Lie-Symmetry Group and Modelling in Non-Isothermal Fluid Mechanics

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Abstract

The symmetry group of the non-isothermal Navier-Stokes equations is used to develop physics-preserving turbulence models for the subgrid stress tensor and the subgrid heat flux. The Reynolds analogy is not used. The theoretical properties of the models are investigated. In particular, their compatibility with the scaling laws of the flow are proven. A numerical test, in the configuration of an air flow in a ventilated and differentially heated room is presented.

Keywords: Symmetry; turbulence modelling; convection; scaling laws.

1. Introduction

Many industrial applications in fluid mechanics involve not only fluid motion but also thermal phenomena. Because of the cost of experimentation, engineers generally use CFD simulations to predict the velocity and the temperature behavior. In the presence of a turbulent flow, the large-eddy simulation approach is one of the most widely used simulation methods. This approach requires modeling a subgrid stress tensor and a subgrid heat flux. Many models exist in the literature but precise predictions can be obtained only whith models wich are compatible with the physics of the flow.

In the isothermal case, Razafindralandy *et al.* [18, 17] developed physics-preserving model for the subgrid stress tensor. It was based on a symmetry approach. Indeed, the symmetries of an equation are very closely linked to the physical properties hidden behind the equation. This connection was made evident by E. Nœther who proved that, for a Lagrangian system, each symmetry group of the Lagrangian action corresponds to a conservation law [12, 11]. In fluid mechanics, the symmetries of the Navier-Stokes equations enabled to compute interesting solutions such as vortex-like solutions [7]. Symmetries was

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also used to study the decay of turbulence [14], the solutions having an eddy-independant energy transfer rate as postulated by Kolmogorov [22], to derive scaling laws [13], *etc.* Finally, we note that on the discrete point of view, symmetries have been integrated by some authors into numerical schemes, in order to build geometric integrators [4]. Especially in the non-isothermal case, Grassi *et al* [6, 8] made use of the symmetry approach to compute self-similar solutions in laminar regime. They used the Navier-Stokes-Fourier equations as non-isothermal model of Navier-Stokes equations. This model differs from the one used in the present paper (equations (1)) in that the temperature and the velocity are decoupled. In addition, it includes an energy-dissipation term in the equation of the temperature.

In this article, we propose to extend the work done in [18, 17] to the non-isothermal case and develop models for the subgrid stress tensor and the subgrid heat flux jointly, using the symmetry approach. Our goal is twofold. First, the preservation of symmetries will lead to physics-preserving models. Next, it avoids the use of the Reynolds analogy. Indeed, most of the time, only the subgrid stress tensor is really modeled. The expression of the the subgrid heat flux is simply deduced by assuming that the ratio between the subgrid viscosity and the subgrid thermal diffusion coefficient, called subgrid Prandtl number, is constant. This analogy limits the scope of the model since they may not correctly take into account the interaction between the dynamics of the flow and the temperature field. As we will see, with the symmetry approach, the subgrid heat flux model appears naturally without using the previous analogy.

This article is structured as follows. In section 2, Lie's algorithm of computing symmetries is presented. The symmetries of the non-isothermal Navier-Stokes equations are then listed. These symmetries are used in section 3 to develop a class of the subgrid models. The properties of these models, such as stability and wall behavior, are studied in section 4. A numerical test is presented in section 5. In Appendix A, the algorithm of symmetry computation is illustrated from an example.

2. The symmetry group of the non-isothermal Navier-Stokes equations

The general presentation of the Lie group theory can be found in [16], followed by the algorithm for the determination of Lie-symmetry groups of a system of differential equations. In the first part of this section, we make a brief reminder and apply the theory to the computation of the symmetries of the Navier-Stokes equations. The calculation will be summarized but interested readers may find in appendix more details on the calculations.

We consider an incompressible and non-isothermal Newtonian fluid flow, governed by the equations

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \dfrac{1}{\rho}\nabla p - \operatorname{div}\,\tau - \beta g\theta\,\boldsymbol{e}_2 = 0 \\ \dfrac{\partial \theta}{\partial t} + \operatorname{div}(\theta\boldsymbol{u}) - \operatorname{div}\,\mathbf{h} = 0 \\ \operatorname{div}\,\boldsymbol{u} = 0 \end{cases} \tag{1}$$

In these equations, $\tau = 2\nu\mathsf{S}$ and $\mathbf{h} = \kappa\nabla\theta$ are respectively the viscous stress tensor and the heat flux, and S is the strain rate tensor. $\boldsymbol{e}_2$ is the ascendant unit vector.

Equations (1) represent an idealized motion where the variation of density, due to the change of temperature, is neglected except in the buoyancy term (Boussinesq hypothesis). The situations where this hypothesis are discussed in [5] where the order of magnitude of each term is analyzed and Reyleigh's model is presented. Moreover, the effect of energy dissipation on the temperature has been neglected before the heat flux term. Indeed, in many engineering fields, such as building or aeronautics, the presence of the energy dissipation in the temperature equation does not significantly change the behaviour of the flow when the variation of temperature remains small.

Consider a set of transformations

$$g_a \;:\; \boldsymbol{q} = (t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \mapsto \widehat{\boldsymbol{q}} = (\widehat{t}, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{u}}, \widehat{p}, \widehat{\theta}), \tag{2}$$

depending continuously on a real parameter a, and having the structure of a (eventually local) Lie group. Transformations g_a act on the phase space J of equations (1).

A transformation (2) is called a Lie-symmetry of (1) if it maps a solution of (1) into another solution. The maximal group formed by Lie-symmetries of the equations is called the Lie-symmetry group of these ones.

The principal tool for the computation of Lie groups is the vector field:

$$\mathbf{X} = \xi_{\boldsymbol{q}} \cdot \frac{\partial}{\partial \boldsymbol{q}} = \xi_t \frac{\partial}{\partial t} + \sum_i \xi_{x_i} \frac{\partial}{\partial x_i} + \sum_i \xi_{u_i} \frac{\partial}{\partial u_i} + \xi_p \frac{\partial}{\partial p} + \xi_\theta \frac{\partial}{\partial \theta} \tag{3}$$

where $\xi_{\boldsymbol{q}} = \xi_{\boldsymbol{q}}(\boldsymbol{q})$ represents the variation of $\widehat{\boldsymbol{q}}$:

$$\xi_{\boldsymbol{q}} = \left.\frac{\partial \widehat{\boldsymbol{q}}}{\partial a}\right|_{a=0}. \tag{4}$$

The vector field $\mathbf{X}$ belongs to the tangent space of the phase space J.

Since the expression of $\widehat{\boldsymbol{q}}$ can be recovered from the knowledge of $\mathbf{X}$ through the exponential map

$$\widehat{\boldsymbol{q}} = \exp(a\mathbf{X}) \cdot \mathbf{q}, \tag{5}$$

$\mathbf{X}$ is called the inifinitesimal generator of the transformation. Note that relation (5) is equivalent to

$$\begin{cases} \dfrac{\partial \widehat{\boldsymbol{q}}}{\partial a} = \xi_q(\widehat{\boldsymbol{q}}), \\ \\ \widehat{\boldsymbol{q}}(a=0) = \mathbf{q}. \end{cases} \tag{6}$$

In other words, the transformation g_a is the flow of the vector field $\mathbf{X}$.

In order to take into consideration the transformation of the partial derivatives involved in equations (1), the phase space J must be prolonged into the second order jet space :

$$J^{(2)} = \big\{\big(t, (x_i), (u_i), p, \theta, (u_{i;\iota}), (u_{i;\iota,\iota'}), (p_{;\iota}), (p_{;\iota,\iota'}), (\theta_{;\iota}), (\theta_{;\iota,\iota'})\big)\big\}. \tag{7}$$

The index i runs from 1 to 3, and $0 \leq \iota \leq \iota' \leq 3$. In this jet space, equations (1) can be seen as a sub-manifold $\mathcal{E}$, and a symmetry of (1) is a transformation keeping $\mathcal{E}$ invariant.

At the same time, the vector field $\mathbf{X}$ is prolonged into the second order vector field

$$\begin{aligned} \mathbf{X}^{(2)} = \mathbf{X} \quad & + \sum_{i,\iota} \xi_{u_{i;\iota}} \frac{\partial}{\partial u_{i;\iota}} + \sum_{i,\iota,\iota'} \xi_{u_{i;\iota,\iota'}} \frac{\partial}{\partial u_{i;\iota,\iota'}} \\ & + \sum_{\iota} \xi_{p_{;\iota}} \frac{\partial}{\partial p_{;\iota}} + \sum_{\iota,\iota'} \xi_{p_{;\iota,\iota'}} \frac{\partial}{\partial p_{;\iota,\iota'}} + \sum_{\iota} \xi_{\theta_{;\iota}} \frac{\partial}{\partial \theta_{;\iota}} + \sum_{\iota,\iota'} \xi_{\theta_{;\iota,\iota'}} \frac{\partial}{\partial \theta_{;\iota,\iota'}}, \end{aligned} \tag{8}$$

belonging to the tangent space of $J^{(2)}$. The first new coefficients are defined as follows :

$$\xi_{u_{i;\iota}} = D_{x_\iota}(\xi_{u_i}) - \sum_{\iota''=0}^{3} \frac{\partial u_i}{\partial x_{\iota''}} D_{x_\iota}(\xi_{x_{\iota''}}) \tag{9}$$

and

$$\xi_{u_{i;\iota,\iota'}} = D_{x_{\iota'}}(\xi_{u_{i;\iota}}) - \sum_{\iota''=0}^{3} \frac{\partial^2 u_i}{\partial x_\iota x_{\iota''}} D_{x'_\iota}(\xi_{x_{\iota''}}) \tag{10}$$

where $x_0 = t$ and D_{x_ι} is the total derivation operator according to x_ι. The other coefficients are defined in a similar way.

For simplicity, we also shorten system (1) to

$$E = 0. \tag{11}$$

According to Lie's theory, a Lie group of transformations, spanned by a vector field $\mathbf{X}$, is a Lie symmetry group of equations (11) if and only if

$$\mathbf{X}^{(2)} \cdot E = 0 \qquad \text{wherever} \qquad E = 0, \tag{12}$$

that is, if the prolonged vector field $\mathbf{X}^{(2)}$ is tangent to the sub-manifold $\mathcal{E}$ at any point of $\mathcal{E}$.

The symmetry criterion (12) provides an algorithmic computation method of the Lie-symmetries of the equations. Indeed, $\mathbf{X}^{(2)} \cdot E$ is a system of polynomials in the local coordinates of $J^{(2)}$. Condition (12) requires that their coefficients vanish. This leads to linear partial differential equations on $\xi_{\mathbf{q}}$ which, after resolution, determine the vector field $\mathbf{X}$. The Lie symmetries are deduced from (5) by exponentiation or from (6).

An interesting example, which describes the algorithm in detail, is provided in Appendix A. It shows how to compute the infinitesimal generators of symmetry for the equations of a thin layer flow.

In our case, $\mathbf{X}$ is a linear combination of the following vector fields:

- $\mathbf{X}_0 = \dfrac{\partial}{\partial t}$, which generates the Lie group of time translations

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t + a, \boldsymbol{x}, \boldsymbol{u}, p, \theta), \tag{13}$$

- $\mathbf{X}_1 = \zeta(t)\dfrac{\partial}{\partial p}$, which corresponds to the infinite-dimension Lie group of pressure translations

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x}, \boldsymbol{u}, p + \zeta(t), \theta), \tag{14}$$

- $\mathbf{X}_2 = \beta g\, x_3 \dfrac{\partial}{\partial p} + \dfrac{1}{\rho}\dfrac{\partial}{\partial \theta}$, which induces the group of pressure-temperature translations

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x}, \boldsymbol{u}, p + a\,\beta g\, x_3, \theta + a\frac{1}{\rho}), \tag{15}$$

- $\mathbf{X}_3 = x_2 \dfrac{\partial}{\partial x_1} - x_1 \dfrac{\partial}{\partial x_2} + u_2 \dfrac{\partial}{\partial u_1} - u_1 \dfrac{\partial}{\partial u_2}$, which spans the 3-dimensional Lie-group of horizontal rotations

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \mathsf{R}\boldsymbol{x}, \mathsf{R}\boldsymbol{u}, p, \theta) \tag{16}$$

  where R is a 2D (constant) rotation matrix,

- $\mathbf{X}_{3+i} = \alpha_i(t)\dfrac{\partial}{\partial x_i} + \dot{\alpha}_i(t)\dfrac{\partial}{\partial u_i} - \rho\, x_i \ddot{\alpha}_i(t)\dfrac{\partial}{\partial p}$, $i = 1, 2, 3$, which are the infinitesimal generators of the infinite-dimensional group of generalised Galilean transformations

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x} + \boldsymbol{\alpha}(t), \boldsymbol{u} + \dot{\boldsymbol{\alpha}}(t), p + \rho\, \boldsymbol{x} \cdot \ddot{\boldsymbol{\alpha}}(t), \theta), \tag{17}$$

- $\mathbf{X}_7 = 2t\dfrac{\partial}{\partial t} + \displaystyle\sum_{j=1}^{3} x_j \dfrac{\partial}{\partial x_j} - \sum_{j=1}^{3} u_j \dfrac{\partial}{\partial u_j} - 2p\dfrac{\partial}{\partial p} - 3\theta\dfrac{\partial}{\partial \theta}$, which generates a first group of scaling transformations

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (\mathrm{e}^{2a}\, t, \mathrm{e}^{a}\, \boldsymbol{x}, \mathrm{e}^{-a}\, \boldsymbol{u}, \mathrm{e}^{-2a}\, p, \mathrm{e}^{-3a}\, \theta). \tag{18}$$

In these expressions, ζ and $\boldsymbol{\alpha}$ are respectively arbitrary scalar and vectorial functions. If one considers transformations which also act on the variables ν and κ, one finds one more vector field

$$\mathbf{X}_8 = \sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial u_j} + 2p\frac{\partial}{\partial p} + \theta\frac{\partial}{\partial \theta} + 2\nu\frac{\partial}{\partial \nu} + 2\kappa\frac{\partial}{\partial \kappa} \tag{19}$$

which spans a second group of scaling transformations

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta, \nu, \kappa) \ \longmapsto \ (t, \mathrm{e}^{a}\, \boldsymbol{x}, \mathrm{e}^{a}\, \boldsymbol{u}, \mathrm{e}^{2a}\, p, \mathrm{e}^{a}\, \theta, \mathrm{e}^{2a}\, \nu, \mathrm{e}^{2a}\, \kappa). \tag{20}$$

The union of the above groups constitutes the Lie-symmetry group of the non-isothermal Navier-Stokes equations (1).

In addition to symmetries (13)-(20), equations (1) own non-Lie symmetries which are

- the (discrete) reflections

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \mapsto (t, \Lambda\boldsymbol{x}, \Lambda\boldsymbol{u}, p, \lambda_3\theta) \tag{21}$$

  where $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i = \pm 1$,

- and the material indifference in the limit of a 2D horizontal flow [2] which is a time-dependent rotation:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \mapsto (t, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{u}}, \widehat{p}, \theta) \tag{22}$$

  with

$$\begin{cases} \widehat{\boldsymbol{x}} = \mathsf{R}(t)\ \boldsymbol{x}, \\ \widehat{\boldsymbol{u}} = \mathsf{R}(t)\ \boldsymbol{u} + \dot{\mathsf{R}}(t)\ \boldsymbol{x}, \\ \widehat{p} = p - 3\omega\boldsymbol{\phi} + \omega^2\|\boldsymbol{x}\|^2/2 \end{cases} \tag{23}$$

  where $\mathsf{R}(t)$ is an horizontal 2D rotation matrix with angle ωt, ω a real parameter, $\boldsymbol{\phi}$ the usual 2D stream function defined by

$$\boldsymbol{u} = \mathrm{curl}(\boldsymbol{\phi}\boldsymbol{e}_3)$$

  if the flow is parallel to the plane $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. The norm symbol $\|\bullet\|$ indicates the Euclidian norm.

The combination of the above symmetries consitutes a group that is called the symmetry group of equations (1). Note that these transformations are symmetries of the equations, not of each solution. Indeed, any particular solution may not have the listed symmetries, especially in turbulent regime. A solution which is invariant under one of these transformations is called self-similar.

In the next section, the symmetry group is used to develop a class of turbulence models.

3. Symmetry and turbulence modelling

In large-eddy simulation approach, one does not directly solve equations (1). Instead, one computes an approximate solution $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ which contains only the large scales of the actual velocity $\boldsymbol{u}$, pressure p and temperature θ of the fluid. The equations of $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ are:

$$\begin{cases} \dfrac{\partial \overline{\boldsymbol{u}}}{\partial t} + \operatorname{div}(\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) + \dfrac{1}{\rho} \nabla \overline{p} - \operatorname{div}(\overline{\tau} - \tau_s) - \beta g \overline{\theta}\, \boldsymbol{e}_2 = 0 \\ \\ \dfrac{\partial \overline{\theta}}{\partial t} + \operatorname{div}(\overline{\theta} \overline{\boldsymbol{u}}) - \operatorname{div}(\overline{\mathbf{h}} - \mathbf{h}_s) = 0 \\ \\ \operatorname{div} \overline{\boldsymbol{u}} = 0. \end{cases} \tag{24}$$

In these equations, $\tau_s = \overline{\boldsymbol{u} \otimes \boldsymbol{u}} - \overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}$ is the subgrid stress tensor and $\mathbf{h}_s = \overline{\theta \boldsymbol{u}} - \overline{\theta} \overline{\boldsymbol{u}}$ the subgrid heat flux, which have to be modelled. A "good" turbulence model is one with which $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ has the same properties as $(\boldsymbol{u}, p, \theta)$ from some point of view. In our approach, we require that $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ has the same symmetry properties as $(\boldsymbol{u}, p, \theta)$. This requirement is essential since, as underlined earlier, the symmetry group contains important physical information on the flow.

More precisely, the model should be such that each symmetry of (1) applied to $(\boldsymbol{u}, p, \theta)$, is also a symmetry of (24) applied to $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$. When it is the case, the model will be called *invariant*. Unfortunately, as analyzed in [19], the major part of existing LES turbulence models do not comply with this requirement. In what follows, we propose new models based on the preservation of the symmetry group of the equations.

It is clear that time translations (13), applied to $(t, \boldsymbol{x}, \overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$, are symmetries of the filtered equations (24) if τ_s and $\mathbf{h}_s$ do not explicitely depend on t. It is also straightforward to check that the pressure-temperature translations, and the galilean transformations remain symmetries of (24) if τ_s and $\mathbf{h}_s$ depend only on $\overline{\mathsf{S}}$ and $\overline{\mathbb{T}} = \nabla \overline{\theta}$:

$$-\tau_s = -\tau_s(\overline{\mathsf{S}}, \overline{\mathbb{T}}), \qquad -\mathbf{h}_s = -\mathbf{h}_s(\overline{\mathsf{S}}, \overline{\mathbb{T}}) \tag{25}$$

From the classical theory of isotropic functions [21, 10], one can deduce that rotations, reflections and the material indifference are symmetries of (24) if the model has the following form

$$\begin{cases} -\tau_s^d &= E_1 \overline{\mathsf{S}} + E_2 \operatorname{Adj}^d \overline{\mathsf{S}} + E_3 (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d \\ & \quad + E_4 \left[\overline{\mathsf{S}} (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})\right]^d + E_5 \left[\overline{\mathsf{S}} (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}}) \overline{\mathsf{S}}\right]^d \\ \\ -\mathbf{h}_s &= E_6 \overline{\mathbb{T}} + E_7 \overline{\mathsf{S}}\, \overline{\mathbb{T}} + E_8 \overline{\mathsf{S}}^2 \overline{\mathbb{T}} \end{cases} \tag{26}$$

where the coefficients E_i are scalar functions of:

$$\chi = \operatorname{tr} \overline{\mathsf{S}}^2, \quad \xi = \det \overline{\mathsf{S}}, \quad \vartheta = \overline{\mathbb{T}}^2, \quad \omega_1 = \overline{\mathbb{T}} \cdot \overline{\mathsf{S}}\, \overline{\mathbb{T}}, \quad \omega_2 = \overline{\mathsf{S}}\, \overline{\mathbb{T}} \cdot \overline{\mathsf{S}}\, \overline{\mathbb{T}}. \tag{27}$$

Adj denotes the adjoint operator, verifying

$$\overline{\mathsf{S}}\ \mathrm{Adj}\,\overline{\mathsf{S}} = (\det \overline{\mathsf{S}}) I_d$$

where I_d is the identity matrix. The superscript $(^d)$ designates the deviatoric part.

Next, equations (24) are invariant under the first scale transformations if

$$\widehat{\tau}_s = \mathrm{e}^{-2a}\,\tau_s \qquad \text{and} \qquad \widehat{\mathbf{h}}_s = \mathrm{e}^{-4a}\,\mathbf{h}_s. \tag{28}$$

This condition implies that

$$E_1(\widehat{\chi}, \widehat{\xi}, \widehat{\vartheta}, \widehat{\omega}_1, \widehat{\omega}_2) = E_1(\chi,\ \xi,\ \vartheta,\ \omega_1,\ \omega_2), \tag{29}$$

that is

$$E_1(\mathrm{e}^{-4a}\,\chi, \mathrm{e}^{-6a}\,\xi, \mathrm{e}^{-8a}\,\vartheta, \mathrm{e}^{-10a}\,\omega_1, \mathrm{e}^{-12a}\,\omega_2) = E_1(\chi,\ \xi,\ \vartheta,\ \omega_1,\ \omega_2). \tag{30}$$

Deriving with respect to a and setting a to zero, we get

$$-4\chi\frac{\partial E_1}{\partial \chi} - 6\xi\frac{\partial E_1}{\partial \xi} - 8\vartheta\frac{\partial E_1}{\partial \vartheta} - 10\omega_1\frac{\partial E_1}{\partial \omega_1} - 12\omega_2\frac{\partial E_1}{\partial \omega_2} = 0. \tag{31}$$

Applying the same procedure to the other coefficients E_i, we get the following conditions:

$$-4\chi\frac{\partial E_i}{\partial \chi} - 6\xi\frac{\partial E_i}{\partial \xi} - 8\vartheta\frac{\partial E_i}{\partial \vartheta} - 10\omega_1\frac{\partial E_i}{\partial \omega_1} - 12\omega_2\frac{\partial E_i}{\partial \omega_2} = s_i E_i \tag{32}$$

where

$$s_1 = 0, \quad s_2 = -\tfrac{1}{2}, \quad s_3 = -\tfrac{3}{2}, \quad s_4 = -2,$$
$$s_5 = -\tfrac{5}{2}, \quad s_6 = 0, \quad s_7 = -\tfrac{1}{2}, \quad s_8 = -1.$$

The characteristic equations are

$$\frac{\mathrm{d}\chi}{\chi} = \frac{\mathrm{d}\xi}{\frac{3}{2}\xi} = \frac{\mathrm{d}\vartheta}{2\vartheta} = \frac{\mathrm{d}\omega_1}{\frac{5}{2}\omega_1} = \frac{\mathrm{d}\omega_2}{3\omega_2} = \frac{\mathrm{d}E_i}{s_i E_i}, \qquad i = 1, ..., 8. \tag{33}$$

From the resolution of these equations, we conclude that the model is invariant under the first scale transformation if

$$E_i(\chi,\ \xi,\ \vartheta,\ \omega_1,\ \omega_2) = \chi^{s_i} E_i'(v_1, v_2, v_3, v_4) \tag{34}$$

where the v_i's are the invariants:

$$v_1 = \frac{\xi}{\chi^{3/2}}, \quad v_2 = \frac{\vartheta}{\chi^2}, \quad v_3 = \frac{\omega_1}{\chi^{5/2}}, \quad v_4 = \frac{\omega_2}{\chi^{3.}}. \tag{35}$$

Finally, the second scale transformations are symmetries of equations (24) if

$$E'_i = \nu F_i, \qquad i = 1, ..., 5 \qquad \text{and} \qquad E'_i = \kappa F_i, \qquad i = 6, ..., 8.$$

To sum up, we get the following class of subgrid models which are consistent with the symmetry group of (1):

$$\begin{cases} -\tau_s^d = \nu F_1 \overline{\mathsf{S}} + \nu \chi^{-1/2} F_2 \ \mathrm{Adj}^d\, \overline{\mathsf{S}} + \nu \chi^{-3/2} F_3 (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d \\ \qquad + \nu \chi^{-2} F_4 [\overline{\mathsf{S}} (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})]^d] + \nu \chi^{-5/2} F_5 \overline{\mathsf{S}} [(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}}) \overline{\mathsf{S}}]^d], \\ -\mathbf{h}_s = \kappa \left(F_6 + \chi^{-1/2} F_7 \overline{\mathsf{S}} + \chi^{-1} F_8 \overline{\mathsf{S}}^2 \right) \overline{\mathbb{T}}. \end{cases} \tag{36}$$

Note that the invariants (35) and the eight arbitrary functions F_i's arise naturally from the symmetry requirement, without extra-hypothesis. In general, the model is strongly coupled, in the sense that the filtered temperature gradient can intervene in the subgrid stress tensor model and, conversely, the filtered strain rate tensor is present in the subgrid heat flux model. However, by the choice of the arbitrary functions, this coupling may be weakened.

Other properties of the models, and choices of the arbitrary functions will be analysed in the next section.

4. Properties and simplification of the models

By construction, models (36) preserve the properties of the actual solution, such as conservation laws, spectral properties, *etc.* In particular, they are compatible with the scaling laws of the mean flow. Indeed, as proved in the next subsection, scaling laws are particular self-similar solutions under the symmetries of the fluctuation equations. Consequently, symmetry preserving turbulence models do not destroy these scaling laws.

4.1. Non-isothermal scaling laws

The present method of computing scaling laws is an extension of the work of Oberlack [13] to the non-isothermal case. Consider a bi-dimensional, parallel, steady turbulent shear flow, driven by a constant pressure gradient K in the streamwise direction x_1. The decomposition of the dependent variables into a mean and a fluctuating parts gives:

$$\boldsymbol{u} = \boldsymbol{U}(x_2) + \boldsymbol{u}', \qquad p = [P(x_2) + K x_1] + p', \qquad \theta = \Theta(x_2) + \theta', \tag{37}$$

with $U_2 = U_3 = 0$. Assume that $\rho = 1$. The equations of the fluctuating parts write

$$\left\{\begin{array}{l}
\dfrac{\partial \boldsymbol{u}'}{\partial t} + \operatorname{div}(\boldsymbol{u}' \otimes \boldsymbol{u}') + U_1 \dfrac{\partial \boldsymbol{u}'}{\partial x_1} + \nabla p' - \nu \Delta \boldsymbol{u}' \\
\qquad + \left[u_2' \dfrac{\mathrm{d}U_1}{\mathrm{d}x_2} - K - \nu \dfrac{\mathrm{d}^2 U_1}{\mathrm{d}x_2^2} \right] \boldsymbol{e}_1 + \left[\dfrac{\mathrm{d}P}{\mathrm{d}x_2} - \beta g(\Theta + \theta') \right] \boldsymbol{e}_2 = 0, \\
\\
\dfrac{\partial \theta'}{\partial t} + \operatorname{div}(\theta' \boldsymbol{u}') - \kappa \Delta \theta' + u_2' \dfrac{\mathrm{d}\Theta}{\mathrm{d}x_2} + U_1 \dfrac{\partial \theta'}{\partial x_1} - \kappa \dfrac{\mathrm{d}^2 \Theta}{\mathrm{d}x_2^2} = 0, \\
\\
\operatorname{div} \boldsymbol{u}' = 0.
\end{array}\right. \tag{38}$$

$\boldsymbol{e}_1$ is the streamwise unit vector.

We look for self-similar solutions in (U_1, Θ), that is, solutions such that

$$\left\{\begin{array}{l}
\widehat{U}_1(\widehat{x}_2) = U_1(x_2), \\
\\
\widehat{\Theta}(\widehat{x}_2) = \Theta(x_2)
\end{array}\right. \tag{39}$$

under a symmetry of equations (38). Lie's theory permits to transform relations (39) into the condition:

$$X \cdot (U_1, \Theta) = 0 \tag{40}$$

where

$$\begin{aligned}
X &= \xi_t \frac{\partial}{\partial t} + \sum_{i=1}^{3} \xi_{x_i} \frac{\partial}{\partial x_i} + \xi_{U_1} \frac{\partial}{\partial U_1} + \xi_P \frac{\partial}{\partial P} + \xi_\Theta \frac{\partial}{\partial \Theta} \\
&+ \sum_{i=1}^{3} \xi_{u_i'} \frac{\partial}{\partial u_i'} + \xi_{p'} \frac{\partial}{\partial p'} + \xi_{\theta'} \frac{\partial}{\partial \theta'} + \xi_\nu \frac{\partial}{\partial \nu} + \kappa \frac{\partial}{\partial \kappa}
\end{aligned} \tag{41}$$

is an infinitesimal generator of Lie-symmetries of (38). Using the Lie-symmetry computation method presented in section 2, applied to (38), we get the components of X:

$$
\begin{aligned}
\xi_t &= [n-2m]\,t + a_0, \\
\xi_{x_2} &= [n-m]\,x_2 + a, \\
\xi_{x_1} &= [n-m]\,x_1 + f_1(t) + a_1, \\
\xi_{x_3} &= [n-m]\,x_3 + f_2(t) + a_2 \\
\xi_{U_1} &= mU_1 - f_3(t) - f_4(t,x_2) + f_5(x_2,U_1,P) + \dot{f}_1(t), \\
\xi_P &= 2mP + \frac{\partial f_4}{\partial t}(t,x_2)x_1 - f_6(t,x_2) + f_7(x_2,U_1,P) \\
&+ \beta g f_8(t,x_1,x_3)x_2 + f_9(t,x_1,x_3), \\
\xi_\Theta &= [3m-n]\,\Theta + f_8(t,x_1,x_3) - f_{10}(x_2), \\
\xi_{u'_1} &= mu'_1 + f_3(t) + f_4(t,x_2) - f_5(x_2,U_1,P), \\
\xi_{u'_2} &= mu'_2, \\
\xi_{u'_3} &= mu'_3 + \dot{f}_2(t), \\
\xi_{p'} &= 2mp' + \left[K(n-3m) - \dot{f}_3(t) - \frac{\partial f_4}{\partial t}(t,x_2)\right]x_1 - \ddot{f}_2(t)x_3 \\
&+ f_6(t,x_2) - f_7(x_2,U_1,P), \\
\xi_{\theta'} &= [3m-n]\,\theta' + f_{10}(x_2), \\
\xi_\nu &= n\nu, \\
\xi_\kappa &= n\kappa.
\end{aligned}
\tag{42}
$$

The a_i's and the f_i's are respectively arbitrary scalars and functions. The self-similar solutions can be obtained from the characteristic equations

$$\frac{\mathrm{d}U_1}{\xi_{U_1}} = \frac{\mathrm{d}x_2}{\xi_{x_2}} = \frac{\mathrm{d}\Theta}{\xi_\Theta}. \tag{43}$$

We do not aim to be exhaustive. Thus, the parameters f_i's are assumed constant. Equations (43) becomes:

$$\frac{\mathrm{d}U_1}{mU_1 + b_1} = \frac{\mathrm{d}x_2}{(n-m)x_2 + a} = \frac{\mathrm{d}\Theta}{(3m-n)\Theta + b_2}. \tag{44}$$

where

$$b_1 = -f_3 - f_4 + f_5 \qquad \text{and} \qquad b_2 = f_8 - f_{10}$$

are constants. According to the values of n and m, we get the following scaling laws.

- If $m = n = 0$, the mean velocity and mean temperature write

$$U_1 = \frac{b_1}{a}(x_2 + b), \qquad \Theta = \frac{b_2}{a}(x_2 + b) \tag{45}$$

where the constants C_i's and b can be determined from a, b_1, b_2 and the integration constants.
Equations (45) represent linear wall laws for both velocity and temperature.

- When $m = 0$ and $n \neq 0$, the mean profiles verify

$$U_1 = C_1 \ln(x_2 + b) + C_3, \qquad \Theta = C_2(x_2 + b)^{-1} + C_4 \tag{46}$$

where the C_i's and b are also constants but do not necessarily have the same values as in (45). In (46), the velocity follows a logarithmic law. The corresponding hyperbolic temperature profile is less common. This scaling law is worthy to be checked experimentally or numerically.

- The third self-similar solution is obtained with $m = n \neq 0$:

$$U_1 = \exp[C_1(x_2 + b)] + C_3, \qquad \Theta = \exp[2C_1(x_2 + b)] + C_4. \tag{47}$$

The velocity law is similar to the exponential law found by Oberlack [13] in the isothermal case, which was confirmed later by DNS [15] in the mid-wake region of a high Reynolds boundary layer. The temperature law is also in an exponential form.

- For the case $m \neq 0$, $n \neq m$ and $n \neq 3m$,

$$U_1 = C_1(x_2 + b)^a + C_3, \qquad \Theta = C_2(x_2 + b)^{2a-1} + C_4 \tag{48}$$

where $a = m/(n - m)$. We recognize algebraic laws for both velocity and temperature. The exponents are closely linked.

When the invariant model developed previously is added to equations (38), the symmetry properties are not violated. We then get the same self-similarity variables as in (45)-(48).

In the next subsection, we impose the stability of the models. This will reduce the degree of freedom of the models.

4.2. Stability of the model

It can be stated that the molecular strain rate tensor τ and the molecular heat flux $\mathbf{h}$ derive from the positive convex potentials $\nu\ \mathrm{tr}\,\overline{\mathsf{S}}^2$ and $\kappa||\overline{\mathbb{T}}||^2/2$, in the sense that

$$\tau = \frac{\partial}{\partial \overline{\mathsf{S}}}\left(\nu\ \mathrm{tr}\,\overline{\mathsf{S}}^2\right) \qquad \text{and} \qquad \mathbf{h} = \frac{\partial}{\partial \mathbb{T}}\left(\frac{1}{2}\kappa||\overline{\mathbb{T}}||^2\right). \tag{49}$$

This ensures, in particular, that the mechanical dissipation, defined as $\mathrm{tr}(\tau\mathsf{S})$, and the thermal dissipation are positive. We assume that, similarly, the subgrid strain tensor τ_s and the subgrid heat flux $\mathbf{h}_s$ derive from convex potentials. This condition leads to the following class of models (see [19]):

$$\begin{cases} -\tau_s^d = \nu \left[2g_m - 3v_1 \dfrac{\partial g_m}{\partial v_1} - 4v_2 \dfrac{\partial g_m}{\partial v_2} - 5v_3 \dfrac{\partial g_m}{\partial v_3} - 6v_4 \dfrac{\partial g_m}{\partial v_4} \right] \overline{\mathsf{S}} \\ \qquad + \nu \left[\chi^{-1/2} \dfrac{\partial g_m}{\partial v_1} \ \mathrm{Adj}^d\, \overline{\mathsf{S}} + \chi^{-3/2} \dfrac{\partial g_m}{\partial v_3} (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d + 2\chi^{-2} \dfrac{\partial g_m}{\partial v_4} [\overline{\mathsf{S}}(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})]^d \right], \\ \\ -\mathbf{h}_s = \kappa \left(\dfrac{\partial g_t}{\partial v_2} + \chi^{-1/2} \dfrac{\partial g_t}{\partial v_3} \overline{\mathsf{S}} + \chi^{-1} \dfrac{\partial g_t}{\partial v_4} \overline{\mathsf{S}}^2 \right) \overline{\mathbb{T}}, \end{cases} \tag{50}$$

where g_m and g_t are arbitrary functions of the invariants v_i's. It can be shown that a model defined as in (50) is stable, in the sense that the L^2-norm of the solution remains bounded [19].

In order to simplify the form of the model, we assume that g_m depends only on $v = v_1$ and $g_t = h_t(v)v_2$, such that v is the only scalar invariant involved in the model. The latter then reads

$$\begin{cases} -\tau_s^d = \nu \left[2g_m(v) - 3v g_m'(v) \right] \overline{\mathsf{S}} + \dfrac{\nu g_m'(v)}{||\overline{\mathsf{S}}||} \ \mathrm{Adj}^d\, \overline{\mathsf{S}}, \\ \\ -\mathbf{h}_s = \kappa h_t(v) \overline{\mathbb{T}}. \end{cases} \tag{51}$$

This reduces the degree of freedom of the model to 2.

In the following section, we study the wall behavior of model (51) and the corresponding subgrid viscosity.

4.3. Subgrid viscosity at the wall

Strictly speaking, model (51) is not a subgrid-viscosity model. However, the effective viscosity, caused by the model, can be defined from the subgrid dissipation as follows.

The molecular dissipation rate ε is linked to the molecular viscosity by the relation

$$\varepsilon = \ \mathrm{tr}[\tau \mathsf{S}] = \nu \ \mathrm{tr}\, \mathsf{S}^2.$$

Similarly, the subgrid scale ν_s is defined by

$$\varepsilon_s = \ \mathrm{tr}[(-\tau_s)\overline{\mathsf{S}}] = \nu_s \ \mathrm{tr}\, \overline{\mathsf{S}}^2.$$

Using (51), it follows:

$$\nu_s = 2\nu g_m(v).$$

Assume that the wall is at $x_2 = 0$. The Taylor expansions of the velocity components are

$$
\begin{aligned}
u_1 &= C_1^1 x_2 + C_1^2 x_2^2 + O(x_2^3) \\
u_2 &= C_2^2 x_2^2 + O(x_2^3) \\
u_3 &= C_3^1 x_2 + C_3^2 x_2^2 + O(x_2^3)
\end{aligned} \tag{52}
$$

It is admitted that ν_s should vanish at the wall and behave as $O(x_2^3)$ near this wall. This behavior is reached if we choose

$$
g_m(v) = C_m(1 - \mathrm{e}^{-v^3}), \tag{53}
$$

since

$$
\overline{\mathsf{S}} = O(1), \qquad \det \overline{\mathsf{S}} = O(x_2), \qquad v = O(x_2).
$$

With the same form for the function h_t, we get

$$
\begin{cases}
-\tau_s^d = C_m \nu \left[\left(2 - 2\,\mathrm{e}^{-v^3} - 9v^3\,\mathrm{e}^{-v^3}\right)\overline{\mathsf{S}} + \dfrac{3v^2\,\mathrm{e}^{-v^3}}{\|\overline{\mathsf{S}}\|}\ \mathrm{Adj}^d\,\overline{\mathsf{S}} \right], \\[2ex]
-\mathbf{h}_s = C_t \kappa (1 - \mathrm{e}^{-v^3}) \overline{\mathbb{T}}.
\end{cases} \tag{54}
$$

where C_m and C_t are constants. With this model, there is then no need to introduce a wall-damping function since the model automatically vanishes at the wall.

A numerical test on model (54) is presented in the next section.

5. Numerical test

Consider an air flow in the ventilated room presented in figure 1 [see 23]. The dimension of the room is 1.04m×1.04m×0.7m. The inlet and outlet heights are respectively 0.018 and 0.024m, and the inlet velocity is 0.57m/s. The floor is heated at 35°C while the other walls are maintained at 15°C. The Reynolds number, based on the inlet height and the inlet velocity, is 678.

The simulation was carried out with the finite volume code *Code_Saturne* [1], on a 86×86×12 grid. The experimental data of Cheeswright *et. al* [3] are taken as reference solution. A Cranck-Nicolson scheme in time is used. The time step is $7{\cdot}10^{-3}$s. In the figures, the mean profiles, computed over 1750 seconds, are presented.

The constants C_m and C_t are taken equal to $(C_s\overline{\delta})^2$ where C_s is the Smagorinsky constant. For the comparison, the usual Smagorinsky model [20] was also used. Recall that the Smagorinsky model reads

$$
\begin{cases}
-\tau_s^d = \nu_{smago}\overline{\mathsf{S}}, \\[2ex]
-\mathbf{h}_s = \kappa_{smago}\overline{\mathbb{T}}.
\end{cases} \tag{55}
$$

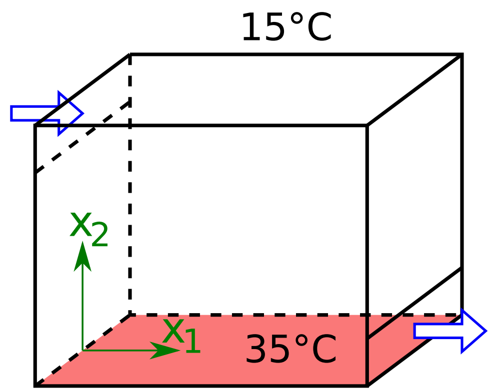

Figure 1: Geometry of the room

where the subgrid viscosity is defined as follows

$$\nu_{smago} = (C_s\bar{\delta})^2\sqrt{2 \ \mathrm{tr}\,\bar{\mathsf{S}}^2} \tag{56}$$

and the subgrid diffusion coefficient κ_{smago} is linked to the subgrid viscosity by the relation:

$$\kappa_{smago} = \frac{\nu_{smago}}{Pr_s} \tag{57}$$

Pr_s is the subgrid Prandtl number, taken equal to 0.699.

Figure 2 shows that the horizontal velocity given by the invariant model (54) fits very well the experimental data. The concordance is particularly striking near the walls. The maximum values near the floor and near the ceilor are correctly predicted. This good agreement can be explained by the non-violation of the wall laws by the invariant model. Almost everywhere, the invariant model provides better results than the classical Smagorinsky model wich seems very dissipative. The vertical velocity, shown on figure 3, right, presents the same trend.

Figures 4 and 5 report the temperature profiles along a vertical and an horizontal lines, passing through the middle of the room. It can be observed on these figures that the invariant model predicts the temperature behavior better than the Smagorinsky model does. However, both models under-estimate the experimental measurements. This may not due to the models but to the (well-known) bad control of boundary conditions during the experimentation. Indeed, some phenomena such as radiation or the variation of temperature at the wall are hard to take into account.

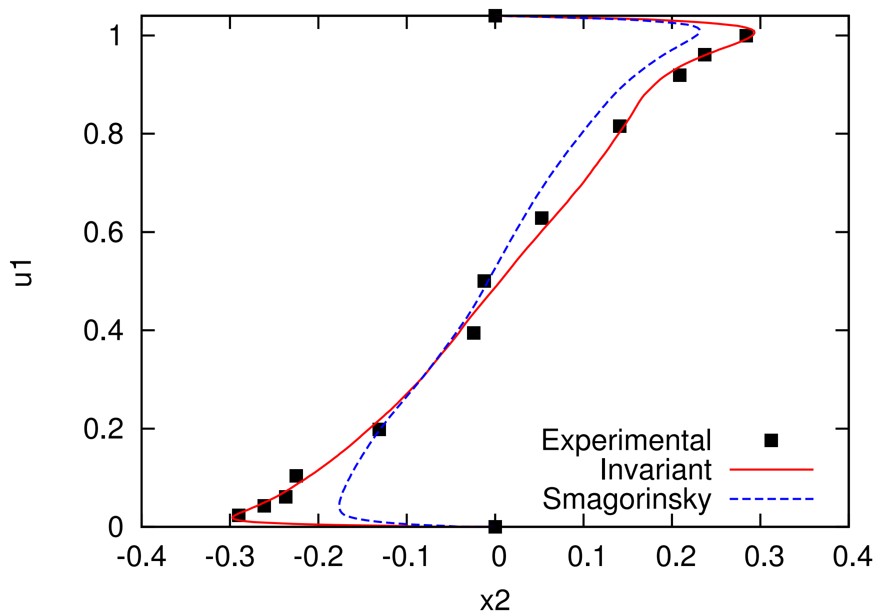

Figure 2: Horizontal mean velocity profile at $x_1 = 0.501$m, at the half-width

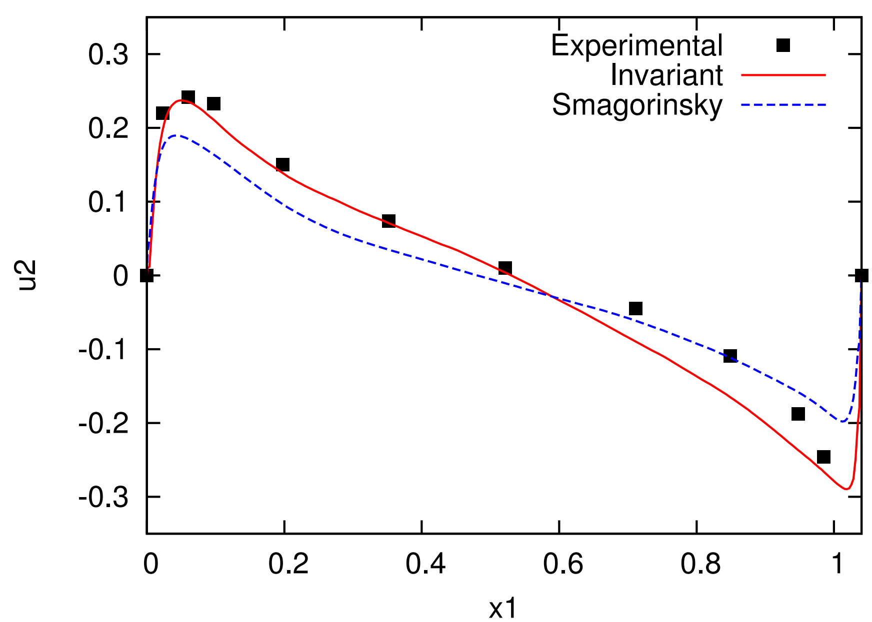

Figure 3: vertical mean velocity profile at $x_2 = 0.501$m, at the half-width

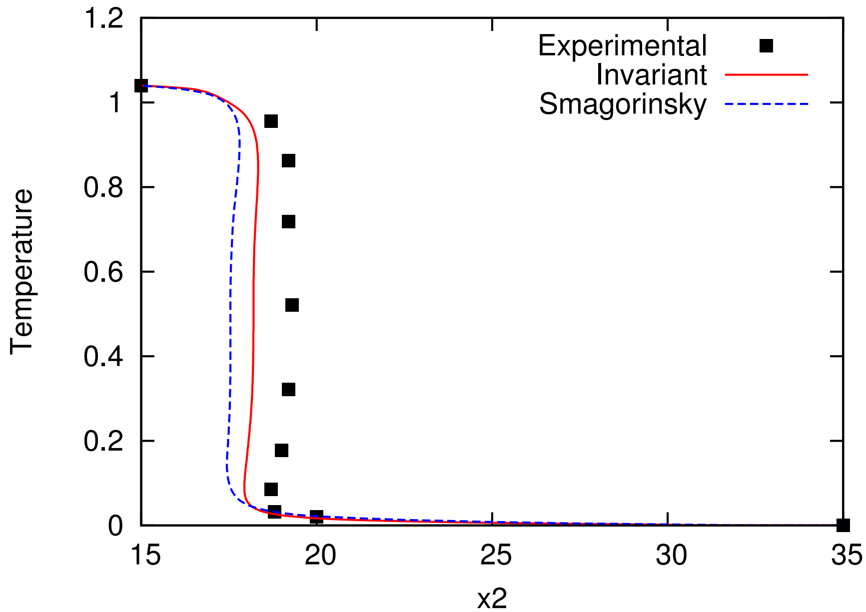

Figure 4: Mean temperature profiles at $x_1 = 0.501$m, at the half-width

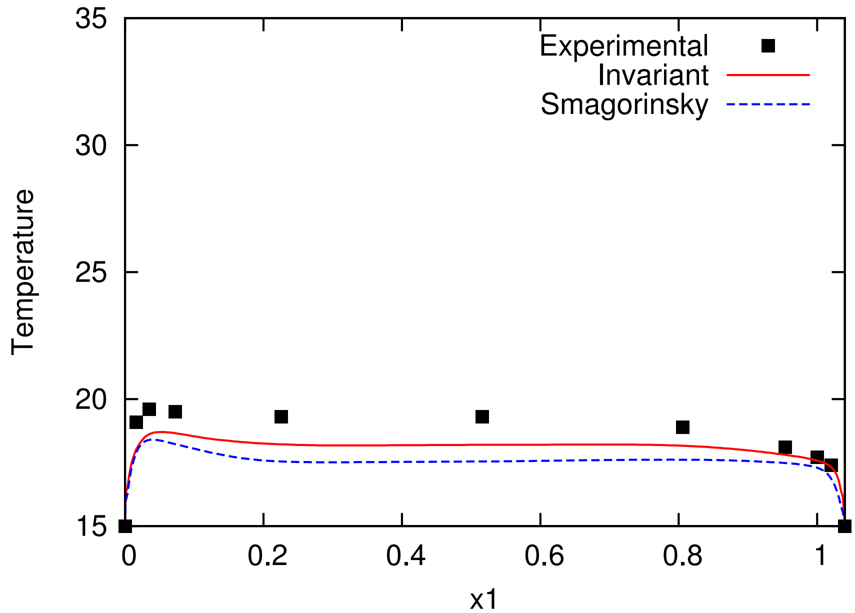

Figure 5: Mean temperature profiles at $x_2 = 0.501$m, at the half-width

6. Conclusion

We showed that the symmetry approach can lead to contructive method for turbulence modelling. The form of model and the non-classical invariants (35) appeared naturally without any other assumption than the invariance under the symmetry group.

However, The symmetry approach does not fix all the parameters of the model. Rather, it leaves some arbitrary functions. This gives some freedom in the developpement, and can be used to ask to the model to fit additionnal requirements. In this article, they were choosen such that the model has a numerical stability and behaves correctly near the wall. The compatibility of the model with the wall laws seems to be confirmed by the numerical test.

Note that the antisymmetric part of grad$\overline{\boldsymbol{u}}$ could be included in the general form (36) of the model in 3D turbulence since the indifference material is a symmetry of the model only in the limit of 2D flow. For simplicity reason, this was not done in this work.

Scaling laws for non-isothermal flows were also derived, in section 4.1, from the symmetry approach. If the velocity laws are familiar, the corresponding temperature behavior is not always common. For example, we saw that a logarithmic velocity profile goes together with an hyperbolic evolution of the temperature. These results have to be compared with experimental or DNS data.

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Appendix A. Example of symmetry computation

In this appendix, we show, from an example, Lie's alogrithm for the determination of the symmetry groups of a system of equations. As we will see, the symmetry group of this example system is not very rich in the sense that the symmetries can be "guessed" without Lie's theory. Rather, the equations have been choosen because of their great interest in fluid mechanics and because they are simple enough that the calculation can be carried out by hand. Some theoretical considerations such as the involutivity are not discussed here.

Consider the equations of a 2D laminar thin shear layer flow:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\[2ex] u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2} \end{cases} \tag{A.1}$$

According to the boundary conditions, these equations can model a boundary layer, a mixing layer or a jet.

A Lie symmetry of (A.1) has the form

$$\boldsymbol{q} = (x, y, u, v) \mapsto (\widehat{x}(\boldsymbol{q}, a), \widehat{y}(\boldsymbol{q}, a), \widehat{u}(\boldsymbol{q}, a), \widehat{v}(\boldsymbol{q}, a)). \tag{A.2}$$

It is caracterized by the infinitesimal generators:

$$\mathbf{X} = \xi_x\frac{\partial}{\partial x} + \xi_y\frac{\partial}{\partial y} + \xi_u\frac{\partial}{\partial u} + \xi_v\frac{\partial}{\partial v}. \tag{A.3}$$

where

$$\xi_x = \left.\frac{\partial \widehat{x}}{\partial a}\right|_{a=0}, \quad \xi_y = \left.\frac{\partial \widehat{y}}{\partial a}\right|_{a=0}, \quad \xi_u = \left.\frac{\partial \widehat{u}}{\partial a}\right|_{a=0}, \quad \xi_v = \left.\frac{\partial \widehat{v}}{\partial a}\right|_{a=0}.$$

The components ξ_x, ξ_y, ξ_u and ξ_v of $\mathbf{X}$ represent respectively the infinitesimal variation of x, y, u and v when transformation (A.2) is applied. They are functions of (x, y, u, v). In what follows, we show how to compute ξ_x, ξ_y, ξ_u and ξ_v and to deduce the symmetry (A.2).

Equations (A.1) define a manifold on the 14-dimensional jet space

$$J^{(2)} = \{(x, y, u, v, u_x, u_y, v_x, v_y, u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, v_{yy})\}.$$

In order to take into account the derivatives involved in the equations, $\mathbf{X}$ is prolonged into the folowing vector field which acts on the jet space $J^{(2)}$:

$$\begin{array}{rcl} \mathbf{X} & = & \xi_x \dfrac{\partial}{\partial x} + \xi_y \dfrac{\partial}{\partial y} + \xi_u \dfrac{\partial}{\partial u} + \xi_v \dfrac{\partial}{\partial v}. \\ & + & \xi_{u_x} \dfrac{\partial}{\partial u_x} + \xi_{u_y} \dfrac{\partial}{\partial u_y} + \xi_{v_x} \dfrac{\partial}{\partial v_x} + \xi_{v_y} \dfrac{\partial}{\partial v_y} \\ & + & \xi_{u_{xx}} \dfrac{\partial}{\partial u_{xx}} + \xi_{u_{xy}} \dfrac{\partial}{\partial u_{xy}} + \xi_{u_{yy}} \dfrac{\partial}{\partial u_{yy}} \\ & + & \xi_{v_{xx}} \dfrac{\partial}{\partial v_{xx}} + \xi_{v_{xy}} \dfrac{\partial}{\partial v_{xy}} + \xi_{v_{yy}} \dfrac{\partial}{\partial v_{yy}} \end{array} \qquad (A.4)$$

The coefficient ξ_{u_x} represents the infinitesimal variation of $\dfrac{\partial u}{\partial x}$ under transformation (A.2). It can be deduced from the infinitesimal variations of x, y and u. More precisely, ξ_{u_x} can be defined as follows (see (9)):

$$\begin{array}{rcl} \xi_{u_x} & = & D_x(\xi_u) - \dfrac{\partial u}{\partial x} D_x(\xi_x) - \dfrac{\partial u}{\partial y} D_x(\xi_y) \\ & = & \dfrac{\partial \xi_u}{\partial x} + u_x \dfrac{\partial \xi_u}{\partial u} + v_x \dfrac{\partial \xi_u}{\partial v} - u_x \left(\dfrac{\partial \xi_x}{\partial x} + u_x \dfrac{\partial \xi_x}{\partial u} + v_x \dfrac{\partial \xi_x}{\partial v} \right) \\ & - & u_y \left(\dfrac{\partial \xi_y}{\partial x} + u_x \dfrac{\partial \xi_y}{\partial u} + v_x \dfrac{\partial \xi_y}{\partial v} \right) \end{array} \qquad (A.5)$$

The components ξ_{u_y} and ξ_{v_y} are defined in similar ways:

$$\begin{array}{rcl} \xi_{u_y} & = & D_y(\xi_u) - \dfrac{\partial u}{\partial x} D_y(\xi_x) - \dfrac{\partial u}{\partial y} D_y(\xi_y) \\ & = & \dfrac{\partial \xi_u}{\partial y} + u_y \dfrac{\partial \xi_u}{\partial u} + v_y \dfrac{\partial \xi_u}{\partial v} - u_x \left(\dfrac{\partial \xi_x}{\partial y} + u_y \dfrac{\partial \xi_x}{\partial u} + v_y \dfrac{\partial \xi_x}{\partial v} \right) \\ & - & u_y \left(\dfrac{\partial \xi_y}{\partial y} + u_y \dfrac{\partial \xi_y}{\partial u} + v_y \dfrac{\partial \xi_y}{\partial v} \right) \end{array} \qquad (A.6)$$

$$\begin{array}{rcl} \xi_{v_y} & = & D_y(\xi_v) - \dfrac{\partial v}{\partial x} D_y(\xi_x) - \dfrac{\partial v}{\partial y} D_y(\xi_y) \\ & = & \dfrac{\partial \xi_v}{\partial y} + u_y \dfrac{\partial \xi_v}{\partial u} + v_y \dfrac{\partial \xi_v}{\partial v} - v_x \left(\dfrac{\partial \xi_x}{\partial y} + u_y \dfrac{\partial \xi_x}{\partial u} + v_y \dfrac{\partial \xi_x}{\partial v} \right) \\ & - & v_y \left(\dfrac{\partial \xi_y}{\partial y} + u_y \dfrac{\partial \xi_y}{\partial u} + v_y \dfrac{\partial \xi_y}{\partial v} \right) \end{array} \qquad (A.7)$$

The component $\xi_{u_{yy}}$ can be recursively written as a function of ξ_x, ξ_y, ξ_u and ξ_v using the formula (see (10)):

$$\xi_{u_{yy}} = D_y(\xi_{u_y}) - u_{yx} D_y(\xi_x) - u_{yy} D_y(\xi_y). \tag{A.8}$$

The other components of $\mathbf{X}^{(2)}$ are not usefull for our equations.

The symmetry criterion (12), applied to equations (A.1), reads

$$\begin{cases} \mathbf{X}^{(2)} \cdot (u_x + v_y) = 0 \\ \\ \mathbf{X}^{(2)} \cdot (u u_x + v u_y - \nu u_{yy}) = 0 \end{cases} \quad \text{when} \quad \begin{cases} u_x + v_y = 0 \\ \\ u u_x + v u_y - \nu u_{yy} = 0. \end{cases} \tag{A.9}$$

Written with the components of $\mathbf{X}^{(2)}$, this criterion becomes:

$$\begin{cases} \xi_{u_x} + \xi_{v_y} = 0 \\ \\ \xi_u u_x + u \xi_{u_x} + \xi_v u_y + v \xi_{u_y} - \nu \xi_{u_{yy}} = 0 \end{cases} \tag{A.10}$$

under the condition that

$$\begin{cases} u_x + v_y = 0 \\ \\ u u_x + v u_y - \nu u_{yy} = 0 \end{cases} \tag{A.11}$$

Since we are dealing with equations of different orders, differential consequences must be considered. The following conditions are then added to (A.11):

$$u_{xx} + v_{xy} = 0 \qquad \text{and} \qquad u_{xy} + v_{yy} = 0. \tag{A.12}$$

System (A.10)-(A.11) is solved as follows. First, (A.10) is converted into PDE's on ξ_x, ξ_y, ξ_u, and ξ_v, via (A.5)-(A.8). Next, νu_{yy} is replaced by $u u_x + v u_y$, u_{xx} by $-v_{xy}$, u_{xy} by $-v_{yy}$ and u_x by $-v_y$ in the new equations. This leads to two polynomials in u_y, v_x, v_y, v_{xx}, v_{xy} and v_{yy}, which equal 0. The first one, which comes from (A.10a), is

$$\begin{aligned} &\left[\frac{\partial \xi_u}{\partial x} + \frac{\partial \xi_v}{\partial y}\right] + \left[\frac{\partial \xi_x}{\partial x} - \frac{\partial \xi_y}{\partial y} - \frac{\partial \xi_u}{\partial u} + \frac{\partial \xi_v}{\partial v}\right] v_y + \left[\frac{\partial \xi_v}{\partial u} - \frac{\partial \xi_y}{\partial x}\right] u_y + \\ &\left[\frac{\partial \xi_u}{\partial v} - \frac{\partial \xi_x}{\partial y}\right] v_x - \left[\frac{\partial \xi_x}{\partial u} + \frac{\partial \xi_y}{\partial v}\right] v_y^2 - \left[\frac{\partial \xi_y}{\partial v} + \frac{\partial \xi_x}{\partial u}\right] u_y v_x = 0 \end{aligned} \tag{A.13}$$

The second polynomial is:

$$
\begin{aligned}
&\left[u\frac{\partial \xi_u}{\partial x}+v\frac{\partial \xi_u}{\partial y}-\nu\frac{\partial^2 \xi_u}{\partial y^2}\right]+\left[\xi_v-u\frac{\partial \xi_y}{\partial x}+v\frac{\partial \xi_y}{\partial y}-\nu\frac{\partial^2 \xi_u}{\partial y\partial u}+\nu\frac{\partial^2 \xi_y}{\partial y\partial y}\right]u_y\\
&+\left[-\xi_u+u\frac{\partial \xi_x}{\partial x}+v\frac{\partial \xi_u}{\partial v}+v\frac{\partial \xi_x}{\partial y}-\nu\frac{\partial^2 \xi_u}{\partial y\partial v}-\nu\frac{\partial^2 \xi_x}{\partial y\partial y}-2u\frac{\partial \xi_y}{\partial y}\right]v_y\\
&+\left[u\frac{\partial \xi_u}{\partial v}\right]v_x+\left[u\frac{\partial \xi_x}{\partial v}\right]v_xv_y+\left[-2u\frac{\partial \xi_x}{\partial u}+v\frac{\partial \xi_x}{\partial v}-\nu\frac{\partial^2 \xi_x}{\partial y\partial v}+2u\frac{\partial \xi_y}{\partial v}\right]v_y^2\\
&+\left[2u\frac{\partial \xi_y}{\partial u}+v\frac{\partial \xi_y}{\partial v}-v\frac{\partial^2 \xi_v}{\partial y\partial v}-\nu\frac{\partial^2 \xi_x}{\partial y\partial u}+\nu\frac{\partial^2 \xi_y}{\partial y\partial v}\right]u_yv_y-\left[u\frac{\partial \xi_y}{\partial v}\right]u_yv_x\\
&+\left[2v\frac{\partial \xi_y}{\partial u}+\nu\frac{\partial^2 \xi_y}{\partial y\partial u}\right]u_y^2+\left[\frac{\partial \xi_u}{\partial v}\right]v_{yy}+2\nu\left[\frac{\partial \xi_x}{\partial y}\right]u_{xy}+2\nu\left[\frac{\partial \xi_x}{\partial u}\right]u_{xy}u_y\\
&+2\nu\left[\frac{\partial \xi_x}{\partial v}\right]u_{xy}v_y-\nu\left[\frac{\partial \xi_x}{\partial v}\right]v_{yy}v_y+\nu\left[\frac{\partial \xi_y}{\partial v}\right]v_{yy}u_y=0
\end{aligned}
\tag{A.14}
$$

Finally, equating the coefficients to zero, one gets the following set of equations:

$$
\begin{gathered}
\frac{\partial \xi_y}{\partial v}=0,\quad \frac{\partial \xi_u}{\partial v}=0,\quad \frac{\partial \xi_x}{\partial y}=0,\quad \frac{\partial \xi_x}{\partial u}=0,\quad \frac{\partial \xi_x}{\partial v}=0,\\
\frac{\partial \xi_u}{\partial x}+\frac{\partial \xi_v}{\partial y}=0,\quad \frac{\partial \xi_x}{\partial x}-\frac{\partial \xi_y}{\partial y}-\frac{\partial \xi_u}{\partial u}+\frac{\partial \xi_v}{\partial v}=0,\quad \frac{\partial \xi_v}{\partial u}-\frac{\partial \xi_y}{\partial x}=0,\\
u\frac{\partial \xi_u}{\partial x}+v\frac{\partial \xi_u}{\partial y}-\nu\frac{\partial^2 \xi_u}{\partial y^2}=0,\quad 2u\frac{\partial \xi_y}{\partial u}-v\frac{\partial^2 \xi_v}{\partial y\partial v}=0\\
-\xi_u+u\frac{\partial \xi_x}{\partial x}-2u\frac{\partial \xi_y}{\partial y}=0,\quad 2v\frac{\partial \xi_y}{\partial u}+\nu\frac{\partial^2 \xi_y}{\partial y\partial u}=0,\\
\xi_v-u\frac{\partial \xi_y}{\partial x}+v\frac{\partial \xi_y}{\partial y}-\nu\frac{\partial^2 \xi_u}{\partial y\partial u}+\nu\frac{\partial^2 \xi_y}{\partial y\partial y}=0.
\end{gathered}
$$

The solution of this system is

$$
\begin{aligned}
\xi_x &= a_1+a_3x, & \xi_y &= a_2-a_4y+f(x),\\
\xi_u &= a_3u+2ua_4, & \xi_v &= a_4v+\dot{f}(x)u,
\end{aligned}
\tag{A.15}
$$

where the a_i's and f are respectively arbitrary constants and function. Using relation (6), one can deduce the symmetry corresponding to each of these parameters. More precisely, a_1 and a_2 lead respectively to the translations

$$(x,y,u,v)\mapsto(x+a_1,y,u,v) \tag{A.16}$$

and

$$(x,y,u,v)\mapsto(x,y+a_2,u,v). \tag{A.17}$$

The parameters a_3 and a_4 generate respectively the scale transformations

$$(x, y, u, v) \mapsto (\mathrm{e}^{a_3}\, x, y, \mathrm{e}^{a_3}\, u, v) \tag{A.18}$$

and

$$(x, y, u, v) \mapsto (x, \mathrm{e}^{-a_4}\, y, \mathrm{e}^{2a_4}\, u, \mathrm{e}^{a_4}\, v). \tag{A.19}$$

The combination of transformations (A.16)-(A.19) constitutes a 4-dimensional Lie-symmetry group of (A.1). At last, $f(x)$ spans the infinite-dimensional Lie-symmetry group of transformations

$$(x, y, u, v) \mapsto (x, y + f(x), u, v + u\dot{f}(x)). \tag{A.20}$$

As mentionned earlier, the cited symmetries could be guessed without Lie's algorithm. This is not always the case. For example, with Burgers'equation, Lie's theory leads to a projective symmetry which is hard to guess.

For more complicated equations, the calculation may be carried out with the help of symbolic software [9].