Anticanonical codes from del Pezzo surfaces with Picard rank one

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Abstract. We construct algebraic geometric codes from del Pezzo surfaces and focus on the ones having Picard rank one and the codes associated to the anticanonical class. We give explicit constructions of del Pezzo surfaces of degree $4, 5$ and 6, compute the parameters of the associated anticanonical codes and study their isomorphisms arising from the automorphisms of the surface. We obtain codes with excellent parameters and some of them turn out to beat the best known codes listed on the database *codetable*.

1. Introduction

The aim of this article is to construct codes from algebraic surfaces. Codes coming from algebraic geometry have seen a growing interest since their original contruction by Goppa. We consider evaluation codes: given an algebraic variety X defined over a finite field $\mathbf{F}_q$, and an effective divisor D on it, we evaluate the global sections in $H^0(X, \mathcal{O}_X(D))$ at the rational points of X. We get a code of length $\sharp X(\mathbf{F}_q)$, the number of rational points, and of dimension $h^0(X, \mathcal{O}_X(D))$ if the evaluation map is injective. It remains to determine the minimum distance.

Most of the algebraic geometry codes studied in the literature are based on algebraic curves, since Riemann-Roch theorem for curves is a powerful tool to estimate the dimension of the space of functions with given poles, and the number of zeroes of such a function never exceeds its number of poles. The study of evaluation codes over higher dimensional varieties is more difficult. On one hand, Riemann-Roch theorem involves the dimensions of the higher cohomology spaces $H^i(X, \mathcal{O}_X(D))$. On the other hand, one has to bound the maximal number of rational points of the schemes defined by global sections of $\mathcal{O}_X(D)$ in order to estimate the minimum distance. Such a scheme is no longer zero-dimensional and might be singular or reducible. As a consequence one has to use new tools for the estimation of its number of rational points.

In the case of surfaces, one first has to determine the intersection pairing. Then, Riemann-Roch theorem often gives a lower bound for the dimension of the space of global sections. We want to use adjunction formula, to get the arithmetic genus of the curves coming from the zeroes of such a section, then to apply the Hasse-Weil-Serre bound on the number of points of irreducible curves. In order to do this, one also has to control their decomposition into irreducible components.

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This can be done working in the Picard lattice of the surface X. In this group, a decomposition of the zero scheme of a global section of D corresponds to writing the class of D as a sum of classes of effective divisors. A very efficient way to control this is to ask the Picard lattice to have rank 1 [Zar07, Lemma 2.1].

We shall work on del Pezzo surfaces. A del Pezzo surface of degree d over an algebraically closed field is obtained from the projective plane $\mathbf{P}^2$ by blowing up $9-d$ points in general position (see Section 3). As a consequence, such surfaces are close to the plane; the arithmetic and geometric theory of these objects has been thoroughly studied (the classical reference is [Man86]) and we can explicit all invariants above: the intersection pairing and the (geometric) Picard lattice have fairly simple descriptions. They come endowed with an action of the Frobenius automorphism, and the (arithmetic) Picard lattice is the sublattice fixed by this action.

1.1. **Our contribution.** We construct del Pezzo surfaces of degree 4, 5 and 6 having Picard rank 1. In each case the Picard lattice is generated by the canonical divisor K_X, and the cone of effective divisors by the anticanonical divisor $-K_X$. We call the code associated to this last divisor *anticanonical*; it has dimension $d+1$ (except for very small values of q), and all zero schemes of global sections are irreducible with arithmetic genus 1 from the adjunction formula. As a consequence, they have at most $q+1+\lfloor 2\sqrt{q}\rfloor$ points from the Hasse-Weil-Serre bound, and this provides very good codes. Some of them turn out to beat the best known codes listed in the database *codetables* [Gra07].

A central role is played by the Frobenius action on the Picard lattice of the surface, i.e. by the conjugacy class of the image of the Frobenius in a certain Weyl group. It gives many properties of the surface, such as the Picard rank or the number of rational points. Moreover, with the help of Galois cohomology, it enables us to determine the $\mathbf{F}_q$-rational automorphisms of the surface; since an automorphism of the surface must preserve the (anti)canonical class, we deduce some automorphisms of the codes.

Along the article, we try to be as constructive as possible. We give explicit descriptions of the anticanonical models. We also use birational morphisms from our surfaces to the projective plane to give explicit constructions of the codes, and Cremona transformations to describe the automorphisms.

1.2. **Related works.** In some sense, this paper fills a gap in the study of algebraic geometric codes constructed from del Pezzo surfaces (even if most of the authors cited below do not mention the fact that they work on del Pezzo surfaces). When X is the projective plane $\mathbf{P}^2$ (the only del Pezzo surface of degree 9 over a finite field), the Picard group is generated by the class L of a line, and the evaluation code associated to $\mathcal{O}(mL)$ is the well-known projective Reed-Muller code of order m (see [Lac88]). There are two types of del Pezzo surfaces of degree 8, one having Picard rank 2 (the hyperbolic quadric, isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$), and the other having Picard rank 1 (the elliptic quadric). Codes over these surfaces have been studied by Edoukou [Edo08], and Couvreur and Duursma [CD13]. In [Cou11, Section 3.2], the second author constructs good codes coming from del Pezzo surfaces of degree 6 having Picard rank 2. In [LS18], Little and Schenck consider anticanonical codes on del Pezzo surfaces of degree 3 and 4 having Picard rank 1. Boguslavsky [Bog98] gives the parameters of anticanonical codes on *split* del Pezzo surfaces, i.e. on surfaces having maximal Picard rank.

Let us finally mention some works on codes on other blowups of the plane [Dav11, Bal13]. These blowups are no longer del Pezzo surfaces: since the blown up points lie on one or two lines, they are not in general position. Moreover the evaluation set is not the set of rational points of the surface, but the torus $\mathbf{G}_m^2(\mathbf{F}_q)$.

1.3. **Outline of the article.** The paper is organized as follows. In Section 3, we recall the necessary material on del Pezzo surfaces, and give the classification of such surfaces of degrees 5 and 6 over a finite field. Then we construct degree 6 del Pezzo surfaces in Section 4; we give the parameters of the associated anticanonical codes, and determine the automorphisms of the surfaces. We construct del Pezzo surfaces of degree 5 and the corresponding codes in Section 5. We give two geometric constructions of these codes: the first one by evaluating linear forms at the rational points of a surface embedded in $\mathbf{P}^5$ and the second one by evaluating particular quintics forms at some rational points of the projective plane. Then, we determine the automorphism group of the surface. Finally, we construct del Pezzo surfaces of degree 4 in Section 6, and determine the parameters of the associated codes.

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2. Context and notation

In the following, we fix $\mathbf{F}_q$ a finite field of characteristic p and an algebraic closure $\overline{\mathbf{F}}_q$. We denote by σ a generator of the absolute Galois group $G := \mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$. The projective space of dimension r over $\mathbf{F}_q$ is denoted by $\mathbf{P}^r$. On a surface X, the intersection product of two divisor classes A, B is denoted by $A \cdot B$ and the self intersection $A \cdot A$ of A is denoted by $A^{\cdot 2}$. Given an effective divisor A, the complete linear system associated to A is denoted by $|A|$, the Picard group of X is denoted by $\mathrm{Pic}(X)$ and the canonical class of X is denoted by K_X.

Given a smooth projective geometrically connected surface X over $\mathbf{F}_q$ and divisor D on X, the code $\mathcal{C}(X(\mathbf{F}_q), D)$ is defined as the image of the map

$$\begin{cases} H^0(X, \mathcal{O}_X(D)) & \longrightarrow & \mathbf{F}_q^n \\ f & \longmapsto & (f(P))_{P \in X(\mathbf{F}_q)}. \end{cases}$$

Note that for the map to be well–defined, one needs to order the rational points of X, which can be done arbitrarily since the choice of another order would provide an isometric code with respect to the Hamming metric. Similarly, the evaluation of a global section of $H^0(X, \mathcal{O}_X(D))$ at a point P depends on the choice of a generator of the stalk of the sheaf at P but choosing another system of generators would provide an isometric code. Since we are mostly interested in the parameters of the code : its dimension and minimum distance, it is sufficient to consider our code up to isometry.

3. del Pezzo surfaces

In this section we collect some definitions and well known facts about del Pezzo surfaces, that we shall use in the sequel. For the proofs and many other results, we refer the reader to [Man86, Chapter 24 sq.].

Definition 3.1. A smooth projective surface X defined over a field k is *del Pezzo* when its anticanonical divisor $-K_X$ is ample. Its *degree* is the self-intersection number $d := K_X^{\cdot 2}$.

We assume $3 \leqslant d \leqslant 7$ in the following. We know from [Man86, Theorems 24.4, 24.5] that such a surface is isomorphic (over the algebraic closure $\overline{k}$) to the blow-up of the projective plane $\mathbf{P}^2$ at $r := 9 - d$ points in *general position*, i.e. such that no three of them are collinear, and no six lie on a conic. In this case, the anticanonical divisor is very ample; the space of its global sections has dimension

$$\dim H^0(X, \mathcal{O}(-K_X)) = d + 1 = 10 - r,$$

and it defines an embedding of X into $\mathbf{P}^d$, whose image has degree d. The image of this embedding is called the *anticanonical model* of X.

If $X' \to X$ is a birational morphism, and X' is a del Pezzo surface, then X is a del Pezzo surface from [Man86, Theorem 24.5.2 (i)].

Let X be as above; it is isomorphic (over $\overline{k}$) to the blowup π of the projective plane at $p_1, \dots, p_r$. Let $E_0 := \pi^* H$ denote the pullback of the class of a line in $\mathbf{P}^2$ and $E_1, \dots, E_r$, denote respectively the class of the exceptional divisor of the blowup at $p_1, \dots, p_r$. From [Man86, Theorems 25.1] (see also [Har77, Proposition V.3.2]), we know that the geometric Picard lattice $\operatorname{Pic}(X \otimes \overline{k})$ is free of rank $r+1$, with basis $E_0, \dots, E_r$. The intersection pairing is defined by

$$E_0^{\cdot 2} = 1,\ E_i^{\cdot 2} = -1,\ i \geqslant 1,\ E_i \cdot E_j = 0,\ i \neq j$$

and the class of the canonical divisor is

$$K_X = -3E_0 + \sum_{i=1}^{r} E_i.$$

In the following, we assume that X is defined over $\mathbf{F}_q$. Associated to X is a representation of G on $\operatorname{Pic}(X \otimes \overline{\mathbf{F}}_q)$ that respects the intersection pairing and the canonical divisor. If $\sigma^* \in \operatorname{Aut}(\operatorname{Pic}(X \otimes \overline{\mathbf{F}}_q))$ is the image of σ under this representation, then we know from a result of Weil [Man86, Theorem 27.1] that the number of rational points of X is given by

$$\sharp X(\mathbf{F}_q) = q^2 + q \operatorname{Tr}(\sigma^*) + 1.$$

Moreover, since X is projective and smooth and the ground field is finite, we have $\operatorname{Pic}(X) = \operatorname{Pic}(X \otimes \overline{\mathbf{F}}_q)^G$, and the Picard rank is the multiplicity of the eigenvalue 1 for σ^*.

Following Dolgachev [Dol12, Section 8.2], we define the lattice $\mathbf{E}_r$ as the orthogonal $K_X^{\perp}$ of the canonical divisor in $\operatorname{Pic}(X \otimes \overline{\mathbf{F}}_q)$. This is a root lattice, whose Weyl group is denoted $W(\mathbf{E}_r)$. The image of the above Galois representation lies in $W(\mathbf{E}_r)$, and it is a finite quotient of G, thus a cyclic subgroup.

Following Manin [Man86], we define the *type* of the del Pezzo surface X as the conjugacy class of the element σ^* in $W(\mathbf{E}_r)$.

3.1. **Isomorphism classes of del Pezzo surfaces of degrees five and six over a finite field.** For a del Pezzo surface of degree 5 or 6 over a perfect field, the types of Manin are actually isomorphism classes (see [Sko01, Theorem 3.1.3] for degree 5, degree 6 is treated in Section 4.3). When the field k is finite, we deduce that the isomorphism classes of such del Pezzo surfaces correspond to the conjugacy classes of elements in the Weyl group $W(\mathbf{E}_r)$.

We deduce the following classification for degree 5 or 6 del Pezzo surfaces over finite fields [Dol12, Section 8.2], [Tre18, Section 3].

3.1.1. *Del Pezzo surfaces of degree* 6. When $d = 6$, the root lattice is $\mathbf{E}_3 = A_1 + A_2$, and its Weyl group is

$$W(A_1) \times W(A_2) \simeq \mathfrak{S}_2 \times \mathfrak{S}_3 \simeq \mathfrak{D}_{12},$$

where $\mathfrak{S}_i$ denotes the symmetric group on i letters and $\mathfrak{D}_{12}$ denotes the dihedral group of order 12, generated by the symmetries $s_i := s_{\alpha_i}$ with respect to the following roots which form a principal system:

$$\alpha_1 = E_0 - E_1 - E_2 - E_3;\ \alpha_2 = E_1 - E_2;\ \alpha_3 = E_2 - E_3.$$

We have $s_1 s_i = s_i s_1$ for $i > 1$ and $(s_2 s_3)^3 = \mathrm{Id}$. Table 1 summarizes the different types of Del Pezzo surfaces with the corresponding Weyl conjugation classes.

Type	Weyl classes	Eigenvalues of σ^*	$\mathrm{Tr}(\sigma^*)$	Picard rank
6_1	$\{Id\}$	$1, 1, 1, 1$	4	4
6_2	$\{s_1\}$	$1, 1, 1, -1$	2	3
6_3	$\{s_2, s_3, s_2 s_3 s_2\}$	$1, 1, 1, -1$	2	3
6_4	$\{s_2 s_3, s_3 s_2\}$	$1, 1, \jmath, \bar{\jmath}$	1	2
6_5	$\{s_1 s_2, s_1 s_3, s_1 s_2 s_3 s_2\}$	$1, 1, -1, -1$	0	2
6_6	$\{s_1 s_2 s_3, s_1 s_3 s_2\}$	$1, -1, \jmath, \bar{\jmath}$	-1	1

TABLE 1. Types of degree 6 del Pezzo surfaces

3.1.2. *Del Pezzo surfaces of degree* 5. When $d = 5$, the root lattice is $\mathbf{E}_4 = A_4$, and its Weyl group is $W(A_4) \simeq \mathfrak{S}_5$ the symmetric group on 5 letters, generated by the symmetries $s_i := s_{\alpha_i}$, $1 \leqslant i \leqslant 4$, with respect to the roots

$$\alpha_1 = E_0 - E_1 - E_2 - E_3;\ \alpha_2 = E_1 - E_2;\ \alpha_3 = E_2 - E_3;\ \alpha_4 = E_3 - E_4.$$

We identify these symmetries respectively to the transpositions $(4, 5), (1, 2), (2, 3)$ and $(3, 4)$. The conjugacy classes in $\mathfrak{S}_5$ identify to the partitions of 5, and we get the classification summarised in Table 2.

We see that there exists exactly one isomorphism class of del Pezzo surface having Picard rank 1 in each degree 5 or 6, corresponding respectively to the types 6_6 and 5_7. We construct explicitely surfaces of these two types in the following sections.

Type	Weyl classes	Eigenvalues of σ^*	$\mathrm{Tr}(\sigma^*)$	Picard rank
5_1	$\{1, 1, 1, 1, 1\}$	$1, 1, 1, 1, 1$	5	5
5_2	$\{2, 1, 1, 1\}$	$1, 1, 1, 1, -1$	3	4
5_3	$\{3, 1, 1\}$	$1, 1, 1, \jmath, \bar{\jmath}$	2	3
5_4	$\{2, 2, 1\}$	$1, 1, 1, -1, -1$	1	3
5_5	$\{3, 2\}$	$1, 1, -1, \jmath, \bar{\jmath}$	0	2
5_6	$\{4, 1\}$	$1, 1, -1, \imath, -\imath$	1	2
5_7	$\{5\}$	$1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$	0	1

TABLE 2. Types of degree 5 del Pezzo surfaces

We end this section with a technical result, that will be useful when we estimate the minimum distance of the codes. It is close to [LS18, Theorem 3.3].

Lemma 3.2. *Assume that X is a del Pezzo surface over $\mathbf{F}_q$, with $\mathrm{Pic}(X) = \mathbf{Z}K_X$. If $C \in |-K_X|$ is an anticanonical curve that is not absolutely irreducible, then we have $\sharp C(\mathbf{F}_q) \leqslant 2$.*

Proof. Let $C = C_1 \cup \cdots \cup C_r$ be the decomposition of C into absolutely irreducible components. Since $C \in |-K_X|$ and $-K_X$ generates the Picard group of X, C is irreducible over $\mathbf{F}_q$. We deduce that we must have $r \geqslant 2$, and the components C_i are cyclically permuted by G. As a consequence, every rational point must be at the intersection of all the C_i's.

Assume that C contains a rational point; from what we have just said, we must have $C_i \cdot C_j \geqslant 1$ for any $i \neq j$. In the Picard group, we get $-K_X = C_1 + \cdots + C_r$, and the arithmetic genera satisfy [Har77, Ex V.1.3]

$$1 = \pi(-K_X) = \sum_{i=1}^{r} \pi(C_i) + \sum_{1 \leqslant i < j \leqslant r} C_i \cdot C_j - (r-1)$$

Since the C_i's are absolutely irreducible, their arithmetic genera are nonnegative and hence, we get

$$r(r-1)/2 \leqslant \sum_{1 \leqslant i < j \leqslant r} C_i \cdot C_j \leqslant r, \quad \text{and} \quad r \leqslant 3.$$

If $r = 2$, then $C_1 \cdot C_2 \leqslant 2$ and C contains at most two rational points. If $r = 3$, then $C_i \cdot C_j = 1$ for any $i \neq j$, and C contains at most one rational point. □

4. Anticanonical codes on some degree six del Pezzo surfaces

In this section, we construct some degree six del Pezzo surfaces with Picard rank one over any finite field, then we determine the parameters of the anticanonical codes on these surfaces.

4.1. **Construction of the surface.** We choose

- $p_1, p_2 = p_1^\sigma$ two conjugate points in $\mathbf{P}^2(\mathbf{F}_{q^2}) \backslash \mathbf{P}^2(\mathbf{F}_q)$,
- $p_3, p_4 = p_3^\sigma, p_5 = p_3^{\sigma^2}$ three conjugate points in $\mathbf{P}^2(\mathbf{F}_{q^3}) \backslash \mathbf{P}^2(\mathbf{F}_q)$.

Recall that five points in the projective plane are in general position when no three of them are collinear. For any prime power q it is possible to choose $p_1, \ldots, p_5$ as above in general position: from [BFL16, Lemma 2.4] it is sufficient to choose them on a smooth conic; such a curve exists for any q. Now if C denotes a smooth conic in the projective plane, we have $\sharp C(\mathbf{F}_{q^i}) = q^i + 1$ for any $i \geqslant 1$, and C has $q^3 - q \geqslant 6$ points defined over $\mathbf{F}_{q^3}$ but not over $\mathbf{F}_q$, and $q^2 - q \geqslant 2$ points defined over $\mathbf{F}_{q^2}$ but not over $\mathbf{F}_q$.

Let $\widetilde{X}$ denote the surface obtained from $\mathbf{P}^2$ after blowing up the points $p_1, \ldots, p_5$, $E_1, \ldots, E_5$ the corresponding exceptional divisors, and $\pi : \widetilde{X} \mapsto \mathbf{P}^2$ the composition of the five blowups. Since $\{p_1, \ldots, p_5\}$ is stable under the action of G, the map π and the surface $\widetilde{X}$ are defined over $\mathbf{F}_q$.

Applying the results in Section 3 to $\widetilde{X}$, we get the following properties. It is a degree 4 del Pezzo surface with geometric Picard lattice $\mathrm{Pic}(\widetilde{X} \otimes \overline{\mathbf{F}}_q) = \oplus_{i=0}^{5} \mathbf{Z}E_i$, and canonical class $K_{\widetilde{X}} = -3E_0 + \sum_{i=1}^{5} E_i$. From our choice of the p_i's, the map

σ^* acts on the E_i's by the permutation $(E_0)(E_1E_2)(E_3E_4E_5)$. As a consequence, the Picard lattice of $\widetilde{X}$ is

$$\mathrm{Pic}(\widetilde{X}) = \mathbf{Z}E_0 + \mathbf{Z}(E_1 + E_2) + \mathbf{Z}(E_3 + E_4 + E_5),$$

and we get $\mathrm{Tr}(\sigma^*) = 1$. Thus $\widetilde{X}$ has $q^2 + q + 1$ points.

In the following, we denote by L the line (p_1p_2), and by C the conic passing through $p_1, \dots, p_5$ in $\mathbf{P}^2$; note that C is unique since we assumed the points to be in general position, and both curves are defined over $\mathbf{F}_q$ from our choice of the p_i's. Let $\widetilde{L}$ and $\widetilde{C}$ denote the respective strict transforms of L and C in $\widetilde{X}$; they are irreducible curves defined over $\mathbf{F}_q$. Their images in $\mathrm{Pic}(\widetilde{X})$ satisfy

$$\widetilde{L} = E_0 - E_1 - E_2,\ \widetilde{C} = 2E_0 - \sum_{i=1}^{5} E_i.$$

We get $\widetilde{L}^{\cdot 2} = \widetilde{C}^{\cdot 2} = -1$, and they have arithmetic genus zero from the adjunction formula. Moreover they are disjoint since $\widetilde{L} \cdot \widetilde{C} = 0$.

From Castelnuovo's contractibility criterion [Bad01, Theorem 3.30], [Man86, Theorem 21.5], there exists a smooth projective surface X, and a birational morphism $\psi : \widetilde{X} \to X$ contracting $\widetilde{L}$ and $\widetilde{C}$ to points $l = \psi(\widetilde{L})$ and $c = \psi(\widetilde{C})$. In other words, the map ψ is the composition of the blowups of X at l and c, and $\widetilde{L}$, $\widetilde{C}$ are the corresponding exceptional divisors. We have the diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ \mathbf{P}^2 & & X \end{tikzcd}$$

Lemma 4.1. *The geometric Picard lattice* $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ *identifies to the sublattice of* $\mathrm{Pic}(\widetilde{X} \otimes \overline{\mathbf{F}}_q)$ *generated by the classes*

$$\begin{aligned} F_0 &= 3E_0 - 2E_1 - \sum_{i=2}^{5} E_i; \\ F_1 &= E_0 - E_1 - E_3; \\ F_2 &= E_0 - E_1 - E_4; \\ F_3 &= E_0 - E_1 - E_5, \end{aligned}$$

which satisfy $F_0^{\cdot 2} = -F_i^{\cdot 2} = 1$, $i \geqslant 1$, *and* $F_i \cdot F_j = 0$, $i \neq j$.

Proof. We use [Har77, Proposition V.3.2]. Since ψ is the composition of two blowups, the map ψ^* from $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ to $\mathrm{Pic}(\widetilde{X} \otimes \overline{\mathbf{F}}_q)$ is an isometry for the intersection pairing; as this pairing is non degenerate, it is an injection, and $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ identifies to its image.

The classes of the exceptional divisors are $\widetilde{L}$ and $\widetilde{C}$; as a consequence, the image $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ is the orthogonal of $\mathbf{Z}\widetilde{L} + \mathbf{Z}\widetilde{C}$ in $\mathrm{Pic}(\widetilde{X} \otimes \overline{\mathbf{F}}_q)$. A class $a_0E_0 - \sum_{i=1}^{5} a_iE_i$ is in the orthogonal if, and only if

$$a_0 = a_1 + a_2 = a_3 + a_4 + a_5.$$

When we look for classes having self-intersection ± 1, we get the condition

$$a_0^2 - \sum_{i=1}^{5} a_i^2 = \pm 1.$$

We get the class F_1, and any class orthogonal to F_1 must satisfy $a_0 = a_1 + a_3$. The class F_2 satisfies the three equalities above, and orthogonality adds the equation $a_0 = a_1 + a_4$. We get F_3, and the last equation $a_0 = a_1 + a_5$. The five equations above finally give F_0. The classes $F_0, \ldots, F_3$ clearly form a basis of $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ from their intersection products. □

We determine the canonical divisor class of X.

Lemma 4.2. *In the Picard lattice of X, the class of the canonical divisor is*

$$K_X = -3F_0 + F_1 + F_2 + F_3.$$

Proof. From [Har77, Proposition V.3.3], we have $K_{\widetilde{X}} = \psi^* K_X + \widetilde{L} + \widetilde{C}$ in $\mathrm{Pic}(\widetilde{X})$. We get $\psi^* K_X = -6E_0 + 3(E_1 + E_2) + 2(E_3 + E_4 + E_5)$, and the result comes from the identification of the lemma above. □

We are ready to prove the main properties of our surface.

Proposition 4.3. *The surface X has the following properties*

(i) it is a degree 6 del Pezzo surface;
(ii) its Picard lattice $\mathrm{Pic}(X)$ has rank one, and is generated by K_X;
(iii) it is defined over $\mathbf{F}_q$, and has $q^2 - q + 1$ rational points.

Proof. As ψ is a birational morphism from a del Pezzo surface, we know from Section 3 that X is a del Pezzo surface. Its degree is $K_X^2 = 6$.

Recall that we have $\mathrm{Pic}(X) = \mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)^G$. From the description of the action of σ^* on the E_i's, we deduce that $D = \sum_{i=0}^{3} a_i F_i$ satisfies $D^\sigma = D$ if, and only if we have $a_1 = a_2 = a_3$ and $a_0 = -3a_1$, i.e. $D = a_1 K_X$. This proves assertion (ii).

Finally, X is defined over $\mathbf{F}_q$ since the canonical class is. To compute the number of its rational points, we use Weil's result from Section 3. The matrix of the action of Frobenius on $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$, with respect to the basis $\{F_0, F_1, F_2, F_3\}$ is

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \end{pmatrix} \tag{4.1}$$

whose trace is -1. □

4.2. **Anticanonical codes.** Here we determine the parameters of the evaluation code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$.

Its length is $\sharp X(\mathbf{F}_q) = q^2 - q + 1$ from Proposition 4.3 (iii). The dimension of the space of global sections is $h^0(X, -K_X) = d + 1 = 7$ from Section 3. We will see below that the evaluation map is injective for any $q \geqslant 4$, and in this case we get a code of dimension 7.

Let $D \in |-K_X|$. Since $-K_X$ generates the Picard lattice of X, the curve D is irreducible over $\mathbf{F}_q$. If it is absolutely irreducible, the adjunction formula gives

$$2p_a(D) - 2 = D \cdot (D + K_X) = -K_X \cdot (-K_X + K_X) = 0.$$

Thus, D has arithmetic genus 1 and contains at most $q + 1 + \lfloor 2\sqrt{q} \rfloor$ rational points.

If $D \in |-K_X|$ is irreducible but not absolutely irreducible, we apply Lemma 3.2. Hence, the maximal number of points of a section comes from an absolutely irreducible one, and the minimum distance is at least $q^2 - 2q - \lfloor 2\sqrt{q} \rfloor$. This number is positive as long as $q \geqslant 4$, and we get

Proposition 4.4. *Assume we have $q \geqslant 4$. Then the code $\mathcal{C}(X, -K_X)$ has parameters*

$$[q^2 - q + 1, 7, \geqslant q^2 - 2q - \lfloor 2\sqrt{q} \rfloor].$$

Parameters of the codes for small values of q are summarized in Table 3. For $q \in \{5, 7, 8, 9\}$ these codes attain the parameters of the best known codes listed in *codetable* databse[Gra07].

q	4	5	7	8	9
$[n, k, d]$	$[13, 7, \geqslant 4]$	$[21, 7, \geqslant 11]$	$[43, 7, \geqslant 30]$	$[57, 7, \geqslant 43]$	$[73, 7, \geqslant 57]$

TABLE 3. Parameters of codes from our degree 6 del Pezzo surface

Remark 4.5. For $q = 4$, `magma` [BCP97] calculations give a minimum distance equal to 5, instead of 4, for all (random) choices of the points p_i we made. Actually the anticanonical system does not carry any maximal elliptic curve, as we shall prove in Section 4.4. This is no longer true when $q \geqslant 5$: the minimum distances we observe are those given in the above Proposition.

4.3. **Automorphisms of the surface.** Asking `magma` for the automorphism groups of these codes, we get respectively groups of order $234, 504, 1548$ for $q = 4, 5, 7$. Among these are the $q - 1$ multiplications by scalars. The aim of this section is to show that the remaining $6(q^2 - q + 1)$ automorphisms come from automorphisms of the surface.

We first describe the geometric group of automorphisms of the split degree 6 del Pezzo surface X_0 obtained by blowing up the projective plane at the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$. Note that X_0 is a toric variety whose maximal torus we note $\mathbf{G}_m^2$. From [Dol12, Theorem 8.4.2], we have

$$\mathrm{Aut}(X_0) = \mathfrak{D}_{12} \ltimes (\overline{\mathbf{F}}_q^*)^2$$

where $\mathfrak{D}_{12}$ is the dihedral group of order 12, which is the Weyl group $W(\mathbf{E}_3)$ of isometries of the geometric Picard lattice of a degree 6 del Pezzo surface.

We first describe the action of these automorphisms on the maximal torus $\mathbf{G}_m^2$; since the group $\mathfrak{D}_{12}$ is generated by the permutations (of the projective coordinates) $g_1 = (12)$, $g_2 = (123)$, and the standard quadratic transform g_3, we get

$$g_1(x, y) = (y, x), \ g_2(x, y) = \left(\frac{y}{x}, \frac{1}{x}\right), \ g_3(x, y) = \left(\frac{1}{x}, \frac{1}{y}\right)$$

and $(a, b) \in (\overline{\mathbf{F}}_q^*)^2$ acts by $(a, b)(x, y) = (ax, by)$.

For any $g \in \mathfrak{D}_{12}$ and $(a, b) \in (\overline{\mathbf{F}}_q^*)^2$, set $g \cdot (a, b) := g \circ (a, b)$ the composition of the above actions. One easily verifies the following assertions

- we have $g(a, b) = g \cdot (a, b) \cdot g^{-1}$, where $g(a, b)$ is the image of (a, b) by the action of g described above.
- the action of σ on $\mathrm{Aut}(X_0)$ is given by ${}^{\sigma}(g \cdot (a, b)) = g \cdot (a^q, b^q)$, since the surface X_0 is split and the action of the Weyl group is defined over $\mathbf{F}_q$.

We now determine $H^1(G, \mathrm{Aut}(X_0))$, which gives the isomorphism classes of degree 6 del Pezzo surfaces over $\mathbf{F}_q$ (all these surfaces become isomorphic over the algebraic closure $\overline{\mathbf{F}}_q$). We have a short exact sequence (with a trivial Galois action on the diedral group)

$$1 \to (\overline{\mathbf{F}}_q^*)^2 \to \mathrm{Aut}(X_0) \to \mathfrak{D}_{12} \to 1$$

which is split by the map $g \mapsto g \cdot (1,1)$. From [Ser94, I.5.5 Proposition 38], we have a map

$$H^1(G, \mathrm{Aut}(X_0)) \to H^1(G, \mathfrak{D}_{12})$$

and, by functoriality of H^1, the splitting above ensures us that this map is surjective. In order to show that it is injective, it is sufficient to show that for any twist ${}_g(\overline{\mathbf{F}}_q^*)^2$, the cohomology group $H^1(G, {}_g(\overline{\mathbf{F}}_q^*)^2)$ vanishes [Ser94, I.5.5 Corollaire 2]. But this is a consequence of Lang's theorem [Ser94, III.2.3 Théorème 1'] since in any case the twist is a smooth connected algebraic group.

Remark 4.6. Note that, since the groups $\mathrm{Aut}(X_0)$ and $\mathfrak{D}_{12}$ are not abelian, the H^1's are not groups but only pointed sets. For this reason, the injectivity of the map cannot be proved using Hilbert 90 Theorem.

We deduce that the set $H^1(G, \mathrm{Aut}(X_0))$ corresponds to the set of conjugacy classes of elements of the group $\mathfrak{D}_{12}$. With this at hand, we are ready to prove the following statement.

Proposition 4.7. *There are $6(q^2 - q + 1)$ automorphisms of the surface X defined over $\mathbf{F}_q$.*

Proof. The eigenvalues of the matrix M of (4.1) are $1, -1, \jmath, \bar{\jmath}$. Therefore, this matrix has order 6 and, from our calculation of $H^1(G, \mathrm{Aut}(X_0))$, the degree 6 del Pezzo surface X constructed above corresponds to an element of order 6 in $\mathfrak{D}_{12}$. There is only one up to conjugacy, and we choose $\gamma := g_2 g_3$. In other words, if $\varphi : X_0 \to X$ is an isomorphism over $\overline{\mathbf{F}}_q$, the cocyle corresponding to X, $c_X = c_\varphi$ sends σ on γ.

Let h_X denote an automorphism of X over $\overline{\mathbf{F}}_q$; from it we construct an automorphism $h = \varphi^{-1} \circ h_X \circ \varphi$ of X_0. Now h_X is defined over $\mathbf{F}_q$ if and only if we have ${}^\sigma h_X = h_X$, that is

$${}^\sigma h = ({}^\sigma\varphi)^{-1} \circ {}^\sigma h_X \circ {}^\sigma\varphi = ({}^\sigma\varphi)^{-1} \circ h_X \circ {}^\sigma\varphi = c_X(\sigma)^{-1} \circ h \circ c_X(\sigma) = \gamma^{-1} \circ h \circ \gamma$$

If we write $h = g \cdot (a,b)$ as above, we get the condition

$$g \cdot (a^q, b^q) = \gamma^{-1} g \cdot (a,b)\gamma = \gamma^{-1} g\gamma \cdot \gamma^{-1}(a,b)\gamma = \gamma^{-1} g\gamma \cdot \left(b, \frac{b}{a}\right)$$

and the automorphisms of X are the $h_X = \varphi \circ g \cdot (a,b) \circ \varphi^{-1}$, where g is in the centralizer of γ, which is the order 6 subgroup of $\mathfrak{D}_{12}$ generated by γ, and $(a,b) \in (\overline{\mathbf{F}}_q^*)^2$ satisfies $b = a^q$, $\frac{b}{a} = b^q$, i.e. $a^{q^2-q+1} = 1$, $b = a^q$. □

4.4. **Improving the minimum distance over the field with four elements.** As we observed in Remark 4.5, the minimum distance of the anticanonical code is one more than the bound given in Proposition 4.4 when $q = 4$. We prove this fact here.

Assume that the anticanonical linear system carries a maximal elliptic curve Z defined over $\mathbf{F}_4$, i.e. with $\sharp Z(\mathbf{F}_4) = N_4(1) = 9$. This curve must be smooth since its geometric genus equals its arithmetic genus.

We begin with a lemma about the automorphisms of the surface X and their action on the curve Z.

Lemma 4.8. *The automorphism group of X satisfies the following properties.*

(1) The group $\mathrm{Aut}(X)$ contains an element of order 13, which permutes cyclically the set $X(\mathbf{F}_4)$.
(2) There exists an $h \in \mathrm{Aut}(X)$ such that $h(Z)$ contains the points l and c.

Proof. The order of $\mathrm{Aut}(X)$ is $78 = 6 \cdot 13$ from Proposition 4.7; thus this group contains an element f of order 13 from a theorem of Cauchy. Since $\sharp X(\mathbf{F}_4) = 13$, either f permutes cyclically the rational points of X, or its fixes all of them.

The automorphism f preserves the exceptional divisors of X; as a consequence, it induces an automorphism on the complementary U of these divisors in X. As we have seen in 4.3, the surface $X \otimes \overline{\mathbf{F}}_4$ is isomorphic to $X_0 \otimes \overline{\mathbf{F}}_4$, and the image of U under such an isomorphism is the maximal torus $\mathbf{G}_m^2$.

Thus f induces an automorphism of X_0, i.e. an element in $\mathfrak{D}_{12} \ltimes (\overline{\mathbf{F}}_4^*)^2$; since f has order 13, its image must lie in $(\overline{\mathbf{F}}_4^*)^2$ and have the form $(a, b) \neq (1, 1)$. Since the automorphism $(x, y) \mapsto (ax, by)$ has no fixed point on the maximal torus, f does not have any fixed point on U.

Now the exceptional divisors of X are the images under ψ of the strict transforms of the lines $(p_i p_j)$, $1 \leqslant i \leqslant 2$, $3 \leqslant j \leqslant 5$. They are cyclically permuted by the action of the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}_4/\mathbf{F}_4)$; any rational point on one of these divisors must lie on all, which does not happen. We get $X(\mathbf{F}_4) = U(\mathbf{F}_4)$, and f has no fixed point in this set. This proves the first assertion.

From above, since c and l lie in $X(\mathbf{F}_4)$, there exists some $1 \leqslant i \leqslant 12$ such that $l = f^i(c)$; replacing f by f^i we assume that $l = f(c)$ in the following. If we write $Z(\mathbf{F}_4) = \{f^{i_1}(c), \ldots, f^{i_9}(c)\}$ for some $0 \leqslant i_1 < \ldots < i_9 \leqslant 12$, then at least two of the i_j's are consecutive, say $i_2 = i_1 + 1$; the automorphism $h = f^{-i_1}$ satisfies the requirements of the second assertion. □

Thus we can assume that Z is a maximal curve containing l and c. Denote by $\widetilde{Z} \subset \widetilde{X}$ its strict transform under ψ; since Z is smooth, it has multiplicity one at l and c, and we have $\psi^* Z = \widetilde{Z} + \widetilde{L} + \widetilde{C}$ in $\mathrm{Pic}(\widetilde{X})$. Moreover ψ induces an isomorphism between the curves Z and $\widetilde{Z}$.

The curve $\widetilde{Z}$ lies in the anticanonical system of $\widetilde{X}$; it is smooth and since for any $1 \leqslant i \leqslant 5$, we have $Z \cdot E_i = (-K_X) \cdot E_i = 1$, then Z is transversal to the exceptional divisors $E_1, \ldots, E_5$. As a consequence, it is isomorphic to its image Y under π, which is a smooth cubic passing through the points $p_1, \ldots, p_5$.

Thus the curve Y is a smooth elliptic curve in $\mathbf{P}^2$ having 9 rational points. Its Frobenius eigenvalues must be equal to -2, and we have

$$\sharp Y(\mathbf{F}_{16}) = 16 + 1 - (-2)^2 - (-2)^2 = 9 = \sharp Y(\mathbf{F}_4).$$

But Y contains p_1 and p_2 which are defined over $\mathbf{F}_{16}$ but not over $\mathbf{F}_4$, a contradiction. We deduce that the anticanonical linear system does not carry any maximal curve over $\mathbf{F}_4$, and the minimum distance of the anticanonical code is at least 5. Consequently, thanks to the Griesmer bound, we get the following result.

Proposition 4.9. *The code $\mathcal{C}(X, -K_X)$ over $\mathbf{F}_4$ has parameters* $[13, 7, 5]$.

5. Anticanonical codes on some degree five del Pezzo surfaces

In this section, we construct some degree five del Pezzo surfaces with Picard rank one over any finite field, then we determine the parameters of the anticanonical codes on these surfaces. Many arguments are similar to those of the preceding section, for this reason we shall skip some proofs.

A new feature here is that we try to be as constructive as possible: we describe explicit constructions of the codes and their automorphisms.

5.1. **Construction of the surface.** We denote by $\mathbf{P}^2$ the projective plane defined over $\mathbf{F}_q$, and we choose

$$p_1, p_2 = p_1^{\sigma}, p_3 = p_1^{\sigma^2}, p_4 = p_1^{\sigma^3}, p_5 = p_1^{\sigma^4}$$

five conjugate points in $\mathbf{P}^2(\mathbf{F}_{q^5}) \backslash \mathbf{P}^2(\mathbf{F}_q)$. We assume that they are in general position, i.e. no three are collinear. This is possible for any q since a smooth conic in $\mathbf{P}^2$ has $q^5 - q$ points defined over $\mathbf{F}_{q^5}$ but not over $\mathbf{F}_q$.

Let $\widetilde{X}$ be the surface obtained by blowing up the plane $\mathbf{P}^2$ at the p_i's. Once again we get a degree 4 del Pezzo surface and denote by E_0 the pullback of the class of a line in $\mathbf{P}^2$ and by $E_1, \ldots, E_5$ the exceptional divisors. The descriptions of the geometric Picard lattice of $\widetilde{X}$, and its intersection pairing are the same as in Section 4, as for its canonical divisor. The difference here is that the map σ^* acts on the E_i's as the permutation $(E_0)(E_1E_2E_3E_4E_5)$; as a consequence, the Picard lattice $\mathrm{Pic}(\widetilde{X})$ has rank 2, and is generated by E_0 and $\sum_{i=1}^5 E_i$.

Let C denote the unique conic passing through $p_1, \ldots, p_5$; it is defined over $\mathbf{F}_q$. Its strict transform $\widetilde{C}$ in $\widetilde{X}$ is an irreducible curve, whose class satisfies

$$\widetilde{C} = 2E_0 - \sum_{i=1}^{5} E_i \in \mathrm{Pic}(\widetilde{X}).$$

Once again this curve has self-intersection -1 and arithmetic genus zero.

Applying Castelnuovo's contractibility criterion, we obtain a smooth surface X by contracting the curve $\widetilde{C}$ in $\widetilde{X}$, and a birational morphism $\psi : \widetilde{X} \to X$. If we set $c = \psi(\widetilde{C})$, then ψ is the blowup of X at c, with exceptional divisor $\widetilde{C}$.

The geometric Picard lattice $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ can be identified to the orthogonal in $\mathrm{Pic}(\widetilde{X} \otimes \overline{\mathbf{F}}_q)$ of the class of $\widetilde{C}$ (see Lemma 4.1). After similar calculations, we get the following "orthonormal" basis for $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$:

$$F_0 = 3E_0 - 2E_1 - \sum_{i=2}^{5} E_i;$$
$$F_i = E_0 - E_1 - E_{i+1}, \quad \text{for} \quad 1 \leqslant i \leqslant 4$$

These classes satisfy $F_0^{\cdot 2} = 1$, $F_i^{\cdot 2} = -1$ for any $1 \leqslant i \leqslant 4$ and $F_i \cdot F_j = 0$ for any $i \neq j$.

Remark 5.1. Note that the classes $F_1, \ldots, F_4$ contain respectively the strict transforms of the lines $(p_1p_2), \ldots, (p_1p_5)$ and the corresponding curves satisfy Castelnuovo's contractibility criterion.

The canonical divisor of X satisfies $\psi^* K_X = -5E_0 + 2\sum_{i=1}^5 E_i$, and we get $K_X = -3F_0 + \sum_{i=1}^4 F_i$ via the above identification. The matrix of the image σ^* of

Frobenius acting on $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ with respect to the basis $\{F_0, F_1, F_2, F_3, F_4\}$ is

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{pmatrix}$$

and has trace zero.

We conclude that X is a degree 5 del Pezzo surface defined over $\mathbf{F}_q$, with Picard lattice $\mathrm{Pic}(X)$ having rank 1 and generated by K_X. It has q^2+1 rational points.

5.2. **Anticanonical codes.** We consider the evaluation code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$. Its length is $\sharp X(\mathbf{F}_q) = q^2+1$, and its dimension is at most $h^0(X, -K_X) = 6$.

The anticanonical class $-K_X$ generates $\mathrm{Pic}(X)$. As a consequence, the sections of its linear system that are defined over $\mathbf{F}_q$ are irreducible over $\mathbf{F}_q$. Let D denote such a section.

- If D is absolutely irreducible, then it has arithmetic genus 1 from the adjunction formula, and has at most $q+1+\lfloor 2\sqrt{q} \rfloor$ rational points from Hasse-Weil-Serre bound.
- If it is not absolutely irreducible, we apply Lemma 3.2.

Once again, the sections having maximal number of zeroes are the absolutely irreducible ones. Moreover the evaluation map is injective for any $q \geqslant 3$, and we deduce the following.

Proposition 5.2. *Assume $q \geqslant 3$. The code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$ has parameters*

$$[q^2+1, 6, \geqslant q^2 - q - \lfloor 2\sqrt{q} \rfloor].$$

Parameters of the code for the small values of q are summarised in Table 5.2.

q	3	4	5	7	8	9
$[n,k,d]$	$[10,6,3]$	$[17,6,8]$	$[26,6,16]$	$[50,6,37]$	$[\mathbf{65},\mathbf{6},\mathbf{51}]$	$[\mathbf{82},\mathbf{6},\mathbf{66}]$

Table 4. Parameters of the codes from our degree 5 del Pezzo surface

For $q \in \{4,5,7\}$ the parameters of the best known codes are reached. For $q \in \{8,9\}$ our codes beat the best known codes listed in [Gra07]; the best known parameters up to now were respectively $[65,6,50]_8$ and $[82,6,65]_9$.

5.3. **Geometric constructions of the code.** In this section we give two geometric descriptions of the code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$.

We first use the anticanonical embedding of the surface X. We get a surface in $\mathbf{P}^5$, and the code can be seen as the code of hyperplane sections on this surface. Then we construct the code as the puncturing of an evaluation code at the rational points of the projective space.

5.3.1. *First description.* The class of the anticanonical divisor $-K_X$ in $\operatorname{Pic}(\widetilde{X})$ is $5E_0 - 2\sum_{i=1}^{5} E_i$. Its sections come from quintics in $\mathbf{P}^2$ having double points at the p_i's, $1 \leqslant i \leqslant 5$. These quintics form a linear system of (projective) dimension 5.

We use [GSHPBS12, Theorem 5], and its proof. We know that this linear system defines a rational map

$$S : \mathbf{P}^2 \dashrightarrow \mathbf{P}^5.$$

Its image is a degree 5 del Pezzo surface whose ten lines are the strict transforms of the lines $(p_i p_j)$, $1 \leqslant i < j \leqslant 5$. The map S has no unassigned base point, thus its only base points are the p_i, $1 \leqslant i \leqslant 5$. In order to resolve them, we consider the surface obtained by their blowup. This is the surface $\widetilde{X}$. Now the image of S is obtained by contracting the strict transform of the conic $(p_1 \dots p_5)$, and this is the anticanonical model of X.

As a consequence, the anticanonical code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$ is the code obtained by evaluating the global sections of the sheaf $\mathcal{O}(1)$ on $\mathbf{P}^5$ (these are linear forms in six variables) at the rational points of the image of S.

5.3.2. *Second description.* We give another description of this code, as the puncturing of an evaluation code at the rational points of the projective plane.

Consider the six dimensional $\mathbf{F}_q$-linear space E of quintics in $\mathbf{P}^2$ having double points at the p_i's, $1 \leqslant i \leqslant 5$. Let h_C denote an equation of the conic $(p_1 \dots p_5)$ in $\mathbf{P}^2$. Then E contains any product $f h_C$, where f is the equation of a cubic through the p_i's. The space of these cubics has dimension 5, of which we fix a basis $\{f_1, \dots, f_5\}$. Finally, we choose a quintic Q in E not containing C. Denote by h_Q an equation of Q. In this way, we get a basis $\mathcal{B} = \{f_1 h_C, \dots, f_5 h_C, h_Q\}$ of E over $\mathbf{F}_q$. Evaluating the elements of E at the rational points $p \in \mathbf{P}^2(\mathbf{F}_q)$, we get a linear code $\mathcal{C}'$ of length $q^2 + q + 1$. The conic C is smooth, and it contains exactly $q + 1$ rational points $r_1, \dots, r_{q+1}$. For any $1 \leqslant i \leqslant q + 1$, the column corresponding to r_i of the generating matrix of $\mathcal{C}'$ associated to $\mathcal{B}$ has five zeroes at the first positions, and a non zero coefficient at the last one (note that Q intersects C only at the p_i's from Bezout's theorem).

From now on, we consider the rational map

$$S : \left\{ \begin{array}{ccc} \mathbf{P}^2 & \dashrightarrow & \mathbf{P}^5 \\ P & \longmapsto & (f_1(P)h_C(P) : \dots : f_5(P)h_C(P) : h_Q(P)) \end{array} \right.$$

associated to the basis $\mathcal{B}$. It has image X from above, and it induces a surjective map on rational points $\mathbf{P}^2(\mathbf{F}_q) \to X(\mathbf{F}_q)$. From the description of S above, the restriction of this map to $\mathbf{P}^2(\mathbf{F}_q) \setminus C(\mathbf{F}_q)$ is injective, and all rational points $r_1, \dots, r_{q+1}$ of C are sent to the point $(0 : \cdots : 0 : 1)$.

Thus the columns of the generating matrix of the code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$ are the ${}^t(f_1(P)h_C(P) \ \dots \ f_5(P)h_C(P) \ h_Q(P))$, where P describes $\mathbf{P}^2(\mathbf{F}_q) \setminus C(\mathbf{F}_q)$, and ${}^t(0 \dots 0 \ 1)$ if $P \in C(\mathbf{F}_q)$. We conclude that the code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$ is obtained from the code $\mathcal{C}'$ by puncturing the positions corresponding to the points $r_2, \dots, r_{q+1}$.

5.4. **Automorphisms.** Our aim here is to show that the surface X admits an order five automorphism defined over $\mathbf{F}_q$, and to describe it explicitely. The proof of the existence is very close to the proof of Proposition 4.7, and we expose it briefly (see also [Sko01, Theorem 3.1.3]).

First denote by X_0 the split degree 5 del Pezzo surface over $\mathbf{F}_q$ defined by the blowup of the projective plane at the points:

$$q_1 := (1:0:0) \quad q_2 := (0:1:0) \quad q_3 := (0:0:1) \quad q_4 := (1:1:1).$$

From [Dol12, Theorem 8.5.6], the geometric automorphism group of X_0, $\mathrm{Aut}(X_0)$, is isomorphic to the group $W(\mathbf{E}_4)$, i.e. to the symmetric group on five letters $\mathfrak{S}_5$ with trivial Galois action (the surface is split).

As a consequence, the set $H^1(G, \mathrm{Aut}(X_0))$ is the set of conjugacy classes of elements of $\mathfrak{S}_5$. The cocycle associated to the surface X constructed above sends σ to γ, any of the (conjugated) order 5 elements in $\mathfrak{S}_5$. As a consequence, the geometric automorphisms of X coming from an automorphism defined over $\mathbf{F}_q$ are exactly those corresponding to the elements in the centralizer of γ in $\mathfrak{S}_5$. But this is the subgroup generated by γ.

We construct explicitely an order 5 automorphism of X defined over $\mathbf{F}_q$; actually we will show that it is induced by a linear automorphism of $\mathbf{P}^5$. We use the preceding discussion. We first determine an isomorphism $\varphi : X_0 \to X$. Now the automorphisms of the surface X_0 are described in [Dol12, Section 8.5.4] and the automorphism we are looking for is conjugated under φ to the automorphism of X_0 which is the image of the Frobenius σ under the cocycle c_φ.

Recall that the surface $\widetilde{X}$ is a degree 4 del Pezzo surface. We consider two geometric markings, which lead to the following diagram

$$\begin{tikzcd}
& \widetilde{X} \arrow[ddl, "\pi"'] \arrow[dr, "\psi"] & & \\
& & X \arrow[dr, "\psi'"] & \\
\mathbf{P}^2 \arrow[rrr, dashed, "\Phi"] & & & \mathbf{P}^2
\end{tikzcd}$$

We get two sequences of contractions from $\widetilde{X}$ to $\mathbf{P}^2$

- π contracts the exceptional divisors $E_1, \dots, E_5$ (the first geometric marking),
- ψ contracts the exceptional divisor $\widetilde{C}$, and ψ' contracts $F_1, \dots, F_4$, sending them respectively to the points q_3, q_1, q_2, q_4.

Note that the group PGL_3 acts transitively on the 4-tuples of points in $\mathbf{P}^2$, which allows the last hypothesis.

We get a birational transform Φ of $\mathbf{P}^2$. Note that Φ is not defined over $\mathbf{F}_q$, since ψ' contracts the family of exceptional divisors $\{F_1, \dots, F_4\}$ which is not stable under the Galois action. The base change matrix (in $\mathrm{Pic}(\widetilde{X} \otimes \overline{\mathbf{F}}_q)$) from $\{E_0, E_1, \dots, E_5\}$ to $\{F_0, F_1, \dots, F_4, \widetilde{C}\}$ is the characteristic matrix of Φ [Dol12, Example 8.2.38]:

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 2 \\ -2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

From the first column (this is the characteristic of Φ [Dol12, 7.5.1]), we deduce that Φ has the form $\Phi(x:y:z) = (h_0 : h_1 : h_2)$, where $\{h_0, h_1, h_2\}$ is a basis of the space of cubics having multiplicity at least two at p_1 and passing through p_2, p_3, p_4, p_5.

One checks that Φ contracts the lines (p_1p_i), $2 \leqslant i \leqslant 5$, and the conic C (actually, for each of these curves, the linear system of the above cubics containing it is one-dimensional). From our assumption on the images of the F_i's, which are the strict transforms of the lines (p_1p_{i+1}), $1 \leqslant i \leqslant 4$, we get the following explicit description of Φ, where ℓ_{ij} is the equation of the line (p_ip_j)

$$\Phi : (x:y:z) \mapsto (u\ell_{12}\ell_{14}\ell_{35} : v\ell_{12}\ell_{13}\ell_{45} : w\ell_{13}\ell_{14}\ell_{25})$$

and the coefficients u, v, w are determined by the condition $\Phi((p_1p_5)) = q_4$. Let $p := \lambda p_1 + \mu p_5$, $(\lambda : \mu) \in \mathbf{P}^1$, denote any point of (p_1p_5); the point $\Phi(p)$ has homogeneous coordinates

$$(\lambda\mu^2 u\ell_{12}(p_5)\ell_{14}(p_5)\ell_{35}(p_1) : \lambda\mu^2 v\ell_{12}(p_5)\ell_{13}(p_5)\ell_{45}(p_1) : \lambda\mu^2 w\ell_{13}(p_5)\ell_{14}(p_5)\ell_{25}(p_1))$$

and we get the following equations defining u, v, w

$$u\ell_{12}(p_5)\ell_{14}(p_5)\ell_{35}(p_1) = v\ell_{12}(p_5)\ell_{13}(p_5)\ell_{45}(p_1) = w\ell_{13}(p_5)\ell_{14}(p_5)\ell_{25}(p_1).$$

Consider the linear system of cubics passing through the points q_1, q_2, q_3, q_4; we get a map

$$S_0 : \left\{ \begin{array}{ccc} \mathbf{P}^2 & \dashrightarrow & \mathbf{P}^5 \\ (x:y:z) & \longmapsto & (y_0 : y_1 : y_2 : y_3 : y_4 : y_5) \end{array} \right.$$

where

$$\begin{array}{lll} y_0 := xy(x-z) & y_1 := xz(x-y) & y_2 := xy(y-z) \\ y_3 := yz(y-x) & y_4 := xz(z-y) & y_5 := yz(z-x) \end{array}$$

and the closure of its image is the anticanonical model of the surface X_0.

Let us make explicit the composition $S_0 \circ \Phi$; write it as the map $(x:y:z) \mapsto (g_0, \ldots, g_5)$. When we compute the first coordinate, we get

$$u\ell_{12}\ell_{14}\ell_{35}v\ell_{12}\ell_{13}\ell_{45}(\ell_{14}(u\ell_{12}\ell_{35} - w\ell_{13}\ell_{25})).$$

Now if $p := \lambda p_1 + \mu p_5$, $(\lambda : \mu) \in \mathbf{P}^1$, is any point of (p_1p_5), we have

$$u\ell_{12}(p)\ell_{35}(p) - w\ell_{13}(p)\ell_{25}(p) = \lambda\mu(u\ell_{12}(p_5)\ell_{35}(p_1) - w\ell_{13}(p_5)\ell_{25}(p_1)) = 0$$

from the equations defining u, v, w, and ℓ_{15} divides $u\ell_{12}\ell_{35} - w\ell_{13}\ell_{25}$. As this last polynomial defines a conic passing through p_2 and p_3, we must have $u\ell_{12}\ell_{35} - w\ell_{13}\ell_{25} = \alpha_0\ell_{15}\ell_{23}$ for some $\alpha_0 \in \overline{\mathbf{F}}_q$.

When we apply the same calculations to the g_i's, $1 \leqslant i \leqslant 5$, and simplify by $\ell_{12}\ell_{13}\ell_{14}\ell_{15}$, we get

$$\begin{array}{ll} g_0 = \alpha_0 uv\ell_{12}\ell_{14}\ell_{23}\ell_{35}\ell_{45}, & g_1 = \alpha_1 uw\ell_{12}\ell_{14}\ell_{25}\ell_{35}\ell_{34}, \\ g_2 = \alpha_2 uv\ell_{12}\ell_{13}\ell_{24}\ell_{35}\ell_{45}, & g_3 = -\alpha_1 vw\ell_{12}\ell_{13}\ell_{34}\ell_{25}\ell_{45}, \\ g_4 = -\alpha_2 uw\ell_{13}\ell_{14}\ell_{24}\ell_{25}\ell_{35}, & g_5 = -\alpha_0 vw\ell_{13}\ell_{14}\ell_{23}\ell_{25}\ell_{45} \end{array}$$

where α_1, α_2 are defined by

$$\alpha_1\ell_{15}\ell_{34} = u\ell_{14}\ell_{35} - v\ell_{13}\ell_{45},\ \alpha_2\ell_{15}\ell_{24} = v\ell_{12}\ell_{45} - w\ell_{14}\ell_{25}.$$

Observe that $\{g_0, \ldots, g_5\}$ is a new basis for the $\overline{\mathbf{F}}_q$-vector space of quintics passing through $p_1, \ldots, p_5$ with multiplicity at least two. As a consequence, there is an element M in $\mathrm{GL}_6(\overline{\mathbf{F}}_q)$ that sends the basis $\{g_0, \ldots, g_5\}$ to $\mathcal{B}\{f_1h_C, \ldots, f_5h_C, h_Q\}$, and it induces a linear automorphism of $\mathbf{P}^5$ whose restriction to X_0 is an isomorphism $\varphi : X_0 \to X$; we get a diagram

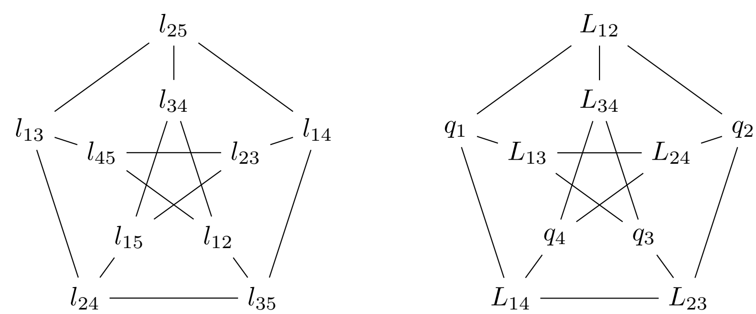
$$\begin{tikzcd} X & X_0 \arrow[l, "\varphi"'] \\ \mathbf{P}^2 \arrow[u, dashed, "S"] \arrow[r, dashed, "\Phi"] & \mathbf{P}^2 \arrow[u, dashed, "S_0"'] \end{tikzcd}$$

Let us write the incidence graphs of the surfaces X and X_0 [Dol12, Section 8.5.1], [Man86, 26.9]. The exceptional lines on X correspond to the following ten classes

$$l_{ij} := E_0 - E_i - E_j, \quad 1 \leqslant i < j \leqslant 5$$

in $\mathrm{Pic}(X \otimes \overline{\mathbf{F}}_q)$ (the notation l_{ij} is chosen to recall that l_{ij} is the image in X of the strict transform in $\widetilde{X}$ of the line $(p_i p_j)$ in $\mathbf{P}^2$). The exceptional divisors of X_0 are labelled from the points q_i and lines $L_{ij} := (q_i q_j)$, $1 \leqslant i < j \leqslant 4$ of $\mathbf{P}^2$ corresponding to the exceptional divisors on X_0.

Both are Petersen graphs, and the automorphism group $\mathrm{Aut}_{\overline{\mathbf{F}}_q}(X)$ is isomorphic to the group of symmetries of these graphs [GSHPBS12, Lemma 13].



Now φ sends each vertex of the right-hand graph to the vertex at the same place of the left-hand one. As a consequence, since the Frobenius automorphism acts as the rotation with angle $\frac{2\pi}{5}$ on the left-hand graph, the cocycle associated to φ sends it to the automorphism acting as the rotation with angle $\frac{2\pi}{5}$ on the right-hand one.

The automorphism of X_0 acting as above on the graph is given in [Dol12, Section 8.5.4]; it comes from the birational map of $\mathbf{P}^2$ (or Cremona transformation)

$$\delta : (x : y : z) \mapsto (x(z-y) : z(x-y) : xz),$$

whose action induces an automorphism of the anticanonical model of X_0 that comes from the following linear action

$$D : (y_0, y_1, y_2, y_3, y_4, y_5) \mapsto (y_0 - y_1 - y_2, y_0 - y_1 + y_4, -y_3 - y_4 + y_5, y_1 - y_4 + y_5, y_4, y_1).$$

From our calculations in Galois cohomology at the beginning of the section, we see that the automorphism of X conjugated to D under the action of φ is defined over $\mathbf{F}_q$. Summarizing, we get the following diagram

$$\begin{tikzcd} \mathbf{P}^2 \arrow[r, dashed, "\Phi"] \arrow[d, dashed, "S"'] & \mathbf{P}^2 \arrow[r, dashed, "\delta"] \arrow[d, dashed, "S_0"] & \mathbf{P}^2 \arrow[d, dashed, "S_0"'] & \mathbf{P}^2 \arrow[l, dashed, "\Phi"'] \arrow[d, dashed, "S"] \\ \mathbf{P}^5 \arrow[rrr, bend right, "A"'] & \mathbf{P}^5 \arrow[l, "M"'] \arrow[r, "D"] & \mathbf{P}^5 \arrow[r, "M"] & \mathbf{P}^5 \end{tikzcd}$$

and an order five automorphism of X defined over $\mathbf{F}_q$ acts on its anticanonical model by the restriction of the linear action $A = M^{-1} \circ D \circ M$ on $\mathbf{P}^5$.

6. Anticanonical codes on some degree four del Pezzo surfaces

In this section, we focus on degree four del Pezzo surfaces and especially those with Picard rank equal to one.

6.1. **Construction of the surface.** Over $\overline{\mathbf{F}}_q$, del Pezzo surfaces of degree 4 are all the blowing up of $\mathbf{P}^2$ in five points in general position. As in degrees 5 or 6, this model may not be defined over $\mathbf{F}_q$. Instead of computing a birational and $\mathbf{F}_q$-rational morphism from $\mathbf{P}^2$ to the considered degree 5 or 6 del Pezzo surface, we adopt a different strategy in degree 4. In fact, it turns out that, following Flynn [Fly09], one can directly compute the anti-canonical model of a degree 4 del Pezzo surface from the Frobenius action on the geometric Picard group, at least when the characteristic is odd. This model, which is defined over the base field, is embedded in $\mathbf{P}^4$ as the intersection of two quadrics. The starting point is still a type of Frobenius action but we have to observe this action from a different point of view to understand Flynn's construction.

So, let X be a degree 4 del Pezzo surface. Geometrically, it is the blowup of $\mathbf{P}^2$ at five points in general position and as before, we denote by E_0 the pullback of the class of a line in $\mathbf{P}^2$ and by $E_1, \ldots, E_5$ the five exceptional divisors. One can prove that X contains exactly ten families of conics whose classes are

$$C_i = E_0 - E_i, \quad \text{and} \quad C_i' = -K_X - C_i = 2E_0 - \sum_{j \neq i} E_j, \quad \text{for} \quad 1 \leqslant i \leqslant 5$$

where $K_X = -3E_0 + \sum_{i=1}^{5} E_i$ is the canonical class [BBFL07, §2,Th 2]. These classes are the only ones satisfying the intersection constraints

$$C \cdot (-K_X) = 2 \qquad \text{and} \qquad C^2 = 0.$$

So the Frobenius σ^* acts on these classes and one can recover the Frobenius action on the whole space $\operatorname{Pic}(X \otimes \overline{\mathbf{F}}_q)$ from this action since one easily checks that the family composed by the classes $\frac{1}{2}\left(-K_X + \sum_{i=1}^{5} C_i\right)$ and the C_i's for $1 \leqslant i \leqslant 5$ is a basis of $\operatorname{Pic}(X \otimes \overline{\mathbf{F}}_q)$.

This action has a strong geometric flavour that we now describe. Let Q_1, Q_2 be two quadrics of $\mathbf{P}^4$ whose intersection defines the anti-canonical embedding of the surface X. Since X is smooth, in the pencil of quadrics of $\mathbf{P}^4$ containing X, i.e. the $\lambda_1 Q_1 - \lambda_2 Q_2$ for $(\lambda_1 : \lambda_2) \in \mathbf{P}^1(\overline{\mathbf{F}}_q)$, there are exactly five singular quadrics [Wit07, §3.3]. The points $(\lambda_1 : \lambda_2) \in \mathbf{P}^1(\overline{\mathbf{F}}_q)$ corresponding to these singular quadrics are conjugate under G since they are the roots of the determinant $\det(\lambda_1 S_1 - \lambda_2 S_2)$, where S_i is the 5×5 symmetric matrix associated to the quadratic form Q_i. The intersection of each singular quadric with its tangent space at a smooth point is the union of two planes. It can be shown that the classes of the intersections of these two planes with the surface X are equal to C_i and C_i' for some $1 \leqslant i \leqslant 5$ [BBFL07, §2,Th 4]. In other terms, to each singular quadric in the pencil containing X corresponds a pair $\{C_i, C_i'\}$.

From the Galois point of view, we deduce how G acts on the ten conics C_i, C_i', $1 \leqslant i \leqslant 5$ [VAV14, §2.4]. First G acts on the set of pairs $\{C_i, C_i'\}$ as it acts on the five conjugates points of $\mathbf{P}^1(\overline{\mathbf{F}}_q)$ associated to the five singular quadrics in the pencil. More precisely, if $(\lambda_1 : \lambda_2) \in \mathbf{P}^1(\overline{\mathbf{F}}_q)$ is one of these points, then the action on the subset of the C_i, C_i''s corresponding to $(\lambda_1 : \lambda_2)$ and all its conjugates is transitive if and only if the two previous planes are not defined over the field of definition $\mathbf{F}_q((\lambda_1 : \lambda_2))$ (but over the quadratic extension of it). In brief, the characterisation of the Frobenius action on the ten classes of conics C_i, C_i', $1 \leqslant i \leqslant 5$ can be reduced to a sequence $d_1[\epsilon_1]\cdots d_r[\epsilon_r]$ with d_i positive integers satisfying $\sum_i d_i = 5$ and with $\epsilon_i \in \{\pm 1\}$. Let us call this data the type of the action. There are three types that lead to a surface with Picard rank equal to one which are listed in Table 5 (see [DD18, Table 3]):

Type	Frobenius action	Eigenvalues of σ^*	$\mathrm{Tr}(\sigma^*)$	Picard rank
4_1	$2[-1]1[-1]1[-1]\,1[-1]$	$1, -1, -1, -1, i, -i$	-2	1
4_2	$4[-1]1[-1]$	$1, -1, \zeta_8 = e^{i\pi/8}, \zeta_8^3, \zeta_8^5, \zeta_8^7$	0	1
4_3	$3[-1]2[-1]$	$1, -1, i, -i, -\jmath, -\jmath^2$	1	1

Table 5. Types of degree 4 del Pezzo surfaces with Picard rank 1.

Unlike the degrees 5 or 6 cases, there are several isomorphism classes in every type of Frobenius action. Flynn develops a very powerful method which, given a type of Frobenius action in the previous sense, computes the anti-canonical model of a del Pezzo surface having this type of action [Fly09, Sko10]. It works as follows, starting from a type $d_1[\epsilon_1]\cdots d_r[\epsilon_r]$. One has to choose r distinct irreducible polynomials $f_1, \ldots, f_r \in \mathbf{F}_q[x]$ of respective degrees $d_1, \ldots, d_r$, and for each $1 \leqslant i \leqslant r$ a non zero element $\delta_i \in \mathbf{F}_q[x]/\langle f_i \rangle$ which is a square or not depending on whether $\epsilon_i = 1$ or -1. Then, we put $f = f_1 \cdots f_r$ and we define $\delta \in \mathbf{F}_q[x]/\langle f \rangle$ in such a way that δ is sent to $(\delta_1, \ldots, \delta_r)$ by the Chinese Remainder isomorphism between $\mathbf{F}_q[x]/\langle f \rangle$ and the product $\prod_i \mathbf{F}_q[x]/\langle f_i \rangle$. Formally, in $\mathbf{F}_q[x]/\langle f \rangle$, we compute the five quadrics $Q_0, \ldots, Q_4$ in $x_0, \ldots, x_4$ such that :

$$\delta \times \left(x_0 + x_1 x + \cdots + x_4 x^4\right)^2 = Q_0(x_0, \ldots, x_4) + \cdots + Q_4(x_0, \ldots, x_4)x^4$$

Then the surface X in $\mathbf{P}^4$, defined by $Q_3(x_0, \ldots, x_4) = Q_4(x_0, \ldots, x_4) = 0$ is a del Pezzo surface of given type.

6.2. **Anticanonical codes.** We now determine the parameters of the evaluation code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$ when X is a del Pezzo surface of degree 4 of Picard rank one. We denote by $\mathrm{Tr}(\sigma^*)$ the trace of the Frobenius morphism, which, as we have noticed, is an element of $\{-2, 0, 1\}$ in our cases. The global sections of the sheaf $\mathcal{O}(-K_X)$ are nothing else than the five coordinate functions $x_0, \ldots, x_4$ and the dimension of the code is thus five. As for the length, it is the number of rational points of X, which is given by $q^2 + 1 + q\,\mathrm{Tr}(\sigma^*)$. The construction of a generator matrix is obvious and only consists in the vertical join of the coordinates vector of the rational points of the surface X:

$$\begin{pmatrix} x_0(p_1) & \cdots & x_0(p_N) \\ \vdots & & \vdots \\ x_4(p_1) & \cdots & x_4(p_N) \end{pmatrix}, \qquad\qquad X(\mathbf{F}_q) = \{p_1, \ldots, p_N\}.$$

Finally, let us compute the minimal distance. Since $-K_X$ is a generator of $\mathrm{Pic}(X)$, any effective divisor $D \in |-K_X|$ must be irreducible over $\mathbf{F}_q$. If D is absolutely irreducible then the (arithmetic) genus $\pi(D)$ of D is 1 since

$$2\pi(D) - 2 = K_X \cdot (D + K_X) = K_X \cdot (-K_X + K_X) = 0.$$

In that case, one has

$$\sharp D(\mathbf{F}_q) \leqslant q + 1 + \lfloor 2\sqrt{q} \rfloor.$$

If D is not absolutely irreducible, then from Lemma 3.2, we have $\sharp D(\mathbf{F}_q) \leqslant 2$. In any case, the number of rational points of D is bounded above by $q + 1 + \lfloor 2\sqrt{q} \rfloor$. We deduce that:

Proposition 6.1. *Assume q odd. Then, the code $\mathcal{C}(X(\mathbf{F}_q), -K_X)$ has parameters*

$$[n, k, d] = \begin{cases} [q^2 - 2q + 1, 5, q^2 - 3q - \lfloor 2\sqrt{q} \rfloor] & \text{for the type } 4_1; \\ [q^2 + 1, 5, q^2 - q - \lfloor 2\sqrt{q} \rfloor] & \text{for the type } 4_2; \\ [q^2 + q + 1, 5, q^2 - \lfloor 2\sqrt{q} \rfloor] & \text{for the type } 4_3. \end{cases}$$

Parameters of such codes for small values of q are listed in Table 6.

	q	3	5	7	9
Type 4_1	$[n, k, d]$		$[16, 5, 6]$	$[36, 5, 23]$	$[64, 5, 48]$
Type 4_2	$[n, k, d]$	$[10, 5, 3]$	$[26, 5, 16]$	$[50, 5, 37]$	$[82, 5, 66]$
Type 4_3	$[n, k, d]$	$[13, 5, 6]$	$[31, 5, 21]$	$[57, 5, 44]$	$[91, 5, 75]$

TABLE 6. Parameters of codes from degree 4 del Pezzo surfaces of rank 1

The codes of the line "Type 4_3" all attain the parameters of the best known codes [Gra07].

Remark 6.2. For $q = 3$ the "Type 4_1" does not exist, since there are not enough non squares in $\mathbf{F}_3$; see also [Tre18, Theorem 1.4 (1)].

Remark 6.3. In fact, the parameters given in proposition 6.1 are also true in even characteristic, at least if the type exists. But, on the contrary the Flynn's construction only works in odd characteristic. Over $\mathbf{F}_8 = \mathbf{F}_2(\zeta)$, by a simple random search, we have found the two quadrics

$$\begin{aligned} Q_1 = {} & \zeta^2 x_0^2 + \zeta^5 x_0 x_1 + \zeta^4 x_1^2 + \zeta x_0 x_2 + \zeta^6 x_1 x_2 + \zeta x_2^2 + \zeta^3 x_0 x_3 \\ & + \zeta^4 x_1 x_3 + \zeta^3 x_2 x_3 + \zeta x_3^2 + x_0 x_4 + \zeta^6 x_1 x_4 + \zeta^6 x_2 x_4 + x_3 x_4 \\ Q_2 = {} & \zeta^3 x_0^2 + \zeta^5 x_0 x_1 + \zeta^2 x_1^2 + \zeta^4 x_2^2 + \zeta^5 x_0 x_3 + \zeta^2 x_1 x_3 + x_2 x_3 + \zeta^5 x_3^2 \\ & + \zeta^4 x_0 x_4 + \zeta x_1 x_4 + \zeta^6 x_2 x_4 + \zeta x_3 x_4 + \zeta^3 x_4^2 \end{aligned}$$

that define a del Pezzo surface of degree 4 over $\mathbf{F}_8$, of "Type 4_3", whose anticanonical code has parameters $[n, k, d] = [73, 5, 59]$.

Let us end this section by displaying a complete example in the type 4_3. We use `magma` for the computations. We start form two polynomials $f_2, f_3 \in \mathbf{F}_q[x]$ of degrees 2 and 3 such that $x \bmod f_2$ and $x \bmod f_3$ are non squares in $\mathbf{F}_{q^2}$ and $\mathbf{F}_{q^3}$. Then the choice $\delta = x$ is good. For example, for $q = 5$, the standard polynomials, $f_2(x) = x^2 + 4x + 2$ and $f_3(x) = x^3 + 3x + 3$ chosen by `magma` to construct $\mathbf{F}_{5^2}$

and $\mathbf{F}_{5^3}$, suit. Their product is $f(x) = x^5 + 4x^4 + 3x + 1$. Using `magma`, we easily compute the two associate quadrics in $\mathbf{F}_5[x_0, \ldots, x_4]$

$$\begin{aligned} Q_3 &= x_1^2 + 2x_0x_2 + 2x_3^2 + 4x_2x_4 + 2x_3x_4 + x_4^2 \\ Q_4 &= 2x_1x_2 + x_2^2 + 2x_0x_3 + 2x_1x_3 \\ &\qquad + 2x_2x_3 + x_3^2 + 2x_0x_4 + 2x_1x_4 + 2x_2x_4 + x_3x_4 + 4x_4^2 \end{aligned}$$

The subvariety of $\mathbf{P}^4$ defined by $Q_1 = Q_2 = 0$ is a degree 4 del Pezzo surface of type 4_3, which contains $31 = 5^2 + 5 + 1$ points. Joining these points in columns, we obtain the generator matrix of the code $\mathcal{C}(X(\mathbf{F}_5), -K_X)$ over $\mathbf{F}_5$:

$$G = \begin{pmatrix} 1304331312042123303230102114231 \\ 0000000001111222233334444412320 \\ 0012224441123012301221113413110 \\ 1340130140331020400141231111100 \\ 1111111111111111111111111100000 \end{pmatrix}$$

This code has parameters $[31, 5, 21]$, as confirmed by `magma`.

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