{t+\tau} \boldsymbol{L}_\xi(s)\mathrm{d}s\Big)\boldsymbol{v}, \boldsymbol{v}\Big) \\ & \geq B\Big(\Big(\tfrac{1}{2n}\sum_{k=0}^{n-1}\boldsymbol{L}_\xi\big(\tfrac{m+k}{n}\tau\big)\Big)\boldsymbol{v}, \boldsymbol{v}\Big) \\ & = B\Big(\boldsymbol{L}_{\boldsymbol{\xi}_{ij}^m/2}\,\boldsymbol{v}, \boldsymbol{v}\Big) \geq \tfrac{\mu}{2}B(\boldsymbol{v}, \boldsymbol{v}), \end{aligned}$$

for all $\boldsymbol{v} \in (\mathbb{R}^d)^N$, where the weights $\boldsymbol{\xi}_{ij}^m$ are defined as in (32). □

Corollary 1. *Suppose that the piecewise constant weights $\xi_{ij}(\cdot)$ take their values in a finite subset $I \subset [0,1]$. Then* (PE) *holds if and only if for all $m \geq 0$, the time-averaged graph which weights $\boldsymbol{\xi}_{ij}^m$ are given by* (32) *is connected.*

Proof : The direct implication of this statement is evident. For the converse, observe that since $I \subset [0,1]$ is a finite set, there exists a finite number of graphs with weights given by (32) which are connected. In particular, the quantity

$$\mu := \min\Big\{\lambda_2(\boldsymbol{L}_{\boldsymbol{\xi}^m}) \text{ s.t. } (\boldsymbol{\xi}_{ij}^m) \text{ are given by (32)}\Big\},$$

is positive and independent from $m \geq 0$. Thus, (PE) holds with parameters $(\tau, \mu) \in \mathbb{R}_+^* \times (0,1]$. □

We now illustrate these general results for piecewise constant communication rates on a simple example with $N = 4$ agents. For $\tau \in \mathbb{R}_+^*$ and $t \geq 0$, define the interactions weights as follows:

$$\begin{aligned} \xi_{14}(t) &= \begin{cases} 1 & \text{if } \lfloor t/\tau \rfloor = 1 \bmod[6], \\ 0 & \text{otherwise}, \end{cases} \\ \xi_{34}(t) &= \begin{cases} 1 & \text{if } \lfloor t/\tau \rfloor = 3 \bmod[6], \\ 0 & \text{otherwise}, \end{cases} \\ \xi_{23}(t) = \xi_{24}(t) &= \begin{cases} 1 & \text{if } \lfloor t/\tau \rfloor = 5 \bmod[6], \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{34}$$

where $\lfloor\cdot\rfloor$ denotes the lower integer part, and we set all the other weights to 0. In this example, we chose $n = 6$ so that our signals are piecewise constant on intervals of the form $[\frac{m\tau}{6}, \frac{(m+1)\tau}{6})$ for any $m \geq 0$.

The weights $\xi_{ij}(\cdot)$ defined in (34) are such that the persistence condition (PE) holds. This can be verified e.g. by computing the smallest positive eigenvalue of the averaged graph-Laplacian matrix $\boldsymbol{L}_{\boldsymbol{\xi}^m}$, where $\boldsymbol{\xi}^m$ is defined as in (32) with $t \in [\frac{m\tau}{6}, \frac{(m+1)\tau}{6})$. In this example, the spectrum of $\boldsymbol{L}_{\boldsymbol{\xi}^m}$ for all $m \geq 0$ is

$$\mathrm{Sp}(\boldsymbol{L}_{\boldsymbol{\xi}^m}) = \{0, \tfrac{1}{6}, \tfrac{1}{2}, \tfrac{2}{3}\},$$

so that (PE) holds with $\tau \in \mathbb{R}_+^*$, $\mu = \lambda_2(\boldsymbol{L}_{\boldsymbol{\xi}^m}) = \frac{1}{6}$.

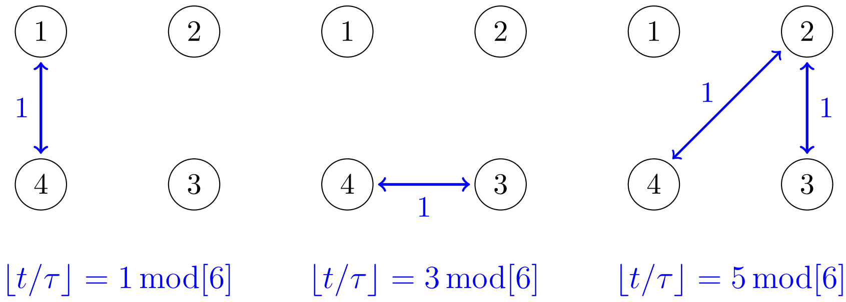

Figure 1: Admissible connections between agents

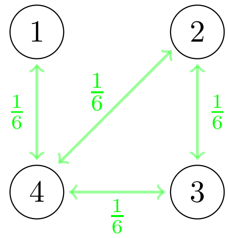

Figure 2: Averaged interaction graph on a time window of the form $[t, t+\tau]$

5 Conclusion and perspectives

In this article, we proved two results of convergence of multi-agent systems under arbitrary communication failures. If communication rates satisfy a persistence of excitation condition, then one has both exponential convergence to consensus for first-order systems (Theorem 1), and quantitative convergence to flocking for Cucker-Smale systems under an additional strong interaction condition (Theorem 2). For the sake of conciseness and readability, we assumed that the initial time of the non-stationary dynamics was fixed and equal to 0. Yet, it could be checked by repeating our argument that both convergence results are *uniform with respect to the initial time*. In the future, we aim to improve our main result Theorem 2 in three directions.

Firstly, we will investigate whether the rather surprising exponent range $\beta \in (0, \frac{1}{2})$ – which is currently necessary in order to ensure that asymptotic flocking occurs – has an intrinsic meaning, or if it is just appearing as a limit of our current choice of Lyapunov function. Answering this question might also pave the way for flocking results with weaker interactions, involving confinement conditions linking the initial state and velocity mean-deviations and the persistence parameters.

Secondly, we will study communication failures defined as the realisations of stochastic processes and try to see under which assumptions and in what sense the convergence towards consensus and flocking can occur (almost surely, in probability, etc...). In this setting, one of the main difficulty will most likely lie in the identificaton of proper stochastic generalisations of (PE).

Lastly, we will investigate whether our dissipative approach, applied here to the mean-square energies – which are L^2-functionals –, can be adapted to L^∞-type Lyapunov functionals in the spirit of [29, 17]. The motivation behind this line of study is that L^2-type functionals do not allow for the study of flocking formation in the mean-field setting as the number of agents N goes to infinity, while L^∞-type functionals do.

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