Predicative proof theory of PDL and basic applications

Lev Gordeev

Predicative proof theory of PDL and basic applications

Draft, Feb. 2019

L. Gordeev

Tübingen University

lew.gordeew@uni-tuebingen.de

1 Extended abstract

Propositional dynamic logic (**PDL**) is presented in Schütte-style mode as one-sided semiformal tree-like sequent calculus $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ with standard cut rule and the omega-rule with principal formulas $[P^*]A$. The omega-rule-free derivations in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ are finite (trees) and sequents deducible by these finite derivations are valid in **PDL**. Moreover the cut-elimination theorem for $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ is provable in Peano Arithmetic (**PA**) extended by transfinite induction up to Veblen's ordinal $\varphi_{\omega}(0)$. Hence (by the cutfree subformula property) such predicative extension of **PA** proves that any given $[P^*]$-free sequent is valid in **PDL** iff it is deducible in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ by a finite cut- and omega-rule-free derivation, while **PDL**-validity of arbitrary star-free sequents is decidable in polynomial space. The former also implies a Herbrand-style conclusion that e.g. a given formula $S = \langle P^* \rangle A \vee Z$ for star-free A and Z is valid in **PDL** iff there is a $k \geq 0$ and a cut- and omega-rule-free derivation of sequent $A, \langle P \rangle^1 A, \cdots, \langle P \rangle^k A, B$ where $\langle P \rangle^i A$ is an abbreviation for $\underbrace{\langle P \rangle \cdots \langle P \rangle}_{i\ times} A$. This eventually leads to PSPACE-decidability of **PDL**-validity of S, provided that P is atomic and A is in a suitable *basic conjunctive normal form*. Furthermore we consider star-free formulas A in dual *basic disjunctive normal form*, and corresponding expansions $S = \langle P^* \rangle A \vee Z$ whose **PDL**-validity problem is known to be EXPTIME-complete. We show that cutfree-derivability in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ (hence **PDL**-validity) of such S is equivalent to plain validity of a suitable "transparent" quantified boolean formula $\widehat{S}$. Hence **EXPTIME** = **PSPACE** holds true iff the validity problem for any $\widehat{S}$ involved is solvable by a polynomial-space deterministic TM. This may reduce the former problem to a more transparent complexity problem in quantified boolean logic. The whole proof can be formalized in **PA** extended by transfinite induction along $\varphi_{\omega}(0)$ – actually in the corresponding primitive recursive weakening, $\mathbf{PRA}_{\varphi_{\omega}(0)}$.

2 Introduction and survey of results

Propositional dynamic logic (**PDL**) was derived by M. J. Fischer and R. Ladner [6], [7] from dynamic logic where it plays the role that classical propositional logic plays in classical predicate logic. Conceptually, it describes the properties of the interaction between programs (as modal operators) and propositions that are independent of the domain of computation. The semantics of **PDL** is based on Kripke frames and comes from that of modal logic. Corresponding sound and complete Hilbert-style formalism was proposed by K. Segerberg [15] (see also [11], [9]). Gentzen-style treatment is more involved. This is because the syntax of **PDL** includes starred programs P^* which make finitary sequential formalism similar to that of (say) Peano Arithmetic with induction (**PA**) that allows no full cut-elimination. In the case of **PA**, however, there is a well-known Schütte-style solution in the form of infinitary (also called semiformal) sequent calculus with Carnap-style omega-rule that allows full cut elimination, provably in **PA** extended by transfinite induction up to Gentzen's ordinal ε_0 (cf. [3], [14]). By the same token, in the case of **PDL**, we introduce Schütte-style semiformal one-sided sequent calculus $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$ whose inferences include the omega-rule with principal formulas $[P^*]A$ and prove cut-elimination theorem using transfinite induction up to Veblen's predicative ordinal $\varphi_\omega(0)$ (that exceeds ε_0, see [17], [5]). The omega-rule-free derivations in $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$ are finite and sequents deducible by these finite derivations are valid in **PDL**. Hence by the cutfree subformula property we conclude that any given $[P^*]$-free sequent is valid in **PDL** iff it is deducible in $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$ by a finite cut- and omega-rule free derivation, which by standard methods enables better structural analysis of the validity of $[P^*]$-free sequent involved. [1] The latter is related to computational complexity of decision problems in **PDL**. Namely, the satisfiability (and hence the validity) problem in **PDL** is known to be EXPTIME-complete (cf. [7], [13]). Actually the EXPTIME-completeness holds for **PDL**-validity of special $[P^*]$-free *basic disjunctive normal expansions* (abbr.: BDNE), whose negations express that satisfying Kripke frames encode accepting computations of polynomial-space alternating TM. Thus the conjecture **EXPTIME** = **PSPACE** holds true iff **PDL**-validity of BDNE is decidable in polynomial space. We show that cutfree-derivability in $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$ (and hence **PDL**-validity) of any given BDNE, S, is equivalent to the validity of a suitable "transparent" quantified boolean formula $\widehat{S}$. Having this we conclude that **EXPTIME** = **PSPACE** holds true iff boolean validity of any $\widehat{S}$ involved is decidable by a polynomial-space deterministic TM. Hence **EXPTIME** = **PSPACE** holds true iff $\widehat{S}$ is equivalid with another quantified boolean formula whose size is polynomial in the size of S, for every $S \in$ BDNE. This may reduce the former problem to a more transparent complexity problem in quantified boolean logic, which will be investigated more deeply elsewhere. The whole proof can be formalized in **PA** extended by transfinite induction along $\varphi_\omega(0)$ – actually in the corresponding primitive recursive weakening, $\mathbf{PRA}_{\varphi_\omega(0)}$.

[1] cf. e.g. Gentzen-style conclusion that any given false equation $\underline{n}=\underline{m}$ (in particular $0 = 1$) is not valid in **PA**, since obviously it has no cutfree derivation.

3 More detailed exposition

3.1 Hilbert-style proof system PDL

Language $\mathcal{L}$

1. Programs PRO (abbr.: P, Q, R, S, possibly indexed):
   (a) include P-variables π_0, π_1, ...(abbr.: p, q, r, possibly indexed),
   (b) are closed under modal connectives ; and $\cup$ and star operation *.
2. Formulas FOR (abbr.: A, B, C, D, F, G, H, E, etc., possibly indexed):
   (a) include F-variables v_0, v_1, ... (abbr.: x, y, z, possibly indexed),
   (b) are closed under implication $\rightarrow$, negation $\neg$ and modal operation $F \hookrightarrow [P]\, F$, where $P \in$ PRO. [2]

Axioms (cf. e.g. [15], [9]): [3]
(D1) *Axioms of propositional logic.*
(D2) $[P]\,(A \rightarrow B) \rightarrow ([P]\,A \rightarrow [P]\,B)$
(D3) $[P]\,(A \wedge B) \leftrightarrow ([P]\,A \wedge [P]\,B)$
(D4) $[P;Q]\,A \leftrightarrow [P][Q]\,A$
(D5) $[P \cup Q]\,A \leftrightarrow [P]A \wedge [Q]\,A$
(D7) $[P^*]\,A \leftrightarrow A \wedge [P]\,[P^*]\,A$
(D8) $[P^*]\,(A \rightarrow [P]\,A) \rightarrow (A \rightarrow [P^*]\,A)$
Inference rules:

(MP) $\dfrac{A \quad A \rightarrow B}{B}$

(G) $\dfrac{A}{[P]\,A}$

3.2 Semiformal sequent calculus $\textsc{Seq}_\omega^{\mathbf{PDL}}$

Definition 1 *The language of* $\textsc{Seq}_\omega^{\mathrm{PDL}}$ *includes* seq-formulas *and* sequents. *Seq-formulas are built up from* literals x *and* $\neg x$ *by propositional connectives* $\vee$ *and* $\wedge$ *and modal operations* $[P]$ *and* $\langle P \rangle$ *for arbitrary* $P \in$ PRO*. Seq-negation* $\overline{F}$ *is defined recursively as follows, for any seq-formula* F*.*

1. $\overline{x} := \neg x$, $\overline{\neg x} := x$,
2. $\overline{A \vee B} := \overline{A} \wedge \overline{B}$, $\overline{A \wedge B} := \overline{A} \vee \overline{B}$.
3. $\overline{\langle P \rangle A} := [P]\overline{A}$, $\overline{[P]A} := \langle P \rangle \overline{A}$.
4. $\overline{\langle P \cup Q \rangle A} := [P \cup Q]\overline{A}$, $\overline{[P \cup Q]A} := \langle P \cup Q \rangle \overline{A}$.

[2]Boolean constants are definable as usual e.g. by $1 := v_0 \rightarrow v_0$ and $0 := \neg 1$.
[3]Standard axiom $(D6)$: $[A?]B \leftrightarrow (A \rightarrow B)$ is obsolete in our ?-free language.

5. $\overline{\langle P;Q\rangle A} := [P;Q]\overline{A}$, $\overline{[P;Q]A} := \langle P;Q\rangle\overline{A}$.

In the sequel we use abbreviations $\langle P\rangle^m := \overbrace{\langle P\rangle \cdots \langle P\rangle}^{m\ times}$ and $[P]^m := \overbrace{[P]\cdots[P]}^{m\ times}$. For any $\chi \in \{0,1\}$, let $(P)_\chi := \begin{cases} [P], & \text{if } \chi = 1, \\ \langle P\rangle, & \text{if } \chi = 0. \end{cases}$ For any $\overrightarrow{P} = P_1, \cdots, P_k$ $(k \geq 0)$ and $f : [1,k] \to \{0,1\}$ let $\left(\overrightarrow{P}\right)_f := (P_1)_{f(1)} \cdots (P_k)_{f(k)}$. By $\left(\overrightarrow{Q}\right)$, $\left\langle\overrightarrow{Q}\right\rangle$ and $\left[\overrightarrow{Q}\right]$ we abbreviate $\left(\overrightarrow{Q}\right)_f$ for arbitrary f, $f \equiv 0$ and $f \equiv 1$, respectively. Formulas from FOR are represented as seq-formulas recursively by $\neg F := \overline{F}$, $F \to G := \overline{F} \vee G$ and, conversely, by $F \vee G := \neg F \to G$, $F \wedge G := \neg(F \to \neg G)$, $\langle P\rangle F := \neg [P] \neg F$. Sequents (abbr.: Γ, Δ, Π, Σ, possibly indexed) are viewed as multisets (possibly empty) of seq-formulas. A sequent $\Gamma = F_1, \cdots, F_n$ is called *valid* iff so is the corresponding disjunction $F_1 \vee \cdots \vee F_n$. *Plain complexity* of a given formula and/or program in $\mathcal{L}$ is its ordinary length (= total number of occurrences of literals and connectives $\vee$, $\wedge$, $\cup$, $;$, $*$).

Definition 2 Ordinal complexity $\mathfrak{o}(-) < \omega^\omega$ *of formulas, programs and sequents in $\mathcal{L}$ is defined recursively as follows, where $\alpha \# \beta$ is the symmetric sum of ordinals α and β.*

1. $\mathfrak{o}(x) = \mathfrak{o}(\neg x) = \mathfrak{o}(p) := 0$.
2. $\mathfrak{o}(A \vee B) = \mathfrak{o}(A \wedge B) := \max\{\mathfrak{o}(A), \mathfrak{o}(B)\} + 1$.
3. $\mathfrak{o}(P \cup Q) := \max\{\mathfrak{o}(P), \mathfrak{o}(Q)\} + 1$, $\quad \mathfrak{o}(P;Q) := \mathfrak{o}(P) \# \mathfrak{o}(Q) + 1$.
4. $\mathfrak{o}(P^*) := \mathfrak{o}(P) \cdot \omega$, $\quad \mathfrak{o}(\langle P\rangle A) = \mathfrak{o}([P]A) := \mathfrak{o}(P) \# \mathfrak{o}(A) + 1$.
5. $\mathfrak{o}(\Gamma) := \sum\{\mathfrak{o}(A) : A \in \Gamma\}$.

Definition 3 $\mathrm{S{\scriptsize EQ}}_\infty^{\mathrm{PDL}}$ *includes the following axiom* (AX) *and inference rules* $(\vee)$, $(\wedge)$, $\langle\cup\rangle$, $[\cup]$, $\langle;\rangle$, $[;]$, $\langle*\rangle$, $[*]$, (GEN), (CUT) *in classical one-sided sequent formalism in the language $\mathcal{L}$. In $[*]$ we allow $\overrightarrow{Q} = \left[\overrightarrow{Q}\right] = \emptyset$.* [4] [5]

(AX)	$x, \neg x, \Gamma$		
$(\vee)$	$\dfrac{A, B, \Gamma}{A \vee B, \Gamma}$	$(\wedge)$	$\dfrac{A, \Gamma \qquad B, \Gamma}{A \wedge B, \Gamma}$
$\langle\cup\rangle$	$\dfrac{\langle P\rangle A, \langle R\rangle A, \Gamma}{\langle P \cup R\rangle A, \Gamma}$	$[\cup]$	$\dfrac{[P]A, \Gamma \qquad [R]A, \Gamma}{[P \cup R]A, \Gamma}$
$\langle;\rangle$	$\dfrac{\langle P\rangle\langle R\rangle A, \Gamma}{\langle P;R\rangle A, \Gamma}$	$[;]$	$\dfrac{[P][R]A, \Gamma}{[P;R]A, \Gamma}$

[4]We assume that all rules exposed have nonempty premises.
[5]$[*]$ has infinitely many premises. Ii is called the ω-rule.

$$\langle * \rangle \quad \frac{\left\langle \overrightarrow{Q} \right\rangle \langle P \rangle^m A, \left\langle \overrightarrow{Q} \right\rangle \langle P^* \rangle A, \Gamma}{\left\langle \overrightarrow{Q} \right\rangle \langle P^* \rangle A, \Gamma} \ (m \geq 0)$$

$$[*] \quad \frac{\cdots \quad \left[\overrightarrow{Q}\right][P]^m A, \Gamma \quad \cdots \ (\forall m \geq 0)}{\left[\overrightarrow{Q}\right][P^*]A, \Gamma}$$

$$(\textsc{Gen}) \quad \frac{A_1, \cdots, A_n}{(p)_{\chi_1} A_1, \cdots, (p)_{\chi_n} A_n, \Gamma} \ (n > 0) \quad \textit{if} \sum_{i=1}^{n} \chi_i = 1.$$

$$(\textsc{Cut}) \quad \frac{C, \Gamma \qquad \overline{C}, \Pi}{\Gamma \cup \Pi}$$

For the sake of brevity we'll drop "seq-" when referring to seq-formulas of $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$. Γ is called *derivable* in $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ if there exists a (tree-like, possibly infinite) $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ *derivation* ∂ with the root sequent Γ (abbr.: $(\partial : \Gamma)$). We assume that $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ *derivations* are well-founded. The simplest way to implement this assumption is to supply nodes x in ∂ with ordinals $ord\,(x)$ such that ordinals of premises are always smaller that the ones of the corresponding conclusions. Having this we let $h\,(\partial) := ord\,(root\,(\partial))$ and call it *the height* of ∂.

In $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$, formulas occurring in Γ and/or Π are called *side formulas*, whereas other (distinguished) ones are called *principal formulas*, of axioms or inference rules exposed. These axioms and inferences, in turn, are called *principal* with respect to their principal formulas. Principal formulas of (Cut) are also called the corresponding *cut formulas*. We'll sometimes specify (Gen) as $(\textsc{Gen})_P$ to indicate principal program P involved.

Theorem 4 (soundness and completeness) $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ *is sound and complete with respect to* ***PDL****. Moreover any* ***PDL****-valid sequent (in particular formula) is derivable in* $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ *using ordinals* $< \omega + \omega =: \omega \cdot 2$.

Proof. The soundness says that any sequent Γ that is derivable in $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ is valid in Kripke-style semantics of **PDL**. It is proved by transfinite induction on $h\,(\partial)$ of well-founded $(\partial : \Gamma)$ involved. [6] Actually it suffices to verify that every inference rule of $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ preserves Kripke validity, which is easy (we omit the details; see also Remark 5 below).

The completeness is proved by deducing in $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ the axioms and inferences $(D1) - (D5)$, $(D7)$, $(D8)$, (MP), (G) of **PDL**.

$(D1)$ is deducible by standard method via extended axiom $(\mathrm{Ax})^+ : F, \overline{F}, \Gamma$ whose finite cutfree derivation is constructed by recursion on plain complexity of F (in particular we pass by (Gen) from $A, \overline{A}$ to $[P]A, \langle P \rangle \overline{A}, \Gamma$).

[6] Plain (finite) induction is sufficient for $[*]$-free derivations.

$(D4)$ and $(D5)$ are trivial, while $(D2)$, $(D3)$, $(D7)$, $(D8)$ are derivable as follows.

$(D2):\ [P]\,(A\to B)\to([P]\,A\to[P]\,B)\overset{\mathcal{L}}{\equiv}\langle P\rangle\left(A\wedge\overline{B}\right)\vee\langle P\rangle\overline{A}\vee[P]B.$

$$\dfrac{\dfrac{\dfrac{\overset{(\mathrm{Ax})^{+}}{A,\overline{A},B}\qquad\overset{(\mathrm{Ax})^{+}}{\overline{B},\overline{A},B}}{A\wedge\overline{B},\overline{A},B}\,(\wedge)}{\langle P\rangle\left(A\wedge\overline{B}\right),\langle P\rangle\overline{A},[P]B}\,(\textsc{Gen})}{\langle P\rangle\left(A\wedge\overline{B}\right)\vee\langle P\rangle\overline{A}\vee[P]B}\,(\vee)$$

$(D3):\ [P]\,(A\wedge B)\leftrightarrow([P]\,A\wedge[P]\,B)$
$\overset{\mathcal{L}}{\equiv}\left(\langle P\rangle\left(\overline{A}\wedge\overline{B}\right)\vee([P]A\wedge[P]B)\right)\wedge\left([P]\,(A\wedge B)\vee\langle P\rangle\overline{A}\vee\langle P\rangle\overline{B}\right).$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(\mathrm{Ax})^{+}}{\overline{A},\overline{B},A}}{\overline{A}\vee\overline{B},A}\,(\vee)}{\langle P\rangle\left(\overline{A}\vee\overline{B}\right),[P]A}\,(\textsc{Gen})\qquad\dfrac{\dfrac{\overset{(\mathrm{Ax})^{+}}{\overline{A},\overline{B},B}}{\overline{A}\vee\overline{B},B}\,(\vee)}{\langle P\rangle\left(\overline{A}\vee\overline{B}\right),[P]B}\,(\textsc{Gen})}{\langle P\rangle\left(\overline{A}\vee\overline{B}\right),[P]A\wedge[P]B}}{\langle P\rangle\left(\overline{A}\vee\overline{B}\right)\vee([P]A\wedge[P]B)}\,(\vee)\quad\&$$

$$\dfrac{\dfrac{\dfrac{\overset{(\mathrm{Ax})^{+}}{A,\overline{A},\overline{B}}\qquad\overset{(\mathrm{Ax})^{+}}{B,\overline{A},\overline{B}}}{A\wedge B,\overline{A},\overline{B}}\,(\wedge)}{[P]\,(A\wedge B),\langle P\rangle\overline{A},\langle\alpha\rangle\overline{B}}\,(\textsc{Gen})}{[P]\,(A\wedge B)\vee\langle P\rangle\overline{A}\vee\langle P\rangle\overline{B}}\,(\vee)$$

$(D7):\ [P^{*}]\,A\leftrightarrow A\wedge[P]\,[P^{*}]\,A$
$\overset{\mathcal{L}}{\equiv}\left(\langle P^{*}\rangle\overline{A}\vee(A\wedge[P][P^{*}]A)\right)\wedge\left([P^{*}]A\vee\overline{A}\vee\langle P\rangle\langle P^{*}\rangle\overline{A}\right).$

$$\dfrac{\dfrac{\dfrac{\overset{(\mathrm{Ax})^{+}}{\overline{A},\langle P^{*}\rangle\overline{A},A}}{\langle P^{*}\rangle\overline{A},A}\,\langle *\rangle}{\langle P^{*}\rangle\overline{A}\vee A}\,(\vee)\qquad\dfrac{\dfrac{\dfrac{\overset{(\mathrm{Ax})^{+}}{\langle P\rangle^{m+1}\overline{A},\langle P^{*}\rangle\overline{A},[P]^{m+1}A}}{\cdots\ \langle P^{*}\rangle\overline{A},[P]^{m+1}A\ \cdots}\,\langle *\rangle}{\langle P^{*}\rangle\overline{A},[P][P^{*}]A}\,[*]}{\langle P^{*}\rangle\overline{A}\vee[P][P^{*}]A}\,(\vee)}{\langle P^{*}\rangle\overline{A}\vee(A\wedge[P][P^{*}]A)}\,(\wedge)\quad\&$$

$$\dfrac{\dfrac{\overset{(\mathrm{Ax})^{+}}{A,\overline{A},\langle P\rangle\langle P^{*}\rangle\overline{A}}\qquad\dfrac{\overset{(\mathrm{Ax})^{+}}{[P]^{m+1}A,\overline{A},\langle P\rangle^{m+1}\overline{A},\langle P\rangle\langle P^{*}\rangle\overline{A}}}{[P]^{m+1}A,\overline{A},\langle P\rangle\langle P^{*}\rangle\overline{A}\quad\cdots}\,\langle *\rangle}{[P^{*}]A,\overline{A},\langle P\rangle\langle P^{*}\rangle\overline{A}}\,[*]}{[P^{*}]A\vee\overline{A}\vee\langle P\rangle\langle P^{*}\rangle\overline{A}}\,(\vee)$$

$(D8):\ [P^{*}]\,(A\to[P]\,A)\to(A\to[P^{*}]\,A)$
$\overset{\mathcal{L}}{\equiv}\langle P^{*}\rangle\left(A\wedge\langle P\rangle\,\overline{A}\right)\vee\overline{A}\vee[P^{*}]\,A.$

$$\boxed{\dfrac{\dfrac{\cdots\ \langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},\overset{\overset{\partial_m}{\Downarrow}}{[P]^m A}\ \cdots}{\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},[P^*]A}\,[*]}{\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big)\vee\overline{A}\vee[P^*]A}\,(\vee)}\ \text{, where:}$$

$$\partial_0 = \boxed{\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},A\quad(\mathrm{Ax})^+}$$

$$\partial_1 = \boxed{\dfrac{\dfrac{\overset{(\mathrm{Ax})^+}{A,\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},[P]A}\quad \overset{(\mathrm{Ax})^+}{\langle P\rangle\overline{A},\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},[P]A}}{A\wedge\langle P\rangle\overline{A},\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},[P]A}\,(\wedge)}{\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},[P]A}\,\langle *\rangle}$$

$$\partial_2 = \boxed{\dfrac{\dfrac{\overset{(\mathrm{Ax})^+}{A,\overline{A},(\cdots)}\qquad \dfrac{\dfrac{\overset{(\mathrm{Ax})^+}{\overline{A},A,PA}\quad \overset{(\mathrm{Ax})^+}{\overline{A},\langle P\rangle\overline{A},[P]A}}{\overline{A},A\wedge\langle P\rangle\overline{A},[P]A}\,(\wedge)}{\langle P\rangle\overline{A},\langle P\rangle\big(A\wedge\langle P\rangle\overline{A}\big),[P]^2A,(\cdots)}\,(\textsc{Gen})}{A\wedge\langle P\rangle\overline{A},\langle P\rangle\big(A\wedge\langle P\rangle\overline{A}\big),(\cdots),\overline{A},[P]^2A}\,(\wedge)}{\langle P^*\rangle\big(A\wedge\langle P\rangle\overline{A}\big),\overline{A},[P]^2A}\,\langle *\rangle}$$

etc. via $(\wedge)$, $\langle *\rangle$ and (Gen).

Obviously these derivations don't use (Gen) and require ordinal assignments $<\omega+\omega$. $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ inferences (MP) and (G) are obviously derivable by (Cut) and (Gen), respectively. These increase ordinals by one, which makes an arbitrary Hilbert-style **PDL** deduction interpretable as a $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ derivation of the height $<\omega\cdot 2$, as required. ■

Remark 5 *The validity of* (Gen) *also follows from that of* $(D1)$, $(D2)$, $(D3)$ *and plain generalzation* (G), *e.g. like this:*

$$\boxed{\begin{array}{c}
\dfrac{A_1,A_2,\cdots,A_n \overset{\mathcal{L}}{\equiv} A_1\vee A_2\vee\cdots\vee A_n}{[P](A_1\vee A_2\vee\cdots\vee A_n)\overset{\mathcal{L}}{\equiv}[P](\neg(A_2\vee\cdots\vee A_n)\to A_1)\underset{(D2)}{\Rightarrow}}\,(\mathrm{G})\\
[P](\neg(A_2\vee\cdots\vee A_n)\to[P]A_1)\overset{\mathcal{L}}{\equiv}[P]A_1\vee\langle P\rangle(A_2\vee\cdots\vee A_n)\\
\underset{(D1)}{\Rightarrow}\ [P]A_1\vee\langle P\rangle(A_2\vee\cdots\vee A_n)\vee\Gamma\ \underset{(D3)}{\Rightarrow}\\
[P]A_1\vee\langle P\rangle A_2\vee\cdots\vee\langle P\rangle A_n\vee\Gamma\overset{\mathcal{L}}{\equiv}[P]A_1,\langle P\rangle A_2,\cdots,\langle P\rangle A_n,\Gamma
\end{array}}$$

3.3 Cut elimination procedure

3.3.1 Auxiliary sequent calculus $\mathbf{SEQ}_{\omega+}^{\mathbf{PDL}}$

Definition 6 $\mathrm{SEQ}_{\omega+}^{\mathrm{PDL}}$ *is a modification of* $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ *that includes the following upgraded inferences* $\langle\cup\rangle$, $[\cup]$, $\langle;\rangle$, $[;]$.

$\langle\cup\rangle \quad \dfrac{\left\langle\overrightarrow{Q}\right\rangle\langle P\rangle A, \left\langle\overrightarrow{Q}\right\rangle\langle R\rangle A, \Gamma}{\left\langle\overrightarrow{Q}\right\rangle\langle P\cup R\rangle A, \Gamma}$	$[\cup] \quad \dfrac{\left[\overrightarrow{Q}\right][P]A, \Gamma \qquad \left[\overrightarrow{Q}\right][R]A, \Gamma}{\left[\overrightarrow{Q}\right][P\cup R]A, \Gamma}$
$\langle;\rangle \quad \dfrac{\left\langle\overrightarrow{Q}\right\rangle\langle P\rangle\langle R\rangle A, \Gamma}{\left\langle\overrightarrow{Q}\right\rangle\langle P;R\rangle A, \Gamma}$	$[;] \quad \dfrac{\left[\overrightarrow{Q}\right][P][R]A, \Gamma}{\left[\overrightarrow{Q}\right][P;R]A, \Gamma}$

Obviously these upgrades are still sound in **PDL** and cut-free derivable in $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$. Hence $\mathrm{SEQ}_{\omega}^{\mathrm{PDL}}$ and $\mathrm{SEQ}_{\omega+}^{\mathrm{PDL}}$ are proof theoretically equivalent.

3.3.2 Derivable refinements

Lemma 7 *The following inferences are derivable in* $\mathrm{SEQ}_{\omega+}^{\mathrm{PDL}}$ minus (CUT). *Moreover, for any inversion* $\dfrac{(\partial : \Delta)}{(\partial^{\circlearrowleft} : \Gamma)}$ *involved we have* $h\left(\partial^{\circlearrowleft}\right) < h\left(\partial\right) + \omega$. *In* $\left(\overrightarrow{\mathrm{GEN}}\right)$ *we assume that* $\overrightarrow{P} = P_1, \cdots, P_k$ $(k > 0)$, $f_1, \cdots, f_n : [1,k] \to \{0,1\}$ *and* $(\forall j \in [1,k]) \sum_{i=1}^{n} f_i(j) = 1$. *Note that* (GEN) *is a particular case of* $\left(\overrightarrow{\mathrm{GEN}}\right)$.

(W) $\dfrac{\Gamma}{\Gamma, \Pi}$ (weakening)	(C) $\dfrac{A, A, \Gamma}{A, \Gamma}$ (contraction)	
$(\vee)^{\circlearrowleft}$ $\dfrac{A\vee B, \Gamma}{A, B, \Gamma}$	$(\wedge)_1^{\circlearrowleft}$ $\dfrac{A\wedge B, \Gamma}{A, \Gamma}$	$(\wedge)_2^{\circlearrowleft}$ $\dfrac{A\wedge B, \Gamma}{B, \Gamma}$
$\langle\cup\rangle^{\circlearrowleft}$ $\dfrac{\left\langle\overrightarrow{Q}\right\rangle\langle P\cup R\rangle A, \Gamma}{\left\langle\overrightarrow{Q}\right\rangle\langle P\rangle A, \left\langle\overrightarrow{Q}\right\rangle\langle R\rangle A, \Gamma}$		
$[\cup]_1^{\circlearrowleft}$ $\dfrac{\left[\overrightarrow{Q}\right][P\cup R]A, \Gamma}{\left[\overrightarrow{Q}\right][P]A, \Gamma}$	$[\cup]_2^{\circlearrowleft}$ $\dfrac{\left[\overrightarrow{Q}\right][P\cup R]A, \Gamma}{\left[\overrightarrow{Q}\right][R]A, \Gamma}$	
$\langle;\rangle^{\circlearrowleft}$ $\dfrac{\left\langle\overrightarrow{Q}\right\rangle\langle P;R\rangle A, \Gamma}{\left\langle\overrightarrow{Q}\right\rangle\langle P\rangle\langle R\rangle A, \Gamma}$	$[;]^{\circlearrowleft}$ $\dfrac{\left[\overrightarrow{Q}\right][P;R]A, \Gamma}{\left[\overrightarrow{Q}\right][P][R]A, \Gamma}$	

$$\boxed{[*]^{\circlearrowleft} \qquad \dfrac{\left[\overrightarrow{Q}\right][P^*]A,\Gamma}{\left[\overrightarrow{Q}\right][P]^m A,\Gamma}\ (m \geq 0)}$$

$$\boxed{\left(\overrightarrow{\mathrm{Gen}}\right) \qquad \dfrac{A_1,\cdots,A_n}{\left(\overrightarrow{P}\right)_{f_1}A_1,\cdots,\left(\overrightarrow{P}\right)_{f_n}A_n,\Gamma}\ (n > 0)}$$

Proof. Induction on proof height and/or formula complexity. Cases (W), (C) are standard. Note that (C) with principal (Gen) is trivial, e.g.

$$\partial : \boxed{\dfrac{(\partial_1 : A, A, B)}{\langle P\rangle A, \langle P\rangle A, [P]\, B, \Gamma}\ (\mathrm{Gen})} \quad \hookrightarrow \quad \partial^{\mathrm{C}} : \boxed{\dfrac{\left(\partial_1^{\mathrm{C}} : A, B\right)}{\langle P\rangle A, [P]\, B, \Gamma}\ (\mathrm{Gen})}.$$

Case $\left(\overrightarrow{\mathrm{Gen}}\right)$ is an obvious iteration of (Gen).

Cases $(\vee)^{\circlearrowleft}$, $(\wedge)_1^{\circlearrowleft}$, $(\wedge)_2^{\circlearrowleft}$ are standard (and trivial) boolean inversions**.**

Case $\langle\cup\rangle^{\circlearrowleft}$ ($[\cup]^{\circlearrowleft}$ analogous)**.** We omit trivial case of principal inversion of $\langle\cup\rangle$ and show only the crucial cases of principal (Gen) (in simple form):

$$\partial : \boxed{\dfrac{(\partial_1 : A, B, C)}{\langle P\cup R\rangle A, \langle P\cup R\rangle B, [P\cup R]C, \Gamma}\ (\mathrm{Gen})} \quad \hookrightarrow \partial^{\langle\cup\rangle^{\circlearrowleft}} :$$

$$\boxed{\dfrac{\dfrac{(\mathrm{Gen})_P\dfrac{(\partial_1 : A, B, C)}{\langle P\rangle A,\langle R\rangle A,\langle P\rangle B,\langle R\rangle B,[P]C,\Gamma}}{\langle P\rangle A,\langle R\rangle A,\langle P\cup R\rangle B,[P]C,\Gamma}\langle\cup\rangle\ \dfrac{\dfrac{(\partial_1 : A, B, C)}{\langle P\rangle A,\langle R\rangle A,\langle P\rangle B,\langle R\rangle B,[R]C,\Gamma}(\mathrm{Gen})_R}{\langle P\rangle A,\langle R\rangle A,\langle P\cup R\rangle B,[R]C,\Gamma}}{\langle P\rangle A,\langle R\rangle A,\langle P\cup R\rangle B,[P\cup R]C,\Gamma}[\cup]},$$

$$\partial : \boxed{\dfrac{(\partial_1 : \langle P\cup R\rangle A, B)}{\langle Q\rangle\langle P\cup R\rangle A, [Q]B, \Gamma}\ (\mathrm{Gen})} \quad \hookrightarrow$$

$$\partial^{\langle\cup\rangle^{\circlearrowleft}} : \boxed{\dfrac{\left(\partial_1^{\langle\cup\rangle^{\circlearrowleft}} : \langle P\rangle A, \langle R\rangle A, B\right)}{\langle Q\rangle\langle P\rangle A, \langle Q\rangle\langle R\rangle A, [Q]B, \Gamma}\ (\mathrm{Gen})}.$$

Case $\langle\,;\rangle^{\circlearrowleft}$ ($[\,;\,]^{\circlearrowleft}$ analogous)**.** As above, we omit trivial case of principal inversion of $\langle\,;\rangle$ and show the crucial cases of principal (Gen) (in simple form):

$$\partial : \boxed{\dfrac{(\partial_1 : A, B, C)}{\langle P;R\rangle A, \langle P;R\rangle B, [P;R]C, \Gamma}\ (\mathrm{Gen})} \quad \hookrightarrow$$

$$\partial^{\langle;\rangle^{\circlearrowleft}} : \boxed{\dfrac{\dfrac{\dfrac{(\partial_1 : A, B, C)}{\langle P\rangle\langle R\rangle A, \langle P\rangle\langle R\rangle B, [P][R]C, \Gamma}\left(\overrightarrow{\mathrm{Gen}}\right)_{R,P}}{\langle P\rangle\langle R\rangle A, \langle P;R\rangle B, [P][R]C, \Gamma}\langle;\rangle}{\langle P\rangle\langle R\rangle A, \langle P;R\rangle B, [P;R]C, \Gamma}\ [;]},$$

$$\partial : \boxed{\dfrac{(\partial_1 : \langle P;R\rangle A, B, C)}{\langle Q\rangle\langle P;R\rangle A, \langle Q\rangle B, [Q]C, \Gamma}\ (\mathrm{Gen})} \quad \hookrightarrow$$

$$\partial^{\langle;\rangle^{\circlearrowleft}} : \boxed{\dfrac{\left(\partial_1^{\langle;\rangle^{\circlearrowleft}} : \langle P\rangle\langle R\rangle A, B, C\right)}{\langle Q\rangle\langle P\rangle\langle R\rangle A, \langle Q\rangle B, [Q]C, \Gamma}\ (\textsc{Gen})}.$$

Case $[*]^{\circlearrowleft}$ is analogous to $(\wedge)_i^{\circlearrowleft}$, via trivial inversion of $[*]$:

$$\partial : \boxed{\dfrac{(\partial_1 : A, B)}{[P^*]A, \langle P^*\rangle B, \Gamma}\ (\textsc{Gen})} \quad \hookrightarrow$$

$$\partial^{[*]^{\circlearrowleft}} : \boxed{\dfrac{\dfrac{(\partial_1 : A, B)}{[P]^m A, \langle P\rangle^m B, \langle P^*\rangle B, \Gamma}\left(\overrightarrow{\textsc{Gen}}\right)\ \underbrace{P \cdots P}_{m}}{[P]^m A, \langle P^*\rangle B, \Gamma}\ \langle * \rangle},$$

$$\partial : \boxed{\dfrac{\left(\partial_1 : \left[\overrightarrow{R}\right][P^*]A, B\right)}{[Q]\left[\overrightarrow{R}\right][P^*]A, \langle Q\rangle B, \Gamma}\ (\textsc{Gen})} \quad \hookrightarrow$$

$$\partial^{[*]^{\circlearrowleft}} : \boxed{\dfrac{\left(\partial_1^{[*]^{\circlearrowleft}} : \left[\overrightarrow{R}\right][P]^m A, B, \Gamma\right)}{[Q]\left[\overrightarrow{R}\right][P]^m A, \langle Q\rangle B, \Gamma}\ (\textsc{Gen})}.$$

Note that (W), (C), $(\vee)^{\circlearrowleft}$, $(\wedge)_1^{\circlearrowleft}$, $(\wedge)_2^{\circlearrowleft}$ don't increase derivation heights. ■

3.3.3 Cut elimination proper

We adapt familiar predicative cut elimination techniques ([14], [4], [12], [8], [2]).

Theorem 8 (Predicative cut elimination) *The following is provable in* ***PA*** *extended by transfinite induction up to Veblen-Feferman ordinal* $\varphi_\omega(0) > \varepsilon_0$. *Any sequent derivable in* $\textsc{Seq}_\omega^{\textsc{pdl}}$ *is derivable in* $\textsc{Seq}_{\omega+}^{\textsc{pdl}}$ *minus* (Cut). *Hence any* ***PDL****-valid sequent (formula) is derivable in the cut-free fraction of* $\textsc{Seq}_{\omega+}^{\textsc{pdl}}$, *and hence also in* $\textsc{Seq}_\omega^{\textsc{pdl}}$ *minus* (Cut).

Proof. Our *cut elimination operator* $\partial \hookrightarrow \mathcal{E}(\partial)$ satisfying $\deg(\mathcal{E}(\partial)) = 0$ is defined for any derivation ∂ in $\textsc{Seq}_{\omega+}^{\textsc{pdl}}$ by simultaneous transfinite recursion on $h(\partial)$ and ordinal cut-degree $\deg(\partial)$.

$$\boxed{\deg(\partial) := \max\{0, \sup\{\mathfrak{o}(C) + 1 : C \textit{ occurs as cut formula in } \partial\}\}}$$

Namely, for any inference rule $(\mathrm{R}) \neq (\textsc{Cut})$ with

$$\boxed{(\partial : \Gamma) = \dfrac{(\partial_1 : \Gamma_1)}{\Gamma}\ (\mathrm{R})}\ ,\ \boxed{(\partial : \Gamma) = \dfrac{(\partial_1 : \Gamma_1) \qquad (\partial_2 : \Gamma_2)}{\Gamma}\ (\mathrm{R})}$$

$$\text{or} \ \boxed{(\partial : \Gamma) = \dfrac{\cdots \quad (\partial_m : \Gamma_m) \quad \cdots \{m \geq 0\}}{\Gamma}\ (\mathrm{R}) = [*]}$$

we respectively let

$$\boxed{(\mathcal{E}(\partial):\Gamma)=\frac{(\mathcal{E}(\partial_1):\Gamma_1)}{\Gamma}\,(\mathrm{R})} \text{ or } \boxed{(\mathcal{E}(\partial):\Gamma)=\frac{(\mathcal{E}(\partial_1):\Gamma_1)\quad(\mathcal{E}(\partial_2):\Gamma_2)}{\Gamma}\,(\mathrm{R})}$$

$$\text{or } \boxed{(\mathcal{E}(\partial):\Gamma)=\frac{\cdots\quad(\mathcal{E}(\partial_m):\Gamma_m)\quad\cdots\{m\geq 0\}}{\Gamma}\;[*]}.$$

Otherwise, if (R) = (CUT) with

$$\boxed{(\partial:\Gamma\cup\Pi)=\frac{(\partial_1:C,\Gamma)\qquad\left(\partial_2:\overline{C},\Pi\right)}{\Gamma\cup\Pi}\;(\mathrm{Cut})}$$

then we stipulate

$$\boxed{(\mathcal{E}(\partial):\Gamma\cup\Pi)=\left(\mathcal{E}\left(\mathcal{R}\left(\frac{(\mathcal{E}(\partial_1):C,\Gamma)\quad\left(\mathcal{E}(\partial_2):\overline{C},\Pi\right)}{\Gamma\cup\Pi}\;(\mathrm{Cut})\right)\right):\Gamma\cup\Pi\right)}$$

with respect to a suitable *cut reduction operation* $\partial\hookrightarrow\mathcal{R}(\partial)$ such that

$$\boxed{\deg(\mathcal{R}(\partial))<\deg(\partial)\text{ if }\deg(\partial_1)=\deg(\partial_2)=0},$$

which makes $\mathcal{E}(\partial)$, $\deg(\mathcal{E}(\partial))=0$, definable by induction on $\deg(\partial)$ and $h(\partial)$.

Now $\mathcal{R}(\partial)$ is defined for any

$$\boxed{(\partial:\Gamma\cup\Pi)=\frac{(\partial_1:C,\Gamma)\qquad\left(\partial_2:\overline{C},\Pi\right)}{\Gamma\cup\Pi}\;(\mathrm{Cut})}$$

by following double induction on ordinal complexity of C and $\max(h(\partial_1),h(\partial_2))$, provided that $\deg(\partial_1)=\deg(\partial_2)=0$.

1. Case $C=L$ and $\overline{C}=\overline{L}$ for $L\in\{x,\neg x\}$. This case is standard. Namely, L is principal left-hand side cut formula only if $(\partial_1:L,\Gamma)$ for $\Gamma=\overline{L},\Gamma'$. But then $\left(\partial_2:\overline{L},\Pi\right)$ infers $\Gamma\cup\Pi=\overline{L},\Gamma',\Pi$ by derivable weakening (W). That is, graphically speaking, $\mathcal{R}(\partial)$ is bottom up constructed by (1) substituting Π for every side formula predecessor of the cut formula L while ascending ∂_1 up to its disappearance due to (GEN) or else principal appearance in (AX) $L,\overline{L},\Gamma'$ followed by (2) adding Γ' to every side formula predecessor of the cut formula $\overline{L}$ while ascending ∂_2.

2. Case $C=A\vee B$ and $\overline{C}=\overline{A}\wedge B$. Use derivable inversions $(\vee)^{\circlearrowleft}$, $(\wedge)_1^{\circlearrowleft}$, $(\wedge)_2^{\circlearrowleft}$:

$$\partial:\boxed{\frac{(\partial_1:A\vee B,\Gamma)\quad\left(\partial_2:\overline{A}\wedge\overline{B},\Pi\right)}{\Gamma\cup\Pi}\;(\mathrm{Cut})}\quad\hookrightarrow$$

$$\mathcal{R}(\partial):=\boxed{\frac{\dfrac{\left(\partial_1^{(\vee)^{\circlearrowleft}}:A,B,\Gamma\right)\quad\left(\partial_2^{(\vee)_1^{\circlearrowleft}}:\overline{A},\Pi\right)}{B,\Gamma\cup\Pi}\;(\mathrm{Cut})\quad\left(\partial_2^{(\vee)_2^{\circlearrowleft}}:\overline{B},\Pi\right)}{\Gamma\cup\Pi}\;(\mathrm{Cut})}.$$

3. Case $C = \left\langle \overrightarrow{Q} \right\rangle \langle P \cup R \rangle A$ and $\overline{C} = \left[\overrightarrow{Q} \right] [P \cup R] \, \overline{A}$. Analogous reduction to (Cut)'s on $\left\langle \overrightarrow{Q} \right\rangle \langle P \rangle A$ and $\left\langle \overrightarrow{Q} \right\rangle \langle R \rangle A$ by derivable inversions $\langle \cup \rangle^{\circlearrowleft}$, $[\cup]_1^{\circlearrowleft}$, $[\cup]_2^{\circlearrowleft}$.

4. Case $C = \left\langle \overrightarrow{Q} \right\rangle \langle P ; R \rangle A$ and $\overline{C} = \left[\overrightarrow{Q} \right] [P ; R] \, \overline{A}$. Immediate reduction to (Cut) on $\left\langle \overrightarrow{Q} \right\rangle \langle P \rangle \langle R \rangle A$ by derivable inversions $\langle \, ; \rangle^{\circlearrowleft}$, $[\, ;]^{\circlearrowleft}$.

5. Case $C = \left\langle \overrightarrow{Q} \right\rangle F$ and $\overline{C} = \left[\overrightarrow{Q} \right] \overline{F}$ where $\overrightarrow{Q} = Q_1 \cdots Q_n$ $(n > 0)$ and $(\forall j \in [1, n]) \left(Q_j = p_j \text{ or } Q_j = P_j^* \right)$, while $F \neq \langle Q \rangle F'$. The reduction is either trivial, if $\partial_1 = (\mathrm{Ax})^+$, or else defined hereditarily with respect to left-hand side non-principal subcases like

$$\partial_1 : \boxed{\dfrac{(\partial_1' : C, \Gamma')}{C, \Gamma} \ (\mathrm{R})}$$ with $\partial_2 : [p] \overline{A}, \Pi$, when we let

$$\mathcal{R}(\partial) := \boxed{\dfrac{(\mathcal{R}(\partial') : \Gamma' \cup \Pi)}{\Gamma \cup \Pi} \ (\mathrm{R})} \text{ for } \partial' : \boxed{\dfrac{(\partial_1' : C, \Gamma') \qquad (\partial_2 : \overline{C}, \Pi)}{\Gamma' \cup \Pi} \ (\textsc{Cut})},$$

or analogous non-principal subcases $\partial_1 : \boxed{\dfrac{(\partial_1' : C, \Gamma') \quad (\partial_1'' : C, \Gamma'')}{C, \Gamma} \ (\mathrm{R})}$,

$$\partial_1 : \boxed{\dfrac{\cdots \quad \left(\partial^{(m)} : C, \Gamma^{(m)}\right) \quad \cdots (\forall m \geq 0)}{C, \Gamma} \ [*]},$$

as well as the following principal subcases 5 (a), 5 (b), 5 (c).

5 (a). $C = \left\langle \overrightarrow{Q} \right\rangle F = \left\langle \overrightarrow{Q'} \right\rangle \langle P^* \rangle A$ and

$$\partial_1 : \boxed{\dfrac{\left(\partial_1' : \left\langle \overrightarrow{Q'} \right\rangle \langle P \rangle^m A, \left\langle \overrightarrow{Q'} \right\rangle \langle P^* \rangle A, \Gamma\right)}{\left\langle \overrightarrow{Q'} \right\rangle \langle P^* \rangle A, \Gamma} \ \langle * \rangle}$$ with $\partial_2 : \left[\overrightarrow{Q'} \right] [P^*] \overline{A}, \Pi$. Let

$\mathcal{R}(\partial) :=$

$$\boxed{\dfrac{\left(\mathcal{R}(\partial') : \left\langle \overrightarrow{Q'} \right\rangle \langle P \rangle^m A, \Gamma \cup \Pi\right) \qquad \left(\partial_2^{[*]^{\circlearrowleft}} : \left[\overrightarrow{Q'} \right] [P]^m \overline{A}, \Pi\right)}{\Gamma \cup \Pi} \ (\textsc{Cut})} \text{ where}$$

$$\partial' : \boxed{\dfrac{\left(\partial_1' : \left\langle \overrightarrow{Q'} \right\rangle \langle P \rangle^m A, \left\langle \overrightarrow{Q'} \right\rangle \langle P^* \rangle A, \Gamma\right) \quad \left(\partial_2 : \left[\overrightarrow{Q'} \right] [P^*] \overline{A}, \Pi\right)}{\left\langle \overrightarrow{Q'} \right\rangle \langle P \rangle^m A, \Gamma \cup \Pi} \ (\textsc{Cut})}.$$

5 (b). $C = \left\langle \overrightarrow{Q} \right\rangle F = \langle P^* \rangle A$ and

$$\partial_1 : \boxed{\dfrac{\left(\partial_1' : A, \overrightarrow{B}, D\right)}{\begin{array}{c}\langle P^* \rangle A, \langle P^* \rangle \overrightarrow{B}, [P^*] D, \Gamma' \\ = \langle P^* \rangle A, \Gamma\end{array}} \ (\textsc{Gen})}$$ with $\partial_2 : [P^*] \overline{A}, \Pi$. Then let

$$\mathcal{R}(\partial) := \boxed{\dfrac{\cdots\ \left(\partial''_m : \langle P^*\rangle \overrightarrow{B}, [P]^m D, \Gamma' \cup \Pi\right)\ \cdots (\forall m \geq 0)}{\begin{array}{c}\langle P^*\rangle \overrightarrow{B}, [P^*]D, \Gamma' \cup \Pi \\ = \Gamma \cup \Pi\end{array}}\ [*]} \quad \text{where}$$

$\partial''_m :=$

$$\boxed{\dfrac{\dfrac{\dfrac{\left(\partial'_1 : A, \overrightarrow{B}, D\right)}{\langle P\rangle^m A, \langle P\rangle^m \overrightarrow{B}, [P]^m D, \langle P^*\rangle \overrightarrow{B}, \Gamma'}\left(\overrightarrow{\text{Gen}}\right) \quad \left(\partial_2^{[*]^{\circlearrowleft}} : [P]^m \overline{A}, \Pi\right)}{\langle P\rangle^m \overrightarrow{B}, \langle P^*\rangle \overrightarrow{B}, [P]^m D, \Gamma' \cup \Pi}(\text{Cut})}{\langle P^*\rangle \overrightarrow{B}, [P]^m D, \Gamma' \cup \Pi}\langle * \rangle}.$$

5 (c). $C = \left\langle \overrightarrow{Q} \right\rangle F = \langle p\rangle A$ and

$$\partial_1 : \boxed{\dfrac{\left(\partial'_1 : A, \overrightarrow{B}, D\right)}{\begin{array}{c}\langle p\rangle A, \langle p\rangle \overrightarrow{B}, [p]D, \Gamma' \\ = \langle p\rangle A, \Gamma\end{array}}(\text{Gen})} \text{ with } \partial_2 : [p]\overline{A}, \Pi. \text{ Then we let}$$

$$\mathcal{R}(\partial) := \boxed{\dfrac{\left(\mathcal{R}'\left(\partial'_1, \partial_2\right) : \langle p\rangle \overrightarrow{B}, [p]D, \Pi\right)}{\begin{array}{c}\langle p\rangle \overrightarrow{B}, [p]D, \Gamma' \cup \Pi \\ = \Gamma \cup \Pi\end{array}}(\text{W})},$$

where $\mathcal{R}'\left(\partial'_1, \partial_2\right)$ is defined by induction on $h(\partial_2)$ – either trivially, if $\partial_2 = (\text{Ax})^+$, or hereditarily, in the non-principal subcases, while in the principal subcases

$$\partial_2 : \boxed{\dfrac{\left(\partial'_2 : \overline{A}, \overrightarrow{G}\right)}{\begin{array}{c}[p]\, A, \langle p\rangle \overrightarrow{G}, \Pi' \\ = [p]A, \Pi\end{array}}(\text{Gen})} \quad \text{and } \partial_2 : \boxed{\dfrac{\left(\partial'_2 : \overrightarrow{G}, H\right)}{\begin{array}{c}[p]\, A, \langle p\rangle \overrightarrow{G}, [p]H, \Pi'' \\ = [p]A, \Pi\end{array}}(\text{Gen})}$$

we respectively let

$$\mathcal{R}'(\partial_1, \partial_2) := \boxed{\dfrac{\dfrac{\left(\partial'_1 : A, \overrightarrow{B}, D\right) \qquad \left(\partial'_2 : \overline{A}, \overrightarrow{G}\right)}{\overrightarrow{B}, D, \overrightarrow{G}}(\text{Cut})}{\begin{array}{c}\langle p\rangle \overrightarrow{B}, [p]D, \langle p\rangle \overrightarrow{G}, \Pi' \\ = \langle p\rangle \overrightarrow{B}, [p]D, \Pi\end{array}}(\text{Gen})} \quad \text{and}$$

$$\mathcal{R}'(\partial_1, \partial_2) := \boxed{\dfrac{\left(\partial'_2 : \overrightarrow{G}, H\right)}{\begin{array}{c}\langle p\rangle \overrightarrow{G}, [p]H, \langle p\rangle \overrightarrow{B}, [p]D, \Pi'' \\ = \langle p\rangle \overrightarrow{B}, [p]D, \Pi\end{array}}(\text{Gen})}, \text{ as desired.}$$

Obviously $\mathcal{R}$ reduces the cut degree of ∂. That is, in each case 1–5 we have $\deg(\mathcal{R}(\partial)) < \deg(\partial) < \omega^\omega$, provided that both ∂_1 and ∂_2 involved are cutffree. Moreover it's readily seen that nodes in $\mathcal{R}(\partial)$ can be augmented with ordinals such that

$$\boxed{h(\mathcal{R}(\partial)) < h(\partial_1) \# h(\partial_2) + \omega < h(\partial) \cdot 2 + \omega}.$$

Having this one can define ordinal assignments also for (slightly modified) cut-free derivations $\mathcal{E}(\partial)$ such that for any ∂ with $\deg(\partial) < \omega^\alpha$ it holds

$$\boxed{h(\mathcal{E}(\partial)) < \varphi(\alpha, h(\partial))},$$

which for $\deg(\partial) < \omega^\omega$ and $h(\partial) < \omega \cdot 2$ (cf. Theorem 4) yields

$$\boxed{h(\mathcal{E}(\partial)) < \sup_{n<\omega} \varphi(n, \omega \cdot 2) = \varphi(\omega, 0) = \varphi_\omega(0)}$$

(see Appendix A for a detailed presentation). It is readily seen that the entire proof is formalizable in $\mathbf{PA}_{\varphi_\omega(0)}$, i.e. $\mathbf{PA}$ extended by schema of transfinite induction along (canonical primitive recursive representation of) ordinal $\varphi_\omega(0)$. [7] ■

Corollary 9 *Let Γ be any sequent that does not contain occurrences $[P^*]$ and suppose that Γ is derivable in $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$. Then Γ is derivable in a subsystem of $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$, called $\mathrm{SEQ}_1^{\mathrm{PDL}}$, that does not contain inferences $[*]$ and/or (CUT). Note that every derivation in $\mathrm{SEQ}_1^{\mathrm{PDL}}$ is finite. Consequently, any given $[P^*]$-free seq-formula is valid in **PDL** iff it is derivable in $\mathrm{SEQ}_1^{\mathrm{PDL}}$.*

Proof. This is obvious by the subformula property of cutfree derivations. ■

Remark 10 *Here and below we argue in $\mathbf{PA}_{\varphi_\omega(0)}$ that is a proper extension of $\mathbf{PA}$, as $\varphi_\omega(0) > \varepsilon_0$. Actually by standard arguments the whole proof is formalizable in the corresponding primitive recursive weakening, $\mathbf{PRA}_{\varphi_\omega(0)}$.*

3.4 Herbrand-style conclusions

Let $\mathcal{L}_0$ be the star-free sublanguage of $\mathcal{L}$. Denote by $\mathrm{SEQ}_0^{\mathrm{PDL}}$ the (finite) $\mathcal{L}_0$-subsystem of $\mathrm{SEQ}_1^{\mathrm{PDL}}$.

Theorem 11 *Let $\Sigma = \langle P^* \rangle A, \Pi$ with $A, \Pi \in \mathcal{L}_0$. Suppose that Σ is derivable in $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$. Then there exists a $k \geq 0$ such that $\widehat{\Sigma}_k := A, \langle p \rangle A, \cdots, \langle p \rangle^k A, \Pi$ is derivable in $\mathrm{SEQ}_0^{\mathrm{PDL}}$.*

Proof. The nontrivial implication $\mathrm{SEQ}_\omega^{\mathrm{PDL}} \vdash \Sigma \Rightarrow \mathrm{SEQ}_0^{\mathrm{PDL}} \vdash \widehat{\Sigma}_k$ follows by standard arguments from the cut elimination theorem by induction on the height of the corresponding finite cutfree proof ∂ of Σ in $\mathrm{SEQ}_\omega^{\mathrm{PDL}}$. Since no $[P^*]$

[7] $\varphi_\omega(0) = D(\omega^{\Omega+\omega})$ according to ordinal notations used in [8].

occurs in Σ, no $\langle P^*\rangle A$ can be principal formula in (Gen). Thus the only crucial case is when some $\langle P^*\rangle A$ is principal formula in

$$\boxed{\langle * \rangle \quad \dfrac{\langle P^*\rangle A, \langle P\rangle^m A, \Sigma}{\langle P^*\rangle A, \Sigma}}$$

which by the induction hypothesis yields k such that $\langle P\rangle^m A, \widehat{\Sigma}_k$ is derivable in $\mathrm{Seq}_0^{\mathrm{PDL}}$. By (C) or (W) this yields the derivability of $\widehat{\Sigma}_{k'}$ for $k' := \max(k, m)$. ■

Remark 12 *By the same token, for any $[P^*]$-free seq-formula F, one can successively replace all subformulas $\langle P^*\rangle A$ by appropriate disjunctions $\bigvee_{i=0}^{k} \langle P\rangle^i A$ such that F is* **PDL***-valid iff the resulting expansion $\widehat{F}$ is derivable in $\mathrm{Seq}_0^{\mathrm{PDL}}$.*

3.4.1 PSPACE refinement

Denote by $\mathcal{L}_{01}$ a sublanguage of $\mathcal{L}_0$ containing only atomic programs $p = \pi_i$ and let $\mathcal{L}_{00}$ be the PRO-free fraction of $\mathcal{L}_{01}$. Note that program operations ";" and "$\cup$" are definable in $\mathcal{L}_{01}$ via $(P;Q)A := (P)(Q)A$, $\langle P \cup Q\rangle A := \langle P\rangle A \vee \langle Q\rangle A$ and $[P \cup Q]A := [P]A \wedge [Q]A$. Let $\mathrm{Seq}_{01}^{\mathrm{PDL}}$ be the following $\mathcal{L}_{01}$-restriction of $\mathrm{Seq}_0^{\mathrm{PDL}}$ (that proves the same $\mathcal{L}_{01}$-sequents as $\mathrm{Seq}_0^{\mathrm{PDL}}$).

(Ax)	$x, \neg x, \Gamma$
$(\vee)$	$\dfrac{A, B, \Gamma}{A \vee B, \Gamma}$
$(\wedge)$	$\dfrac{A, \Gamma \qquad B, \Gamma}{A \wedge B, \Gamma}$
(Gen)	$\dfrac{A_1, \cdots, A_n}{(p)_{\chi_1} A_1, \cdots, (p)_{\chi_n} A_n, \Gamma}$ $(n > 0)$ if $\sum_{i=1}^{n} \chi_i = 1.$

Note that any $\mathcal{L}_{00}$-formula A is derivable in $\mathrm{Seq}_{01}^{\mathrm{PDL}}$ iff it is valid in propositional logic, and hence, by contraposition, $\mathrm{Seq}_{01}^{\mathrm{PDL}} \nvdash A$ iff $\models \neg A$ (i.e. $\neg A$ is satisfiable).

Lemma 13 (p-inversion) *Suppose that $[p]A_1, \cdots, [p]A_j, \langle p\rangle B_1, \cdots, \langle p\rangle B_k, \Gamma$, where $\Gamma = (q_1)C_1, \cdots, (q_l)C_l, \Pi$ for $q_j \neq p$, and $\Pi \in \mathcal{L}_{00}$, is derivable in $\mathrm{Seq}_{01}^{\mathrm{PDL}}$. Then so is either Γ or $A_i, B_1, \cdots, B_k$, for some $i \in [1, j]$, without increasing the height of the former derivation.*

Proof. By straightforward induction on the derivation height. In the crucial principal case we have

$$\boxed{(\mathrm{Gen}) \quad \dfrac{A_i, \Delta}{[p]A_1, \cdots, [p]A_k, \langle p\rangle B_1, \cdots, \langle p\rangle B_l}}$$

where $0 < i \leq k$ and $\Delta \subseteq B_1, \cdots, B_l$, which by derivable (W) yields the required derivability of $A_i, B_1, \cdots, B_l$. ■

Theorem 14 *The derivability in* $\mathrm{S{\scriptsize EQ}}_{01}^{\mathrm{PDL}}$ *is a PSPACE problem.*

Proof. For the sake of brevity we consider $\mathcal{L}_{01}$ formulas containing at most one atomic program $p = \pi_0$. Furthermore, we refine the notion of $\mathrm{S{\scriptsize EQ}}_{01}^{\mathrm{PDL}}$ derivability by asserting that a sequent $\Delta \neq (\mathrm{A{\scriptsize X}})$ is the conclusion of a rule (R) if one of the following *priority conditions* 1–3 is satisfied.

1. $(\mathrm{R}) = (\vee)$.

2. $(\mathrm{R}) = (\wedge)$ and no disjunction $A \vee B$ occurs as formula in Δ; thus Δ is not a conclusion of any $(\vee)$.

3. No disjunction $A \vee B$ or conjunction $A \wedge B$ occurs as formula in Δ. Thus Δ is not a conclusion of any $(\vee)$ or $(\wedge)$, i.e. $\Delta = (p)_{\xi_1} F_1, \cdots, (p)_{\xi_n} F_n$ for $\sum_{i=1}^{n} \xi_i \geq 1$. In this case we stipulate that Δ is the conclusion of (R) if one of the following two conditions holds:

   (a) $\sum_{i=1}^{n} \xi_i = 1$ and $F_1, \cdots, F_n$ is the premise of $(\mathrm{R}) = (\mathrm{G{\scriptsize EN}})$.

   (b) $\sum_{i=1}^{n} \xi_i > 1$ and there exists $j \in [1, n]$ with $\xi_j = 1$ such that either $\Delta^{(j)} := F_j \cup \{F_l \in \Delta : \xi_l = 0\}$ or $\Delta^{(-j)} := \Delta \setminus \{F_j\}$ is the premise of (R). (Note that we have $(\mathrm{R}) = (\mathrm{G{\scriptsize EN}})$ and $(\mathrm{R}) = (\mathrm{W})$ in the former and in the latter case, respectively.)

Having this we consider derivations in the refined $\mathrm{S{\scriptsize EQ}}_{01}^{\mathrm{PDL}}$ as at most binary-branching trees ∂ whose nodes are labeled with sequents of $\mathcal{L}_{01}$. Actually, for any given $\mathcal{L}_{01}$-sequent Σ it will suffice to fix one distinguished *proof search tree* ∂_0 with root sequent Σ that is defined by bottom-up recursion while applying the conditions 1–3 in a chosen order as long as possible. It is readily seen by inversions in Lemmata 7, 13 that Σ is derivable in $\mathrm{S{\scriptsize EQ}}_{01}^{\mathrm{PDL}}$ iff ∂_0 proves Σ, i.e. every maximal path in ∂_0 is locally correct with respect to 1–3. Moreover, by the obvious subformula property we conclude that the depth, $d(\partial_0)$, and maximum sequent length, $\max\{|\Delta| : \Delta \in \partial_0\}$, of ∂_0 are both proportional to $|\Sigma|$. Hence every maximal path in ∂_0 can be encoded by a $\mathcal{L}_{01}$-string of the length proportional to $|\Sigma|$ whose local correctness is verifiable by TM in $\mathcal{O}(|\Sigma|)$ space. The corresponding universal verification runs by counting all maximal paths successively, still in $\mathcal{O}(|\Sigma|)$ space, which completes the proof. ■

Remark 15 *Arguing along more familiar lines we can turn* ∂_0 *into a Boolean circuit with (binary) AND, OR and (unary) ID gates, where* $ID(x) := x$ *for* $x \in \{0, 1\}$*, such that AND, OR and ID correspond to the above conditions 2,* *3 (b) and 1 and/or 3 (a), respectively. The corresponding truth evaluations* $val(-)$ *are defined as usual via* $val(\Delta) := 1$ (**true**) *iff* $\Delta = (\mathrm{Ax})$, *for every leaf* Δ. *Then* $val(\Sigma) = 1$ *iff* ∂_0 *proves* Σ, *as required.* [8]

3.4.2 Special cases

Recall that by (a particular case of) Theorem 11, for any $\Sigma = \langle p^* \rangle A, \Pi$ with $A \in \mathcal{L}_{01}, \Pi \in \mathcal{L}_{00}$ the following holds. Suppose that Σ is derivable in $\mathrm{S}\textsc{eq}_{\omega}^{\mathrm{PDL}}$. Then there exists a $k \geq 0$ such that $\widehat{\Sigma}_k := A, \langle p \rangle A, \cdots, \langle p \rangle^k A, \Pi$ is derivable in $\mathrm{S}\textsc{eq}_{01}^{\mathrm{PDL}}$. It turns out that in some cases it's possible to estimate the minimum k and hence the corresponding $\left| \widehat{\Sigma}_k \right|$.

Definition 16 *Let* $p = \pi_0$ *be fixed. Call* basic conjunctive normal form *(abbr.:* BCNF*) any* $\mathcal{L}_{01}$*-formula* $\bigwedge_{i=1}^{m} \left(B_i \vee \langle p \rangle C_i \vee \bigvee_{j=1}^{n_i} [p] D_{i,j} \right)$ *for* $m > 0$, $n_i \geq 0$ *and* B_i, C_i, $D_{i,j} \in \mathcal{L}_{00} \cup \{\emptyset\}$. *Formulas* $\langle p^* \rangle A \vee Z$ *for* $A \in \mathrm{BCNF}$ *and* $Z \in \mathcal{L}_{00}$ *are called* basic conjunctive normal expressions *(abbr.:* BCNE*).*

Theorem 17 *Let* $A = \bigwedge_{i=1}^{m} \left(B_i \vee \langle p \rangle C_i \vee \bigvee_{j=1}^{n_i} [p] D_{i,j} \right) \in \mathrm{BCNF}$, $k \geq 0$, $\widehat{A}_k :=$ $A, \langle p \rangle A, \cdots, \langle p \rangle^k A$ *and* $\widehat{\Sigma}_k := \widehat{A}_k, \Pi$ *for* $\Pi \in \mathcal{L}_{00}$. *If* $\widehat{\Sigma}_k$ *is derivable in* $\mathrm{S}\textsc{eq}_{01}^{\mathrm{PDL}}$ *then so is* $\widehat{\Sigma}_{n+1}$ *too, where* $n = \sum_{i=1}^{m} n_i$.

Proof. For $i \in [1, m]$ let $\Delta_i := \{[p] D_{i,j} : 1 \leq j \leq n_i\}$. So Lemma 13 yields

$$\begin{aligned} \vdash \widehat{\Sigma}_0 &\Leftrightarrow \vdash A, \Pi \Leftrightarrow \bigwedge_{i=1}^{m} \vdash B_i, \langle p \rangle C_i, \Delta_i, \Pi \\ &\Leftrightarrow \left(\bigwedge_{i=1}^{m} \right) \left(\vdash B_i, \Pi \vee \left(\bigvee_{j=1}^{n_i} \right) \vdash C_i, D_{i,j} \right) \end{aligned}$$

where " $\vdash$" stands for " $\mathrm{S}\textsc{eq}_{01}^{\mathrm{PDL}} \vdash$", and hence

$$\boxed{\nvdash \widehat{\Sigma}_0 \Leftrightarrow \nvdash A, \Pi \Leftrightarrow \left(\bigvee_{i=1}^{m} \right) \left(\nvdash B_i, \Pi \wedge \left(\bigwedge_{j=1}^{n_i} \right) \nvdash C_i, D_{i,j} \right)}.$$

By the same token, for any $s \geq 0$ we let $\langle p \rangle \widehat{A}_s := \langle p \rangle \left(A \vee \langle p \rangle A \vee \cdots \vee \langle p \rangle^s A \right)$

[8] This proof is dual to familiar proof of polynomial space solvability of QSAT (cf. e,g. [10]).

and arrive at

$$\begin{aligned}
&\vdash\ \widehat{\Sigma}_{s+1} \Leftrightarrow \vdash\ \widehat{A}_{s+1}, \Pi \Leftrightarrow \vdash A, \langle p\rangle \widehat{A}_s, \Pi \\
\Leftrightarrow\ & \bigwedge_{i=1}^{m} \vdash B_i, \langle p\rangle C_i, \Delta_i, \langle p\rangle \widehat{A}_s, \Pi \\
\Leftrightarrow\ & \left(\bigwedge_{i=1}^{m}\right)\left(\vdash B_i, \Pi \vee \left(\bigvee_{j=1}^{n_i}\right) \vdash \widehat{A}_s, C_i, D_{i,j}\right)
\end{aligned}$$

which yields

$$\boxed{\nvdash \widehat{\Sigma}_{s+1} \Leftrightarrow \nvdash \widehat{A}_{s+1}, \Pi \Leftrightarrow \left(\bigvee_{i=1}^{m}\right)\left(\nvdash B_i, \Pi \wedge \left(\bigwedge_{j=1}^{n_i}\right) \nvdash \widehat{A}_s, C_i, D_{i,j}\right)}.$$

Thus for any $k \geq 0$, the assertion $\nvdash \widehat{\Sigma}_k$ is equivalent to the existence of a labeled rooted *refutation tree* T_k of the height $k+1$ such that the following conditions 1–3 hold, where sequents $\ell(x)$ are the labels of $x \in T_k$ (ρ being the root).

1. $\ell(\rho) = \Pi$.
2. $\nvdash \ell(x)$ holds for every leaf $x \in T_k$.
3. For any inner node $x \in T_k$ there exists $i \in [1, m]$ such that x has $m_i + 1$ ordered children: x_0 (the *son*) with label $\ell(x_0) = B_i, \ell(x)$ and $x_1, \cdots, x_{m_i}$ (the *daughters*) labeled $\ell(x_j) = C_i, D_{i,j}$, respectively; moreover x_j ($j \geq 0$) is a leaf iff it is either a son or else a daughter of the depth $k+1$.

Since daughters are subsequents of their sons, condition 2 is equivalent to

2*. $\nvdash \ell(x)$ holds for every node $x \in T_k$.

Now if $k \leq n+1$ then $\widehat{\Sigma}_k \subseteq \widehat{\Sigma}_{n+1}$, and hence $\vdash \widehat{\Sigma}_k$ implies $\vdash \widehat{\Sigma}_{n+1}$. Furthermore, from $\nvdash \widehat{\Sigma}_{n+1}$ we'll infer $(\forall s > n) \nvdash \widehat{\Sigma}_s$ and conclude by contraposition that $(\exists k) \vdash \widehat{\Sigma}_k$ implies (in fact is equivalent to) $\vdash \widehat{\Sigma}_{n+1}$, as required. So assume $\nvdash \widehat{\Sigma}_{n+1}$. We prove the existence of the refutation trees T_s, $s > n$, by recursion on s. Basis case $k = n+1$ holds by the assumption. To pass from T_s to T_{s+1} we argue as follows. Let $x \in T_s$ be any leaf-daughter and $\theta = (\rho, y_1, \cdots, y_s = x)$ the corresponding maximal path, in T_s. Since θ contains at most $n < s$ different labels $\ell(y_\imath) = C_i, D_{i,j}$ ($i \in [1, m]$, $j \in [1, n_i]$), there exist a (say, minimal) pair $0 < r < t < s$ such that $\ell(y_r) = \ell(y_t)$. Let $T_{(s,x,r,t)}$ be a tree that arises from T_s by substituting its subtree rooted in y_r for a one rooted in y_t. $T_{(s,x,r,t)}$ is higher than T_s – so let $T^{(x)}_{s+1}$ be a subtree of $T_{(s,x,r,t)}$ consisting of the nodes of the depths $\leq s+1$. Proceeding this way successively with respect to all leaf-daughters $x \in T_s$ while keeping in mind condition 2* we eventually obtain a refutation tree T_{s+1} of the height $s+1$, as required. ■

By Remark 10 and Theorem 11, the following are provable in $\mathbf{PRA}_{\varphi_\omega(0)}$.

Corollary 18 *Let $A \in$ BCNF, n and Π be as above. Then $\Sigma := \langle p^* \rangle A, \Pi$ is derivable in* $\mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}}$ *iff* $\widehat{\Sigma}_{n+1} := \widehat{A}_{n+1}, \Pi$ *is derivable in* $\mathrm{S{\scriptstyle EQ}}_{01}^{\mathrm{PDL}}$.

Corollary 19 *Let $S \in$ BCNE. Problem* **PDL** $\vdash S$, *i.e.* **PDL**-*validity of S, is solvable by deterministic TM in $\mathcal{O}\left(|S|^2\right)$ space.*

Proof. For A as above we have $n < |A|$, and hence $\left|\widehat{A}_{n+1}\right| = \mathcal{O}\left(|A|^2\right)$. This yields $\left|\widehat{A}_{n+1}, Z\right| = \mathcal{O}\left(|A|^2 + |Z|\right) = \mathcal{O}\left(|S|^2\right)$. Now by Theorem 4 followed by Theorems 11, 17 we have

$$\boxed{\mathbf{PDL} \vdash S \Leftrightarrow \mathrm{S{\scriptstyle EQ}}_{\omega}^{\mathrm{PDL}} \vdash S \Leftrightarrow \mathrm{S{\scriptstyle EQ}}_{01}^{\mathrm{PDL}} \vdash \widehat{A}_{n+1}, Z}$$

while problem $\mathrm{S{\scriptstyle EQ}}_{01}^{\mathrm{PDL}} \vdash \widehat{A}_{n+1}, Z$ is solvable in $\mathcal{O}\left(\left|\widehat{A}_{n+1}, Z\right|\right) = \mathcal{O}\left(|S|^2\right)$ space.
■

Definition 20 *Call* basic disjunctive normal form *(abbr.:* BDNF*) any $\mathcal{L}_{01}$-formula $\bigwedge_{i=1}^{m} \left(B_i \vee \langle p \rangle C_i \vee \bigvee_{j=1}^{n_i} [p] D_{i,j}\right)$ for $m > 0$, $n_i \geq 0$ and B_i, C_i, $D_{i,j}$ $\in \mathcal{L}_{00} \cup \{\emptyset\}$. Formulas $\langle p^* \rangle A \vee Z$ for $A \in$ BDNF and $Z \in \mathcal{L}_{00}$ are called* basic disjunctive normal expressions *(abbr.:* BDNE*).*

Problem 21 *Let $S \in$ BDNE. Is problem* **PDL** $\vdash S$ *solvable by deterministic TM in $|S|$-polynomial space?*

3.4.3 More on BDNE

PDL-satisfiability problem for certain statements $\mathrm{A{\scriptstyle CCEPTS}}_{M,x} = [p^*] V \wedge W$ for $V \in$ BCNF, $W \in \mathcal{L}_{00}$ – expressing that satisfying Kripke frames encode accepting computations of polynomial-space alternating TM – is known to be EXPTIME-complete (cf. [1] and [9]: Theorem 8.5, et al; see also [16]). Hence so is also the dual **PDL**-validity problem for the corresponding negations $S := \overline{\mathrm{A{\scriptstyle CCEPTS}}_{M,x}} = \langle p^* \rangle A \vee Z \in$ BDNE. [9] So the affirmative solution to Problem 21 would infer **EXPTIME** = **PSPACE** (and vice versa, since general **PDL**-validity is EXPTIME-complete).

Now consider a given $S := \langle p^* \rangle A \vee Z$ for $A \in$ BDNF and $E \in \mathcal{L}_{00}$, where for brevity we let $A = F_0 \vee \bigvee_{i=1}^{s} (F_i \wedge [p]\, G_i) \vee \bigvee_{i=s+1}^{t} (F_i \wedge \langle p \rangle H_i)$, cf. Footnote 9 and Appendix B. We wish to present the assertion **PDL** $\vdash S$ in a suitable "transparent" quantified boolean form. To this end, by de Morgan laws, we first convert $A \in$ BDNF to $R = \bigwedge_{\xi \in \Xi} R_\xi \in$ BCNF, where $R_\xi = B_\xi \vee \langle p \rangle C_\xi \vee \bigvee_{j \in J_\xi} [p] D_{\xi,j}$

[9] It suffices to set $A := F_0 \vee \bigvee_{i=1}^{s} (F_i \wedge [p]\, G_i) \vee \bigvee_{i=s+1}^{t} (F_i \wedge \langle p \rangle\, H_i)$, see Appendix B.

for $\Xi := \{\xi = (\xi(1), \cdots, \xi(t))\}$ with $\xi(i) \in \{1, 2\}$, for every $1 \le i \le t$, while

$$\begin{aligned} B_\xi &:= F_0 \vee \bigvee \{F_i : 1 \le i \le t \wedge \xi(i) = 1\}, \\ C_\xi &:= \bigvee \{H_i : s < i \le t \wedge \xi(i) = 2\}, \\ D_{\xi,j} &:= G_j \text{ for } j \in J_\xi := \{i : 1 \le i \le s \wedge \xi(i) = 2\}. \end{aligned}$$

Clearly $\mathbf{PDL} \vdash A \leftrightarrow R$ (also by **PDL**-equivalence $\langle p \rangle H \vee \langle p \rangle H' \leftrightarrow \langle p \rangle (H \vee H')$). Note that $|\Xi| = 2^t$ and $|R_\xi| < |A|$, for every $\xi \in \Xi$.

By the cut-elimination theorem $\mathbf{PDL} \vdash S$ is equivalent to $\mathrm{SEQ}_\omega^{\mathrm{PDL}} \vdash \langle p^* \rangle R, Z$, which by Theorem 17 is equivalent to $\mathrm{SEQ}_{01}^{\mathrm{PDL}} \vdash \widehat{R}_{n+1}, Z$, where

$$\widehat{R}_{n+1} = R, \langle p \rangle R, \cdots, \langle p \rangle^{n+1} R$$

for $n := \sum_{\xi \in \Xi} |J_\xi| < s \cdot |\Xi| = s2^t$. Hence arguing as in the proof of Theorem 17 we arrive at

$$\boxed{\mathbf{PDL} \vdash S \Leftrightarrow \mathrm{SEQ}_{01}^{\mathrm{PDL}} \vdash \widehat{R}_{n+1}, Z \Leftrightarrow f(s2^t + 1, Z) = 1}$$

where f is boolean-valued function defined for every $i \ge 0$ and formula $X \in \mathcal{L}_{00}$, $|X| < |S|$, by the following recursive clauses 1–2, where "$\vdash Y$" stands for plain boolean validity of Y, for any given $Y \in \mathcal{L}_{00}$ (see above Ch. 3.4.1).

1. $$\boxed{f(0, X) = 1 \Leftrightarrow \bigwedge_{\xi \in \Xi} \left(\vdash (B_\xi \vee X) \vee \bigvee_{j \in_\xi} \vdash (C_\xi \vee D_{\xi,j}) \right)}$$

2. $$\boxed{f(i+1, X) = 1 \Leftrightarrow \bigwedge_{\xi \in \Xi} \left(\vdash (B_\xi \vee X) \vee \bigvee_{j \in J_\xi} f(i, C_\xi \vee D_{\xi,j}) = 1 \right)}$$

Moreover, every "$\vdash Y$" involved is expressible in quantified boolean logic as $\forall x_1 \cdots \forall x_q Y$, where $\{x_1, \cdots, x_q\}$ is the set of propositional variables occurring in Y. Having this done, by recursion on i according to clauses 1–2 we obtain a desired quantified boolean formula $\widehat{S}$ such that

$$\boxed{\mathbf{PDL} \vdash S \ \Leftrightarrow f(s2^t + 1, Z) = 1 \Leftrightarrow \mathbf{QBL} \vdash \widehat{S}}$$

(**QBL** being the canonical proof system for valid quantified boolean formulas).

Conclusion 22 *For any $S \in$ BDNE there exists a "transparent" quantified boolean formula $\widehat{S}$ that is valid iff S is valid in **PDL**. Moreover, since **PDL**-validity of* BDNE *is EXPTIME-complete,* **EXPTIME** = **PSPACE** *holds true iff the validity problem for any $\widehat{S}$ involved is solvable by a polynomial-space deterministic TM.*

Remark 23 *The size of $\widehat{S}$ is exponential in that of S,* [10] *while quantified boolean validity (and/or satisfiability) is known to be PSPACE-complete (cf. e.g. [10]). Hence* **EXPTIME = PSPACE** *holds iff $\widehat{S}$ is equivalid with another quantified boolean formula whose size is polynomial in the size of S, for every $S \in$* BDNE. *This interrelation will be investigated more deeply elsewhere.*

References

[1] P. Blackburn, M. de Rijke, Y. Venema, **Modal Logic**, Cambridge UP (2001)

[2] W. Buchholz, *A simplified version of local predicativity*, in **Proof Theory: A Selection of Papers from the Leeds Proof Theory Programme 1990**, Cambridge UP, 114–147 (1992)

[3] R. Carnap, *Logische Syntax der Sprache* (1934) (Engl. translation: *The logical Syntax of Language*, NY: Humanities, 1937)

[4] S. Feferman, *Systems of predicative analysis I*, JSL **29**, 1–30 (1968)

[5] S. Feferman, *Systems of predicative analysis II: Representations of ordinals*, JSL **33**, 193–220 (1968)

[6] M. J. Fischer, R. E. Ladner, *Propositional modal logic of programs*, Proc. 9tg Symp. theory of Comput, ACM, 286–294 (1977)

[7] M. J. Fischer, R. E. Ladner, *Propositional Dynamic Logic of Regular Programs*, Journal of computer and system sciences **18**, 194–211 (1979)

[8] L. Gordeev, *Proof-theoretic analysis: weak systems of functions and classes*, APAL **38**, 1–122 (1988)

[9] D. Harel, D. Kozen, J. Tiurin, **Dynamic Logic**, MIT Press (2000)

[10] C. H. Papadimitriou, **Computational Complexity**, Addison-Wesley PC (1995)

[11] R. Parikh, *The completeness of propositional dynamic logic*, Math. Foundations of Computer Science, LN in Comp. Sci. **64**, 403–415 (1978)

[12] W. Pohlers, **Proof Theory: An Introduction**, LN in Mathematics **1409**, Springer (1989)

[13] V. Pratt, *A near-optimal method for reasoning about action*, Journal of Computer and System Sciences **20**, 231–254 (1980)

[14] K. Schütte, **Proof theory**, *Grundlehren der mathematischen Wissenschaften* **225**, Springer (1977)

[10] This is in contrast to analogous polynomial BCNE case, see Corollary 19.

[15] K. Segerberg, *A completeness theorem in the modal logic of programs*, Notices of AMS **24**, 522 (1977)

[16] E. Spaan, *Complexity of Modal Logics*, PhD Thesis, Amsterdam (1993)

[17] O. Veblen, *Continuous increasing functions of finite transfinite ordinals*, Transactions of AMS **9**, 280–292 (1908)

4 Appendix A: Ordinal assignments

4.1 Ordinal arithmetic

We use basic properties 1–8 of Veblen's ordinals (abbr.: $\alpha, \beta, \gamma, \delta$) ([17], [5], [12]).

1. Basic relation $<$ is linearly ordered.
2. Symmetric sum is associative and commutative.
3. $0 < 1 = \omega^0$, $\omega = \omega^1$, $\omega^\beta = \varphi(0, \beta)$.
4. $\alpha \mathbin{+\!\!\!+} 0 = \alpha$, $\alpha < \beta \to \alpha \mathbin{+\!\!\!+} \gamma < \beta \mathbin{+\!\!\!+} \gamma \mathbin{+\!\!\!+} \delta$.
5. $\alpha < \beta \to \varphi(\alpha, \gamma) < \varphi(\beta, \gamma) \wedge \varphi(\gamma, \alpha) < \varphi(\gamma, \beta)$.
6. $\alpha \leq \beta < \varphi(\gamma, \delta) \to \alpha \mathbin{+\!\!\!+} \beta < \varphi(\gamma, \delta)$.
7. $\alpha < \beta \wedge \gamma < \varphi(\beta, \delta) \to \varphi(\alpha, \gamma) < \varphi(\beta, \delta) \wedge \varphi(\alpha, \varphi(\beta, \delta)) = \varphi(\beta, \delta)$.
8. $\alpha \leq \omega^\alpha$, $0 < \alpha \to \omega^{\varphi(\alpha, \beta)} = \varphi(\alpha, \beta)$.

$\varphi(\alpha, \beta)$ is also denoted by $\varphi_\alpha(\beta)$. Note that $\varepsilon_0 = \varphi_1(0) < \varphi_\omega(0) < \Gamma_0$.

In the rest of this chapter we freely use these properties without explicit references.

4.2 Cut elimination $\partial \hookrightarrow \mathcal{E}(\partial)$

For the sake of brevity we'll slightly refine our inductive definition of $\mathcal{E}(\partial)$. To this end we upgrade $\mathcal{R}$ to $\mathcal{R}^+ : \left(\partial \vdash^{\beta}_{\rho + \omega^a} \Delta\right) \hookrightarrow \left(\mathcal{R}^+(\rho, \alpha, \partial) \vdash^{\varphi(\alpha, \beta)}_{\rho} \Delta\right)$. That is, for any $\rho > 0$, α and $(\partial : \Delta)$ with $\deg(\partial) < \rho + \omega^a$ we define $(\mathcal{R}^+(\rho, \alpha, \partial) : \Delta)$ such that $\deg(\mathcal{R}^+(\rho, \alpha, \partial)) < \rho$ and $h(\mathcal{R}^+(\rho, \alpha, \partial)) < \varphi(\alpha, h(\partial))$. Then for any ∂ with cuts we let

$$\boxed{\mathcal{E}(\partial) := \mathcal{R}^+(1, \alpha, \partial), \text{ where } \alpha := \min\left\{\beta : \deg(\partial) < \omega^\beta\right\}}$$

and conclude that $\deg(\mathcal{E}(\partial)) = 0$ and $h(\mathcal{E}(\partial)) < \varphi(\alpha, h(\partial))$.

Now $\mathcal{R}^+(\rho, \alpha, \partial)$ is defined for any ∂ with $\deg(\partial) < \rho + \omega^a$ as follows by double induction on α and $h(\partial)$. Let (R) be the lowermost inference in ∂. If (R) is not a (Cut) on C with $\mathfrak{o}(C) + 1 \geq \rho$ then $\mathcal{R}^+(\rho, \alpha, \partial)$ arises from

∂ by substituting $\mathcal{R}^+(\rho,\alpha,\partial_i)$ for the lowermost subdeductions ∂_i (recall that $h(\partial_i) < h(\partial)$). Otherwise, we have

$$(\partial : \Gamma \cup \Pi) = \frac{(\partial_1 : C, \Gamma) \qquad \left(\partial_2 : \overline{C}, \Pi\right)}{\Gamma \cup \Pi} \;(\textsc{Cut})$$

where $\rho \leq \mathfrak{o}(C) + 1 \leq \deg(\partial) < \rho + \omega^a$. Let

$$\left(\widehat{\partial} : \Gamma \cup \Pi\right) := \frac{(\mathcal{R}^+(\rho,\alpha,\partial_1) : C, \Gamma) \qquad \left(\mathcal{R}^+(\rho,\alpha,\partial_2) : \overline{C}, \Pi\right)}{\Gamma \cup \Pi} \;(\textsc{Cut})$$

and consider two cases.

Case $\alpha = 0$**.** Let $\mathcal{R}^+(\rho,\alpha,\partial) = \mathcal{R}^+(\rho,0,\partial) := \mathcal{R}\left(\widehat{\partial}\right)$. Recall that

$$\deg\left(\mathcal{R}\left(\widehat{\partial}\right)\right) < \deg\left(\widehat{\partial}\right) = \mathfrak{o}(C) + 1 \leq \deg(\partial) < \rho + 1$$

and hence $\deg(\mathcal{R}^+(\rho,\alpha,\partial)) = \deg\left(\mathcal{R}\left(\widehat{\partial}\right)\right) < \rho$. On the other hand

$$\begin{aligned} h\left(\mathcal{R}\left(\widehat{\partial}\right)\right) &< h\left(\mathcal{R}^+(\rho,\alpha,\partial_1)\right) \# h\left(\mathcal{R}^+(\rho,\alpha,\partial_2)\right) + \omega \\ &\leq \omega^{h\left(\mathcal{R}^+(\rho,\alpha,\partial_1)\right)} \# \omega^{h\left(\mathcal{R}^+(\rho,\alpha,\partial_2)\right)} + \omega \\ &< \omega^{h\left(\widehat{\partial}\right)} = \varphi\left(0, h\left(\widehat{\partial}\right)\right) \end{aligned}$$

which yields $h(\mathcal{R}^+(\rho,\alpha,\partial)) = h\left(\mathcal{R}\left(\widehat{\partial}\right)\right) < \varphi\left(0, h\left(\widehat{\partial}\right)\right)$, as desired.

Case $\alpha > 0$**.** Thus $\omega^a = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ for $\alpha > \alpha_1 \geq \cdots \geq \alpha_n$ (by Cantor's normal form). In this case we apply inductive hypotheses successively for $\alpha_1, \cdots, \alpha_n$ and let

$$\mathcal{R}^+(\rho,\alpha,\partial) := \mathcal{R}^+\left(\rho,\alpha_1,\mathcal{R}^+\left(\rho_1,\alpha_2,\cdots,\mathcal{R}^+\left(\rho_{n-1},\alpha_n,\partial\right)\right)\right)$$

where $\rho_0 := \rho$ and $(\forall i > 0)\, \rho_{i+1} := \rho_i + \omega^{\alpha_{i+1}}$. Then $\deg(\mathcal{R}^+(\rho,\alpha,\partial)) < \rho$ and $h(\mathcal{R}^+(\rho,\alpha,\partial)) < \varphi(\alpha_1, \cdots, \varphi(\alpha_n, h(\partial))) < \varphi(\alpha, h(\partial))$, as desired.

4.3 Formalization

We fix a chosen "canonical" primitive recursive ordinal representation

$$\boxed{\mathcal{O} = \left\langle 0, 1, \omega, <, +, \#, \omega^{(-)}, \varphi(-,-)\right\rangle}$$

(also known as *system of ordinal notations*) in the language of **PA** that is supposed to be well-ordered by $<$ up to $\varphi_\omega(0)$ (at least). To formalize the latter assumption we extend standard formalism of **PA** by the transfinite induction axiom (schema) for arbitrary arithmetical formulas, $\mathrm{TI}_{\mathcal{O}}(\varphi_\omega(0))$. The extended proof system is abbreviated by $\mathbf{PA}_{\varphi_\omega(0)}$. Derivations ∂ used in the proofs are interpreted as primitive recursive trees whose nodes x are labeled with sequents and ordinals $ord\,(x) < \varphi_\omega(0)$. Having this it is easy to formalize in $\mathbf{PA}_{\varphi_\omega(0)}$ the whole cut elimination proof; note that the operators $\mathcal{R}$, $\mathcal{R}^+$ and $\mathcal{E}$ involved are constructively defined and $\mathrm{TI}_{\mathcal{O}}\,(\varphi_\omega(0))$ is used in the corresponding termination-and-correctness proofs only. Actually we can restrict $\mathrm{TI}_{\mathcal{O}}\,(\varphi_\omega(0))$ to primitive recursive induction formulas thus reducing $\mathbf{PA}_{\varphi_\omega(0)}$ to $\mathbf{PRA}_{\varphi_\omega(0)}$.

5 Appendix B: Formula $\mathbf{Accepts}_{M,x}$ [11]

5.1 Semantics

Consider a given polynomial-space-bounded k-tape alternating Turing machine M on a given input x of length n with blanks over M's input alphabet; $\vdash$ and $\dashv$ are the left and right endmarkers, respectively. Formula $\textsc{Accepts}_{M,x}$ involves the single atomic program $\textsc{Next}$, atomic propositions $\textsc{Symbol}_i^a$ and $\textsc{State}_i^q$ for each symbol a in M's tape alphabet, q a state of M's finite control, and $0 \leq i \leq n$, and an atomic proposition $\textsc{Accept}$. Then $\textsc{Accepts}_{M,x}$ has the property that any satisfying Kripke frame encodes an accepting computation of M on x. In any such Kripke frame, states u represent configurations of M occurring in the computation tree of M on input $x = x_1, \cdots, x_n$; the truth values of $\textsc{Symbol}_i^a$ and $\textsc{State}_i^q$ at state u give the tape contents, current state, and tape head position in the configuration corresponding to u. The truth value of $\textsc{Accept}$ will be $\mathbf{1}$ iff the computation beginning in state u is an accepting computation according to the rules of alternating Turing machine acceptance. Then M accepts x iff $\textsc{Accepts}_{M,x}$ is satisfiable. $\textsc{Accepts}_{M,x}$ is EXPTIME-complete (cf. [9]: Theorem 8.5) and hence so is the negation $\overline{\textsc{Accepts}_{M,x}}$.

5.2 Formal definition

Let Γ be M's tape alphabet and Q the set of states; there is a distinguished start-state $s \in Q$ and left/right annotations $\ell, r \notin Q$. Let $U \subseteq Q$ and $E \subseteq Q$ be the sets of universal and existential states, respectively. Thus $U \cup E = Q$ and $U \cap E = \emptyset$. For each pair $(q, a) \in Q \times \Gamma$ let $\Delta\,(q, a)$ be the set of all triples describing a possible action when scanning a in state q. Working in $\mathcal{L}$ we let

$$\boxed{\textsc{Accepts}_{M,x} := \textsc{Acc} \wedge \textsc{Start} \wedge [\textsc{Next}^*](\textsc{Config} \wedge \textsc{Move} \wedge \textsc{Acceptance})}$$

where $\textsc{Acc(ept)}$, $\textsc{State}_{(-)}^{(-)}$, $\textsc{Symbol}_{(-)}^{(-)} \in \mathit{VAR}$, $\textsc{Next} \in \mathit{PRO}$ while $\textsc{Start}$, $\textsc{Config}$, $\textsc{Move}$ and $\textsc{Acceptance}$ are defined as follows.

1. $\textsc{Start} := \textsc{State}_0^s \wedge \bigwedge_{1 \leq i \leq n} \textsc{Symbol}_i^{x_i} \wedge \bigwedge_{n+1 \leq i \leq n^k} \textsc{Symbol}_i^{\square}$.
2. $\textsc{Config} :=$

[11] This is a recollection of [9]: 8.2.

$$\bigwedge_{0\le i\le n+1}\bigvee_{a\in\Gamma}\left(\textsc{Symbol}_i^a\wedge\bigwedge_{a\neq b\in\Gamma}\overline{\textsc{Symbol}}_i^b\right)\wedge\textsc{Symbol}_0^{\vdash}\wedge\overline{\textsc{Symbol}}_{n+1}^{\dashv}\wedge$$
$$\bigvee_{0\le i\le n+1}\bigvee_{q\in Q}\textsc{State}_i^q\wedge\bigwedge_{0\le i\le n+1}\bigvee_{q\in Q\cup\{\ell,r\}}\left(\textsc{State}_i^q\wedge\bigwedge_{q\neq p\in Q\cup\{l,r\}}\overline{\textsc{State}}_i^p\right)\wedge$$
$$\bigwedge_{0\le i\le n}\bigwedge_{q\in Q\cup\{\ell\}}\left(\overline{\textsc{State}}_i^q\vee\textsc{State}_{i+1}^{\ell}\right)\wedge\bigwedge_{0\le i\le n+1}\bigwedge_{q\in Q\cup\{r\}}\left(\overline{\textsc{State}}_i^q\vee\textsc{State}_{i-1}^{r}\right).$$

3. $\textsc{Move} :=$
$$\bigwedge_{0\le i\le n+1}\left(\overline{\textsc{State}}_i^{\ell}\vee\overline{\textsc{State}}_i^{r}\vee\bigwedge_{a\in\Gamma}\left(\overline{\textsc{Symbol}}_i^a\vee[\textsc{Next}]\,\textsc{Symbol}_i^a\right)\right)\wedge$$
$$\bigwedge_{0\le i\le n+1}\bigwedge_{\substack{a\in\Gamma\\ q\in Q}}\left(\begin{array}{c}\overline{\textsc{Symbol}}_i^a\vee\overline{\textsc{State}}_i^q\vee\\ \left(\begin{array}{c}\displaystyle\bigwedge_{(p,b,d)\in\Delta(q,a)}\langle\textsc{Next}\rangle\left(\textsc{Symbol}_i^b\wedge\textsc{State}_{i+d}^p\right)\wedge\\ {[\textsc{Next}]}\left(\displaystyle\bigvee_{(p,b,d)\in\Delta(q,a)}\left(\textsc{Symbol}_i^b\wedge\textsc{State}_{i+d}^p\right)\right)\end{array}\right)\end{array}\right).$$

4. $\textsc{Acceptance} :=$
$$\left(\bigwedge_{0\le i\le n+1}\bigwedge_{q\in E}\overline{\textsc{State}}_i^q\vee\left(\left(\textsc{Acc}\vee[\textsc{Next}]\,\overline{\textsc{Acc}}\right)\wedge\left(\overline{\textsc{Acc}}\vee\langle\textsc{Next}\rangle\,\textsc{Acc}\right)\right)\right)\wedge$$
$$\left(\bigwedge_{0\le i\le n+1}\bigwedge_{q\in U}\overline{\textsc{State}}_i^q\vee\left(\left(\textsc{Acc}\vee\langle\textsc{Next}\rangle\,\overline{\textsc{Acc}}\right)\wedge\left(\overline{\textsc{Acc}}\vee[\textsc{Next}]\,\textsc{Acc}\right)\right)\right).$$

Hence

$$\boxed{\begin{array}{c}\overline{\textsc{Accepts}}_{M,x}=\\ \overline{\textsc{Acc}}\vee\overline{\textsc{Start}}\vee\langle\textsc{Next}^*\rangle\left(\overline{\textsc{Config}}\vee\overline{\textsc{Move}}\vee\overline{\textsc{Acceptance}}\right)\end{array}}$$

is equivalent to $\boxed{\langle p^*\rangle A\vee Z}$ for $\boxed{p=\textsc{Next},\ E=\overline{\textsc{Acc}}\vee\overline{\textsc{Start}}\in\mathcal{L}_{00}}$ and

$$\boxed{\begin{array}{c}A=F_0\vee(F_1\wedge[p]\,G_1)\vee(F_2\wedge[p]\,G_2)\vee\displaystyle\bigvee_{\alpha\in R}(F_\alpha\wedge[p]\,G_\alpha)\vee(F_3\wedge\langle p\rangle\,G_3)\\ \vee(F_4\wedge\langle p\rangle\,G_4)\vee\displaystyle\bigvee_{\beta\in T}(F_\beta\wedge\langle p\rangle\,G_\beta)\vee\bigvee_{\gamma\in S}(F_\gamma\wedge\langle p\rangle\,G_\gamma)\in\mathrm{BDNF}\end{array}}$$

where:
$R=\{\alpha=(i,a,q,(p,b,d))\in[n+1]\times\Gamma\times Q\times\Delta(q,a)\}$,
$T=\{\beta=(i,a)\in[n+1]\times\Gamma\}$,
$S=\{\gamma=(i,a,q)\in[n+1]\times\Gamma\times Q\}$,
$$F_0=\bigvee_{0\le i\le n+1}\bigwedge_{a\in\Gamma}\left(\overline{\textsc{Symbol}}_i^a\vee\bigvee_{a\neq b\in\Gamma}\textsc{Symbol}_i^b\right)\vee\overline{\textsc{Symbol}}_0^{\vdash}\vee\textsc{Symbol}_{n+1}^{\dashv}$$
$$\vee\bigwedge_{0\le i\le n+1}\bigwedge_{q\in Q}\overline{\textsc{State}}_i^q\vee\bigvee_{0\le i\le n+1}\bigwedge_{q\in Q\cup\{\ell,r\}}\left(\overline{\textsc{State}}_i^q\vee\bigvee_{q\neq p\in Q\cup\{l,r\}}\textsc{State}_i^p\right)\vee$$
$$\bigvee_{0\le i\le n}\bigvee_{q\in Q\cup\{\ell\}}\left(\overline{\textsc{State}}_i^q\wedge\textsc{State}_{i+1}^{\ell}\right)\vee\bigvee_{0\le i\le n+1}\bigvee_{q\in Q\cup\{r\}}\left(\overline{\textsc{State}}_i^q\wedge\textsc{State}_{i-1}^{r}\right),$$

$F_1 = \bigwedge_{0 \le i \le n+1} \bigwedge_{q \in E} \overline{\textsc{State}_i^q} \wedge \textsc{Acc}$, $G_1 = \overline{\textsc{Acc}}$,
$F_2 = \bigwedge_{0 \le i \le n+1} \bigwedge_{q \in U} \overline{\textsc{State}_i^q} \wedge \overline{\textsc{Acc}}$, $G_2 = \textsc{Acc}$,
$F_3 = \bigwedge_{0 \le i \le n+1} \bigwedge_{q \in E} \overline{\textsc{State}_i^q} \wedge \overline{\textsc{Acc}}$, $G_3 = \textsc{Acc}$,
$F_4 = \bigwedge_{0 \le i \le n+1} \bigwedge_{q \in U} \overline{\textsc{State}_i^q} \wedge \textsc{Acc}$, $G_4 = \overline{\textsc{Acc}}$,
$F_\alpha = \textsc{Symbol}_i^a \wedge \overline{\textsc{State}_i^q}$, $G_\alpha = \overline{\textsc{Symbol}_i^b} \vee \overline{\textsc{State}_{i+d}^p}$,
$F_\beta = \textsc{State}_i^\ell \wedge \textsc{State}_i^r \wedge \textsc{Symbol}_i^a$, $G_\beta = \overline{\textsc{Symbol}_i^a}$,
$F_\gamma = \textsc{Symbol}_i^a \wedge \overline{\textsc{State}_i^q}$, $G_\gamma = \bigvee_{(p,b,d) \in \Delta(q,a)} \left(\overline{\textsc{Symbol}_i^b} \vee \overline{\textsc{State}_{i+d}^p} \right)$.
Note that $|\langle p^* \rangle A \vee E|$ is at most quadratic in $|\textsc{Accepts}_{M,x}|$.
