Proofs of the properties presented in the article
"Watershed hierarchies"
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Proofs of the properties presented in the article "Watersheding hierarchies"

This document contains the proofs of the properties and theorem presented in the following conference article:


A Proof of Property 3

Property 16. Let $P$ be a map from $\mathcal{R}(B)$ to $\mathbb{R}^+$. The map $P$ is an extinction map for $w$ if and only if the following statements hold true:

1. $\text{range}(P) = \{0, \ldots, n\}$;
2. for any two minima $M_1$ and $M_2$ if $P(M_1) = P(M_2)$, then $M_1 = M_2$; and
3. for any region $R$ of $B$, we have $P(R) = \lor\{P(M)\text{ such that } M\text{ is a minimum of } w\text{ included in } R\}$.

We prove the forward and backward implications of Property 3 in Property 17 and Property 18, respectively.

Property 17. Let $P$ be a map from $\mathcal{R}(B)$ to $\mathbb{R}^+$. If the map $P$ is an extinction map for $w$, then the following statements hold true:

1. $\text{range}(P) = \{0, \ldots, n\}$;
2. for any two minima $M_1$ and $M_2$ if $P(M_1) = P(M_2)$, then $M_1 = M_2$; and
3. for any region $R$ of $B$, we have $P(R) = \lor\{P(M)\text{ such that } M\text{ is a minimum of } w\text{ included in } R\}$.

Proof. Let $P$ be an extinction map. Then there is a sequence $S = (M_1, \ldots, M_n)$ of minima of $w$ such that $P$ is the extinction map for $S$. By Definition 2, for any minimum $M_i$, for $i$ in $\{1, \ldots, n\}$, we have $P(M_i) = i$ because $M_i$ is the only minimum of $w$ included in $M_i$. Therefore, for any two distinct minima $M_i$ and $M_j$, for $i$ in $\{1, \ldots, n\}$, we have $P(M_i) \neq P(M_j)$, which proves the condition 2 of Property 17. Since $w$ has $n$ minima, the extinction value of any region which includes a minimum (i.e. any non leaf region) is in the set $\{1, \ldots, n\}$. For any leaf region $R$ which do not include any minimum of $w$, we have $P(R) = 0$ by Definition 2. Therefore, the range of $P$ is $\{0, \ldots, n\}$, which corresponds to the first condition of Property 17. The third condition of Property 17 is part of the Definition 2, so its proof is trivial. $\square$

Property 18. Let $P$ be a map from $\mathcal{R}(B)$ to $\mathbb{R}^+$ such that:

1. $\text{range}(P) = \{0, \ldots, n\}$;
2. for any two minima $M_1$ and $M_2$ if $P(M_1) = P(M_2)$, then $M_1 = M_2$; and
3. for any region $R$ of $B$, we have $P(R) = \vee \{P(M) \; \text{such that} \; M \; \text{is a minimum of} \; w \; \text{included in} \; R\}$.

The map $P$ is an extinction map for $w$.

Proof. Let $P$ be a map from $\mathcal{R}(B)$ to $\mathbb{R}^+$ for which the statements 1, 2 and 3 hold true. To prove that $P$ is an extinction map, we have to show that there exists a sequence $S$ of $n$ pairwise distinct minima of $w$ such that, for any region $R$ of $B$, the value $P(R)$ is the extinction value of $R$ for $S$.

Let $S = (M_1,\ldots,M_n)$ be a sequence of $n$ pairwise distinct minima of $w$ ordered in non-decreasing order for $P$, i.e., for any two distinct minima $M_i$ and $M_j$, for $i$ and $j$ in $\{1,\ldots,n\}$, if $i < j$ then $P(M_i) \leq P(M_j)$.

By the statement 2, the sequence $S$ is unique. By the statement 3, for any region $R$ of $B$ such that there is no minimum of $w$ included in $R$, $P(R) = \vee\{\} = 0$, so $P(R)$ is the extinction value of $R$ for $S$.

Since $w$ has $n$ minima, for any minimum $M$ of $w$, the value $P(M)$ is in $\{1,\ldots,n\}$. Otherwise, if there existed a minimum $M'$ of $w$ such that $P(M') = 0$, then there would be a value $i$ in $\{1,\ldots,n\}$ such that for any minimum $M''$ of $w$ the value $P(M'')$ is different from $i$. Consequently, the range of $P$ would be $\{0,\ldots,n\} \setminus \{i\}$, which contradicts the statement 1. Therefore, for any minimum $M_i$, for $i$ in $\{1,\ldots,n\}$, we have that $P(M_i) = i$, so $P(M_i)$ is the extinction value of $M_i$ for $S$.

It follows that, by the statement 3, for any region $R$ such that there is a minimum of $B$ included in $R$, the value $P(R)$ is the maximum value $i$ in $\{1,\ldots,n\}$ such that $M_i$ is included in $R$.

Thus, for any region $R$ of $B$, the value $P(R)$ is the extinction value of $R$ for $S$. Therefore, the map $P$ is an extinction map of $w$. \hfill \Box

B Proof of Property 8

Property 19. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. The map $f$ is the saliency map of a hierarchical watershed of $(G,w)$ if and only if the map $\xi_f$ is an extinction map.

We prove the forward and backward implications of Property 8 in Property 20 and Property 29, respectively.

Property 20. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $f$ is the saliency map of a hierarchical watershed, then $\xi_f$ is an extinction map.

By Theorem 3 of [11], if $f$ is the saliency map of a hierarchical watershed, then $f$ is a one-side increasing map, which implies that:

1. range$(f) = \{0,\ldots,n-1\}$;
2. for any $u$ in $E$, $f(u) > 0$ if and only if $u \in WS(w)$, where $WS(w)$ is the set of watershed-cut edges of $w$; and
3. for any \( u \) in \( E \), there exists a child \( R \) of \( R_u \) such that \( f(u) \geq \vee \{f(v) \mid v \in R \} \).

In order to prove Property 20, we prove that three conditions for \( \xi_f \) to be an extinction map (Property 3) are satisfied in Properties 21, 22 and 26.

**Property 21.** Let \( f \) be the saliency map of a hierarchical watershed and let \( \xi_f \) be the approximated extinction map of \( f \). Then \( \text{range}(\xi_f) = \{0, \ldots, n\} \).

**Proof.** We need to prove that:

1. for any \( i \) in \( \{0, \ldots, n\} \), there is a region \( R \) of \( B \) such that \( \xi_f(R) = i \); and
2. for any region \( R \) of \( B \), we have \( \xi_f(R) \) in \( \{0, \ldots, n\} \).

**Proof of 1:**

Let \( R = V \), then \( \xi_f(R) = \vee^f(R) + 1 \). Since the range of \( f \) is \( \{0, \ldots, n - 1\} \), we have \( \vee^f(V) = \vee \{f(u) \mid R_u \subseteq V \} = n - 1 \). Then, \( \xi_f(R) = n - 1 + 1 = n \).

Let \( R \) be a leaf region of \( B \). Then the building edge of \( \text{parent}(R) \) is not in \( WS(w) \). Let \( u \) be the building edge of \( \text{parent}(R) \). Since \( f \) is a one-side increasing map and since \( u \) is not in \( WS(w) \), we have \( f(u) = 0 \). Since \( R \) is a leaf region, \( R \) is not a dominant region for \( f \) and then, by Definition 7, we have \( \xi_f(R) = f(u) = 0 \).

Now, we have to prove that, for any \( i \) in \( \{1, \ldots, n - 1\} \), there is a region \( R \) of \( B \) such that \( \xi_f(R) = i \). Since \( w \) has \( n \) minima, we can conclude that \( w \) has \( n - 1 \) watershed-cut edges. Since the range of \( f \) is \( \{0, \ldots, n - 1\} \) and the weight of the watershed-cut edges of \( w \) is strictly greater than zero, we can conclude that the watershed-cut edges of \( w \) have pairwise distinct weights for \( f \) from 1 to \( n - 1 \). Given any watershed-cut edge \( u \), we can affirm that exactly one of the children of \( R_u \) is a dominant region of \( f \) because, given the children \( X \) and \( Y \) of \( R_u \), we know that both \( X \) and \( Y \) are not leaf regions and we have either \( X \prec_{(f,w)} Y \) or \( Y \prec_{(f,w)} X \). Let \( Y \) be the child of \( R_u \), which is not a dominant region for \( f \). By Definition 7, we have \( \xi_f(Y) = f(u) \). Therefore, for any \( i \) in \( \{1, \ldots, n - 1\} \), there is a watershed-cut edge \( u \) in \( WS(w) \) such that \( f(u) = i \) and such that there is a child \( Y \) of \( R_u \) such that \( \xi_f(Y) = f(u) = i \).

**Proof of 2:**

Let \( R \) be a region of \( B \) and let \( u \) be the building edge of the parent of \( R \) if \( R \neq V \). By Definition 7, the value \( \xi_f(R) \) is: \( \vee^f(V) + 1, f(u) \) or \( \xi_f(\text{parent}(R)) \).

It is enough to prove that \( \vee^f(V) + 1 \) and \( f(u) \) are in \( \{0, \ldots, n\} \). Since the range of \( f \) is \( \{0, \ldots, n - 1\} \), it is clear that \( f(u) \) is in \( \{0, \ldots, n\} \). We can see that \( \vee^f(V) = n - 1 \) because any region of \( B \) is included in \( V \). Then \( \vee^f(V) + 1 = n \) which is in \( \{0, \ldots, n\} \).

**Property 22.** Let \( f \) be the saliency map of a hierarchical watershed and let \( \xi_f \) be the approximated extinction map of \( f \). For any two minima \( M_1 \) and \( M_2 \) of \( w \), if \( \xi_f(M_1) = \xi_f(M_2) \), then \( M_1 = M_2 \).

As established by Theorem 3 of [11], if \( f \) is the saliency map of a hierarchical watershed of \( (G, w) \), then \( f \) is one-side increasing for \( B \). To prove Property 22, we first present the Properties 23, 24 and 25.
Property 23. Let \( f \) be a one-side increasing map and let \( \xi_f \) be the approximated extinction map of \( f \). For any region \( X \) of \( B \), we denote by \( u_X \) the building edge of \( X \). For any region \( X \) such that there is a minimum of \( w \) strictly included in \( X \), there is a child \( Y \) of \( X \) such that:

- \( \xi_f(Y) = \xi_f(X) \);
- \( \xi_f(\text{sibling}(Y)) = f(u_X) \); and
- there is a minimum of \( w \) included in \( Y \).

Proof. Let \( X \) be a region such that there is at least one minimum of \( w \) strictly included in \( X \). By the definition of dominant regions (Definition 6), at most one of the children of \( X \) is a dominant region of \( B \). Since there is at least one minimum of \( w \) strictly included in \( X \), then there is a minimum included in at least one child of \( X \). Therefore, there is a child \( Y \) of \( X \) such that \( Y \) is a dominant region of \( B \) and, therefore \( \xi_f(Y) = \xi_f(X) \) (condition 2 of Definition 7). Thus, \( \text{sibling}(Y) \) is not a dominant region of \( B \) and \( \xi(\text{sibling}(Y)) = f(u) \), where \( u \) is the building edge of \( X \) (condition 3 of Definition 7).

Property 24. Let \( u \) be any watershed edge of \( w \) and let \( f \) be a one-side increasing map. There is a minimum \( M \) of \( w \) such that \( \xi_f(M) = f(u) \).

Proof. Let \( u \) be a watershed-cut edge of \( w \) and let \( f \) be a one-side increasing map. By Property 23, there is a child \( X_1 \) of \( R_u \) such that \( \xi_f(X_1) = f(u) \). Since \( u \) is a watershed edge, \( X_1 \) cannot be a leaf node. If \( X_1 \) is a minimum of \( w \), then the property holds true. Otherwise, by Property 23, there is a child \( X_2 \) of \( X_1 \) such that \( \xi_f(X_2) = \xi_f(X_1) = f(u) \) and such that there is a minimum of \( w \) included in \( X_2 \). We can see that we define a sequence \( (X_1, \ldots, X_p) \) where \( X_p \) is a minimum of \( w \) and such that \( \xi_f(X_p) = \cdots = \xi_f(X_1) = f(u) \) and \( X_i \subset X_{i-1} \) for any \( i \) in \( \{2, \ldots, p\} \). Therefore, there is a minimum \( X_p \) included in \( R_u \) such that \( \xi_f(X_p) = f(u) \).

Property 25. Let \( X \) be a non-leaf region of \( B \). There exists a minimum \( M \) of \( w \) such that \( \xi_f(M) = \xi_f(X) \).

Proof. If \( X \) is a minimum of \( w \), then the proof is trivial. Otherwise, there is a minimum of \( w \) strictly contained in \( X \). By Property 23, there is a child \( X_1 \) of \( X \) such that \( \xi_f(X_1) = \xi_f(X) \) and such that there is a minimum of \( w \) included in \( X_1 \). If \( X_1 \) is a minimum of \( w \), then the property holds true. Otherwise, by Property 23, there is a child \( X_2 \) of \( X_1 \) such that \( \xi_f(X_2) = \xi_f(X_1) = \xi_f(X) \) and such that there is a minimum of \( w \) included in \( X_2 \). We can see that we define a sequence \( (X_1, \ldots, X_p) \) where \( X_p \) is a minimum of \( w \) and such that \( \xi_f(X_p) = \cdots = \xi_f(X_1) = \xi_f(X) \) and \( X_i \subset X_{i-1} \) for any \( i \) in \( \{2, \ldots, p\} \). Therefore, there is a minimum \( X_p \) included in \( X \) such that \( \xi_f(X_p) = \xi_f(X) \).

Proof (Property 22).

Let \( f \) be a one-side increasing map for \( B \) and let \( \xi_f \) be the estimated extinction map for \( f \). We need to prove that, for any two minima \( M_1 \) and \( M_2 \) of \( w \), if \( \xi_f(M_1) = \xi_f(M_2) \), then \( M_1 = M_2 \). By Property 24, we know that for
any watershed edge \( u \) of \( w \), there is a minimum \( M \) such that \( \xi_f(M) = f(u) \). By Property 25, we can say that there is a minimum \( M \) of \( w \) such that \( \xi_f(V) = n \). Since the range of \( f \) for the set of watershed edges is \( \{1, \ldots, n-1\} \), we can conclude, by Properties 24 and 25, that the range of \( \xi_f \) for the set of minima of \( w \) is \( \{1, \ldots, n\} \). Since \( w \) has \( n \) minima, it implies that the values \( \xi_f(M_1) \) and \( \xi_f(M_2) \) should be distinct for any pair \( (M_1, M_2) \) of distinct minima of \( w \).

**Property 26.** Let \( f \) be the saliency map of a hierarchical watershed and let \( \xi_f \) be the estimated extinction map for \( f \). For any region \( R \) in \( \mathcal{B}(\mathcal{B}) \), we have \( \xi_f(R) = \vee\{\xi_f(M) \mid M \text{ is a minimum of } w \text{ included in } R\} \).

As established by Theorem 3 of [11], if \( f \) is the saliency map of a hierarchical watershed of \( (G, w) \), then \( f \) is one-side increasing for \( \mathcal{B} \). To prove this lemma, we introduce properties 27 and 28.

**Property 27.** Let \( f \) be a one-side increasing map and let \( X \) be a region of \( \mathcal{B} \). Then \( \xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\} \).

**Proof.** Let \( X \) be a region of \( \mathcal{B} \). We will prove that this property holds true in all the cases of the definition of approximated extinction maps (Definition 7).

1. If \( X = V \), then \( \xi_f(X) = n \) (first case of Definition 7). Since the range of \( f \) is \( \{0, \ldots, n-1\} \), we have \( \xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\} \).
2. If \( X \) is non-dominant region of \( \mathcal{B} \) and \( X \) is a leaf region of \( \mathcal{B} \). Then \( \{f(v) \mid R_v \subseteq X\} = \emptyset \). Since \( \emptyset = 0 \), we have \( \xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\} \).
3. If \( X \) is non-dominant region and a non-leaf region of \( \mathcal{B} \), then sibling(X) is a dominant region of \( \mathcal{B} \), which implies that \( \vee\{f(v) \mid R_v \subseteq X\} \leq \vee\{f(v) \mid R_v \subseteq \text{sibling(X)}\} \) by the definition of dominant regions. Since \( f \) is a one-side increasing map, then \( f(\text{parent}(X)) \geq \vee\{f(v) \mid R_v \text{ is included in } Z\} \) for a child \( Z \) of parent(X). Consequently, \( f(\text{parent}(X)) \geq \vee\{f(v) \mid R_v \text{ is a descendant of } X\} \) and, therefore, \( \xi_f(X) = f(\text{parent}(X)) \geq \vee\{f(v) \mid R_v \text{ is a descendant of } X\} \).
4. If \( X \) is a dominant region of \( \mathcal{B} \), then \( \xi_f(X) = \xi_f(\text{parent}(X)) \). We will prove that \( \xi_f(X) \geq \vee\{f(v) \mid R_v \text{ is a descendant of } X\} \) by induction.
   - **Base step:** if parent(X) is V, then \( \xi_f(X) = \xi_f(V) = n \) and our property holds true.
   - **Inductive step:** if the property holds for parent(X), then we have to show that it holds for \( X \) as well. If \( \xi_f(\text{parent}(X)) \geq \vee\{f(v) \mid R_v \text{ is a descendant of } \text{parent}(X)\} \) then \( \xi_f(X) = \xi_f(\text{parent}(X)) \geq \vee\{f(v) \mid R_v \text{ is a descendant of } X\} \) because every descendant of \( X \) is a descendant of parent(X) as well.

**Property 28.** Let \( X \) be a non-leaf region of \( \mathcal{B} \). Then, for any region \( Y \) such that \( Y \subseteq X \), the value \( \xi_f(Y) \) is in \( \{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\} \).

**Proof.** By induction:
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– Base step: if $X$ is a minimum of $w$. Let $u$ be the building edge of $X$. For any child $Y$ of $X$, we can affirm that $Y$ is a non-dominant region of $B$ and then $\xi_f(Y) = f(u)$, which is equal to zero because $f$ is a one-side increasing map. Thus, $\xi_f(Y)$ is in $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$.

– Inductive step: if $X$ is not a minimum and the property holds for both children of $X$. By Property 23, we know that there is a child $Y$ of $X$ such that $\xi_f(Y) = f(u_X)$ and $\xi_f(sibling(Y)) = \xi_f(X)$. Therefore, for any region $Y$ such that $Y \subseteq X$, the value $\xi_f(Y)$ is in $\{\xi_f(Y), 0\} \cup \{f(u) \mid R_u \subseteq Y\} \cup \{\xi_f(sibling(Y)), 0\} \cup \{f(u) \mid R_u \subseteq sibling(Y)\} \cup \{\xi_f(X)\}$ which is equivalent to $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$.

Proof (Property 26). We can now prove that, for any region $R$ of $B$, we have $\xi_f(R) = \lor\{\xi_f(M)\}$ such that $M$ is a minimum of $w$ included in $R$. Given a region $X$ of $B$:

– If there is no minimum of $w$ included in $X$, then $X$ is a leaf region and $X$ is a non-dominant region of $B$. Then $\xi_f(Y) = f(u)$ (third condition of Definition 7), which is equal to zero because $f$ is a one-side increasing map. Therefore, $\xi_f(X) = \lor\{\xi_f(M)\}$ such that $M$ is a minimum of $w$ included in $R) = \lor\{0\} = 0$

– Otherwise, for any region $Y \subseteq X$, $\xi_f(Y)$ is in $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$ by Property 28. By Property 27, $\xi_f(X) \geq \{f(v) \mid R_v \subseteq X\}$. Therefore, $\xi_f(X) \geq \xi_f(Y)$. Then, $\xi_f$ is increasing on the hierarchy $B$, i.e., for any region $X$, we have $\xi_f(X) = \lor\{\xi_f(Y) \mid Y \subseteq X\}$. By Property 23, there is a minimum $M$ of $w$ such that $\xi_f(X) = \xi_f(M)$. Hence, $\xi_f(X) = \lor\{\xi_f(Y) \mid Y \subseteq X$ and $Y$ is a minimum of $w\}$.

Property 29. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map, then $f$ is the saliency map of a hierarchical watershed.

If $\xi_f$ is an extinction map, then, by Property 3, we have:

– range($P$) = $\{0, \ldots, n\}$;
– for any two distinct minima $M_1$ and $M_2$, $P(M_1) \neq P(M_2)$; and
– for any region $R$ of $B$, we have $P(R) = \lor\{P(M)\}$ such that $M$ is a minimum of $w$ included in $R$).

Now we have to prove that $f$ is a one-side increasing map. We need to prove that the three conditions for $f$ to be a one-side increasing map are satisfied.

Property 30. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map, then $\text{range}(f) = \{0, \ldots, n-1\}$.

Proof. We need to prove that:

1. for any $i$ in $\{0, \ldots, n-1\}$, there is an edge $u$ in $E$ such that $f(u) = i$; and
2. for any edge $u$ in $E$, we have $f(u)$ in $\{0, \ldots, n-1\}$.
Proof of 1:

For $i = 0$: Since $\xi_f$ is an extinction map, for any leaf region $R$ of $B$, we have $\xi_f(R) = \bigvee \{\xi_f(M)\}$ such that $M$ is a minimum of $w$ included in $R$. Let $R$ be a leaf region. Since $R$ is not a dominant region for $f$, this means that $\xi_f(R) = f(u)$, where $u$ is the building edge of parent$(R)$, and, since $\xi_f(R) = 0$, this implies that there exists an edge $u$ in $E$ such that $f(u) = 0$.

For $i$ in $\{1, \ldots, n-1\}$: Since $\xi_f$ is an extinction map, then range$(\xi_f) = \{0, \ldots, n\}$. Then, for any $i$ in $\{1, \ldots, n-1\}$ there is a region $R$ of $B$ such that $\xi_f(R) = i$. Let $i$ be any value in $\{1, \ldots, n-1\}$ and let $R$ be a region of $B$ such that $\xi_f(R) = i$. If $R$ is not a dominant region for $f$, then $\xi_f(R) = f(u)$, where $u$ is the building edge of the parent of $R$ and, then, we can affirm that there exists an edge in $E$ whose weight for $f$ is $i$. Otherwise, if $R$ is a dominant region for $f$, then $\xi_f(R) = \xi_f($parent$(R))$. If parent$(R)$ is not a dominant region for $f$, then $\xi_f($parent$(R)) = \xi_f(v)$, where $v$ is the building edge of the parent of parent$(R)$ and we have our property. Otherwise, if parent$(R)$ is a dominant region of $B$, then $\xi_f($parent$(R)) = \xi_f($parent$(($parent$(R))))$. We can see that, at some point, we will have $\xi_f(R) = \xi_f($parent$(\ldots($parent$(R)))$) = $f(y)$ for an edge $y$ in $E$.

Proof of 2: By contradiction, let us assume that there is an edge $u$ in $E$ such that $f(u)$ is not in $\{0, \ldots, n-1\}$. We can affirm that any non leaf region of $B$ has a child which is not a dominant region for $f$. So, we can affirm that there is a child $X$ of $R_u$ such that $\xi_f(X) = f(u)$. Since $\xi_f$ is an extinction map, the range of $\xi_f$ is $\{0, \ldots, n\}$. Then, $\xi_f(X) = f(u)$ should be in $\{0, \ldots, n\}$ as well. Therefore, the only value that $f(u)$ could have and that is not in $\{0, \ldots, n-1\}$ is $n$. So, let us assume that $f(u) = n$. In this case, we would have $\bigvee f(V) = n + 1$, which contradicts the fact that range$(\xi_f) = \{0, \ldots, n\}$. Therefore, we may conclude that, for any edge $u$ in $E$, we have $f(u)$ in $\{0, \ldots, n-1\}$.

Property 31. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map, then for any $u$ in $E$, $f(u) > 0$ if and only if $u \in WS(w)$.

Proof. If $\xi_f$ is an extinction map, then only the leaf nodes do not include any minimum of $w$, which implies that only the leaf nodes has a value equal to zero for $\xi_f$. We can say that any leaf region $R$ is not a dominant region of $B$, and then $\xi_f(R) = f(u)$ where $u$ is the building edge of the parent of $R$. We can say that an edge is a watershed-cut edge if and only if it has no leaf regions as children. This implies that $\xi_f(R) = 0$ if and only if the parent of $R$ is not a watershed-cut edge.

Property 32. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map, then for any $u$ in $E$, there exists a child $R_u$ of $R_u$ such that $f(u) \geq \bigvee f(v)$ such that $R_u$ is included in $R$.

In order to prove Property 32, we first present properties 33 and 34.

Property 33. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map then, for any region $R$ of $B$, $\xi_f(R) \geq \bigvee \{\xi_f(X) \mid X \subseteq R\}$.
Proof. The proof is straightforward if we consider the third condition of Property 3.

Property 34. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( \xi_f \) be the approximated extinction map of \( f \). If \( \xi_f \) is an extinction map then, for any region \( R \) of \( B \), \( \xi_f(R) \geq f(u) \), where \( u \) is the building edge of \( R \).

Proof. Let \( R \) be a region of \( B \):

- If \( R = \emptyset \), then \( \xi_f(R) = \bigvee f(V) + 1 \), which means that \( \xi_f(R) > f(u) \).
- If \( R \) is not a dominant region, then \( \xi_f(R) = f(v) \) where \( v \) is the building edge of the parent of \( R \). By Property 33, we know that \( \xi_f(parent(R)) \geq \bigvee \{ \xi_f(X) \mid X \subseteq R \} \), which means that \( \xi_f(parent(R)) \geq \xi_f(R) = f(v) \), so the property holds for the parent of \( R \).

Proof (Property 32).

Let \( R \) be any region of \( B \). We have \( \xi_f(R) \geq \{ \xi_f(X) \mid X \subseteq R \} \). Then, we have \( \xi_f(R) \geq \{ f(u) \mid u \) is the building edge of \( X \subseteq R \} \). If \( R \) is a non-dominant region, then \( \xi_f(R) = f(v) \) where \( v \) is the building edge of the parent of \( R \). So, we will have \( f(v) \geq \{ f(u) \mid u \) is the building edge of \( X \subseteq R \} \). Since \( R \) is a child of \( R_v \), we can say that there is a child of \( R_v \) such that the property holds.

Proof (Property 8).

Properties 20 and 29 correspond to the forward and backward implications of Property 8.

C Proof of Property 10

Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( S_f \) be the estimated sequence of minima of \( f \). If \( f \) is the saliency map of a hierarchical watershed, then \( \xi_f \) is an extinction map by Property 8. First, we have to prove that \( \xi_f \) is the extinction map for the sequence \( S_f \). Then, we have to prove that, for any edge \( u \), we have \( f(u) = \min\{ \xi_f(X) \mid X \subseteq R_u \} \).

Property 35. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( S_f \) be the estimated sequence of minima of \( f \). If \( \xi_f \) is an extinction map, then \( \xi_f \) is the extinction map for the estimated sequence of minima \( S_f \) of \( f \).

Proof. If \( \xi_f \) is an extinction map, then for any two distinct minima \( M_1 \) and \( M_2 \), we have \( \xi_f(M_1) \neq \xi_f(M_2) \). If we prove that \( \xi_f(M) > 0 \) for any minimum \( M \), then we prove that the range of \( \xi_f \) for the set of minima is \( \{1, \ldots, n\} \). Let \( M \) be a minimum of \( w \) and let us assume that \( \xi_f(M) = 0 \). If \( M \) is a not a dominant region of \( B \), then \( \xi_f(M) = f(v) = 0 \) where \( v \) is the building edge of parent\((M)\). However, if \( M \) is not a dominant region of \( B \), this implies that none of the children of \( R_v \) is a leaf region and then \( v \) is a watershed-cut edge. Since \( f \) is one-side increasing, this implies that \( f(v) \) is strictly greater than 0. If \( M \) is a dominant region of \( B \), this implies that \( \xi_f(M) = \xi_f(parent(M)) \). Since \( \xi_f \)
is an extinction map, only the leaf regions of $B$ has a zero value for $\xi_f$. Since the parent of $M$ is not a leaf region, we can affirm that $\xi_f(\text{parent}(M)) > 0$ and, therefore, $\xi_f(M) > 0$. So, we just proved that the range of $\xi_f$ for the set of minima is $\{1, \ldots, n\}$. Since $\xi_f$ is an extinction map, we can say for any region $R$, we have $\xi_f(R) = \vee \{\xi_f(M) \mid M \text{ is a minimum}\}$. So, $\xi_f$ is the extinction map for $S_f$.

**Property 36.** Let $f$ be a map from $E$ into $\mathbb{R}^+$. Let $S_f$ be the estimated sequence of minima of $f$. If $\xi_f$ is an extinction map, then for any edge $u$, we have $f(u) = \min \{\xi_f(R_u), f(u)\}$.

Proof. Any region has a child which is dominant and a child which is not a dominant region of $B$. This implies that, for any edge $u$, there is a child $X$ of $u$ such that $\xi_f(X) = \xi_f(R_u)$ and another child $Y$ such that $\xi_f(Y) = f(u)$. By Property 34, we have $\xi_f(R_u) \geq f(u)$, which implies that $f(u) = \min \{\xi_f(X) \mid X \subseteq R_u\} = \min \{\xi_f(R_u), f(u)\}$. □

**D Proof of Theorem 12**

**Theorem 37.** Let $f$ be a map from $E$ into $\mathbb{R}^+$. The watersheding $\omega(f)$ of $f$ is the saliency map of the hierarchical watershed of $(G,w)$ for the estimated sequence of minima for $f$.

Proof. By Property 8, the map $\xi_f$ is an extinction map and, by Property 35, the map $\xi_f$ is an extinction map for the estimated sequence of minima $S_f$ for $f$. Therefore, by the Definition 11 of watersheding, the watersheding $\omega(f)$ of $f$ is the saliency map of the hierarchical watershed of $(G,w)$ for the estimated sequence of minima for $f$.

**E Proof of Property 13**

**Property 38.** Let $f$ be a map from $E$ into $\mathbb{R}^+$. The watersheding $\omega(\omega(f))$ of $\omega(f)$ is equal to $\omega(f)$.

Proof. By Theorem 12, we know that $\omega(f)$ is the saliency map of a hierarchical watershed. By Property 10, we know that $\xi_{\omega(f)}$ is the extinction map for the estimated sequence of minima $S_{\omega(f)}$ for $\omega(f)$. By Property 36, we know that $\omega(f)(u) = \min \{\xi_{\omega(f)}(X) \mid X \subseteq R_u\}$. By Definition 11, we have $\omega(\omega(f)) = \min \{\xi_{\omega(f)}(X) \mid X \subseteq R_u\} = \omega(f)(u)$. □

**F Proof of Property 15**

**Property 39.** Let $H$ be a hierarchy and let $f$ be the saliency map of $H$. The hierarchy $H$ is a hierarchical watershed of $(G,w)$ if and only if $\omega(f) = f$. 

Proof. Let \( \mathcal{H} \) be a hierarchical watershed and let \( f \) be the saliency map of \( \mathcal{H} \). By Property 14, we may say that \( \omega(f) = f \).

On the other hand, let \( \omega(f) = f \). Then, for any edge \( u \), we have \( f(u) = \omega(f)(u) = \min \{ P(R) \mid R \text{ is a child of } R_u \} \), where \( P \) is the extinction map for \( S_f \). Therefore, by the backward implication of Property 8, we can say that \( f \) is the saliency map of the hierarchical watershed for \( S_f \). \( \square \)