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Abstract

Lewis' Logic $\mathbb{V}$ is the fundamental logic of counterfactuals. Its proof theory is here investigated by means of two sequent calculi based on the connective of comparative plausibility. First, a labelled calculus is defined on the basis of Lewis' sphere semantics. This calculus has good structural properties and provides a decision procedure for the logic. An internal calculus, recently introduced, is then considered. In this calculus, each sequent in a derivation can be interpreted directly as a formula of $\mathbb{V}$. In spite of the fundamental difference between the two calculi, a mutual correspondence between them can be established in a constructive way. In one direction, it is shown that any derivation of the internal calculus can be translated into a derivation in the labelled calculus. The opposite direction is considerably more difficult, as the labelled calculus comprises rules which cannot be encoded by purely logical rules. However, by restricting to derivations in normal form, derivations in the labelled calculus can be mapped into derivations in the internal calculus. On a general level, these results aim to contribute to the understanding of the relations between labelled and internal proof systems for logics belonging to the realm of modal logic and its extensions, a topic still relatively unexplored.

Keywords: Counterfactual logic, sequent calculus, labelled sequent calculi, translation between calculi.

1 Introduction

In 1973 David Lewis presented, in a short but dense monograph, a family of logics for counterfactual reasoning. Lewis denoted counterfactual implication with the modal connective $A > B$, the meaning of which is "if A had been the case, then B would have been the case". A counterfactual implication $A > B$ is true if B is true in any state of affairs differing minimally from the actual one and in which A is true. Standard possible worlds semantics had to be generalized to capture the counterfactual conditional; to this aim Lewis proposed *sphere semantics*, a new semantics of topological flavour, which

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later inspired a wealth of new endeavours in the field of modal logic and its applications [19].

Lewis' development was essentially axiomatic, and it took one decade before the first (quite complex) Gentzen-style proof system for the logic of counterfactuals was proposed [11]; the calculus is non-standard, as it contains infinitely many rules. The meaning explanation of the counterfactual conditional is not truth-functional in the conventional sense; for this reason, one cannot find natural deduction or sequent calculus rules as one does for the classical connectives. Labelled sequent calculi offer an answer to the problem of developing a proof-theoretic semantics for logics based on possible worlds semantics [17] and have been shown indeed apt to capture any modal logic characterized by first-order conditions on their Kripke frames [1]. Recently, the labelled approach has been extended to deal also with neighbourhood semantics [13,14]. For the logic of counterfactuals, labelled sequent calculi have been proposed both for a generalized relational semantics, based on ternary accessibility relations [16] and for preferential conditional logic based on a broader version of neighbourhood semantics [15].

Parallel to these developments, alternative inference styles, not directly referring to formalized semantics in their rules, have been developed by a number of authors. In such calculi, rather than adding labels for elements of the characteristic semantic structures, one enriches the structure of sequents with new structural connectives (cf. [11,3], and more recently [9,18,7,6]). These calculi are usually referred to as "internal," since their sequents can be directly interpreted in the language of the corresponding logic.

Natural questions arise on the relationships between the two different inference styles of labelled and internal sequent calculi: What is the expressive power of internal calculi in relation to labelled calculi? Can one find an equally general way of generating internal calculi?

A summary of the relationships between various calculi for normal propositional modal logics was presented in [20, pp. 116, 206] together with a conjecture of a general interpretability of tree hypersequents into labelled calculi. This conjecture has been confirmed by in [8] through an embedding of nested sequent calculi and tree-hypersequent calculi into a suitable subclass of labelled calculi, those in which the relational structure forms a tree. A similar result has been proved for nested sequent calculi and labelled tableaux [2].

The question we address in this article is whether we can establish a translation between the labelled and the internal calculi for the logics of counterfactuals. As is often the case in logic, the problem is better tackled from the analysis of a significant case study: thus, we shall focus on logic $\mathbb{V}$, the most basic system among the counterfactual logics presented by Lewis.

Following Lewis, in Section 2 we define the language of $\mathbb{V}$ taking as primitive the comparative plausibility operator $A \preccurlyeq B$, meaning "A is at least as plausible as B", instead of the counterfactual conditional. The two connectives are interdefinable, but the former has a simpler explanation: $A \preccurlyeq B$ is true at x if every sphere of x that meets B also meets A. Since this condition is universal, it is readily translated into left and right labelled rules. Then, we add rules for "meet", the existential forcing relation, and rules for the propositional base as well as rules for spheres, all these latter unchanged with respect to the calculus presented in [15]. The resulting calculus, **G3V**, is a new and non-trivial proof system for $\mathbb{V}$ (Section 3). The calculus has all the typical structural properties of the G3-family of sequent calculi (hence the name) and enjoys a simple completeness proof via a proof-or-countermodel construction. This latter yields the finite model property, and thus an effective decision procedure, for the logic.

In Section 4, the internal sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, introduced in [5], is recalled. Differently from the labelled calculus, each $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequent can be directly interpreted as a formula of the language. The translation from the internal to the labelled calculus (Section 5) can be directly specified by adding to the labelled calculus a few admissible rules. The converse direction (Section 6) is far from immediate. At each step of inference, labelled calculi display many relational formulas that cannot be directly translated into the language of $\mathbb{V}$. This overload has somehow to be disciplined to transform a labelled derivation into an internal one. The core of the procedure lies in the identification of a "normal form" for **G3V** derivations, basically corresponding to the requirement that the relational structure of a derivation forms a tree, and in the Jump lemma (Lemma 6.5), that allows to focus on particular subsets of a labelled sequent, namely, to formulas labelled with the same world label. We shall show that, thanks to these restriction, we are able to translate **G3V** derivations into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations.

Both directions of the translation are defined by means of an inductive procedure. The present paper should give enough support to the claim that, at least the present case study, internal and labelled sequent calculi can be considered as two sides of the same coin.

2 Logic $\mathbb{V}$

The language of $\mathbb{V}$ is defined as: $A := P \mid \bot \mid \neg A \mid A \vee A \mid A \wedge A \mid A \supset A \mid A \preccurlyeq A$ where $\preccurlyeq$ is the comparative plausibility operator. A formula $A \preccurlyeq B$ means "A is at least as plausible as B". The counterfactual conditional operator $>$ can be defined in terms of $\preccurlyeq$: $A > B \;\equiv\; (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$ [2]. Thus, for a counterfactual conditional $A > B$ to be true, it must be that either A is impossible, or that $A \wedge \neg B$ is less plausible than $A \wedge B$.

An axiomatization of $\mathbb{V}$ is given extending classical propositional logic with the following axioms and inference rules:

(CPR) $\dfrac{\vdash B \supset A}{\vdash A \preccurlyeq B}$ (CPA) $(A \preccurlyeq A \vee B) \vee (B \preccurlyeq A \vee B)$

(TR) $(A \preccurlyeq B) \wedge (B \preccurlyeq C) \supset (A \preccurlyeq C)$ (CO) $(A \preccurlyeq B) \vee (B \preccurlyeq A)$

Following [10], we define a semantics for $\mathbb{V}$ in terms of sphere models.

[2] And, vice versa, the comparative plausibility operator can be defined in terms of the counterfactual conditional: $A \preccurlyeq B \;\equiv\; ((A \vee B) > \bot) \vee \neg((A \vee B) > \neg A)$.

Definition 2.1 A *sphere model* is a triple $\mathcal{M} = \langle W, S, [\![\ ]\!] \rangle$ where W is a non-empty set of possible words; S is a function $S : W \to \mathcal{P}(\mathcal{P}(W))$, and $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the propositional evaluation. We assume S to satisfy the properties of *Non-emptiness*: $\forall \alpha \in S(x) \,.\, \alpha \neq \emptyset$ and of *Nesting*: $\forall \alpha, \beta \in S(x) \,.\, \alpha \subseteq \beta \ \vee \ \beta \subseteq \alpha$.

We write $w \Vdash A$ to denote truth of formula A at world w. Truth conditions for Boolean combinations of formulas are the standard ones. As for comparative plausibility, a formula $A \preccurlyeq B$ is true at a world x if for all the spheres $\alpha \in S(x)$, if α contains a world that satisfies B then it must also contain a world that satisfies A. Using the existential forcing relation from [15] $\alpha \Vdash^{\exists} A$ iff $\exists y \in \alpha.\ y \Vdash A$ the truth condition for comparative plausibility [3] can be formally stated as follows: $x \Vdash A \preccurlyeq B$ iff $\forall \alpha \in S(x).\ \alpha \Vdash^{\exists} B \to \alpha \Vdash^{\exists} A$.

A formula A is *valid in a sphere model* $\mathcal{M}$ if for all $w \in W$, $w \Vdash A$. We say that A is *valid* if A is valid in every sphere model.

3 The labelled calculus G3V

The calculus **G3V** (Figure 1) displays two sorts of labels: $x, y, z, \ldots$ for worlds, and $a, b, c, \ldots$ for spheres. All formulas occurring in a sequent are labelled: an expression $x : A$ means that world x forces A and $a \Vdash^{\exists} A$ means that sphere a contains a world that forces A. The propositional rules can be found in [12].

Initial sequents $\quad x : p, \Gamma \Rightarrow \Delta, x : p \qquad x : \bot, \Gamma \Rightarrow \Delta$

Rules for local forcing

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ L\Vdash^{\exists}\ (x \text{ fresh}) \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ R\Vdash^{\exists}$$

Propositional rules: rules of **G3K**

Rules for comparative plausibility

$$\frac{a \Vdash^{\exists} B, a \in S(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\ \mathsf{R}\preccurlyeq\ (a \text{ fresh})$$

$$\frac{a \in S(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} B \qquad a \Vdash^{\exists} A, a \in S(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}{a \in S(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\preccurlyeq$$

Rules for inclusion and nesting

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ \mathsf{L}\subseteq$$

$$\frac{a \subseteq b, a \in S(x), b \in S(x), \Gamma \Rightarrow \Delta \qquad b \subseteq a, a \in S(x), b \in S(x), \Gamma \Rightarrow \Delta}{a \in S(x), b \in S(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}$$

Fig. 1. Rules of **G3V**

[3] Truth condition for the (defined) conditional operator is the following: $x \Vdash A > B$ iff $\forall \alpha \in S(x).\ \alpha \nVdash^{\exists} A$ *or* $\exists \beta \in S(x).\ \beta \Vdash^{\exists} A\ \&\ \beta \Vdash^{\forall} A \supset B$, for $\alpha \Vdash^{\forall} A$ iff $\forall y \in \alpha.\ y \Vdash A$.

Theorem 3.1 (Soundness) *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* **G3V***, then it is valid in all sphere models.*

Proof. As in [15], we need to define the notion of *realization*, which interprets labelled sequents in sphere semantics; soundness is then proved by straightforward induction on the height of the derivation. Given a sphere model $\mathcal{M} = \langle W, I, [\![\ ]\!] \rangle$, a set S of world labels, and a set N of sphere labels, an SN-*realization* over $\mathcal{M}$ is a pair of functions (ρ, σ) such that $\rho : S \to W$ is a function that assigns to each world label $x \in S$ an element $\rho(x) \in W$, and $\sigma : N \to \mathcal{P}(W)$ is a function that assigns to each sphere label $a \in N$ a sphere $\sigma(a) \in S(w)$, for $w \in W$. *Satisfiability* of a formula $\mathcal{F}$ under an SN-realization is defined by cases as follows: $\mathcal{M} \vDash_{\rho,\sigma} a \in S(x)$ if $\sigma(a) \in S(\rho(x))$; $\mathcal{M} \vDash_{\rho,\sigma} a \subseteq b$ if $\sigma(a) \subseteq \sigma(b)$; $\mathcal{M} \vDash_{\rho,\sigma} x \in a$ if $\rho(x) \in \sigma(a)$; $\mathcal{M} \vDash_{\rho,\sigma} x : P$ if $\rho(x) \in [\![P]\!]$, for P atomic[4]; $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ if $\sigma(a) \Vdash^{\exists} A$; $\mathcal{M} \vDash_{\rho,\sigma} x : A \preccurlyeq B$ if for all $\alpha \in S(\rho(x))$, if $\alpha \Vdash^{\exists} B$ then $\alpha \Vdash^{\exists} A$. A sequent $\Gamma \Rightarrow \Delta$ is *valid in* $\mathcal{M}$ *under the* (ρ, σ) *realization* iff $\mathcal{M} \nvDash_{\rho,\sigma} F$ for all $F \in \Gamma$ or $\mathcal{M} \vDash_{\rho,\sigma} G$ for some $G \in \Delta$. A sequent is *valid in* $\mathcal{M}$ if it is valid under any (ρ, σ) realization. □

The calculus enjoys admissibility of all the structural rules. For admissibility of weakening, invertibility of all the rules and admissibility of contraction, the proof is an extension of the proofs in [15]. In order to prove the admissibility of cut, we need a substitution lemma, spelled out as in [15] with the addition of the clause for comparative plausibility, namely $x : A \preccurlyeq B[x/y] \equiv y : A \preccurlyeq B$. Then, we need a suitable definition of weight of a labelled formula.

Definition 3.2 Given a labelled formula $\mathcal{F}$, we define the pure part $p(\mathcal{F})$ and the label part $l(\mathcal{F})$ of $\mathcal{F}$ as follows: $p(x : A) = p(a \Vdash^{\exists} A) = A$; $l(x : A) = x$; $l(a \Vdash^{\exists} A) = a$. The *weight* of a labelled formula is defined as an ordered pair $\langle w(p(\mathcal{F}), w(l(\mathcal{F})) \rangle$ where

- for all world labels. x, $w(x) = 0$; for all sphere labels a, $w(a) = 1$;
- $w(P) = w(\bot) = 1$; $w(A \circ B) = w(A) + w(B) + 1$ for $\circ$ conjunction, disjunction or implication; $w(A \preccurlyeq B) = w(A) + w(B) + 2$.

Theorem 3.3 *The rule of cut is admissible in* **G3V**.

Proof. 3.3 Primary induction on the weight of formulas and secondary induction on the sum of heights of derivations of the premisses. The proof is by cases, according to the last rules applied in the derivations of the premisses of Cut. We only show the case in which both occurrences of the cut formula are principal and derived by R $\preccurlyeq$ and L $\preccurlyeq$ respectively.

$$\dfrac{\dfrac{\overset{(1)}{b \in S(x), b \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, b \Vdash^{\exists} A}}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\,\mathsf{R}\preccurlyeq \qquad \dfrac{(2) \qquad (3)}{a \in S(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta'}\,\mathsf{L}\preccurlyeq}{\Gamma, a \in S(x), \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

[4] This definition can be extended in the standard way to the propositional formulas of the language.

where (2) $= a \in S(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} B$ and (3) $= a \Vdash^{\exists} A, a \in S(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta'$ are the left and right premisses of $\mathsf{L}\preccurlyeq$. In (1), substitute variable b with variable a. The derivation is transformed as follows:

$$\dfrac{\overset{(2)}{a \in S(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} B} \quad \dfrac{(1)[a/b] \qquad (3)}{\Gamma, a \Vdash^{\exists} B, a \in S(x), \Gamma' \Rightarrow \Delta, \Delta'}\ Cut}{\dfrac{\Gamma, a \in S(x), a \in S(x), \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{\Gamma, a \in S(x), \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{Ctr}}\ Cut$$

where the upper occurrence of cut has smaller sum of heights than the original cut, and the lower occurrence of cut is applied to formulas of smaller weight than the cut formula. □

The following rule is needed to define the translation in Section 5. The proof is by easy induction on the height of the derivation.

Lemma 3.4 *Rule* $\mathsf{Mon}\exists$ *is admissible.*

$$\dfrac{b \subseteq a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A, b \Vdash^{\exists} A}{b \subseteq a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{Mon}\exists$$

Remark 3.5 Notational convention: given multisets of formulas $\Gamma = \{A_1, \ldots, A_m\}$ and $\Sigma = \{S_1, \ldots, S_k\}$, we shall write $x : \Gamma$ and $a \Vdash^{\exists} \Sigma$ as abbreviations for $x : A_1, \ldots, x : A_m$ and $a \Vdash^{\exists} S_1, \ldots, a \Vdash^{\exists} S_k$ respectively.

Theorem 3.6 (Syntactic completeness) *If a formula A is valid in $\mathbb{V}$, sequent $\Rightarrow x : A$ is derivable in* **G3V**.

Proof. Using completeness for the axiomatic system, it suffices to show that the axioms and inference rules of $\mathbb{V}$ are derivable in **G3V**. By means of example, derivation of axiom (CO) in which we have omitted the right premiss of Nes.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\triangledown}{a \subseteq b, a \in S(x), b \in S(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, y : B, b \Vdash^{\exists} B}}{a \subseteq b, a \in S(x), b \in S(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{R}\Vdash^{\exists}}{a \subseteq b, a \in S(x), b \in S(x), y \in a, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{L}\subseteq}{a \subseteq b, a \in S(x), b \in S(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{L}\Vdash^{\exists}}{a \in S(x), b \in S(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{Nes}}{a \in S(x), a \Vdash^{\exists} B \Rightarrow a \Vdash^{\exists} A, x : B \preccurlyeq A}\ \mathsf{R}\preccurlyeq}{\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A}\ \mathsf{R}\preccurlyeq}{\Rightarrow x : A \preccurlyeq B \vee B \preccurlyeq A}\ \mathsf{R}\vee$$

□

Bottom-up proof search in **G3V** comes to an end with the adoption the following strategy: In any derivation branch (i) no rule can be applied to an initial sequent and (ii) no redundant application of any rule is allowed, where the standard notion of *redundant application* of a rule is defined as follows. Let $\mathcal{B} = S_0, S_1, \ldots$ be a derivation branch with S_i sequent $\Gamma_i \Rightarrow \Delta_i$ for $i = 1, 2, \ldots$. An application of a **G3V** rule (R) to S_i is *redundant* if whenever the branch up to S_i contains the conclusion of that application of (R) then it also contains at least one premiss of that application of (R). We say that a branch

$\mathcal{B} = S_0, S_1, \ldots$ to be *saturated* if every **G3V** rule is redundant, and $\mathcal{B}$ to be *open* if it does not contain an initial sequent.

To check the validity of a formula A, we try to build a derivation with root $\Rightarrow x : A$ according to the strategy. We show that the process of building such a derivation always comes to an end in a finite number of steps. We actually prove the result for a more general case: the one in which the sequent at the root of the derivation is *simple*. This notion will be useful in the following sections.

Definition 3.7 A sequent $\Gamma \Rightarrow \Delta$ is *simple* if (i) it contains only one world label x, (ii) if Γ is non-empty, then x occurs in Γ, (iii) for all neighbourhood labels a occurring in $\Gamma \Rightarrow \Delta$, $a \in S(x)$ occurs in Γ and (iv) $x \in a$ does not occur in $\Gamma \Rightarrow \Delta$.

Definition 3.8 Given a branch $\mathcal{B} = S_0, S_1, \ldots$ where $S_i = \Gamma_i \Rightarrow \Delta_i$ for $i = 1, 2, \ldots$ and let $\Pi_{\mathcal{B}} = \bigcup_i \Gamma_i$. We define the following relations:

- $x \rightarrow_{\Pi_{\mathcal{B}}} a$ if $a \in S(x)$ occurs in $\Pi_{\mathcal{B}}$;
- $a \rightarrow_{\Pi_{\mathcal{B}}} y$ if for some $S_i = \Gamma_i \Rightarrow \Delta_i$, $y \in a$ occurs in Γ_i and y does not occur in any S_j with $j < i$;
- $x \rightarrow_{\Pi_{\mathcal{B}}} y$ if there exists an a such that $x \rightarrow_{\Pi_{\mathcal{B}}} a$ and $a \rightarrow_{\mathcal{B}} y$;
- $x \rightarrow^*_{\Pi_{\mathcal{B}}} y$ is the transitive closure of $x \rightarrow_{\Pi_{\mathcal{B}}} y$.

Lemma 3.9 *Let $\mathcal{B} = S_0, S_1, \ldots$ be any branch of a derivation of a simple sequent $\Gamma \Rightarrow \Delta$ where x_0 is the only world label appearing in S_0. Then: (a) for every label x occurring in any sequent of $\mathcal{B}$, it holds that $x_0 \rightarrow^*_{\Pi_{\mathcal{B}}} x$; (b) the relation $\rightarrow^*_{\Pi_{\mathcal{B}}}$ forms a tree $\mathcal{T}_{x_0}$ with root x_0; (c) if $x \rightarrow^*_{\Pi_{\mathcal{B}}} y$ then $m(y) < m(x)$, where for a world label u, $m(u)$ is the maximal modal degree of all formulas C such that $u : C$ occurs in any sequent of $\mathcal{B}$; (d) If $\mathcal{B}$ is built according to the strategy, then for every x the set $\{x \mid x \rightarrow_{\Pi_{\mathcal{B}}} a\}$ is finite, and for every a the set $\{y \mid a \rightarrow_{\mathcal{B}} y\}$ is finite, whence the tree $\mathcal{T}_{x_0}$ is finite[5].*

The proof of this lemma relies on the fact that world and sphere labels are introduced analysing only *once* (by the irredundancy restriction) the subformulas of the sequent at the root.

Theorem 3.10 (Termination) *Let $\Gamma \Rightarrow \Delta$ be a simple sequent. Proof search for $\Gamma \Rightarrow \Delta$ always terminates.*

Proof. We prove that any derivation of $\Gamma \Rightarrow \Delta$ built according to the strategy is finite. Let $\mathcal{B} = S_0, S_1, S_k, S_{k+1} \ldots$ be any branch with $S_0 = \Gamma \Rightarrow \Delta$; by Lemma 3.9 the set of world labels and the set of sphere labels occurring in $\mathcal{B}$ are both finite. Since all pure formulas occurring in $\mathcal{B}$ are subformulas of the root sequent $\Gamma \Rightarrow \Delta$, also the set of labelled formulas that may occur in the *whole* $\mathcal{B}$ is finite. Whence by the strategy (no redundant application of the rules), any sequent S_i is finite and the branch $\mathcal{B} = S_0, S_1, S_k, S_{k+1} \ldots$ must

[5] As a matter of fact it can be proved that the sets $\{x \mid x \rightarrow_{\Pi_{\mathcal{B}}} a\}$, $\{y \mid a \rightarrow_{\mathcal{B}} y\}$ and the tree $\mathcal{T}_{x_0}$ are not only finite, but bounded in size by some function of the size of the sequent $\Gamma \Rightarrow \Delta$ at the root.

come to an end after a finite number steps, that is it must be $\mathcal{B} = S_0, \ldots, S_k$, where either S_k is an initial sequent or the whole $\mathcal{B}$ is saturated. We have shown that every derivation of $\Gamma \Rightarrow \Delta$ is finite. $\square$

Termination yields a decision procedure: to check provability of formula A, build a proof search tree $\mathcal{D}$ with root $\Rightarrow x : A$. By the previous theorem, $\mathcal{D}$ is finite: either every branch of $\mathcal{D}$ terminates with an initial sequent, and $\mathcal{D}$ is a derivation of A, or $\mathcal{D}$ contains an open saturated branch. In the former case A is provable; in the latter case it is not, and it is possible to extract a countermodel of A from the open branch. Thus we can give an alternative (semantic) completeness proof for **G3V**. Observe that the following theorem combined with soundness of **G3V** provides a constructive proof of the *finite model property* of $\mathbb{V}$.

Theorem 3.11 (Semantic completeness) *If a formula A is not derivable in* **G3V***, there is a* finite *sphere model $\mathcal{M}$ such that A is not valid in $\mathcal{M}$.*

Proof. By Theorem 3.10 any derivation $\mathcal{D}$ with root S_0 sequent $\Rightarrow x_0 : A$ contains *a finite open branch* $S_0, \ldots, S_k$, with S_i sequent $\Gamma_i \Rightarrow \Delta_i$. Define a model $\mathcal{M} = (W, S, [\![\ ]\!])$, where $W = \{x \mid x \text{ occurs in } \bigcup_i^k \Gamma_i\}$. Given a sphere label a, we define a sphere $\alpha_a = \{y \in W \mid y \in a \text{ occurs in } \bigcup_i^k \Gamma_i\}$, and $S(x) = \{\alpha_a \mid a \in S(x) \text{ occurs in } \bigcup_i^k \Gamma_i\}$. For any atom P, $[\![P]\!] = \{x \in W \mid x : P \text{ occurs in } \bigcup_i^k \Gamma_i\}$. We consider the SN-realisation (ρ, σ) where $\rho(x) = x$ and $\sigma(a) = \alpha_a$. For all $i = 0, \ldots, k$ and for any formula $\mathcal{F}$ we show that if $\mathcal{F}$ occurs in Γ_i then $\mathcal{M} \vDash_{\rho,\sigma} \mathcal{F}$ and if $\mathcal{F}$ occurs in Δ_i, then $\mathcal{M} \nvDash_{\rho,\sigma} \mathcal{F}$. In particular for labelled formulas $\mathcal{F}$ of the form $x : B$ and $a \Vdash^{\exists} B$ we proceed by induction of on the weight $w(\mathcal{F})$. Since $x_0 : A$ occurs in Δ_0 A is not valid in $\mathcal{M}$. $\square$

4 Internal sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

The sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ for $\mathbb{V}$ was proposed in [5]; we recall here the basic notions. Sequents of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ are composed of formulas and *blocks*, where for formulas $A_1, ..., A_n, B$, a block is a syntactic structure $[A_1, .., A_n \lhd B]$, representing the disjunction $(A_1 \preccurlyeq B) \vee \cdots \vee (A_n \preccurlyeq B)$.

Definition 4.1 A *block* is an expression of the form $[\Sigma \lhd B]$, where Σ is a multiset of formulas and B is a formula. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ is a multiset of formulas and Δ is a multiset of formulas and blocks.
The *formula interpretation* of a sequent is given by:

$$\iota(\Gamma \Rightarrow \Delta', [\Sigma_1 \lhd B_1], \ldots, [\Sigma_n \lhd B_n]) := \bigwedge \Gamma \rightarrow \bigvee \Delta' \vee \bigvee_{1 \leq i \leq n} \bigvee_{A \in \Sigma_i} (A \preccurlyeq B_i)$$

We write $[\Theta, \Sigma \lhd B]$ for $[(\Theta, \Sigma) \lhd B]$, with Θ, Σ denoting multiset union.

Rules of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ are shown in Figure 2. Proofs of admissibility of weakening and contraction, invertibility of all rules and admissibility of cut can be found in [5], as well as proof of admissibility of the following rule (needed in Section 6).

Initial sequents $p, \Gamma \Rightarrow \Delta, p \qquad \Gamma, \bot \Rightarrow \Delta$

Propositional rules (standard)

Rules for comparative plausibility

$$\frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B} \; \mathsf{R}\preccurlyeq^{\mathsf{i}}$$

$$\frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]} \; \mathsf{L}\preccurlyeq^{\mathsf{i}}$$

Rules for blocks

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]} \; \mathsf{Com}^{\mathsf{i}}$$

$$\frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \; \mathsf{Jump}$$

Fig. 2. Rules of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

Lemma 4.2 *Weakening inside blocks is admissible in* $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$.

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd C]}{\Gamma \Rightarrow \Delta, [A, \Sigma \lhd C]} \; \mathsf{Wk_B}$$

Example 4.3 Derivation in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ of axiom (CO):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\triangledown}{B \Rightarrow A, B}}{\Rightarrow [A, B \lhd B], [B \lhd A]} \; \mathsf{Jump} \qquad \dfrac{\overset{\triangledown}{A \Rightarrow A, B}}{\Rightarrow [A \lhd B], [A, B \lhd A]} \; \mathsf{Jump}}{\Rightarrow [A \lhd B], [B \lhd A]} \; \mathsf{Com}^{\mathsf{i}}}{\Rightarrow [A \lhd B], B \preccurlyeq A} \; \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A} \; \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow (A \preccurlyeq B) \vee (B \preccurlyeq A)} \; \vee_R$$

5 From the internal to the labelled calculus

In this section and in the next one we shall present a mutual translation between calculi $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ and **G3V**. Since we aim at translating derivations, we introduce a notation to represent derivations, applicable to both calculi.

Definition 5.1 Let INIT be a **G3V** / $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ initial sequent, and let SEQ denote a **G3V** / $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequent $\Gamma \Rightarrow \Delta$. Let R be a **G3V** / $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ rule. A derivation is the following object, where (1) and (2) are sequents:

$$\mathcal{D} \; : \; \overset{\triangledown}{\text{INIT}} \quad ; \quad \dfrac{\overset{\mathcal{D}_1}{(1)}}{\text{SEQ}} \; R \quad ; \quad \dfrac{\overset{\mathcal{D}_1}{(1)} \quad \overset{\mathcal{D}_2}{(2)}}{\text{SEQ}} \; R$$

In this section, we show how $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations can be translated into derivations in **G3V**. We first define t, translation for sequents; then, we specify a function taking as argument $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations and producing in output derivations in **G3V**

+ Wk + Ctr + Mon∃, and prove that the the translation specified by the function is correct.

Definition 5.2 Given a world label x, a list of countably many sphere labels $\bar{a} = a_1\, a_2\, \ldots a_n$ and multisets of formulas $\Gamma, \Delta, \Sigma_1, \ldots, \Sigma_n$, define:

- $t(\Gamma)^x \;:=\; x : F_1, ..., x : F_k$
- $t(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd B_1], ..., [\Sigma_n \lhd B_n])^{x,\bar{a}} := a_1 \in S(x), ..., a_n \in S(x),\ a_1 \Vdash^{\exists} B_1, ..., a_n \Vdash^{\exists} B_n, t(\Gamma)^x \Rightarrow t(\Delta)^x, a_1 \Vdash^{\exists} \Sigma_1, ..., a_n \Vdash^{\exists} \Sigma_n$

The translation takes as parameter one world label, x, and sphere labels $\bar{a}$: the idea is that for each block $[\Sigma_i \lhd B_i]$ we introduce a new sphere label a_i such that $a_i \in S(x)$, and formulas $a_i \Vdash^{\exists} B_i$ in the antecedent and $a_i \Vdash^{\exists} \Sigma_i$ in the consequent. These formulas correspond to the semantic condition for a block i.e., a disjunction of $\preccurlyeq$ formulas in sphere models.

We now describe function $\{\ \}^{x,\bar{a}}$ that takes as input a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation $\mathcal{D}$ and produces as output a **G3V** derivation $\{\mathcal{D}\}^{x,\bar{a}}$. The parameters of the function are the labels x and $\bar{a}$; these are the world and sphere labels used to translate the root sequent of $\mathcal{D}$. For $\bar{a} = (a_1 \ldots a_n)$, we write $\bar{a}b$ to denote the list $(a_1 \ldots a_n\ b)$. The function for propositional rules is immediate: from a translation of the premiss(es) derive a translation of the conclusion applying the corresponding **G3V** rule. For R rule of a calculus, we denote by $R(n)$ n applications of R.

(init) $$\left\{\overset{\triangledown}{\mathrm{INIT}}\right\}^{x,\bar{a}} \rightsquigarrow \overset{\triangledown}{t(\mathrm{INIT})^{x,\bar{a}}}$$

(R $\preccurlyeq^{\mathsf{i}}$) $$\left\{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \Delta, [A \lhd B]\end{array}}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}\right\}^{x,\bar{a}} \rightsquigarrow \dfrac{\begin{array}{c}\{\mathcal{D}_1\}^{x,\bar{a}\,b}\\ t(\Gamma \Rightarrow \Delta, [A \lhd B])^{x,\bar{a}\,b}\end{array}}{t(\Gamma \Rightarrow \Delta, A \preccurlyeq B)^{x,\bar{a}}}\ \mathsf{R}\preccurlyeq$$

(L $\preccurlyeq^{\mathsf{i}}$) $$\left\{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A \preccurlyeq B, \Gamma \Rightarrow \Delta\, [\Sigma, B \lhd C]\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]\end{array}}{A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{L}\preccurlyeq^{\mathsf{i}}\right\}^{x,\bar{a}\,b} \rightsquigarrow$$

$$\dfrac{\begin{array}{c}\{\mathcal{D}_1\}^{x,\bar{a}\,b}\\ t(A \preccurlyeq B, \Gamma \Rightarrow \Delta\, [\Sigma, B \lhd C])^{x,\bar{a}\,b}\end{array} \quad \dfrac{\begin{array}{c}\{\mathcal{D}_2\}^{x,\bar{a}\,b\,c\,[c/b]}\\ t(A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C])^{x,\bar{a}\,b\,c\,[c/b]}\end{array}}{b \in S(x), b \Vdash^{\exists} A, b \Vdash^{\exists} C, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma}\ \mathsf{Ctr}}{t(A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A])^{x,\bar{a}\,b}}\ \mathsf{L}\preccurlyeq$$

(Com$^{\mathsf{i}}$) $$\left\{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B]\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma, \Pi \lhd B]\end{array}}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1 \lhd B]}\ \mathsf{Com}^{\mathsf{i}}\right\}^{x,\bar{a}\,b\,c} \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\{\mathcal{D}_1\}^{x,\bar{a}\,b\,c}\\ t(\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B])^{x,\bar{a}\,b\,c}\end{array}}{b \subseteq c, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2, c \Vdash^{\exists} \Sigma_2}\ \mathsf{Wk}}{b \subseteq c, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2}\ \mathsf{Mon}\exists \quad \dfrac{\dfrac{\begin{array}{c}\{\mathcal{D}_2\}^{x,\bar{a}\,b\,c}\\ t(2)^{x,\bar{a}\,b\,c}\end{array}}{(2')}\ \mathsf{Wk}}{(2'')}\ \mathsf{Mon}\exists}{t(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1 \lhd B])^{x,\bar{a}\,b\,c}}\ \mathsf{Nes}$$

(2) $\Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma, \Pi \lhd B]$;
(2′) $b \subseteq c, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2$;
(2″) $c \subseteq b, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2$

(Jump)

$$\left\{\begin{array}{c}\mathcal{D}_1\\ \dfrac{x:\Sigma\Rightarrow x:A}{\Gamma\Rightarrow\Delta,[\Sigma\lhd A]}\,\mathsf{Jump}\end{array}\right\}^{x,\bar{a}\,b} \rightsquigarrow \dfrac{\dfrac{\dfrac{\begin{array}{c}t\{\mathcal{D}_1\}^{x\,[x/y]}\\ t(x:\Sigma\Rightarrow x:A)^{x\,[x/y]}\end{array}}{y\in b, b\in S(x), y:A, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, y:\Sigma, b\Vdash^{\exists}\Sigma}\,Wk}{y\in b, b\in S(x), y:A, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma}\,\mathsf{R}\Vdash^{\exists}(n)}{t(\Gamma\Rightarrow\Delta,[\Sigma\lhd A])^{x,\bar{a}\,b}}\,\mathsf{L}\Vdash^{\exists}$$

Theorem 5.3 *Let* $\mathcal{D}$ *be a* $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ *derivation of* $\Gamma\Rightarrow\Delta$. *Then* $\{\mathcal{D}\}^{x,\bar{a}}$ *is a derivation of* $t(\Gamma\Rightarrow\Delta)^{x,\bar{a}}$ *in* **G3V**.

Proof. By induction on the height h of the derivation of the sequent. If $h=0$, $\Gamma\Rightarrow\Delta$ is a $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ initial sequent, and $t(\Gamma\Rightarrow\Delta)^{x,\bar{a}}$ is a **G3V** initial sequent. If $h>0$, $\Gamma\Rightarrow\Delta$ must have been derived applying a rule of $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$. All cases easily follow applying the clauses of the procedure described above.

[$\mathsf{R}\preccurlyeq^{\mathsf{i}}$] The translation of the premiss of $\mathsf{R}\preccurlyeq^{\mathsf{i}}$ is the **G3V** sequent $t(\Gamma\Rightarrow\Delta, A\preccurlyeq B)^{x,\bar{a}} = b\in S(x), b\Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists} A$. Applying $\mathsf{R}\preccurlyeq$ we obtain the translation of the conclusion: $t(\Gamma\Rightarrow\Delta, A\preccurlyeq B)^{x,\bar{a}} = t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, x:A\preccurlyeq B$.

[$\mathsf{L}\preccurlyeq^{\mathsf{i}}$] The translations of the premisses are the **G3V** sequents: $t(A\preccurlyeq B,\Gamma\Rightarrow\Delta,[\Sigma,B\lhd C])^{x,\bar{a}\,b} = b\in S(x), b\Vdash^{\exists} C, x:A\preccurlyeq B, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma, b\Vdash^{\exists} B$; $t(A\preccurlyeq B,\Gamma\Rightarrow\Delta,[\Sigma\lhd A],[\Sigma\lhd C])^{x,\bar{a}\,b\,c} = b\in S(x), c\in S(x), b\Vdash^{\exists} A, c\Vdash^{\exists} C, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma, c\Vdash^{\exists}\Sigma$. We substitute the sphere label c with b in the second sequent, obtaining $b\in S(x), b\in S(x), b\Vdash^{\exists} A, c\Vdash^{\exists} C, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma, b\Vdash^{\exists}\Sigma$. After application of contraction, application of $\mathsf{L}\preccurlyeq$ to this sequent and to the translation of the first premiss yields sequent $b\in S(x), b\Vdash^{\exists} C, x:A\preccurlyeq B, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma$. This sequent is the translation of the $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ sequent $A\preccurlyeq B,\Gamma\Rightarrow\Delta,[\Sigma\lhd C]$, with parameters $x,\bar{a}\,b$.

[$\mathsf{Com}^{\mathsf{i}}$] The translations of the premisses are the **G3V** sequents: $t(\Gamma\Rightarrow\Delta,[\Sigma_1,\Sigma_2\lhd A],[\Sigma_2\lhd B])^{x,\bar{a}\,b\,c} = b\in S(x), c\in S(x), b\Vdash^{\exists} A, c\Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma_1, b\Vdash^{\exists}\Sigma_2, c\Vdash^{\exists}\Sigma_2$; $t(\Gamma\Rightarrow\Delta,[\Sigma_1\lhd A],[\Sigma_1,\Sigma_2\lhd B])^{x,\bar{a}\,b\,c} = b\in S(x), c\in S(x), b\Vdash^{\exists} A, c\Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma_1, c\Vdash^{\exists}\Sigma_2, c\Vdash^{\exists}\Sigma_2$. We add by weakening $b\subseteq c$ to the translation of the first premiss and $c\subseteq b$ to the translation of the second, and apply rule $\mathsf{Mon}\exists$ to both. A final application of Nes yields the desired sequent: $t(\Gamma\Rightarrow\Delta,[\Sigma_1\lhd A],[\Sigma_2\lhd b])^{x,\bar{a}\,b\,c} = b\in S(x), c\in S(x), b\Vdash^{\exists} A, c\Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma_1, c\Vdash^{\exists}\Sigma_2$.

[Jump] The translation of the premiss of Jump is the sequent $t(x:\Sigma\Rightarrow x:A)^{x} = x:\Sigma\Rightarrow x:A$. We substitute x with a fresh world label y, and apply the transformations described above; we obtain the sequent $b\in S(x), b\Vdash^{\exists} A, t(\Gamma)^{x,\bar{a}}\Rightarrow t(\Delta)^{x,\bar{a}}, b\Vdash^{\exists}\Sigma$, which is the translation of the sequent $\Gamma\Rightarrow\Delta,[\Sigma\lhd A]$, the conclusion of Jump, with parameters $x,\bar{a}\,b$.

□

Example 5.4 This **G3V** derivation is obtained translating the $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ derivation of Example 4.3. In the application of Nes only the left premiss is shown.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\nabla}{y : B \Rightarrow y : A, y : B}}{a \in S(x), b \in S(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B, y : A, y : B}\,\mathsf{Wk}}{a \in S(x), b \in S(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B}\,\mathsf{R}\Vdash^{\exists}(2)}{a \in S(x), b \in S(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B}\,\mathsf{L}\Vdash^{\exists}}{a \subseteq b, a \in S(x), b \in S(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B}\,\mathsf{Wk}}{a \subseteq b, a \in S(x), b \in S(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\,\mathsf{Mon}\exists}{a \in S(x), b \in S(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\,\mathsf{Nes}}{a \in S(x), a \Vdash^{\exists} B \Rightarrow x : B \preccurlyeq A, a \Vdash^{\exists} A}\,\mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A}\,\mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow x : A \preccurlyeq B \vee B \preccurlyeq A}\,R\vee$$

6 From the labelled to the internal calculus

The inverse translation takes care of translating **G3V** derivations into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations. With respect to the previous translation, we are faced with an additional difficulty: there are **G3V** derivable sequents that cannot be translated into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequents or, equivalently, there are *more* **G3V** derivable sequents than $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivable sequents. For this reason, proving that if $t(\Gamma \Rightarrow \Delta)^x$ is derivable in **G3V** then $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ would not work: in the **G3V** derivation of $t(\Gamma \Rightarrow \Delta)^x$ there could occur some sequents that are not in the range of the translation t.

Thus, we need a more complex proof strategy. After defining a translation s for sequents, we shall introduce the notion of normal form derivations in **G3V**: the idea is that we cannot translate *any* derivation, but only those constructed following a certain order of application of the rules. We will prove that any derivation in **G3V** can be transformed into a normal form derivation. Then, we shall prove the fundamental Jump lemma, which allows us to “skip” the sequents that we cannot translate in the translation of a derivation. Finally, we shall define a function to translate derivations from **G3V** to $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$.

Definition 6.1 Let $\Gamma \Rightarrow \Delta$ be a sequent of the form

$$\mathcal{R}^{a_1}, \ldots, \mathcal{R}^{a_n}, a_1 \in S(x), ..., a_n \in S(x), a_1 \Vdash^{\exists} A_1, ..., a_n \Vdash^{\exists} A_n, x : \Gamma^P \\ \Rightarrow x : \Delta^P, a_1 \Vdash^{\exists} \Sigma_1, ..., a_n \Vdash^{\exists} \Sigma_n$$

where: a) each $\mathcal{R}^{a_i}$ contains zero or more inclusions $a_i \subseteq a_j$ for $1 \leqslant i \leqslant j \leqslant n$; b) Γ^P and Δ^P are composed only of propositional and $\preccurlyeq$ formulas; c) for each a_i, there is exactly one formula $a_i \Vdash^{\exists} A_i$ in the antecedent, and at least one formula $a_i \Vdash^{\exists} B_i$ in the consequent. The translation s takes as parameter a world label x, label of Γ^P and Δ^P, and is defined as

$$s(\Gamma \Rightarrow \Delta)^x := \Gamma^P \Rightarrow \Delta^P, \Pi$$

where: Γ^P is obtained from $x : \Gamma^P$ by removing the label x, Δ^P is obtained from $x : \Delta^P$ by removing the label x, and Π contains n blocks $[\{\Sigma_1\} \lhd A_1], ..., [\{\Sigma_n\} \lhd A_n]$, where each $\{\Sigma_i\}$ is the multiset union $\{\Sigma_i\} = \Sigma_i \cup \bigcup \{\Sigma_j \mid a_i \subseteq a_j \textit{ occurs in } \mathcal{R}^{a_i}\}$.

For instance, consider the translation s of the following sequent:

$$s(a_1 \subseteq a_2, a_2 \subseteq a_3, a_1 \subseteq a_3, a_1 \in S(x), a_2 \in S(x), a_3 \in S(x), a_1 \Vdash^{\exists} A_1, a_2 \Vdash^{\exists} A_2,\ a_3 \Vdash^{\exists} A_3, x : \Gamma \Rightarrow x : \Delta, a_1 \Vdash^{\exists} \Sigma_1, a_2 \Vdash^{\exists} \Sigma_2, a_3 \Vdash^{\exists} \Sigma_3)^x := \\ := s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma_1, \Sigma_2, \Sigma_3 \lhd A_1], [\Sigma_2, \Sigma_3 \lhd A_2], [\Sigma_3 \lhd A_3]$$

Intuitively, s re-assembles the blocks from formulas labelled with the same sphere label. Furthermore, for each inclusion $a_i \subseteq a_j$ we add to the corresponding block also formulas Σ_j such that $a_j \Vdash^{\exists} \Sigma_j$ occurs in the consequent of the labelled sequent. Thus, each block in the internal calculus consists of $\preccurlyeq$-formulas relative to some sphere i.e., labelled with the same sphere label in **G3V**.

We now introduce the notion of normal form derivations and state the Jump lemma.

Definition 6.2 Given a world label x and a sequent $\Gamma \Rightarrow \Delta$, the sequent is *saturated with respect to variable* x (is x-saturated) if: *a*) if $x : A$ belongs to $\Gamma \cup \Delta$, then A is atomic or $A \equiv B \preccurlyeq C$ and $x : B \preccurlyeq C$ does not belong to Δ; *b*) if $x : A \preccurlyeq B$ and $a \in S(x)$ occur in Γ, either $a \Vdash^{\exists} A$ occurs in Δ or $a \Vdash^{\exists} B$ occurs in Γ; *c*) if $a \in S(x)$ and $b \in S(x)$ occur in Γ, either $a \subseteq b$ or $b \subseteq a$ occurs in Γ. A sequent is *x-hypersaturated with respect to variable x* if, for all $a \in S(x)$, the following hold: *a*) for each $a \in S(x)$, no formulas $a \Vdash^{\exists} B$ occurs in Γ; *b*) for each $a \in S(x)$, $y \in a$ occurring in Γ and for each $a \Vdash^{\exists} B$ occurring in Δ, there is a formula $y : B$ occurring in Δ; *c*) for each $a \in S(x)$, $b \in S(x)$, $a \subseteq b$ and $y \in a$ occurring in Γ, there is a formula $y \in b$ occurring in Γ. Given a branch $\mathcal{B}$ of a derivation of $\Gamma \Rightarrow \Delta$, we say that $\mathcal{B}$ is in normal form with respect to x if from the root sequent $\Gamma \Rightarrow \Delta$ upwards the following holds: first all propositional and $\preccurlyeq$ rules are applied, until an x-saturated sequent is reached, and then rules $\mathsf{R}\Vdash^{\exists}$, $\mathsf{L}\Vdash^{\exists}$ and $\mathsf{L}\subseteq$ are applied to the x-saturated sequent, until a sequent which is x-hypersaturated is reached. We say that a derivation of $\Gamma \Rightarrow \Delta$ is *in normal form* with respect to x if all its branches are in normal form. [6]

Lemma 6.3 *Given a* **G3V** *derivable sequent* $\Gamma \Rightarrow \Delta$ *which is the result of a translation* t *and a variable* x *occurring in it, we can transform any derivation of* $\Gamma \Rightarrow \Delta$ *into a derivation in normal form with respect to variable* x.

Proof. Induction on the height of the derivation of $\Gamma \Rightarrow \Delta$. Let $\to^*_{\Pi_\mathcal{B}}$ be the relation between world labels occurring in the union of all antecedents of a branch, as in Definition 3.8. If the sequent is an axiom, we are done. If the height of the derivation is greater than zero, we proceed by cases: if there are no labels y different from x such that $x \to^*_{\Pi_\mathcal{B}} y$, then x is the only label in the branch. The derivation of $\Gamma \Rightarrow \Delta$ will use only propositional rules; thus, the branch is in normal form with respect to x. If there is some label y such that

[6] Recall Definition 3.7; if a **G3V** derivable sequent $\Gamma \Rightarrow \Delta$ is the result of a translation t (i.e. there exists a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequent $\Gamma^I \Rightarrow \Delta^I$ such that $t(\Gamma^I \Rightarrow \Delta^I)^{x,\bar{a}} = \Gamma \Rightarrow \Delta$), the sequent is *simple*, and the labels occurring in its derivation form a tree according to the relation $\to^*_{\Pi_\mathcal{B}}$ (Lemma 3.9). This result is not unexpected: as in the case of [8], we are able to translate only tree-form sequents.

$x \rightarrow^*_{\Pi_{\mathcal{B}}} y$, transform each branch of the derivation of $\Gamma \Rightarrow \Delta$ as follows. Sequent $\Gamma \Rightarrow \Delta$ contains at most one world label and possibly some sphere labels $a, b \ldots$ such that $a \in S(x), b \in S(x) \ldots$. Labels $y \in a$ might be introduced only by $\mathsf{L} \Vdash^{\exists}$. If some rules are applied to formulas $a \Vdash^{\exists} A$ or to formulas $y \in a, a \subseteq b$ or to formulas $y : A$, when there are still some rules (non-redundantly) applicable to formulas $x : A$ or $a \in S(x), b \in S(x)$, apply *first* the rules for $x : A$ and $a \in S(x), b \in S(x)$, until the x-saturated sequent is reached. Similarly, if some rules are applied to a formula $y : A$ when there are still some rules which can be (non-redundantly) applied to formulas $a \Vdash^{\exists} A$ or to formulas $y \in a, a \subseteq b$, apply these latter rules until an x-hypersaturated sequent is reached, before proceeding to apply rules for $y : A$. In both cases, permuting the rules in the derivation does not represent a problem: rules applicable to $x : A$ and to $y : A$ involve different active formulas. As for rules applicable to $a \Vdash^{\exists} A$ with $a \in S(x)$, observe that the normal form "respects" the order in which labels are generated in the tree. For instance, the normal form with respect to x requires that rule $\mathsf{R} \preccurlyeq$, generating spheres $a \in S(x)$, has to be applied to $x : C \preccurlyeq D$ *before* rules $\mathsf{R} \Vdash^{\exists}$ or $\mathsf{L} \Vdash^{\exists}$ might be applied to $b \Vdash^{\exists} C$ or $b \Vdash^{\exists} D$. To obtain a normal form derivation with respect to x, we have to apply the procedure to all branches; we might also have to add some rules to obtain the x-saturated and x-hypersaturated sequents. □

Definition 6.4 We give here a simplified version of Definition 3.8. Given a multiset of labelled formulas Π, define: *a)* $x \rightarrow_{\Pi} a$ if $a \in S(x)$ occurs in Π; *b)* $a \rightarrow_{\Pi} y$ if $y \in a$ occurs in Π; *c)* $x \rightarrow_{\Pi} y$ if there exists an a such that $x \rightarrow_{\Pi} a$ and $a \rightarrow_{\Pi} y$ occur in Π. Let $W_{\Pi}(x)$ be the reflexive and transitive closure of $x \rightarrow_{\Pi} y$: $W_{\Pi}(x) = \{y \mid x \rightarrow^*_{\Pi} y\}$. Let $N_{\Pi}(x) = \{b \mid \exists u \,.\, x \rightarrow^*_{\Pi} u \textit{ and } u \rightarrow_{\Pi} b\}$. These sets represents respectively the set of world labels accessible from a world label x, and the set of sphere labels accessible from a world label x occurring in Π. Define Σ^{Π}_x as the union of the sets:

$$\begin{aligned} \Sigma^{\Pi}_x = \; & \{u : F \mid u : F \textit{ occurs in } \Sigma \textit{ and } u \in W_{\Pi}(x)\} \cup \\ & \cup \{a \Vdash^{\exists} B \mid a \Vdash^{\exists} B \textit{ occurs in } \Sigma \textit{ and } a \in N_{\Pi}(x)\} \cup \\ & \cup \{b \in S(y) \mid b \in S(y) \textit{ occurrs in } \Pi,\; b \in N_{\Pi}(x) \textit{ and } y \in W_{\Pi}(x)\} \cup \\ & \cup \{a \subseteq b \mid a \subseteq b \textit{ occurs in } \Pi \textit{ and } a, b \in N_{\Pi}(x)\} \cup \\ & \cup \{z \in a \mid z \in a \textit{ occurs in } \Pi \textit{ and } a \in N_{\Pi}(x)\}. \end{aligned}$$

Lemma 6.5 (Jump lemma) *Let $\Gamma \Rightarrow \Delta$ be a derivable* **G3V** *sequent. If the labels occurring in $W_{\Gamma}(x)$ have a tree structure, for each label x occurring in the sequent, it holds that either 1) sequent $\Gamma^{\Gamma}_x \Rightarrow \Delta^{\Gamma}_x$ or 2) sequent $\Gamma - \Gamma^{\Gamma}_x \Rightarrow \Delta - \Delta^{\Gamma}_x$ is derivable, with the same derivation height.*

Proof. To simplify the notation, we write Γ^* for Γ^{Γ}_x and Δ^* for Δ^{Γ}_x respectively. The proof is by induction on the height of the derivation, and by distinction of cases. If $\Gamma \Rightarrow \Delta$ is an initial sequent, it has the form $u : P, \Gamma' \Rightarrow \Delta', u : P$. If $u \in W^{\Gamma}_x$, we have that $u : P, \Gamma^* \Rightarrow \Delta^*, u : P$ is and initial sequent, hence derivable, and we are in case 1. If $u \notin W^{\Gamma}_x$, then $u : P \in \Gamma - \Gamma^{\Gamma}_x$, and we obtain case 2. For the propositional rules, we show only the case of $L \rightarrow$.

$$\frac{\Gamma \Rightarrow \Delta, u : B \quad u : A, \Gamma \Rightarrow \Delta}{u : A \rightarrow B, \Gamma \Rightarrow \Delta} \; L\rightarrow$$

Suppose $u \in W_x^\Gamma$. We have to show that either 1) the sequent $u : A \rightarrow B, \Gamma^* \Rightarrow \Delta^*$ is derivable, or that 2) sequent $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*$ is derivable. By inductive hypothesis applied to both premisses, we have that either a) $\Gamma^* \Rightarrow \Delta^*, u : B$ is derivable or b) $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*$ is derivable, and that either c) $u : A, \Gamma^* \Rightarrow \Delta^*$ is derivable or d) $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*$ is derivable. If *a*) and *c*) are derivable, we apply $L \rightarrow$ and obtain the derivable sequent $u : A \rightarrow B, \Gamma^* \Rightarrow \Delta^*$ (which is case 1). If *a*) and *d*) are derivable, *d*) is already the sequent corresponding to case 2 of the statement; the same holds if *b*) and *c*) are derivable, and if *b*) and *d*) are derivable. If $u \notin W_x^\Gamma$, we want to show that either 1) $\Gamma^* \Rightarrow \Delta^*$ id derivable or that 2) $\Gamma - \Gamma^*, u : A \rightarrow B \Rightarrow \Delta - \Delta^*$ is derivable. Again, by inductive hypothesis we have that either *a*) $\Gamma - \Gamma^*, u : B \Rightarrow \Delta - \Delta^*$ or *b*) $\Gamma^* \Rightarrow \Delta^*$ and either *c*) $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*, u : A$ or *d*) $\Gamma^* \Rightarrow \Delta^*$ are derivable. If *a*) and *c*) are derivable, we obtain case 2; otherwise we are in case 1.
If $\Gamma \Rightarrow \Delta$ has been derived by $\mathsf{L} \Vdash^\exists$, we have:

$$\frac{\Gamma, y \in a, y : B \Rightarrow \Delta}{\Gamma, a \Vdash^\exists B \Rightarrow \Delta} \; \mathsf{L}\Vdash^\exists$$

where y does not occur in Γ and Δ. If $a \in N_\Gamma(x)$ then $y \in W_\Gamma(x)$; by inductive hypothesis, we have that either *a*) $\Gamma^*, y : B, y \in a \Rightarrow \Delta^*$ is derivable, or *b*) $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta*$ is derivable. In the former case, a step of $\mathsf{L} \Vdash^\exists$ gives that $\Gamma^*, a \Vdash^\exists B \Rightarrow \Delta^*$ is derivable. If *b*) is derivable, we already have our desired sequent (case 2). If $a \notin N_\Gamma(x)$, then $y \notin W_\Gamma(x)$. By inductive hypothesis, either $\Gamma^* \Rightarrow \Delta^*$ is derivable, and we are done, or $\Gamma - \Gamma^*, y : B, y \in a \Rightarrow \Delta - \Delta^*$ is derivable. Apply $\mathsf{L} \Vdash^\exists$ to obtain the sequent $\Gamma - \Gamma^*, a \Vdash^\exists B \Rightarrow \Delta - \Delta^*$.
For the remaining cases: $\mathsf{R} \preccurlyeq$ is similar to $\mathsf{L} \Vdash^\exists$. Rules Nes and $\mathsf{L} \preccurlyeq$ are similar to $L \rightarrow$. $\mathsf{R} \Vdash^\exists$ and $\mathsf{L} \subseteq$ are immediate, since they do not introduce new labels and have just one premiss. □

Example 6.6 Suppose that the following sequent is derivable.

$$a \in S(x), b \in S(x), y \in a, y : B, z \in a, z : C \Rightarrow a \Vdash^\exists A, b \Vdash^\exists B, z : B, y : A$$

Consider label z: $N_z^\Gamma = \varnothing$ and $W_z^\Gamma = \{z\}$. Thus, Γ_z^Γ coincides with $z : C$, and Δ_z^Γ coincides with $z : B$, and either $z : C \Rightarrow z : A$ is derivable, or the rest of the sequent is derivable, namely $a \in S(x), b \in S(x), y \in a, y : B, z \in a \Rightarrow a \Vdash^\exists A, y : A, b \Vdash^\exists B$.

Lemma 6.7 *If a sequent* $a \Vdash^\exists B, a \Vdash^\exists C, \Gamma \Rightarrow \Delta$ *is derivable in* **G3V**, *then either* $a \Vdash^\exists B, \Gamma \Rightarrow \Delta$ *or* $a \Vdash^\exists C, \Gamma \Rightarrow \Delta$ *are derivable in* **G3V** *with same derivation height.*

Proof. By induction on the height of the derivation. The only relevant case is $R = \mathsf{L} \Vdash^\exists$, applied to one of the formulas $a \Vdash^\exists A$ or $a \Vdash^\exists B$.

$$\frac{y \in a, y : A, a \Vdash^\exists B, \Gamma \Rightarrow \Delta}{a \Vdash^\exists A, a \Vdash^\exists B, \Gamma \Rightarrow \Delta} \; \mathsf{L}\Vdash^\exists$$

Apply the Jump lemma to the premiss, obtaining that either 1) $y : A \Rightarrow$ is derivable, or 2) $y \in a, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta$ is derivable. In the former case, apply weakening and $\mathsf{L}\Vdash^{\exists}$ to obtain a derivation of $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$. In the latter case, by invertibility of $\mathsf{L}\Vdash^{\exists}$ we have that sequent $y \in a, w \in a, w : B, \Gamma \Rightarrow \Delta$ is derivable, for some $w \notin \Gamma, \Delta$. We substitute variable y with variable w. The substitution does not affect other formulas than $y \in a$, since $y, w \notin \Gamma, \Delta$. Contraction and $\mathsf{L}\Vdash^{\exists}$ give the sequent $a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta$. □

In order to define a translation $[\![\ ]\!]^x$ for **G3V** normal form derivations we need to define a sub-translation $[\ ]^x$, that takes care of the translation of a derivation from the root sequent up to the x-saturated sequents. Theorem 6.9 will take care of translating the upper part of the normal form derivations: from x-saturated sequents to x-hypersaturated sequents, making an essential use of the Jump lemma.

The translation $[\ ]^x$ takes as parameter a world label x, the one used to translate the root sequent. For $\mathsf{L}\preccurlyeq$ and Nes we explicitly define sets of inclusions that might occur in the sequent: $\mathcal{R}^a = \{a \subseteq c_1, \ldots, a \subseteq c_n\}$; $\mathcal{R}^b = \{b \subseteq d_1, \ldots, b \subseteq d_k\}$. We will also use the corresponding mulisets of formulas: $\{\Omega\} = \{c \Vdash^{\exists} \Omega \,|\, c \Vdash^{\exists} \Omega \textit{ occurs in } \Delta \textit{ and } a \subseteq c \textit{ occurs in } \mathcal{R}^a\}$ and $\{\Xi\} = \{d \Vdash^{\exists} \Xi \,|\, d \Vdash^{\exists} \Xi \textit{ occurs in } \Delta \textit{ and } b \subseteq d \textit{ occurs in } \mathcal{R}^b\}$.

$$\text{(init)} \qquad \Big[\overset{\triangledown}{\text{INIT}}\Big]^x \rightsquigarrow s(\overset{\triangledown}{\text{INIT}})^x$$

$$(\mathsf{R}\preccurlyeq) \quad \left[\dfrac{\begin{array}{c}\mathcal{D}_1\\ a \in S(x), a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A\end{array}}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\,\mathsf{R}\preccurlyeq\right]^x \rightsquigarrow \dfrac{\begin{array}{c}[\mathcal{D}_1]^x\\ s(a \in S(x), a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A)^x\end{array}}{s(\Gamma \Rightarrow \Delta, x : A \preccurlyeq B)^x}\,\mathsf{R}\preccurlyeq^{\mathsf{i}}$$

$$(\mathsf{L}\preccurlyeq) \quad \left[\dfrac{\begin{array}{c}\mathcal{D}_1\\ (1)\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ (2)\end{array}}{\mathsf{Conc}}\,\mathsf{L}\preccurlyeq\right]^x \rightsquigarrow \dfrac{\begin{array}{c}[\mathcal{D}_1]^x\\ s(1)^x\end{array} \qquad \dfrac{\begin{array}{c}[\mathcal{D}^-]^x\\ s(S_1)^x\end{array}}{Q}\,\mathsf{Wk}}{s(\mathsf{Conc})^x}\,\mathsf{L}\preccurlyeq^{\mathsf{i}}$$

$(1) = a \in S(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, \mathcal{R}^a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, a \Vdash^{\exists} B$
$(2) = a \in S(x), a \Vdash^{\exists} A, a \Vdash^{\exists} C, x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$
$\mathsf{Conc} = a \in S(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, \mathcal{R}^a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$
$S_1 = a \in S(x), a \Vdash^{\exists} A, x : A \preccurlyeq B, \mathcal{R}^a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$ (from (2) by Lemma 6.7)
$Q = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd C], [\Sigma, \{\Omega\} \lhd A]$

$$(\mathsf{Nes}) \quad \left[\dfrac{\begin{array}{c}\mathcal{D}_1\\ (1)\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ (2)\end{array}}{\mathsf{Conc}}\,\mathsf{Nes}\right]^x \rightsquigarrow \dfrac{\dfrac{\begin{array}{c}[\mathcal{D}_1]^x\\ s(1)^x\end{array}}{P}\,\mathsf{Wk_B} \qquad \dfrac{\begin{array}{c}[\mathcal{D}_2]^x\\ s(2)^x\end{array}}{Q}\,\mathsf{Wk_B}}{s(\mathsf{Conc})^x}\,\mathsf{Com}^{\mathsf{i}}$$

The underlined formulas are added by $\mathsf{Wk_B}$.
$(1) = a \subseteq b, a \in S(x), b \in S(x), \mathcal{R}^a, \mathcal{R}^b, a \Vdash^{\exists} A, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Pi$
$(2) = b \subseteq a, a \in S(x), b \in S(x), \mathcal{R}^a, \mathcal{R}^b, a \Vdash^{\exists} A, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Pi$
$\mathsf{Conc} = a \in S(x), b \in S(x), \mathcal{R}^a, \mathcal{R}^b, a \Vdash^{\exists} A, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Pi$
$\mathrm{P} = s(\Gamma)^x \Rightarrow s(\Delta)^x, \Big[\Sigma, \Pi, \{\Omega\}, \underline{\{\Xi\}} \lhd A\Big], [\Pi, \{\Xi\} \lhd B]$
$\mathrm{Q} = s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd A], \Big[\Sigma, \Pi, \underline{\{\Omega\}}, \{\Xi\} \lhd B\Big]$

Lemma 6.8 *Let $\mathcal{D}^S$ be a* **G3V** *derivation of $\Gamma \Rightarrow \Delta$ from x-saturated sequents $\Gamma_1^S \Rightarrow \Delta_1^S, \ldots, \Gamma_n^S \Rightarrow \Delta_n^S$; then $[\mathcal{D}^S]^x$ is a derivation of $s(\Gamma \Rightarrow \Delta)^x$ in $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ from sequents $s(\Gamma_1^{S^-} \Rightarrow \Delta_1^{S^-})^x, \ldots, s(\Gamma_n^{S^-} \Rightarrow \Delta_n^{S^-})^x$, where for each $\Gamma_i^{S^-} \Rightarrow \Delta_i^{S^-}$ it* *holds that* $\Gamma_i^{S^-} \cup \Delta_i^{S^-} \subseteq \Gamma_i^S \cup \Delta_i^S$.

Proof. By distinction of cases, and by induction on the height of the derivation. If $h = 0$, $\Gamma \Rightarrow \Delta$ is a **G3V** initial sequent, and its translation $s(\Gamma \Rightarrow \Delta)$ is a $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ initial sequent. The propositional cases are obtained applying the corresponding $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ rule to the translation(s) of the premiss(es).
[$\mathsf{R} \preccurlyeq$] Translation of the premiss: $s(a \in S(x), a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A)^x = s(\Gamma)^x \Rightarrow s(\Delta)^x, [A \lhd B]$; translation of the conclusion: $s(\Gamma \Rightarrow \Delta, x : A \preccurlyeq B)^x = s(\Gamma)^x \Rightarrow s(\Delta)^x, A \preccurlyeq B$.
[$\mathsf{L} \preccurlyeq$] $s(1)^x = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, B, \{\Omega\} \lhd C]$. The right premiss (2) cannot be translated, since it features in the antecedent *two* formulas with the same sphere label: $a \Vdash^{\exists} A$ and $a \Vdash^{\exists} C$. By Lemma 6.7 we have that we have that either sequent $S_1 = a \in S(x), a \Vdash^{\exists} A, x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$ or $S_2 = a \in S(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$ are derivable (possibly both). However, sequent S_2 is the same sequent as the conclusion of $\mathsf{L} \preccurlyeq$; thus, if we replace the right premiss with this sequent, the application of $\mathsf{L} \preccurlyeq$ would be useless, and we can ignore the case. Thus, replace the right premiss with S_1. Let $\mathcal{D}^-$ be the derivation for S_1: for Lemma 6.7 the derivation has height less or equal than $\mathcal{D}_2$, derivation of (2). Moreover, observe that $\mathcal{D}^-$ is a *subderivation* of $\mathcal{D}_2$: it displays the same formulas (and the same rules) except for formula $a \Vdash^{\exists} A$ (and the rules applied to it). Let $\Gamma_1^{S^-} \Rightarrow \Delta_1^{S^-}, \dots, \Gamma_k^{S^-} \Rightarrow \Delta_k^{S^-}$ be the x-saturated sequents from which $\mathcal{D}^-$ is derived. Each of these sequent is composed of *less* formulas than the x- saturated sequents $\Gamma_1^S \Rightarrow \Delta_1^S, \dots, \Gamma_k^S \Rightarrow \Delta_k^S$ from which S_2 was derived. This is the reason why we translate x-saturated sequents which are composed of not exactly the same formulas of the original x-saturated sequents, but of a subset of them. Apply the translation to S_1: $s(S_1)^x = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd C]$. Add the missing block to $s(S_1)^x$ by weakening. Application of $\mathsf{L} \preccurlyeq^{\mathsf{i}}$ yields the translation of the conclusion of $\mathsf{L} \preccurlyeq$: $s(\mathsf{Conc})^x = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd C]$.
[Nes] Sequent $s(1)^x$ is $s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \Pi, \{\Omega\} \lhd A], [\Pi, \{\Xi\} \lhd B]$; sequent $s(2)^x$ is $s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd A], [\Sigma, \Pi, \{\Xi\} \lhd B]$; sequent $s(\mathsf{Conc})^x$ is $s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd A], [\Pi, \{\Xi\} \lhd B]$.
Sets $\{\Omega\}$ and $\{\Xi\}$ account for the formulas to be added inside blocks, in correspondence with inclusions in $\mathcal{R}^a$ and $\mathcal{R}^b$ (see Definition 6.1). If both sets $\mathcal{R}^a$ and $\mathcal{R}^b$ are empty, $\{\Omega\}$ and $\{\Xi\}$ are empty there is no needed to apply $\mathsf{Wk_B}$ to either of the premisses. If $\mathcal{R}^a$ is not empty, $\{\Omega\}$ is not empty; we need to apply $\mathsf{Wk_B}$ to the second block of $s(2)^x$; Similarly, if $\mathcal{R}^b$ is not empty, $\{\Xi\}$ is not empty; we need to apply $\mathsf{Wk_B}$ to the first block of $s(1)^x$. If both $\{\Omega\}$ and $\{\Xi\}$ are not empty, combine the two above strategies. □

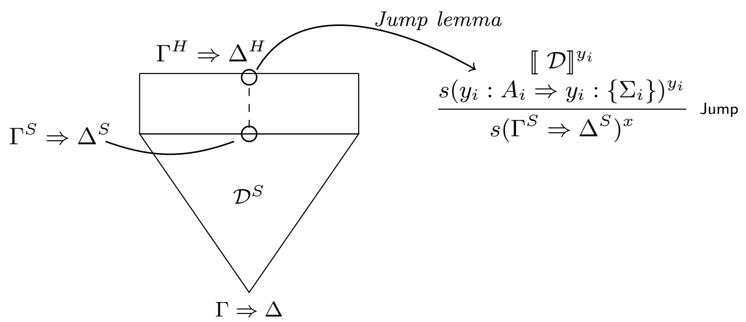
We are finally ready to define the full translation for derivations.

Theorem 6.9 *Let* $\mathcal{D}$ *be a* **G3V** *derivation of* $\Gamma \Rightarrow \Delta$ *in normal form with respect to some* x *in* $\Gamma \cup \Delta$. *Then* $[\![\mathcal{D}]\!]^x$ *is a* $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ *derivation of* $s(\Gamma \Rightarrow \Delta)^x$.

Proof. By induction on the height of the derivation. Since $\mathcal{D}$ is in normal form with respect to x, it will contain a subderivation $\mathcal{D}^S$ of $\Gamma \Rightarrow \Delta$ from x-saturated sequents $\Gamma_1^S \Rightarrow \Delta_1^S, \dots, \Gamma_n^S \Rightarrow \Delta_n^S$. Apply translation $[\ ]^x$ to $\mathcal{D}^S$, and obtain a derivation of $s(\Gamma \Rightarrow \Delta)^x$ from $s(\Gamma_1^{S^-} \Rightarrow \Delta_1^{S^-})^x, \dots, s(\Gamma_n^{S^-} \Rightarrow \Delta_n^{S^-})^x$ (Lemma 6.8). Each x-saturated sequent $\Gamma_i^S \Rightarrow \Delta_i^S$ has the form[7]: $\mathcal{R}^{a_1}, \dots, \mathcal{R}^{a_n},\ a_1 \in S(x), \dots, a_n \in S(x), a_1 \Vdash^{\exists} A_1, \dots, a_n \Vdash^{\exists} A_n, x : \Gamma^P \Rightarrow$ $\Rightarrow x : \Delta^P, a_1 \Vdash^{\exists} \Sigma_1, \dots, a_n \Vdash^{\exists} \Sigma_n$. Its translation according to s and x is: $s(\Gamma_i^S \Rightarrow \Delta_i^S)^x = \Gamma^P \Rightarrow \Delta^P, [\{\Sigma_1\} \lhd A_1], \dots, [\{\Sigma_n\} \lhd A_n]$. For each $\Gamma_i^S \Rightarrow \Delta_i^S$, apply the following transformation: go up in the derivation until the x-hypersaturated sequent $\Gamma_i^H \Rightarrow \Delta_i^H$ is reached. The sequent will have the following form: $\mathcal{R}^{a_1}, \dots, \mathcal{R}^{a_n}, a_1 \in S(x), \dots, a_n \in S(x), \{y_1 \in a_1\}, \dots, \{y_n \in a_n\}, y_1 : A_1, \dots, y_n : A_n,\ x : \Gamma^P \Rightarrow x : \Delta^P, y_1 : \{\Sigma_1\}, \dots y_n : \{\Sigma_n\}, a_1 \Vdash^{\exists} \Sigma_1, \dots, a_n \Vdash^{\exists} \Sigma_n$ where $\{y_i \in a_i\}$ is a shorthand for $y_i \in a_i \cup \{y_i \in a_j \,|\, a_i \subseteq a_j \in \mathcal{R}^{a_i}$ *and* $y \in a_i\}$. By Jump lemma either $y_1 : A_1 \Rightarrow y_1 : \{\Sigma_1\}$ is derivable, or the following sequent is derivable: $\mathcal{R}^{a_1}, \dots, \mathcal{R}^{a_n}, a_1 \in S(x), \dots, a_n \in S(x), \{y_1 \in a_1\}, , \dots, \{y_n \in a_n\}, y_2 : A_2 \dots y_n : A_n,\ x : \Gamma^P \Rightarrow x : \Delta^P, y_2 : \{\Sigma_2\}, \dots y_n : \{\Sigma_n\}, a_1 \Vdash^{\exists} \Sigma_1, \dots, a_n \Vdash^{\exists} \Sigma_n$.

If $y_1 : A_1 \Rightarrow y_1 : \{\Sigma_1\}$ is not derivable, apply the Jump lemma to the above sequent, and iterate the procedure until a derivable sequent $y_i : A_i \Rightarrow y_i : \{\Sigma_i\}$ is found, for some $1 \leqslant i \leqslant n$. The existence of such a derivable sequent is guaranteed by the Jump lemma, and by the fact that a) the x-hypersaturated sequent is derivable and b) the x-hypersaturated sequent is not derivable in virtue of the part $x : \Gamma^P \Rightarrow x : \Delta^P$. If this was the case, the proof search would have stopped way before, since only propositional rules would have been applied in the derivation.

Suppose $y_i : A_i \Rightarrow y_i : \{\Sigma_i\}$ is derivable; $s(y_i : A_i \Rightarrow y_i : \{\Sigma_i\})^y = A_i \Rightarrow \{\Sigma_i\}$. Application of Jump to this sequent yields the translation of the x-saturated sequent $s(\Gamma^S \Rightarrow \Delta^S)^x = \Gamma^P \Rightarrow \Delta^P, [\{\Sigma_1\} \lhd A_1], \dots, [\{\Sigma_i\} \lhd A_i] \dots, [\{\Sigma_n\} \lhd A_n]$. Then, $[\![\ ]\!]^{y_i}$ has to be recursively invoked to translate, with variable y_i as a parameter, the derivation of sequent $y_i : A_i \Rightarrow y_i : \{\Sigma_i\}$; of smaller height than derivation of $\Gamma \Rightarrow \Delta$.



□

Example 6.10 Consider the **G3V** derivation in the proof of Theorem 3.6. As it is, the derivation is *not* in normal form with respect to x: we have to

[7] If rule $\mathsf{L} \preccurlyeq$ has been employed in $\mathcal{D}^S$, instead of the x-saturated sequent we choose its "smaller" version $\Gamma_i^{S-} \Rightarrow \Delta_i^{S-}$, since translation s is possibly not applicable to $\Gamma_i^S \Rightarrow \Delta_i^S$. Refer to case $\mathsf{L} \preccurlyeq$ of the proof of Lemma 6.8 for details. In any case, the proof strategy remains the same.

saturate its upper part with respect to rules $\mathsf{L} \Vdash^{\exists}$, $\mathsf{R} \Vdash^{\exists}$ and $\mathsf{L} \subseteq$, to the effect that the two upper sequents become two x-hypersaturated sequents. Then, the part of the derivation up to the x-saturated sequents (in this case: premisses of Nesting) is translated employing $[\ ]^x$. Then, we apply $[\![\ ]\!]^x$: consider the left premiss of Nes (x-saturated), and go up to the x-hypersaturated sequent $y \in a, y \in b, z \in a, a \subseteq b, y : B, z : A \Rightarrow y : A, z : B, a \Vdash^{\exists} A, b \Vdash^{\exists} B$. From Lemma 6.5, we have that either a) $y : B \Rightarrow y : A, y : B$ is derivable, or b) $y \in a, y \in b, z \in a, z : A \Rightarrow z : B, a \Vdash^{\exists} A, b \Vdash^{\exists} B$ is derivable. Sequent a) is derivable. Thus, we translate sequent $y : B \Rightarrow y : A, y : B$ in the internal sequent calculus, obtaining $B \Rightarrow A, B$. An application of Jump allows us to obtain the sequent which is the leftmost application of $\mathsf{Com}^{\mathsf{i}}$. A similar reasoning is applied to translate the right premiss of Nes.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\nabla}{B \Rightarrow A, B}}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ \mathsf{Jump} \qquad \Rightarrow [A \lhd B], [A, B \lhd A]}{\Rightarrow [A \lhd B], [B \lhd A]}\ \mathsf{Com}^{\mathsf{i}}}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \mathsf{R}\preccurlyeq}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \mathsf{R}\preccurlyeq}{\Rightarrow A \preccurlyeq B \vee B \preccurlyeq A}\ L\vee$$

7 Conclusions

In this paper we defined **G3V**, a proof-theoretically well-behaved labelled calculus that provides an effective decision procedure for logic $\mathbb{V}$. Then, we have considered the calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, the only internal and standard calculus known the logic [5]. We have shown that it is possible to translate directly derivations of the internal calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ into derivations of the labelled calculus **G3V**. The opposite mapping is considerably more complex: we are able to translate derivations of the labelled calculus into derivations of the internal calculus provided (i) they satisfy a kind of normal form, and (ii) the relation between labels is essentially tree-like. It is worth noticing that this latter requirement is analogous to the tree-like restriction needed for mapping labelled calculi for standard modal logic [8] into nested sequent ones.

The present results are the first attempt to relate two basically different types of calculi for logics well beyond standard modal logics; despite their syntactic difference the two calculi are intrinsically related.

Many issues deserve to be further investigated. First, we aim at analysing the computational cost of the translation, namely what is the size of translated derivations with respect to the size of the input ones. Then, we can use the mapping to transfer results and properties of one calculus to the other: in one direction, syntactic cut-elimination and countermodel extraction, relatively easy to prove in the labelled calculus can be inherited in the internal calculus, for which these results are more difficult to prove. In the opposite direction, complexity bound and interpolation should be provable directly for the internal calculus, similarly to [9]. These results could be transferred to the labelled calculus, for which they are presently not known. Furthermore, since the mappings are given by functional procedures, we are interested in implementing an automated translation between derivations. Finally, the present results concern only logic $\mathbb{V}$: they could be extended to the other logics of the Lewis' cube for which internal calculi exist [6].

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