Proofs of the properties presented in the article "Watersheding hierarchies"
Deise Santana Maia, Jean Cousty, Laurent Najman, B Perret

To cite this version:
Deise Santana Maia, Jean Cousty, Laurent Najman, B Perret. Proofs of the properties presented in the article "Watersheding hierarchies". [Research Report] Université Paris-Est, LIGM UMR CNRS 8049, France; ESIEE Paris. 2019. hal-02087042

HAL Id: hal-02087042
https://hal.archives-ouvertes.fr/hal-02087042
Submitted on 16 Apr 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Proofs of the properties presented in the article "Watersheding hierarchies"
3. for any region \( R \) of \( B \), we have \( P(R) = \vee \{ P(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R \} \).

The map \( P \) is an extinction map for \( w \).

Proof. Let \( P \) be a map from \( \mathcal{R}(B) \) to \( \mathbb{R}^+ \) for which the statements 1, 2 and 3 hold true. To prove that \( P \) is an extinction map, we have to show that there exists a sequence \( S \) of \( n \) pairwise distinct minima of \( w \) such that, for any region \( R \) of \( B \), the value \( P(R) \) is the extinction value of \( R \) for \( S \).

Let \( S = (M_1, \ldots, M_n) \) be a sequence of \( n \) pairwise distinct minima of \( w \) ordered in non-decreasing order for \( P \), i.e., for any two distinct minima \( M_i \) and \( M_j \), for \( i \) and \( j \) in \( \{1, \ldots, n\} \), if \( i < j \) then \( P(M_i) \leq P(M_j) \).

By the statement 2, the sequence \( S \) is unique. By the statement 3, for any region \( R \) of \( B \) such that there is no minimum of \( w \) included in \( R \), \( P(R) = \vee \{ \} = 0 \), so \( P(R) \) is the extinction value of \( R \) for \( S \).

Since \( w \) has \( n \) minima, for any minimum \( M \) of \( w \), the value \( P(M) \) is in \( \{1, \ldots, n\} \). Otherwise, if there existed a minimum \( M' \) of \( w \) such that \( P(M') = 0 \), then there would be a value \( i \) in \( \{1, \ldots, n\} \) such that for any minimum \( M'' \) of \( w \) the value \( P(M'') \) is different from \( i \). Consequently, the range of \( P \) would be \( \{0, \ldots, n\} \setminus \{i\} \), which contradicts the statement 1. Therefore, for any minimum \( M_i \), for \( i \) in \( \{1, \ldots, n\} \), we have that \( P(M_i) = i \), so \( P(M_i) \) is the extinction value of \( M_i \) for \( S \).

It follows that, by the statement 3, for any region \( R \) such that there is a minimum of \( B \) included in \( R \), the value \( P(R) \) is the maximum value \( i \) in \( \{1, \ldots, n\} \) such that \( M_i \) is included in \( R \).

Thus, for any region \( R \) of \( B \), the value \( P(R) \) is the extinction value of \( R \) for \( S \). Therefore, the map \( P \) is an extinction map of \( w \).

\( \square \)

B Proof of Property 8

Property 19. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( \xi_f \) be the approximated extinction map of \( f \). The map \( f \) is the saliency map of a hierarchical watershed of \((G, w)\) if and only if the map \( \xi_f \) is an extinction map.

We prove the forward and backward implications of Property 8 in Property 20 and Property 29, respectively.

Property 20. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( \xi_f \) be the approximated extinction map of \( f \). If \( f \) is the saliency map of a hierarchical watershed, then \( \xi_f \) is an extinction map.

By Theorem 3 of [11], if \( f \) is the saliency map of a hierarchical watershed, then \( f \) is a one-side increasing map, which implies that:

1. range(\( f \)) = \( \{0, \ldots, n - 1\} \);
2. for any \( u \) in \( E \), \( f(u) > 0 \) if and only if \( u \in WS(w) \), where \( WS(w) \) is the set of watershed-cut edges of \( w \); and
3. for any \( u \) in \( E \), there exists a child \( R \) of \( R_u \) such that \( f(u) \geq \vee \{ f(v) \} \) such that \( R_v \) is included in \( R \).

In order to prove Property 20, we prove that three conditions for \( \xi_f \) to be an extinction map (Property 3) are satisfied in Properties 21, 22 and 26.

**Property 21.** Let \( f \) be the saliency map of a hierarchical watershed and let \( \xi_f \) be the approximated extinction map of \( f \). Then range(\( \xi_f \)) = \( \{0, \ldots, n\} \).

*Proof.* We need to prove that:
1. for any \( i \) in \( \{0, \ldots, n\} \), there is a region \( R \) of \( B \) such that \( \xi_f(R) = i \); and
2. for any region \( R \) of \( B \), we have \( \xi_f(R) \) in \( \{0, \ldots, n\} \).

*Proof of 1:*
Let \( R = V \), then \( \xi_f(R) = \vee f(R) + 1 \). Since the range of \( f \) is \( \{0, \ldots, n-1\} \), we have \( \vee f(V) = \vee \{ f(u) \mid R_u \subseteq V \} = n - 1 \). Then, \( \xi_f(R) = n - 1 + 1 = n \).

Let \( R \) be a leaf region of \( B \). Then the building edge of parent(\( R \)) is not in \( WS(w) \). Let \( u \) be the building edge of parent(\( R \)). Since \( f \) is a one-side increasing map and since \( u \) is not in \( WS(w) \), we have \( f(u) = 0 \). Since \( R \) is a leaf region, \( R \) is not a dominant region for \( f \) and then, by Definition 7, we have \( \xi_f(R) = f(u) = 0 \).

Now, we have to prove that, for any \( i \) in \( \{1, \ldots, n-1\} \), there is a region \( R \) of \( B \) such that \( \xi_f(R) = i \). Since \( w \) has \( n \) minima, we can conclude that \( w \) has \( n-1 \) watershed-cut edges. Since the range of \( f \) is \( \{0, \ldots, n-1\} \) and the weight of the watershed-cut edges of \( w \) is strictly greater than zero, we can conclude that the watershed-cut edges of \( w \) have pairwise distinct weights for \( f \) from 1 to \( n-1 \).

Given any watershed-cut edge \( u \), we can affirm that exactly one of the children of \( R_u \) is a dominant region of \( f \) because, given the children \( X \) and \( Y \) of \( R_u \), we know that both \( X \) and \( Y \) are not leaf regions and we have either \( X \prec_{\langle f, w \rangle} Y \) or \( Y \prec_{\langle f, w \rangle} X \). Let \( Y \) be the child of \( R_u \), which is not a dominant region for \( f \). By Definition 7, we have \( \xi_f(Y) = f(u) \). Therefore, for any \( i \) in \( \{1, \ldots, n-1\} \), there is a watershed-cut edge \( u \) in \( WS(w) \) such that \( f(u) = i \) and such that there is a child \( Y \) of \( R_u \) such that \( \xi_f(Y) = f(u) = i \).

*Proof of 2:*
Let \( R \) be a region of \( B \) and let \( u \) be the building edge of the parent of \( R \) if \( R \neq V \). By Definition 7, the value \( \xi_f(R) \) is: \( \vee f(V) + 1 \), \( f(u) \) or \( \xi_f(\text{parent}(R)) \). It is enough to prove that \( \vee f(V) + 1 \) and \( f(u) \) are in \( \{0, \ldots, n\} \). Since the range of \( f \) is \( \{0, \ldots, n-1\} \), it is clear that \( f(u) \) is in \( \{0, \ldots, n\} \). We can see that \( \vee f(V) = n - 1 \) because any region of \( B \) is included in \( V \). Then \( \vee f(V) + 1 = n \) which is in \( \{0, \ldots, n\} \).

**Property 22.** Let \( f \) be the saliency map of a hierarchical watershed and let \( \xi_f \) be the approximated extinction map of \( f \). For any two minima \( M_1 \) and \( M_2 \) of \( w \), if \( \xi_f(M_1) = \xi_f(M_2) \), then \( M_1 = M_2 \).

As established by Theorem 3 of [11], if \( f \) is the saliency map of a hierarchical watershed of \( (G, w) \), then \( f \) is one-side increasing for \( B \). To prove Property 22, we first present the Properties 23, 24 and 25.
**Property 23.** Let \( f \) be a one-side increasing map and let \( \xi_f \) be the approximated extinction map of \( f \). For any region \( X \) of \( B \), we denote by \( u_X \) the building edge of \( X \). For any region \( X \) such that there is a minimum of \( w \) strictly included in \( X \), there is a child \( Y \) of \( X \) such that:

- \( \xi_f(Y) = \xi_f(X) \);
- \( \xi_f(\text{sibling}(Y)) = f(u_X) \); and
- there is a minimum of \( w \) included in \( Y \).

**Proof.** Let \( X \) be a region such that there is at least one minimum of \( w \) strictly included in \( X \). By the definition of dominant regions (Definition 6), at most one of the children of \( X \) is a dominant region of \( B \). Since there is at least one minimum of \( w \) strictly included in \( X \), then there is a minimum included in at least one child of \( X \). Therefore, there is a child \( Y \) of \( X \) such that \( Y \) is a dominant region of \( B \) and, therefore \( \xi_f(Y) = \xi_f(X) \) (condition 2 of Definition 7). Thus, \( \text{sibling}(Y) \) is not a dominant region of \( B \) and \( \xi(\text{sibling}(Y)) = f(u) \), where \( u \) is the building edge of \( X \) (condition 3 of Definition 7).

**Property 24.** Let \( u \) be any watershed edge of \( w \) and let \( f \) be a one-side increasing map. There is a minimum \( M \) of \( w \) such that \( \xi_f(M) = f(u) \).

**Proof.** Let \( u \) be a watershed-cut edge of \( w \) and let \( f \) be a one-side increasing map. By Property 23, there is a child \( X_1 \) of \( R_u \) such that \( \xi_f(X_1) = f(u) \). Since \( u \) is a watershed edge, \( X_1 \) cannot be a leaf node. If \( X_1 \) is a minimum of \( w \), then the property holds true. Otherwise, by Property 23, there is a child \( X_2 \) of \( X_1 \) such that \( \xi_f(X_2) = \xi_f(X_1) = f(u) \) and such that there is a minimum of \( w \) included in \( X_2 \). We can see that we define a sequence \( (X_1, \ldots, X_p) \) where \( X_p \) is a minimum of \( w \) and such that \( \xi_f(X_p) = \cdots = \xi_f(X_1) = f(u) \) and \( X_i \subset X_{i-1} \) for any \( i \) in \( \{2, \ldots, p\} \). Therefore, there is a minimum \( X_p \) included in \( R_u \) such that \( \xi_f(X_p) = f(u) \).

**Property 25.** Let \( X \) be a non-leaf region of \( B \). There exists a minimum \( M \) of \( w \) such that \( \xi_f(M) = \xi_f(X) \).

**Proof.** If \( X \) is a minimum of \( w \), then the proof is trivial. Otherwise, there is a minimum of \( w \) strictly contained in \( X \). By Property 23, there is a child \( X_1 \) of \( X \) such that \( \xi_f(X_1) = \xi_f(X) \) and such that there is a minimum of \( w \) included in \( X_1 \). If \( X_1 \) is a minimum of \( w \), then the property holds true. Otherwise, by Property 23, there is a child \( X_2 \) of \( X_1 \) such that \( \xi_f(X_2) = \xi_f(X_1) = \xi_f(X) \) and such that there is a minimum of \( w \) included in \( X_2 \). We can see that we define a sequence \( (X_1, \ldots, X_p) \) where \( X_p \) is a minimum of \( w \) and such that \( \xi_f(X_p) = \cdots = \xi_f(X_1) = \xi_f(X) \) and \( X_i \subset X_{i-1} \) for any \( i \) in \( \{2, \ldots, p\} \). Therefore, there is a minimum \( X_p \) included in \( X \) such that \( \xi_f(X_p) = \xi_f(X) \).

**Proof (Property 22).**

Let \( f \) be a one-side increasing map for \( B \) and let \( \xi_f \) be the estimated extinction map for \( f \). We need to prove that, for any two minima \( M_1 \) and \( M_2 \) of \( w \), if \( \xi_f(M_1) = \xi_f(M_2) \), then \( M_1 = M_2 \). By Property 24, we know that for
Proofs of the properties presented in the article "Watersheding hierarchies" V
any watershed edge \( u \) of \( w \), there is a minimum \( M \) such that \( \xi_f(M) = f(u) \). By Property 25, we can say that there is a minimum \( M \) of \( w \) such that \( \xi_f(V) = \xi_f(V) = n \). Since the range of \( f \) for the set of watershed edges is \( \{1, \ldots, n-1\} \), we can conclude, by Properties 24 and 25, that the range of \( \xi_f \) for the set of minima of \( w \) is \( \{1, \ldots, n\} \). Since \( w \) has \( n \) minima, it implies that the values \( \xi_f(M_1) \) and \( \xi_f(M_2) \) should be distinct for any pair \( (M_1, M_2) \) of distinct minima of \( w \). \( \square \)

**Property 26.** Let \( f \) be the saliency map of a hierarchical watershed and let \( \xi_f \) be the estimated extinction map for \( f \). For any region \( R \) in \( \mathcal{B} \), we have \( \xi_f(R) = \forall \{\xi_f(M) \mid \text{such that } M \text{ is a minimum of } w \} \text{ included in } R \} \).

As established by Theorem 3 of [11], if \( f \) is the saliency map of a hierarchical watershed of \((G, w)\), then \( f \) is one-side increasing for \( \mathcal{B} \). To prove this lemma, we introduce properties 27 and 28.

**Property 27.** Let \( f \) be a one-side increasing map and let \( X \) be a region of \( \mathcal{B} \). Then \( \xi_f(X) \geq \forall \{f(v) \mid R_v \subseteq X \} \).

**Proof.** Let \( X \) be a region of \( \mathcal{B} \). We will prove that this property holds true in all the cases of the definition of approximated extinction maps (Definition 7).

1. If \( X = V \), then \( \xi_f(X) = n \) (first case of Definition 7). Since the range of \( f \) is \( \{0, \ldots, n-1\} \), we have \( \forall \{f(v) \mid R_v \subseteq X \} \).
2. If \( X \) is non-dominant region of \( \mathcal{B} \) and \( X \) is a leaf region of \( \mathcal{B} \). Then \( \forall \{f(v) \mid R_v \subseteq X \} = \emptyset \). Since \( \emptyset = 0 \), we have \( \forall \{f(v) \mid R_v \subseteq X \} \).
3. If \( X \) is non-dominant region and a non-leaf region of \( \mathcal{B} \), then sibling(X) is a dominant region of \( \mathcal{B} \), which implies that \( \forall \{f(v) \mid R_v \subseteq X \} \leq \forall \{f(v) \mid R_v \subseteq \text{sibling(X)} \} \) by the definition of dominant regions. Since \( f \) is a one-side increasing map, then \( f(u_{\text{parent}(X)}) \geq \forall \{f(v) \mid R_v \subseteq Z \} \) for a child \( Z \) of parent(X). Consequently, \( f(u_{\text{parent}(X)}) \geq \forall \{f(v) \mid R_v \subseteq \text{dominant region of } X \} \) \text{ and, therefore, } \xi_f(X) = f(u_{\text{parent}(X)}) \geq \forall \{f(v) \mid R_v \subseteq \text{dominant region of } X \} \).
4. If \( X \) is a dominant region of \( \mathcal{B} \), then \( \xi_f(X) = \xi_f(\text{parent(X)}) \). We will prove that \( \xi_f(X) \geq \forall \{f(v) \mid R_v \subseteq X \} \) by induction.
   
   - Base step: if parent(X) is V, then \( \xi_f(X) = \xi_f(V) = n \) and our property holds true.
   
   - Inductive step: if the property holds for parent(X), then we have to show that it holds for \( X \) as well. If \( \xi_f(\text{parent}(X)) \geq \forall \{f(v) \mid R_v \subseteq X \} \), then \( \xi_f(X) = \xi_f(\text{parent}(X)) \geq \forall \{f(v) \mid R_v \subseteq X \} \).

**Property 28.** Let \( X \) be a non-leaf region of \( \mathcal{B} \). Then, for any region \( Y \) such that \( Y \subseteq X \), the value \( \xi_f(Y) \) is in \( \{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X \} \).

**Proof.** By induction:
– Base step: if $X$ is a minimum of $w$. Let $u$ be the building edge of $X$. For any child $Y$ of $X$, we can affirm that $Y$ is a non-dominant region of $B$ and then $\xi_f(Y) = f(u)$, which is equal to zero because $f$ is a one-side increasing map. Thus, $\xi_f(Y)$ is in $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$.

– Inductive step: if $X$ is not a minimum and the property holds for both children of $X$. By Property 23, we know that there is a child $Y$ of $X$ such that $\xi_f(Y) = f(u_X)$ and $\xi_f(sibling(Y)) = \xi_f(X)$. Therefore, for any region $Y$ such that $Y \subseteq X$, the value $\xi_f(Y)$ is in $\{\xi_f(Y), 0\} \cup \{f(u) \mid R_u \subseteq Y\} \cup \{\xi_f(sibling(Y)), 0\} \cup \{f(u) \mid R_u \subseteq sibling(Y)\} \cup \{\xi_f(X)\}$ which is equivalent to $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$.

Proof (Property 26). We can now prove that, for any region $R$ of $B$, we have $\xi_f(R) = \vee\{\xi_f(M)\}$ such that $M$ is a minimum of $w$ included in $R$. Given a region $X$ of $B$:

– If there is no minimum of $w$ included in $X$, then $X$ is a leaf region and $X$ is a non-dominant region of $B$. Then $\xi_f(Y) = f(u)$ (third condition of Definition 7), which is equal to zero because $f$ is a one-side increasing map. Therefore, $\xi_f(X) = \vee\{\xi_f(M)\}$ such that $M$ is a minimum of $w$ included in $R$ = $\vee 0 = 0$

– Otherwise, for any region $Y \subseteq X$, $\xi_f(Y)$ is in $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$ by Property 28. By Property 27, $\xi_f(X) \geq \{f(v) \mid R_v \subseteq X\}$. Therefore, $\xi_f(Y) \geq \xi_f(Y)$. Then, $\xi_f$ is increasing on the hierarchy $B$, i.e., for any region $X$, we have $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X\}$. By Property 23, there is a minimum $M$ of $w$ such that $\xi_f(X) = \xi_f(M)$. Hence, $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X \text{ and } Y \text{ is a minimum of } w\}$.

Property 29. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map, then $f$ is the saliency map of a hierarchical watershed.

If $\xi_f$ is an extinction map, then, by Property 3, we have:

– $\text{range}(P) = \{0, \ldots, n\}$;
– for any two distinct minima $M_1$ and $M_2$, $P(M_1) \neq P(M_2)$; and
– for any region $R$ of $B$, we have $P(R) = \vee\{P(M)\}$ such that $M$ is a minimum of $w$ included in $R$.

Now we have to prove that $f$ is a one-side increasing map. We need to prove that the three conditions for $f$ to be a one-side increasing map are satisfied.

Property 30. Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map, then $\text{range}(f) = \{0, \ldots, n-1\}$.

Proof. We need to prove that:

1. for any $i$ in $\{0, \ldots, n-1\}$, there is an edge $u$ in $E$ such that $f(u) = i$; and
2. for any edge $u$ in $E$, we have $f(u)$ in $\{0, \ldots, n-1\}$.
Proof of 1:

For \( i = 0 \): Since \( \xi_f \) is an extinction map, for any leaf region \( R \) of \( B \), we have \( \xi_f(R) = \vee\{\xi_f(M) \mid M \text{ is a minimum of } w \text{ included in } R\} = 0 \). Let \( R \) be a leaf region. Since \( R \) is not a dominant region for \( f \), this means that \( \xi_f(R) = f(u) \), where \( u \) is the building edge of \( \text{parent}(R) \), and, since \( \xi_f(R) = 0 \), this implies that there exists an edge \( u \) in \( E \) such that \( f(u) = 0 \).

For \( i \) in \( \{1, \ldots, n-1\} \): Since \( \xi_f \) is an extinction map, then \( \text{range}(\xi_f) = \{0, \ldots, n\} \). Then, for any \( i \) in \( \{1, \ldots, n-1\} \) there is a region \( R \) of \( B \) such that \( \xi_f(R) = i \). Let \( i \) be any value in \( \{1, \ldots, n-1\} \) and let \( R \) be a region of \( B \) such that \( \xi_f(R) = i \). If \( R \) is not a dominant region for \( f \), then \( \xi_f(R) = f(u) \), where \( u \) is the building edge of \( \text{parent} \) of \( R \) and, then, we can affirm that there exists an edge in \( E \) whose weight for \( f \) is \( i \). Otherwise, if \( R \) is a dominant region for \( f \), then \( \xi_f(R) = \xi_f(\text{parent}(R)) \). If \( \text{parent}(R) \) is not a dominant region for \( f \), then \( \xi_f(\text{parent}(R)) = \xi_f(v) \), where \( v \) is the building edge of \( \text{parent} \) of \( \text{parent}(R) \) and we have our property. Otherwise, if \( \text{parent}(R) \) is a dominant region of \( B \), then \( \xi_f(\text{parent}(R)) = \xi_f(\text{parent}(\text{parent}(R))) \). We can see that, at some point, we will have \( \xi_f(R) = \xi_f(\text{parent} \ldots (\text{parent}(R))) = f(y) \) for an edge \( y \) in \( E \).

Proof of 2: By contradiction, let us assume that there is an edge \( u \) in \( E \) such that \( f(u) \) is not in \( \{0, \ldots, n-1\} \). We can affirm that any non leaf region of \( B \) has a child which is not a dominant region for \( f \). So, we can affirm that there is a child \( X \) of \( R_u \) such that \( \xi_f(X) = f(u) \). Since \( \xi_f \) is an extinction map, the range of \( \xi_f \) is \( \{0, \ldots, n\} \). Then, \( \xi_f(X) = f(u) \) should be in \( \{0, \ldots, n\} \) as well. Therefore, the only value that \( f(u) \) could have and that is not in \( \{0, \ldots, n-1\} \) is \( n \). So, let us assume that \( f(u) = n \). In this case, we would have \( \vee\{V\} = n+1 \), which contradicts the fact that \( \text{range}(\xi_f) = \{0, \ldots, n\} \). Therefore, we may conclude that, for any edge \( u \) in \( E \), we have \( f(u) \) in \( \{0, \ldots, n-1\} \).

Property 31. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( \xi_f \) be the approximated extinction map of \( f \). If \( \xi_f \) is an extinction map, then for any \( u \) in \( E \), \( f(u) > 0 \) if and only if \( u \in \text{WS}(w) \).

Proof. If \( \xi_f \) is an extinction map, then only the leaf nodes do not include any minimum of \( w \), which implies that only the leaf nodes has a value equal to zero for \( \xi_f \). We can say that any leaf region \( R \) is not a dominant region of \( B \), and then \( \xi_f(R) = f(u) \) where \( u \) is the building edge of the parent of \( R \). We can say that an edge is a watershed-cut edge if and only if it has no leaf regions as children. This implies that \( \xi_f(R) = 0 \) if and only if the parent of \( R \) is not a watershed-cut edge.

Property 32. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( \xi_f \) be the approximated extinction map of \( f \). If \( \xi_f \) is an extinction map, then for any \( u \) in \( E \), there exists a child \( R_u \) of \( R_u \) such that \( f(u) \geq \vee\{f(v) \mid R_u \text{ is included in } R \} \).

In order to prove Property 32, we first present properties 33 and 34.

Property 33. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( \xi_f \) be the approximated extinction map of \( f \). If \( \xi_f \) is an extinction map then, for any region \( R \) of \( B \), \( \xi_f(R) \geq \vee\{\xi_f(X) \mid X \subseteq R \} \).
Proof. The proof is straightforward if we consider the third condition of Property 3.

**Property 34.** Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $\xi_f$ be the approximated extinction map of $f$. If $\xi_f$ is an extinction map then, for any region $R$ of $B$, $\xi_f(R) \geq f(u)$, where $u$ is the building edge of $R$.

Proof. Let $R$ be a region of $B$:

- If $R = V$, then $\xi_f(R) = \bigvee f(V) + 1$, which means that $\xi_f(R) > f(u)$.
- If $R$ is not a dominant region, then $\xi_f(R) = f(v)$ where $v$ is the building edge of the parent of $R$. By Property 33, we know that $\xi_f(\text{parent}(R)) \geq \bigvee \{\xi_f(X) \mid X \subseteq R\}$, which means that $\xi_f(\text{parent}(R)) \geq \xi_f(R) = f(v)$, so the property holds for the parent of $R$.

Proof (Property 32).

Let $R$ be any region of $B$. We have $\xi_f(R) \geq \{\xi_f(X) \mid X \subseteq R\}$. Then, we have $\xi_f(R) \geq \{f(u) \mid u$ is the building edge of $X \subseteq R\}$. If $R$ is a non-dominant region, then $\xi_f(R) = f(v)$ where $v$ is the building edge of the parent of $R$. So, we will have $f(v) \geq \{f(u) \mid u$ is the building edge of $X \subseteq R\}$. Since $R$ is a child of $R_v$, we can say that there is a child of $R_v$ such that the property holds.

Proof (Property 8).

Properties 20 and 29 correspond to the forward and backward implications of Property 8.

**C Proof of Property 10**

Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $S_f$ be the estimated sequence of minima of $f$. If $f$ is the saliency map of a hierarchical watershed, then $\xi_f$ is an extinction map by Property 8. First, we have to prove that $\xi_f$ is the extinction map for the sequence $S_f$. Then, we have to prove that, for any edge $u$, we have $f(u) = \min\{\xi_f(X) \mid X \subseteq R_u\}$.

**Property 35.** Let $f$ be a map from $E$ into $\mathbb{R}^+$ and let $S_f$ be the estimated sequence of minima of $f$. If $\xi_f$ is an extinction map, then $\xi_f$ is the extinction map for the estimated sequence of minima $S_f$ of $f$.

Proof. If $\xi_f$ is an extinction map, then for any two distinct minima $M_1$ and $M_2$, we have $\xi_f(M_1) \neq \xi_f(M_2)$. If we prove that $\xi_f(M) > 0$ for any minimum $M$, then we prove that the range of $\xi_f$ for the set of minima is $\{1, \ldots, n\}$. Let $M$ be a minimum of $w$ and let us assume that $\xi_f(M) = 0$. If $M$ is a not a dominant region of $B$, then $\xi_f(M) = f(v) = 0$ where $v$ is the building edge of parent($M$). However, if $M$ is not a dominant region of $B$, this implies that none of the children of $R_v$ is a leaf region and then, $v$ is a watershed-cut edge. Since $f$ is one-side increasing, this implies that $f(v)$ is strictly greater than 0. If $M$ is a dominant region of $B$, this implies that $\xi_f(M) = \xi_f(\text{parent}(M))$. Since $\xi_f$
is an extinction map, only the leaf regions of \( \mathcal{B} \) has a zero value for \( \xi_f \). Since the parent of \( M \) is not a leaf region, we can affirm that \( \xi_f(\text{parent}(M)) > 0 \) and, therefore, \( \xi_f(M) > 0 \). So, we just proved that the range of \( \xi_f \) for the set of minima is \( \{1, \ldots, n\} \). Since \( \xi_f \) is an extinction map, we can say for any region \( R \), we have \( \xi_f(R) = \vee\{\xi_f(M) \mid M \text{ is a minimum}\} \). So, \( \xi_f \) is the extinction map for \( S_f \).

Property 36. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \) and let \( S_f \) be the estimated sequence of minima of \( f \). If \( \xi_f \) is an extinction map, then for any edge \( u \), we have \( f(u) = \min\{\xi_f(R_u), f(u)\} \).

Proof. Any region has a child which is dominant and a child which is not a dominant region of \( \mathcal{B} \). This implies that, for any edge \( u \), there is a child \( X \) of \( u \) such that \( \xi_f(X) = \xi_f(R_u) \) and another child \( Y \) such that \( \xi_f(Y) = f(u) \). By Property 34, we have \( \xi_f(R_u) \geq f(u) \), which implies that \( f(u) = \min\{\xi_f(X) \mid X \subseteq R_u\} = \min\{\xi_f(R_u), f(u)\} \).

D Proof of Theorem 12

Theorem 37. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \). The watersheding \( \omega(f) \) of \( f \) is the saliency map of the hierarchical watershed of \((G, w)\) for the estimated sequence of minima for \( f \).

Proof. By Property 8, the map \( \xi_f \) is an extinction map and, by Property 35, the map \( \xi_f \) is an extinction map for the estimated sequence of minima \( S_f \) for \( f \). Therefore, by the Definition 11 of watersheding, the watersheding \( \omega(f) \) of \( f \) is the saliency map of the hierarchical watershed of \((G, w)\) for the estimated sequence of minima for \( f \).

E Proof of Property 13

Property 38. Let \( f \) be a map from \( E \) into \( \mathbb{R}^+ \). The watersheding \( \omega(\omega(f)) \) of \( \omega(f) \) is equal to \( \omega(f) \).

Proof. By Theorem 12, we know that \( \omega(f) \) is the saliency map of a hierarchical watershed. By Property 10, we know that \( \xi_{\omega(f)} \) is the extinction map for the estimated sequence of minima \( S_{\omega(f)} \) for \( \omega(f) \). By Property 36, we know that \( \omega(f)(u) = \min\{\xi_{\omega(f)}(X) \mid X \subseteq R_u\} \). By Definition 11, we have \( \omega(\omega(f)) = \min\{\xi_{\omega(f)}(X) \mid X \subseteq R_u\} = \omega(f)(u) \).

F Proof of Property 15

Property 39. Let \( \mathcal{H} \) be a hierarchy and let \( f \) be the saliency map of \( \mathcal{H} \). The hierarchy \( \mathcal{H} \) is a hierarchical watershed of \((G, w)\) if and only if \( \omega(f) = f \).
Proof. Let $H$ be a hierarchical watershed and let $f$ be the saliency map of $H$. By Property 14, we may say that $\omega(f) = f$.

On the other hand, let $\omega(f) = f$. Then, for any edge $u$, we have $f(u) = \omega(f)(u) = \min\{P(R) \mid R \text{ is a child of } R_u\}$, where $P$ is the extinction map for $S_f$. Therefore, by the backward implication of Property 8, we can say that $f$ is the saliency map of the hierarchical watershed for $S_f$. \qed