Gaussian lower bounds for the Boltzmann equation without cut-off

Cyril Imbert, Clément Mouhot, Luis Silvestre

▶ **To cite this version:**

Cyril Imbert, Clément Mouhot, Luis Silvestre. Gaussian lower bounds for the Boltzmann equation without cut-off. SIAM Journal on Mathematical Analysis, 2020, 52 (3), pp.2930-2944. 10.1137/19M1252375 . hal-02078069v2

HAL Id: hal-02078069
https://hal.science/hal-02078069v2

Submitted on 4 Mar 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GAUSSIAN LOWER BOUNDS FOR THE BOLTZMANN EQUATION WITHOUT CUT-OFF

CYRIL IMBERT, CLÉMENT MOUHOT, AND LUIS SILVESTRE

Abstract. The study of positivity of solutions to the Boltzmann equation goes back to Carleman [15], and the initial argument of Carleman was developed in [28, 26, 13, 12], but the appearance of a lower bound with Gaussian decay had remained an open question for long-range interactions (the so-called non-cutoff collision kernels). We answer this question and establish such Gaussian lower bound for solutions to the Boltzmann equation without cutoff, in the case of hard and moderately soft potentials, with spatial periodic conditions, and under the sole assumption that hydrodynamic quantities (local mass, local energy and local entropy density) remain bounded. The paper is mostly self-contained, apart from the L^∞ upper bound and weak Harnack inequality on the solution established respectively in [30, 22] and [24].

Contents

1. Introduction 1
2. Preliminaries 6
3. Lower bounds 10
References 14

1. Introduction

1.1. **The Boltzmann equation.** We consider the *Boltzmann equation* ([25, 11])

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \tag{1.1}$$

on a given time interval $I = [0,T]$, $T \in (0,+\infty]$, x in the flat torus $\mathbb{T}^d$, and $v \in \mathbb{R}^d$.

The unknown $f = f(t,x,v) \geq 0$ represents the time-dependent probability density of particles in the phase space, and $Q(f,f)$ is the *collision operator*, i.e. a quadratic integral operator modelling the interaction between particles:

$$Q(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}} \left[f(v'_*) f(v') - f(v_*) f(v) \right] B(|v-v_*|, \cos\theta) \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

where the pre-collisional velocities v'_* and v' are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \quad \text{and} \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$$

Date: March 4, 2020.

LS is supported in part by NSF grants DMS-1254332 and DMS-1362525. CM is partially supported by ERC grant MAFRAN.

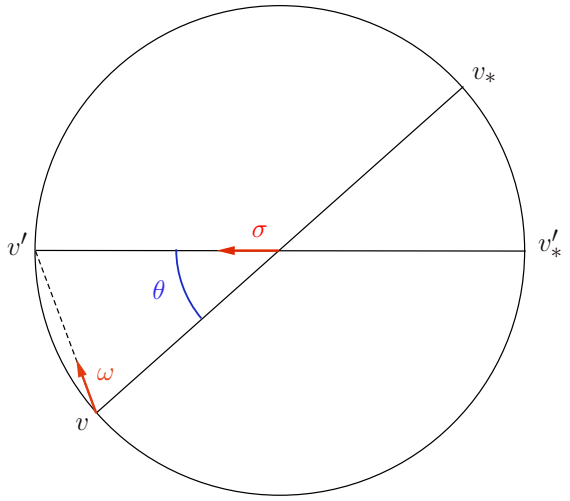

FIGURE 1. The geometry of the binary collision.

and the *deviation angle* θ is defined by (see Figure 1)

$$\cos\theta := \frac{v - v_*}{|v - v_*|} \cdot \sigma \qquad \left(\text{and} \quad \sin(\theta/2) := \frac{v' - v}{|v' - v|} \cdot \sigma = \omega \cdot \sigma \right).$$

It is known since Maxwell [25] that as soon as the interaction between particles is long-range, the so-called *grazing collisions* are predominant, and this results in a singularity of the collision kernel B at small θ. In particular when particles interact microscopically via repulsive inverse-power law potentials, the kernel B has a non integrable singularity around $\theta \sim 0$, commonly known as the *non-cutoff* case, and has the following general product form

$$B(r, \cos\theta) = r^\gamma b(\cos\theta) \quad \text{with} \quad b(\cos\theta) \approx_{\theta\sim 0} |\theta|^{-(d-1)-2s} \tag{1.2}$$

with $\gamma > -d$ and $s \in (0,1)$. In dimension $d = 3$ and for an inverse-power law potential $\Phi(r) = r^{-\alpha}$ with $\alpha \in (1, +\infty)$ then the exponents in (1.2) are

$$\gamma = \frac{\alpha - 4}{\alpha} \quad \text{and} \quad s = \frac{1}{\alpha}.$$

In dimension $d = 3$, it is standard terminology to denote *hard potentials* the case $\alpha > 4$, *Maxwellian molecules* the case $\alpha = 4$, *moderately soft potentials* the case $\alpha \in (2, 4)$ and *very soft potentials* the case $\alpha \in (1, 2)$. By analogy we denote in any dimension $d \geq 2$ *hard potentials* the case $\gamma > 0$, *Maxwellian molecules* the case $\gamma = 0$, *moderately soft potentials* the case $\gamma < 0$ and $\gamma + 2s \in [0, 2]$ and *very soft potentials* the remaining case $\gamma < 0$ and $\gamma + 2s \in (-d, 0)$.

1.2. **The program of conditional regularity.** The Boltzmann equation (1.1) is the main and oldest equation of statistical mechanics. It describes the dynamics of a gas at the *mesoscopic level*, between the *microscopic* level of the many-particle (and thus very high dimension) dynamical system following the trajectories of each particle, and the *macroscopic* level of fluid mechanics governed by Euler and Navier-Stokes equations.

The dynamical system of Newton equations on each particle is out of reach mathematically and contains way more information than could be handled. Regarding the macroscopic level, the well-posedness and regularity of solutions to the Euler and Navier-Stokes equations are still poorly understood in dimension 3 (with or without incompressibility condition). The state of the art on the Cauchy problem for the Boltzmann equation is similar to that of the 3D incompressible Navier-Stokes equations, which is not surprising given that the Boltzmann equation "contains" the fluid mechanical equations as formal scaling limits. Faced with this difficulty, Desvillettes and Villani initiated in [16] an *a priori* approach, where solutions with certain properties are assumed to exist and studied. We follow this approach but refine it by assuming only controls of natural local hydrodynamic quantities. This means that we focus on the specifically kinetic aspect of the well-posedness issue.

Our result in this paper is conditional to the following bounds:

$$0 < m_0 \leq \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v \leq M_0, \tag{1.3}$$

$$\int_{\mathbb{R}^d} f(t,x,v)|v|^2\,\mathrm{d}v \leq E_0, \tag{1.4}$$

$$\int_{\mathbb{R}^d} f(t,x,v)\log f(t,x,v)\,\mathrm{d}v \leq H_0 \tag{1.5}$$

for some constants $M_0 > m_0 > 0$, $E_0 > 0$, $H_0 > 0$.

The first equation (1.3) implies that the mass density is bounded above and below on the spatial domain and that there is no vaccum. It would be desirable to relax its lower bound part to only $\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \geq m_0 > 0$, i.e. averaged in space. The equation (1.4) implies that the energy density is bounded above on the spatial domain, and the equation (1.5) implies that the entropy density is bounded above on the spatial domain (note that the energy bound implies that it is also bounded below). These conditions are satisfied for perturbative solutions close to the Maxwellian equilibrium (see for instance the recent work [19] and references therein in the hard spheres case and [18, 4, 3, 5, 7] in the non-cutoff case) but it is an oustandingly difficult problem to prove them in general.

Previous results of conditional regularity include, for the Boltzmann equation and under the conditions above, the proof of L^∞ bounds in [30], the proof of a weak Harnack inequality and Hölder continuity in [24], the proof of polynomially decaying upper bounds in [22], and the proof of Schauder estimates to bootstrap higher regularity estimates in [23]. In the case of the closely related Landau equation, which is a nonlinear diffusive approximation of the Boltzmann equation, the L^∞ bound was proved in [31, 17], the Harnack inequality and Hölder continuity were obtained in [32, 17], decay estimates were obtained in [14], and Schauder estimates were established in [20] (see also [21]). The interested reader is refereed to the short review [27] of the conditional regularity program.

1.3. **The main result.** Let us define the notion of classical solutions we will use.

Definition 1.1 (Classical solutions to the Boltzmann equation)**.** *Given $T \in (0, +\infty]$, we say that a function $f : [0,T] \times \mathbb{T}^d \times \mathbb{R}^d \to [0, +\infty)$ is a* ***classical solution to the Boltzmann equation*** (1.1) *if*

- *it is differentiable in t and x and twice differentiable in v everywhere;*
- *the equation* (1.1) *holds classically at every point in $[0,T] \times \mathbb{T}^d \times \mathbb{R}^d$.*

The main result is then:

Theorem 1.2. *Assume that $\gamma + 2s \in [0, 2]$ (hard and moderately soft potentials). Let $f \geq 0$ be a solution to* (1.1) *according to Definition 1.1 that satisfies the hydrodynamic bounds* (1.3)-(1.4)-(1.5) *for all $t \in [0,T]$ and $x \in \mathbb{T}^d$. Then, there exists $a(t) > 0$ and $b(t) > 0$ depending on t, s, γ, d, m_0, M_0, E_0 and H_0 only so that*

$$\forall\, t > 0,\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \geq a(t) e^{-b(t)|v|^2}.$$

Remarks 1.3.

(1) *The bound does not depend on the size domain of periodicity in x; this is due to the fact that the hydrodynamic bounds* (1.3)-(1.4)-(1.5) *are uniform in space. The periodicity assumption is made for technical conveniency and could most likely be removed.*

(2) *The requirement $\gamma + 2s \geq 0$ could be relaxed in our proof at the expense of assuming $f(t,x,v) \leq K_0$ for some constant $K_0 > 0$ (note that the functions $a(t)$ and $b(t)$ would then depend on K_0). However when $\gamma + 2s < 0$, this L^∞ bound is only proved when assuming more than* (1.3)-(1.4)-(1.5) *on the solutions, namely some $L^\infty_{t,x} L^p_v$ bounds with $p > 0$, see* [30].

1.4. **Previous results of lower bound and comparison.** The emergence and persistence of lower bounds for the Boltzmann equation is one of the most classical problems in the analysis of kinetic equations. It is a natural question: it advances the understanding of how the gas fills up the phase space, and how it relaxes towards the Maxwellian state; and such lower bounds are related to coercivity properties of the collision operator.

The study of lower bounds was initiated in the case of short-range interactions (namely hard spheres) and for spatially homogeneous solutions: Carleman [15] proved the generation of lower bounds of the type $f(t,v) \geq C_1 e^{-C_2|v|^{2+\epsilon}}$ for constants $C_1, C_2 > 0$ and $t \geq t_0 > 0$, with $\epsilon > 0$ as $t_0 > 0$ as small as wanted. He considered classical solutions with polynomial pointwise estimates of decay (such estimates are also proved in his paper) and assumed that the initial data has already a minoration over a ball in velocity. This was later significantly improved in [28]: the authors proved in this paper that spatially homogeneous solutions with finite mass, energy and entropy are bounded from below by a Maxwellian $C_1 e^{-C_2|v|^2}$, where the constants depend on the mass, energy and entropy; they obtained the optimal Maxwellian decay by refining the calculations of Carleman but also got rid of the minoration assumptions on the initial data through a clever use of the iterated gain part of the collision kernel. Note that, in this spatially homogeneous setting and for hard spheres, the assumption of finite entropy could probably be relaxed in the latter statement by using the non-concentration estimate on the iterated gain part of the collision operator proved later in [1]. Finally, still for hard spheres, the optimal Maxwelllian lower bound was extended to spatially

inhomogeneous solutions in the torus satisfying the hydrodynamic bounds (1.3)-(1.4)-(1.5) in [26], and to domains with boundaries in [13, 12]. The paper [26] also proved the first lower bounds in the non-cutoff case, however they were poorer than Maxwellian ($C_1 e^{-C_2|v|^\beta}$ with $\beta > 2$ not necessarily close to 2) and required considerably stronger a priori assumptions on the solutions than (1.3)-(1.4)-(1.5). The latter point is due to the fact that the proof of the lower bound in [26] in the non-cutoff case is based on a decomposition of the collision between grazing and non-grazing collisions and treating the former as mere error terms. Thus, Theorem 1.2 is a significant improvement over the result of [26] in the non-cutoff case. Our proof here uses coercivity properties at grazing collisions, as pioneered by [30], instead of treating them as error terms.

1.5. **Method of proof.** The well established pattern for proving lower bounds goes back to Carleman [15] and follows the collision process: (1) establish a minoration on a ball on the time interval considered, (2) spread iteratively this lower bound through the collision process, i.e. using coercivity properties of the collision operator. The step (1), i.e. the minoration on a ball $v \in B_1$, is deduced here from the weak Harnack inequality as in [24], see Proposition 3.1 below (and [24, Theorem 1.3]). The spreading argument of step (2) is then performed in Lemma 3.4. The geometric construction in Lemma 3.4 resembles the iterative spreading of lower bounds in the cut-off case, as in [26]. The key difference is the way we handle the singularity in the integral kernel. In [26], a priori assumptions of smoothness of the solution are used to remove a neighbourhood around $\theta = 0$ in the collision integral and treat it as an error term. Here instead we use coercivity and sign properties of this singular part of the collision operators, as developed and used in [30, 23, 22]; because of the fractional derivative involved we use also a barrier method to justify the argument; it is inspired from [29], and was recently applied to the Boltzmann equation in [30, 22].

1.6. **A note on weak solutions.** No well-posedness results are known for the Boltzmann equation without perturbative conditions or special symmetry. This is true both for strong and weak notions of solutions. As far as the *existence* is concerned, the unconditional existence of solutions is only known for *renormalized solutions with defect measure*, see [9]. Current results on *uniqueness* of solutions require significant regularity assumptions (see for example [8]). It is thus not surprising that it is rather inconvenient to prove estimates for the inhomogeneous Boltzmann equation in any context other than that of classical solutions. Our main result in Theorem 1.2 is presented as an a priori estimate on classical solutions. The estimate does not depend quantitatively on the smoothness of the solution f. Some computations in the proof however require a qualitative smoothness assumption of f so the quantities involved make sense.

Let us be more precise, and discuss the two parts of the proof of Theorem 1.2 described in the previous section. Part (1) is established thanks to the weak Harnack inequality from [24] (see Proposition 3.1). The qualitative conditions necessary for this step, as stated in [24], is that $f \in L^2([0,T] \times \mathbb{T}^d, L^\infty_{loc} \cap H^s_{loc}(\mathbb{R}^d))$ solves the equation (1.1) in the sense of distributions. Part (2) consists in expanding the lower bound from $v \in B_1$ to larger values of $|v|$ and is based on comparison principles with certain barrier functions. The notion of solution that is compatible with these methods is that of *viscosity super-solutions*. In this context, it would be defined in the following way.

Denote $\underline{f}$ the lower semicontinuous envelope of f. We say a function $f : C([0,T] \times \mathbb{T}^d, L^1_2(\mathbb{R}^d))$ is a viscosity super-solution of (1.1) if whenever there is a C^2 function φ for which $\underline{f} - \varphi$ attains a local minimum at some point $(t_0, x_0, v_0) \in (0,T] \times \mathbb{T}^d \times \mathbb{R}^d$, then the following inequality holds

$$\begin{aligned}(\varphi_t + v \cdot \nabla_x \varphi)(t_0,x_0,v_0) \geq & \int_{B_\varepsilon(v_0)} (\varphi(t_0,x_0,v') - \varphi(t_0,x_0,v_0)) K_f(t_0,x_0,v_0,v')\, \mathrm{d}v' \\ & + \int_{\mathbb{R}^d \setminus B_\varepsilon(v_0)} (f(t_0,x_0,v') - f(t_0,x_0,v_0)) K_f(t_0,x_0,v_0,v')\, \mathrm{d}v' \\ & + Q_{ns}(f,\varphi)(t_0,x_0,v_0).\end{aligned}$$

Here K_f is the Boltzmann kernel written in (2.4), Q_{ns} is the non-singular term written below in (2.1), $\varepsilon > 0$ is any small number so that the minimum of $\underline{f} - \varphi$ in $B_\varepsilon(v_0)$ is attained at v_0. Following the methods in [10] or [29] for instance, one should have no problem reproducing our proofs in this paper in the context of such viscosity solutions.

It is unclear how the notion of viscosity solution compares with the notion of renormalized solutions. We are not aware of any work on viscosity solutions in the context of the Boltzmann equation. If one tries to adapt the proofs in this paper to the renormalized solutions with defect measure of [6], it seems that one would face serious technical difficulties. In order to keep this paper cleaner, and to make it readable for the largest possible audience, we believe that it is most convenient to restrict our analysis to classical solutions.

1.7. **Plan of the paper.** Section 2 introduces the decomposition of the collision operator adapted to the non-cutoff setting and recalls key estimates on it. Section 3 proves the main statement: we first recall the result of [24] providing a minoration on a ball, then introduce our new argument for the spreading step, and finally complete the proof that follows readily from the two latter estimates.

1.8. **Notation.** We denote $a \lesssim b$ (respectively $a \gtrsim b$) for $a \leq Cb$ (respectively $a \geq Cb$) when the constant $C > 0$ is independent from the parameters of the calculation; when it depends on such parameters it is indicated as an index, such as $a \lesssim_{M_0} b$. We denote $B_R(v_0)$ the ball of $\mathbb{R}^d$ centered at v_0 and with radius R, and we omit writing the center when it is 0, as in $B_R = B_R(0)$.

2. Preliminaries

2.1. **Decomposition of the collision operator.** It is standard since the discovery of the so-called "cancellation lemma" [2] to decompose the Bolzmann collision operator $Q(f,f)$ into singular and non-singular parts as follows:

$$\begin{aligned}& Q(f_1,f_2)(v) \\ &= \int_{\mathbb{R}^d \times \mathbb{S}} \Big[f_1(v'_*) f_2(v') - f_1(v_*) f_2(v) \Big] B \, \mathrm{d}v_* \, \mathrm{d}\sigma \\ &= \int f_1(v'_*) \big[f_2(v') - f(v) \big] B \, \mathrm{d}v_* \, \mathrm{d}\sigma + f_2(v) \int \big[f_1(v'_*) - f_1(v_*) \big] B \, \mathrm{d}v_* \, \mathrm{d}\sigma \\ &=: Q_s(f_1,f_2) + Q_{ns}(f_1,f_2)\end{aligned}$$

where "s" stands for "singular" and "ns" stands for "non-singular". The part Q_{ns} is indeed non-singular: Given $v \in \mathbb{R}^d$, the change of variables $(v_*, \sigma) \mapsto (v'_*, \sigma)$ has Jacobian $\mathrm{d}v'_*\mathrm{d}\sigma = 2^{d-1}(\cos\theta/2)^2\,\mathrm{d}v_*\mathrm{d}\sigma$, which yields (same calculation as [2, Lemma 1])

$$Q_{ns}(f_1, f_2)(v) = f_2(v)\int_{\mathbb{R}^d}\int_{\mathbb{S}}\big[f_1(v'_*) - f_1(v_*)\big]B\,\mathrm{d}v_*\,\mathrm{d}\sigma =: f_2(v)(f_1 * S)(v) \tag{2.1}$$

with

$$\begin{aligned} S(u) &:= \Big|\mathbb{S}^{d-2}\Big|\int_0^{\frac{\pi}{2}}(\sin\theta)^{d-2}\Big[(\cos\theta/2)^{-d}B\left(\frac{|u|}{\cos\theta/2}, \cos\theta\right) - B(|u|, \cos\theta)\Big]\,\mathrm{d}\theta \\ &= \Big|\mathbb{S}^{d-2}\Big|\,|u|^\gamma\int_0^{\frac{\pi}{2}}(\sin\theta)^{d-2}\Big[(\cos\theta/2)^{-d-\gamma} - 1\Big]b(\cos\theta)\,\mathrm{d}\theta \\ &=: C_S|u|^\gamma \end{aligned}$$

where we have used the precise form (1.2) of the collision kernel in the second line. The constant $C_S > 0$ is finite, positive and only depends on b, d, and γ. The first term Q_s is an elliptic non-local integral operator of order $2s$ (see [2, 30] and the many other subsequent works revealing this fact), and the second term $Q_{ns}(f, f)$ is a lower order term that happens to be nonnegative.

Observe that $Q_{ns} \geq 0$, and thus we can remove this lower order term and the function f is a supersolution of the following equation,

$$f_t + v\cdot\nabla_x f \geq Q_s(f, f), \tag{2.2}$$

where

$$Q_s(f, f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\big[f_2(v') - f_2(v)\big]f_1(v'_*)b(\cos\theta)\,\mathrm{d}v_*\,\mathrm{d}\sigma.$$

We change variables (Carleman representation [15]) according to $(v_*, \sigma) \mapsto (v', v'_*)$, with Jacobian $\mathrm{d}v_*\mathrm{d}\sigma = 2^{d-1}|v - v'|^{-1}|v - v_*|^{-(d-2)}\,\mathrm{d}v'\mathrm{d}v'_*$ (see for instance [30, Lemma A.1]) and we deduce

$$Q_s(f_1, f_2)(v) = \text{p.v.}\int_{\mathbb{R}^d}K_{f_1}(v, v')\big[f_2(v') - f_2(v)\big]\,\mathrm{d}v', \tag{2.3}$$

where

$$\begin{aligned} K_{f_1}(t, x, v, v') &:= \frac{2^{d-1}}{|v' - v|}\int_{v'_*\in\, v+(v'-v)^\perp} f_1(t, x, v'_*)|v - v_*|^{\gamma-(d-2)}b(\cos\theta)\,\mathrm{d}v'_* \\ &:= \frac{1}{|v' - v|^{d+2s}}\int_{v'_*\in\, v+(v'-v)^\perp} f_1(t, x, v'_*)|v - v'_*|^{\gamma+2s+1}\tilde{b}(\cos\theta)\,\mathrm{d}v'_* \end{aligned} \tag{2.4}$$

and where we have used the assumption (1.2) to write

$$2^{d-1}b(\cos\theta) = |v - v'|^{-(d-1)-2s}|v - v_*|^{(d-2)-\gamma}|v - v'_*|^{\gamma+2s+1}\tilde{b}(\cos\theta)$$

with $\tilde{b}$ smooth on $[0, \pi]$ and stricly positive on $[0, \pi)$. The notation p.v. denotes the Cauchy principal value around the point v. It is needed only when $s \in [1/2, 1)$.

2.2. **Estimates on the collision operator.** We start with a simple estimate from above the kernel K_f as in [30, Lemma 4.3].

Proposition 2.1 (Upper bounds for the kernel)**.** *For any $r > 0$, the following inequality holds*

$$\forall\, t \in [0,\infty),\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad \begin{cases} \displaystyle\int_{B_r(v)} |v-v'|^2 K_f(t,x,v,v')\,\mathrm{d}v' \lesssim \Lambda(t,x,v) r^{2-2s}, \\ \displaystyle\int_{\mathbb{R}^d\setminus B_r(v)} K_f(t,x,v,v')\,\mathrm{d}v' \lesssim \Lambda(t,x,v) r^{-2s} \end{cases}$$

where

$$\Lambda(t,x,v) = \int_{\mathbb{R}^d} f(t,x,v'_*)|v-v'_*|^{\gamma+2s}\,\mathrm{d}v'_*.$$

Remark 2.2. Note that if $\gamma + 2s \in [0,2]$ and (1.3-1.5) hold, then

$$\Lambda(t,x,v) \lesssim_{M_0,E_0} (1+|v|)^{\gamma+2s}.$$

Proof of Proposition 2.1. To prove the first inequality we write (omitting t,x)

$$\begin{aligned}
&\int_{B_r(v)} |v-v'|^2 K_f(v,v')\,\mathrm{d}v' \\
&= \int_{B_r(v)} |v-v'|^{2-d-2s} \int_{v'_*\in\, v+(v'-v)^\perp} f(v'_*)|v-v'_*|^{\gamma+2s+1}\tilde{b}(\cos\theta)\,\mathrm{d}v'_*\,\mathrm{d}v' \\
&= \int_{\omega\in\mathbb{S}^{d-1}} \left(\int_{u=0}^{r} u^{1-2s}\,\mathrm{d}u\right) \int_{v'_*\in\, v+(v'-v)^\perp} f(v'_*)|v-v'_*|^{\gamma+2s+1}\tilde{b}(\cos\theta)\,\mathrm{d}v'_*\,\mathrm{d}v' \\
&\lesssim r^{2-2s} \int_{\omega\in\mathbb{S}^{d-1}} \int_{\tilde{\omega}\in\mathbb{S}^{d-1}} \delta_0(\omega\cdot\tilde{\omega}) \int_{\tilde{u}=0}^{+\infty} \tilde{u}^{d-1+\gamma+2s} f(\tilde{u}\tilde{\omega})\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{\omega}\,\mathrm{d}\omega \\
&\lesssim r^{2-2s} \int_{v'_*\in\mathbb{R}^d} f(v'_*)|v-v'_*|^{\gamma+2s}\,\mathrm{d}v'_*.
\end{aligned}$$

The proof of the second inequality is similar:

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus B_r(v)} K_f(v,v')\,\mathrm{d}v' \\
&= \int_{\mathbb{R}^d\setminus B_r(v)} |v-v'|^{-d-2s} \int_{v'_*\in\, v+(v'-v)^\perp} f(v'_*)|v-v'_*|^{\gamma+2s+1}\tilde{b}(\cos\theta)\,\mathrm{d}v'_*\,\mathrm{d}v' \\
&= \int_{\omega\in\mathbb{S}^{d-1}} \left(\int_{u=r}^{+\infty} u^{-1-2s}\,\mathrm{d}u\right) \int_{v'_*\in\, v+(v'-v)^\perp} f(v'_*)|v-v'_*|^{\gamma+2s+1}\tilde{b}(\cos\theta)\,\mathrm{d}v'_*\,\mathrm{d}v' \\
&\lesssim r^{-2s} \int_{\omega\in\mathbb{S}^{d-1}} \int_{\tilde{\omega}\in\mathbb{S}^{d-1}} \delta_0(\omega\cdot\tilde{\omega}) \int_{\tilde{u}=0}^{+\infty} \tilde{u}^{d-1+\gamma+2s} f(\tilde{u}\tilde{\omega})\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{\omega}\,\mathrm{d}\omega \\
&\lesssim r^{-2s} \int_{v'_*\in\mathbb{R}^d} f(v'_*)|v-v'_*|^{\gamma+2s}\,\mathrm{d}v'_*.
\end{aligned}$$

This concludes the proof. □

The latter bounds are useful to estimate $Q_s(f,\varphi)$ for a C^2 barrier function φ.

Lemma 2.3 (Upper bound for the linear Boltzmann operator)**.** *Let φ be a bounded, C^2 function in $\mathbb{R}^d$. The following inequality holds*

$$|Q_s(f,\varphi)| = \left|\text{p.v.} \int_{\mathbb{R}^d} \big[\varphi(v') - \varphi(v)\big] K_f(t,x,v,v')\, \mathrm{d}v'\right| \leq \Lambda(t,x,v) \|\varphi\|^{1-s}_{L^\infty(\mathbb{R}^d)} [\varphi]^s_{\dot{C}^2(\mathbb{R}^d)}$$

where $\Lambda(t,x,v)$ is the same quantity as in Proposition 2.1 and

$$[\varphi]_{\dot{C}^2(\mathbb{R}^d)} := \sup_{v' \neq v} \frac{|\varphi(v') - \varphi(v) - (v'-v)\cdot\nabla\varphi(v)|}{|v'-v|^2} \lesssim \|\nabla^2\varphi\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. We decompose the domain of integration in $Q_s(f,\varphi)$ between $B_r(v)$ and $\mathbb{R}^d \setminus B_r(v)$, for an arbitrary radius $r > 0$ to be specified later. Due to the symmetry of the kernel $K_f(t,x,v,v+w) = K_f(t,x,v,v-w)$, we have that

$$\text{p.v.} \int_{B_r(v)} (v'-v)\cdot\nabla\varphi(v) K_f(t,x,v,v')\, \mathrm{d}v' = 0.$$

Therefore

$$\begin{aligned}
&\left|\text{p.v.} \int_{B_r(v)} [\varphi(v') - \varphi(v)] K_f(t,x,v,v')\, \mathrm{d}v'\right| \\
&= \left|\int_{B_r(v)} [\varphi(v') - \varphi(v) - (v'-v)\cdot\nabla\varphi(v)] K_f(t,x,v,v')\, \mathrm{d}v'\right|, \\
&\leq [\varphi]_{\dot{C}^2(\mathbb{R}^d)} \int_{B_r(v)} |v'-v|^2 K_f(t,x,v,v')\, \mathrm{d}v', \\
&\lesssim [\varphi]_{\dot{C}^2(\mathbb{R}^d)} \Lambda(t,x,v) r^{2-2s},
\end{aligned}$$

where we have used Proposition 2.1 in the last line.

Regarding the rest of the domain $\mathbb{R}^d \setminus B_r(v)$, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d\setminus B_r(v)} [\varphi(v') - \varphi(v)] K_f(t,x,v,v')\, \mathrm{d}v'\right| &\leq 2\|\varphi\|_{L^\infty} \int_{\mathbb{R}^d\setminus B_r(v)} K_f(t,x,v,v')\, \mathrm{d}v', \\
&\lesssim \|\varphi\|_{L^\infty} \Lambda(t,x,v) r^{-2s}.
\end{aligned}$$

Adding both inequalities above, we get

$$Q_1(f,\varphi) \lesssim \Lambda(t,x,v) \left([\varphi]_{C^2(v)} r^{2-2s} + \|\varphi\|_{L^\infty} r^{-2s}\right).$$

We conclude the proof choosing $r := \left(\|\varphi\|_{L^\infty(\mathbb{R}^d)} / [\varphi]_{\dot{C}^2(\mathbb{R}^d)}\right)^{1/2}$. □

2.3. **Pointwise upper bound on the solution.** The following L^∞ bound is one of the main results in [30]. We state the slightly refined version in [22, Theorem 4.1], which includes in particular the limit case $\gamma + 2s = 0$.

Proposition 2.4 (Global upper bound – [30, 22])**.** *Assume $\gamma + 2s \in [0,2]$. Let $f \geq 0$ be a solution to (1.1) according to Definition 1.1 that satisfies the hydrodynamic bounds (1.3-1.4-1.5) for all $t \in [0,T]$ and $x \in \mathbb{T}^d$.*

Then, there exists a non-increasing function $b(t) > 0$ on $(0,T]$ depending on m_0, M_0, E_0 and H_0 only so that

$$\forall\, t \in (0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \le b(t).$$

3. Lower bounds

Recall that we prove the appearance of a lower bound on a ball thanks to a weak Harnack inequality, then spread it iteratively using the mixing properties of the geometry of collision and coercivity estimates on the collision operator.

3.1. Weak Harnack inequality and initial plateau.

In [24], two of the authors obtain a weak Harnack inequality for the linear Boltzmann equation. This weak Harnack inequality implies a local lower bound for the nonlinear Boltzmann equation. It is stated in the following proposition. Note that Proposition 2.4 gives us control of $\|f\|_{L^\infty}$ in terms of the other parameters.

Proposition 3.1 (Local minoration – [24, Theorem 1.3])**.** *Let $f \ge 0$ be a solution to (1.1) according to Definition 1.1 that is L^∞ and satisfies the hydrodynamic bounds (1.3-1.4-1.5) for all $t \in [0,T]$ and $x \in \mathbb{T}^d$.*

Then, for any $R > 0$, there is a nondecreasing function $a : (0,\infty) \to (0,\infty)$ depending on s, γ, d, m_0, M_0, E_0 and H_0 and R only, such that

$$\forall\, v \in B_R(0), \quad f(t,x,v) \ge a(t).$$

Remark 3.2. Note that this result implies, in particular, that any solutions as in the statement satisfies $f(t,x,v) > 0$ for every $(t,x,v) \in (0,T] \times \mathbb{T}^d \times \mathbb{R}^d$.

Remark 3.3. Note that this result holds for $\gamma + 2s < 0$ conditionally to the L^∞ bound. However this L^∞ bound is proved only when $\gamma + 2s \in [0,2]$ (see [30, 22]).

3.2. Spreading lemma around zero.

Lemma 3.4 (Spreading around zero)**.** *Consider $T_0 \in (0,1)$. Let $f \ge 0$ be a supersolution of (2.2) so that (1.3-1.4) hold, and such that $f \ge \ell\, \mathbf{1}_{v \in B_R}$ on $[0,T_0]$ for some $\ell > 0$ and $R \ge 1$.*

Then, there is a constant $c_s > 0$ depending only on d, s, M_0, E_0 (but not on m_0 or H_0) such that for any $\xi \in (0, 1-2^{-1/2})$ so that $\xi^q R^{d+\gamma} \ell < 1/2$ with $q = d + 2(\gamma + 2s + 1)$, one has

$$\forall\, t \in [0,T_0],\ x \in \mathbb{T}^d,\ v \in B_{\sqrt{2}(1-\xi)R}, \quad f(t,x,v) \ge c_s \xi^q R^{d+\gamma} \ell^2 \min\left(t, R^{-\gamma} \xi^{2s}\right).$$

The proof of Lemma 3.4 combines, at its core, the spreading argument of the cut-off case that goes back to Carleman [15], used here in the form developed in [26], coercivity estimates on the collision operator at grazing collisions developed in [30, 24, 22], and finally a barrier argument similar to [29, Theorem 5.1].

Proof of Lemma 3.4. Given $\xi \in (0, 1 - 2^{-1/2})$ as in the statement, consider a smooth function φ_ξ valued in $[0,1]$ so that $\varphi_\xi = 1$ in $B_{\sqrt{2}(1-\xi)}$ and $\varphi_\xi = 0$ outside $B_{\sqrt{2}(1-\xi/2)}$ and $\|D^2 \varphi_\xi\|_{L^\infty(\mathbb{R}^d)} \lesssim \xi^{-2}$. Define $\varphi_{R,\xi}(v) := \varphi_\xi(v/R)$. Observe that $\|\varphi_{R,\xi}\|_{L^\infty(\mathbb{R}^d)} = 1$ and $\|D^2 \varphi_{R,\xi}\|_{L^\infty(\mathbb{R}^d)} \lesssim (R\xi)^{-2}$.

Apply Lemma 2.3 and Remark 2.2 to get

$$|Q_1(f,\varphi_{R,\xi})| \lesssim \Lambda(t,x,v)(R\xi)^{-2s} \lesssim_{M_0,E_0} R^{\gamma+2s}(R\xi)^{-2s} \lesssim_{M_0,E_0} R^\gamma \xi^{-2s}. \tag{3.1}$$

Define the following barrier function

$$\tilde{\ell}(t) := \alpha \xi^q R^{d+\gamma} \ell^2 \left(\frac{1 - e^{-CR^\gamma \xi^{-2s} t}}{CR^\gamma \xi^{-2s}} \right)$$

for some $\alpha \in (0,1)$ to be chosen small enough later and where $C \geq 1$ is the constant in (3.1) depending on M_0, E_0. Our goal is to prove that

$$\forall\, t \in [0,T_0],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f \geq \tilde{\ell}(t)\varphi_{R,\xi}.$$

Thanks to Remark 3.2 we can assume that $f > 0$ everywhere on $[0,T_0] \times \mathbb{T}^d \times \mathbb{R}^d$: it is true for any $t \in (0,T_0]$ and we can shift the solution f in time by $f(t+\varepsilon,x,v)$ and prove the lower bound independently of $\varepsilon > 0$.

Let us prove that $f(t,x,v) > \tilde{\ell}(t)\varphi_{R,\xi}(v)$ for all $t \in [0,T_0] \times \mathbb{T}^d \times \mathbb{R}^d$. This inequality holds at $t = 0$ since $\tilde{\ell}(0) = 0$. If the inequality was not true, then there would be a first crossing point $(t_0,x_0,v_0) \in [0,T_0] \times \mathbb{T}^d \times \operatorname{supp}\varphi$ (the crossing point cannot be outside the support of φ since $f > 0$) so that $f(t_0,x_0,v_0) = \tilde{\ell}(t_0)\varphi_{R,\xi}(v_0)$ and $f(t,x,v) \geq \tilde{\ell}(t)\varphi_{R,\xi}(v)$ for all $t \in [0,t_0]$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$.

The smallness condition imposed on ξ in the statement implies that $\tilde{\ell}(t) \leq \ell/2$ for all $t \in [0,T_0]$, and thus $v_0 \notin B_R$ since $\varphi \equiv 1$ and $f \geq \ell$ in B_R. Moreover $v_0 \in B_{\sqrt{2}R(1-\xi/2)}$ since $\varphi = 0$ outside the latter ball. The contact point also satisfies the extremality and monotonocity (in time) conditions $\nabla_x f(t_0,x_0,v_0) = 0$ and $\tilde{\ell}'(t_0)\varphi_{R,\xi}(v_0) \geq \partial_t f(t_0,x_0,v_0)$.

Using the fact that f is a supersolution of (2.2), we thus get

$$\tilde{\ell}'(t_0)\varphi_{R,\xi}(v_0) \geq Q_s(f,f)(t_0,x_0,v_0). \tag{3.2}$$

We decompose $Q_s(f,f)$ as

$$Q_s(f,f)(t_0,x_0,v_0) = Q_s\left(f, f - \tilde{\ell}(t)\varphi_{R,\xi}\right)(t_0,x_0,v_0) + \tilde{\ell}(t) Q_s\left(f,\varphi_{R,\xi}\right)(t_0,x_0,v_0),$$

$$= \int_{\mathbb{R}^d} \left[f(t_0,x_0,v') - \tilde{\ell}(t_0)\varphi_{R,\xi}(v') \right] K_f(t_0,x_0,v_0,v')\, \mathrm{d}v' + \tilde{\ell}(t_0) Q_s\left(f,\varphi_{R,\xi}\right)(t_0,x_0,v_0).$$

We omit t_0, x_0 in f and K_f from now on to unclutter equations. The barrier satisfies

$$\tilde{\ell}'(t) = \alpha \xi^q R^{d+\gamma} \ell^2 - CR^\gamma \xi^{-2s} \tilde{\ell}(t),$$

and plugging the last equation into (3.2), and using (3.1), gives

$$\alpha \xi^q R^{d+\gamma} \ell^2 - C\xi^{-2s} R^\gamma \tilde{\ell}(t_0) \geq \int_{\mathbb{R}^d} \left[f(v') - \tilde{\ell}(t_0)\varphi_{R,\xi}(v') \right] K_f(v_0,v')\, \mathrm{d}v' - C\xi^{-2s} R^\gamma \tilde{\ell}(t_0).$$

We cancel out the last term and obtain

$$\alpha \xi^q R^{d+\gamma} \ell^2 \gtrsim$$

$$\int_{\mathbb{R}^d} \int_{v_*' \in v_0 + (v'-v_0)^\perp} \left[f(v') - \tilde{\ell}(t_0)\varphi_{R,\xi}(v') \right] f(v_*') \frac{|v'-v_*'|^{\gamma+2s+1}}{|v_0 - v'|^{d+2s}} \tilde{b}(\cos\theta)\, \mathrm{d}v_*'\, \mathrm{d}v'.$$

Since the integrand is non-negative, we bound the integral from below by restricting the domain of integration to $v' \in B_R$ and $v'_* \in B_R$ (balls centered at zero):

$$\alpha \xi^q R^{d+\gamma} \ell^2 \gtrsim$$
$$\int_{v' \in B_R} \int_{v'_* \in v_0 + (v'-v_0)^\perp} \mathbf{1}_{B_R}(v'_*) \left[f(v') - \tilde{\ell}(t_0) \varphi_{R,\xi}(v') \right] f(v'_*) \frac{|v'-v'_*|^{\gamma+2s+1}}{|v_0 - v'|^{d+2s}} \tilde{b}(\cos\theta) \, \mathrm{d}v'_* \, \mathrm{d}v'.$$

On this domain of integration, we have $f(v'_*) \geq \ell$, and the assumption $\xi^q R^{d+\gamma} \ell < 1/2$ implies that $f(v') - \tilde{\ell}(t_0)\varphi_{R,\xi}(v') \geq \ell - \tilde{\ell} \geq \ell/2$, thus

$$\alpha \xi^q R^{d+\gamma} \ell^2 \gtrsim \ell^2 R^{-d-2s} \int_{v' \in B_R} \int_{v'_* \in v_0 + (v'-v_0)^\perp} \mathbf{1}_{B_R}(v'_*) |v'-v'_*|^{\gamma+2s+1} \tilde{b}(\cos\theta) \, \mathrm{d}v'_* \, \mathrm{d}v'.$$

Observe that since $|v_0| \in [R, \sqrt{2}R(1-\xi/2)]$, the volume of $v' \in B_R$ such that the distance between 0 and the line (vv') is more than $R(1-\xi/2)$, is $O(R^d \xi^{(d+1)/2})$. For v' in this region $\mathcal{C}_R$ (see shaded region in Figure 2) the $(d-1)$-dimensional volume of $v'_* \in B_R$ such that $(v'_* - v_0) \perp (v' - v_0)$ is $O(R^{d-1}\xi^{(d-1)/2})$. Finally removing the $v' \in B_{\xi^2 R}(v_0)$ (which does not change the volume estimates above) ensures that $|v' - v'_*| \geq \xi^2 R$. Therefore,

$$\alpha \xi^q R^{d+\gamma} \ell^2 \gtrsim \ell^2 R^{-d-2s} \int_{v' \in \mathcal{C}_R} \int_{v' \in \mathcal{C}^*_R} |v'-v'_*|^{\gamma+2s+1} \tilde{b}(\cos\theta) \, \mathrm{d}v'_* \, \mathrm{d}v' \gtrsim \xi^q R^{d+\gamma} \ell^2$$

with $q = d + 2(\gamma + 2s + 1)$, and where we have used the deviation angles $\theta \sim \pi/2$ (non-grazing collisions) for which $\tilde{b}$ is positive.

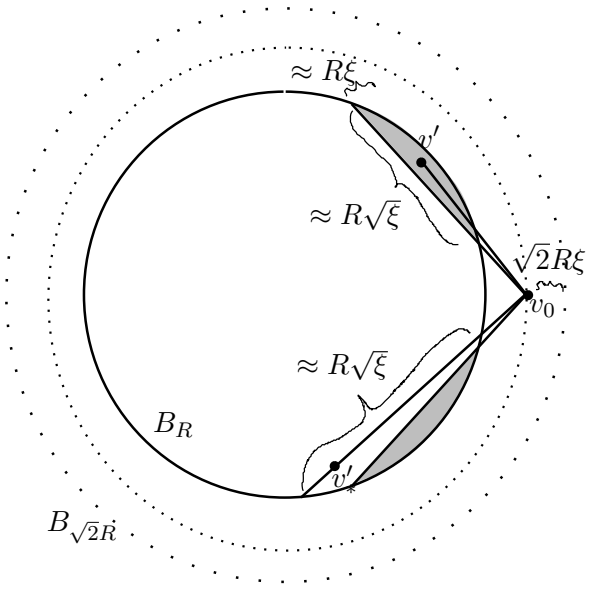

FIGURE 2. The binary collisions spreading the lower bound.

We have hence finally obtained the inequality

$$\alpha \xi^q R^{d+\gamma} \ell^2 \geq \beta \xi^q R^{d+\gamma} \ell^2$$

for some $\beta > 0$ independent of the free parameter $\alpha \in (0,1)$, which is absurd.

The resulting lower bound is

$$\begin{aligned}
&\forall\, t \in [0,T_0],\ x \in \mathbb{T}^d,\ v \in B_{\sqrt{2}(1-\xi)R},\\
&f(t,x,v) \geq \tilde{\ell}(t)\varphi_{R,\xi}(v)\\
&\qquad \geq \alpha \xi^q R^{d+\gamma}\ell^2 \frac{1-e^{-CR^\gamma \xi^{-2s}t/2}}{CR^\gamma \xi^{-2s}} \geq c\xi^q R^{d+\gamma}\ell^2 \min\left(t, R^{-\gamma}\xi^{2s}\right)
\end{aligned}$$

which concludes the proof. □

Remark 3.5. The estimate in Lemma 3.4 is most likely not optimal in terms of the power of ξ, as one could estimate better the factor $|v'-v'_*|^{\gamma+2s+1}|v_0-v'|^{-d-2s}$. However we do not search for optimality here since the power in ξ plays no role in the proof of Proposition 3.6 below.

3.3. **Proof of the Gaussian lower bound.** Theorem 1.2 is a direct consequence of the following proposition.

Proposition 3.6 (Gaussian lower bound)**.** *Consider $T_0 \in (0,1)$. Assume that $f : [0,T_0]\times\mathbb{T}^d\times\mathbb{R}^d \to [0,+\infty)$ is a solution to (1.1) according to Definition 1.1 that satisfies the hydrodynamic bounds (1.3-1.4-1.5) for all $t \in [0,T_0]$ and $x \in \mathbb{T}^d$.*

Then there are $a, b > 0$ depending on d, s, m_0, M_0, E_0, H_0 and T_0 so that

$$\forall\, x \in \mathbb{T}^d, v \in \mathbb{R}^d, \quad f(T_0,x,v) \geq a e^{-b|v|^2}.$$

Proof. Define the following sequences:

$$\begin{cases} T_n := \left(1-\frac{1}{2^n}\right)T_0, \quad n \geq 1,\\ \xi_n = \frac{1}{2^{n+1}}, \quad n \geq 1,\\ R_{n+1} = \sqrt{2}(1-\xi_n)R_n, \quad n \geq 1, \quad R_0 = 1. \end{cases}$$

Observe that $2^{n/2} \lesssim R_n \leq 2^{n/2}$ since $\Pi_{n=1}^{+\infty}(1-2^{-n}) < +\infty$.

Proposition 3.1 implies that $f \geq \ell_0$ for $t \in [T_0/2, T_0] = [T_1, T_0]$, $x \in \mathbb{T}^d$, $v \in B_1 = B_{R_0}$, and for some $\ell_0 > 0$, which initialises our induction.

We then construct inductively a sequence of lower bounds $\ell_n > 0$ so that $f \geq \ell_n$ for $t \in [T_{n+1}, T_0]$, $x \in \mathbb{T}^d$ and $v \in B_{R_n}$. We apply Lemma 3.4 repeatedly to obtain the successive values of ℓ_n. Observe that $\xi_n^q R_n^{d+\gamma}\ell_n < (2^{-n})^{q-(d+\gamma)/2} < 1/2$, so the smallness assumption on ξ of Lemma 3.4 holds through the iteration. The sequence of lower bounds ℓ_n satisfies the induction

$$\begin{aligned}
\ell_{n+1} &= c_s \xi_n^q R_n^{d+\gamma}\ell_n^2 \min\left(T_{n+1}-T_n, R_n^{-\gamma}\xi_n^{2s}\right),\\
&= c_s \xi_n^q R_n^{d+\gamma}\ell_n^2 \min\left(2^{-n-1}T_0, R_n^{-\gamma}\xi_n^{2s}\right) \geq c 2^{-Cn}\ell_n^2 T_0
\end{aligned}$$

for some constants $c, C > 0$, which results in $\ell_n \geq u^{2^n}$ for some $u \in (0,1)$. This implies the Gaussian decay. □

Remark 3.7. Note that the proof of Lemma 3.4 applies just as well in the cut-off case when b is integrable, and covers actually all physical interactions. This is a manifestation of the fact that the collisions used to spread the lower bound are those with

non-grazing angles $\theta \sim \pi/2$. In our notation, the short-range interactions correspond to $s < 0$. The most important such short-range interaction is that of hard spheres in dimension $d = 3$, corresponds to $\gamma = 1$ and $s = -1$. Proposition 3.1 is taken from [24], which applies exclusively to the non-cutoff case. In the proof of Theorem 1.2, we used Proposition 3.1 to establish the lower bound in the initial ball B_1. This initial step would be different in the cut-off case. The estimates in the rest of the iteration carry through and the conclusion does not depend on s being positive.

References

[1] Fredrik Abrahamsson. Strong L^1 convergence to equilibrium without entropy conditions for the Boltzmann equation. *Comm. Partial Differential Equations*, 24(7-8):1501–1535, 1999.

[2] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg. Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.*, 152(4):327–355, 2000.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. The Boltzmann equation without angular cutoff in the whole space: II, Global existence for hard potential. *Anal. Appl. (Singap.)*, 9(2):113–134, 2011.

[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. Global existence and full regularity of the Boltzmann equation without angular cutoff. *Comm. Math. Phys.*, 304(2):513–581, 2011.

[5] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential. *J. Funct. Anal.*, 262(3):915–1010, 2012.

[6] R. Alexandre and C. Villani. On the Boltzmann equation for long-range interactions. *Comm. Pure Appl. Math.*, 55(1):30–70, 2002.

[7] Radjesvarane Alexandre, Yoshinori Morimoto, Seiji Ukai, Chao-Jiang Xu, and Tong Yang. Uniqueness of solutions for the non-cutoff Boltzmann equation with soft potential. *Kinet. Relat. Models*, 4(4):919–934, 2011.

[8] Radjesvarane Alexandre, Yoshinori Morimoto, Seiji Ukai, Chao-Jiang Xu, and Tong Yang. Uniqueness of solutions for the non-cutoff Boltzmann equation with soft potential. *Kinet. Relat. Models*, 4(4):919–934, 2011.

[9] Radjesvarane Alexandre and Cédric Villani. On the Boltzmann equation for long-range interactions. *Communications on pure and applied mathematics*, 55(1):30–70, 2002.

[10] Guy Barles and Cyril Imbert. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(3):567–585, 2008.

[11] Ludwig Boltzmann. Weitere studien uber das wärmegleichgewicht unter gasmolekulen. *Kais. Akad. Wiss. Wien Math. Naturwiss. Classe*, 66:275–370, 1872.

[12] Marc Briant. Instantaneous exponential lower bound for solutions to the Boltzmann equation with Maxwellian diffusion boundary conditions. *Kinet. Relat. Models*, 8(2):281–308, 2015.

[13] Marc Briant. Instantaneous filling of the vacuum for the full Boltzmann equation in convex domains. *Arch. Ration. Mech. Anal.*, 218(2):985–1041, 2015.

[14] Stephen Cameron, Luis Silvestre, and Stanley Snelson. Global a priori estimates for the inhomogeneous Landau equation with moderately soft potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(3):625–642, 2018.

[15] Torsten Carleman. Sur la théorie de l'équation intégrodifférentielle de Boltzmann. *Acta Math.*, 60(1):91–146, 1933.

[16] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.*, 159(2):245–316, 2005.

[17] François Golse, Cyril Imbert, Clément Mouhot, and Alexis F. Vasseur. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(1):253–295, 2019.

[18] Philip T. Gressman and Robert M. Strain. Global classical solutions of the Boltzmann equation without angular cut-off. *J. Amer. Math. Soc.*, 24(3):771–847, 2011.

[19] M. P. Gualdani, S. Mischler, and C. Mouhot. Factorization for non-symmetric operators and exponential h-theorem. *Mémoire de la Société Mathématique de France*, 2018.
[20] Christopher Henderson and Stanley Snelson. C^∞ smoothing for weak solutions of the inhomogeneous Landau equation. *Preprint arXiv:1707.05710, to appear in* Archive for Rational Mechanics and Analysis, 2019.
[21] Cyril Imbert and Clément Mouhot. A toy nonlinear model in kinetic theory. *arXiv e-prints*, page arXiv:1801.07891, Jan 2018.
[22] Cyril Imbert, Clément Mouhot, and Luis Silvestre. Decay estimates for large velocities in the Boltzmann equation without cutoff. *J. Éc. polytech. Math.*, 7:143–184, 2020.
[23] Cyril Imbert and Luis Silvestre. The Schauder estimate for kinetic integral equations. *arXiv e-prints*, page arXiv:1812.11870, Dec 2018.
[24] Cyril Imbert and Luis Silvestre. The weak Harnack inequality for the Boltzmann equation without cut-off. *J. Eur. Math. Soc. (JEMS)*, 22(2):507–592, 2020.
[25] J. C. Maxwell. On the dynamical theory of gases. *J. Philosophical Transactions of the Royal Society of London*, 157:49–88, 1867.
[26] Clément Mouhot. Quantitative lower bounds for the full Boltzmann equation. I. Periodic boundary conditions. *Comm. Partial Differential Equations*, 30(4-6):881–917, 2005.
[27] Clément Mouhot. De Giorgi–Nash–Moser and Hörmander theories: new interplays. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 2467–2493. World Sci. Publ., Hackensack, NJ, 2018.
[28] Ada Pulvirenti and Bernt Wennberg. A Maxwellian lower bound for solutions to the Boltzmann equation. *Comm. Math. Phys.*, 183(1):145–160, 1997.
[29] Luis Silvestre. Regularity estimates for parabolic integro-differential equations and applications. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pages 873–894. Kyung Moon Sa, Seoul, 2014.
[30] Luis Silvestre. A new regularization mechanism for the Boltzmann equation without cut-off. *Comm. Math. Phys.*, 348(1):69–100, 2016.
[31] Luis Silvestre. Upper bounds for parabolic equations and the Landau equation. *J. Differential Equations*, 262(3):3034–3055, 2017.
[32] Wendong Wang and Liqun Zhang. The C^α regularity of weak solutions of ultraparabolic equations. *Discrete Contin. Dyn. Syst.*, 29(3):1261–1275, 2011.

(C. Imbert) CNRS & Department of Mathematics and Applications, École Normale Supérieure (Paris), 45 rue d'Ulm, 75005 Paris, France
E-mail address: Cyril.Imbert@ens.fr

(C. Mouhot) University of Cambridge, DPMMS, Centre for Mathematical Sciences, Wilberforce road, Cambridge CB3 0WA, UK
E-mail address: C.Mouhot@dpmms.cam.ac.uk

(L. Silvestre) Mathematics Department, University of Chicago, Chicago, Illinois 60637, USA
E-mail address: luis@math.uchicago.edu