Linear stability of vertical natural convection

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The equations governing the velocity $\mathbf{u}$ and pressure P in natural convection in Newtonian fluids with small temperature differences (of scale R) were derived by Oberbeck (1879). In dimensionless form:

$$\begin{aligned}\left(\frac{\mathrm{D}}{\mathrm{D}t} - \frac{1}{R}\nabla^2\right)\mathbf{u} &= -\nabla P + \frac{2}{R}T\hat{\boldsymbol{\jmath}} &(1)\\ \left(\sigma\frac{\mathrm{D}}{\mathrm{D}t} - \frac{1}{R}\nabla^2\right)T &= \frac{2m^4}{R}\mathbf{u}\cdot\hat{\boldsymbol{\jmath}} &(2)\\ \nabla\cdot\mathbf{u} &= 0. &(3)\end{aligned}$$

Here we work not with the temperature but with its difference T from a reference stable stratification with gradient $2m^4/\sigma R$. Fluids are distinguished by the Prandtl number σ (≈ 0.7 for air and 7 for water) and $\hat{\boldsymbol{\jmath}}$ is the unit vector in the y or vertical direction.

It is convenient to split the velocity into mean and disturbance parts and, for two-dimensional problems, to represent the velocity disturbance by a stream-function (Stuart, 1958), and similarly for the temperature:

$$\begin{aligned}\mathbf{u} &= \left\{U - \frac{\partial\psi}{\partial y}\right\}\hat{\boldsymbol{\imath}} + \left\{V + \frac{\partial\psi}{\partial x}\right\}\hat{\boldsymbol{\jmath}} &(4)\\ T &= \Theta(x) + \theta(x,y) &(5)\end{aligned}$$

where ψ and θ have zero vertical mean: e.g. $\Theta = \int T\mathrm{d}y$. If U is uniform and $V = V(x)$, mass conservation (3) is automatic.

Steady solutions with $U = 0$ and independent of y are governed by (Ostroumov, 1958, p. 56)

$$\begin{aligned}V''(x) &= -2\Theta(x) &(6)\\ \Theta''(x) &= 2m^4V(x) &(7)\\ \psi = \theta &= 0 &(8)\end{aligned}$$

and include for $m = 0, V(\pm1) = 0, \Theta(\pm1) = \mp1$

$$\begin{aligned}V(x) &= (x^3 - x)/3 &(9)\\ \Theta(x) &= -x &(10)\end{aligned}$$

(Waldmann, 1938); for $m > 0, V(\pm1) = 0, \Theta(\pm1) = \mp1$

$$\begin{aligned}V(x) &= \{\sinh m(1-x)\sin m(1+x) &(11)\\ &\quad - \sinh m(1+x)\sin m(1-x)\}/m^2d\\ \Theta(x) &= \{\cosh m(1-x)\cos m(1+x) &(12)\\ &\quad - \cosh m(1+x)\cos m(1-x)\}/d\end{aligned}$$

where $d = \cosh 2m - \cos 2m$ (Ostroumov, 1958, p. 57); and for $m > 0, V(0) = 0, \Theta(0) = 1, V(\infty) \to 0, \Theta(\infty) \to 0$

$$\begin{aligned}V(x) &= \mathrm{e}^{-x}\sin x &(13)\\ \Theta(x) &= \mathrm{e}^{-x}\cos x &(14)\end{aligned}$$

(Prandtl, 1952, pp. 422–425). In the last case, the half-space has one less length scale than the slot so that m is disposable. The three base solutions are illustrated in figure 1.

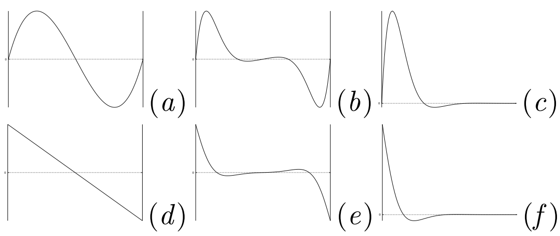

Figure 1: Steady vertical natural convection velocity (a–c) and temperature (d–f) for unstratified slot (a, d), stratified slot with $m = 5$ (b, e), and stratified half-space (c–f).

The Waldmann (9–10) and Prandtl (13–14) solutions can be considered as the $m \to 0$ and $m \to \infty$ limits of the stratified slot solution (12–13), with an appropriate rescaling of x in the latter case.

For small disturbances, the equations for ψ and θ have coefficients independent of t and y so that they can be expanded in normal modes (Drazin and Reid, 1981, p. 11); products of $\exp\{\mathrm{i}\alpha(y - ct)\}$ and functions of x, where α is the (real) wavenumber and c the (complex) wave speed. The equations

$$\begin{aligned}&[(\alpha R)^{-1}(\alpha^2 - D^2)^2 - \{V(\alpha^2 - D^2) + V''\}]\psi\\ &\qquad + 2\mathrm{i}(\alpha R)^{-1}D\theta = -c(\alpha^2 - D^2)\psi \quad (15)\end{aligned}$$

$$\begin{aligned}&[\mathrm{i}\sigma\Theta' + 2m^4(\alpha R)^{-1}D]\psi\\ &\qquad + [\mathrm{i}\sigma V + (\alpha R)^{-1}(\alpha^2 - D^2)]\theta = \mathrm{i}\sigma c\theta \quad (16)\end{aligned}$$

with boundary conditions $\psi = \psi' = \theta = 0$, where $D \equiv \mathrm{d}/\mathrm{d}x$, were derived by Gill and Davey (1969).

Studies of the linear stability of the three base solutions began with Gershuni (1953), Birikh et al. (1969), and Gill and Davey (1969), respectively.

Since the three base solutions are each independent of y, the temperature derivative at the walls, and therefore the heat flux, is too. This means that the same solutions apply to the case of uniform heat flux; apart from Lietzke (1955), this appears to have been little appreciated. It is significant because the fixed temperature condition in the stratified cases implies a wall temperature increasing linearly with height, which seems unnatural and difficult to impose in an experiment. While the base solutions are unchanged, the stability behaviour, particularly for 'thermal' modes is different; the thermal boundary condition being replaced with $\theta' = 0$.

Equations (15)–(16) were discretized by an interior collocation method (Villadsen and Stewart, 1967), leading to a generalized algebraic eigenvalue problem of the form $\mathsf{L}q = c\mathsf{M}q$. Taking care to only impose the no-slip condition $\psi' = 0$ on the viscous operator $(\alpha^2 - D^2)^2$, the 'mass matrix' M is nonsingular and spurious eigenvalues are avoided (Weideman and Reddy, 2000). Thus, $\mathsf{L}q = c\mathsf{M}q$ can be replaced by the standard algebraic eigenvalue problem $\mathsf{M}^{-1}\mathsf{L}q = cq$ and solved by, e.g., the QR algorithm (Watkins, 2000).

For given α and R, stability requires that the spectrum of c lies in the lower half complex plane (Drazin and Reid, 1981, p. 11). Marginal stability curves (tracked by the adaptive skirting algorithm of McBain 2003) for the stratified half-space are presented in figure 2 for $\sigma = 0.7$ and 7, and Dirichlet ($\theta = 0$) and Neumann ($\theta' = 0$) conditions. In all four cases the margins have two lobes, caused by the competition of two modes of instability: the 'hydrodynamic' (higher wavenumber) and 'thermal' (lower α). The latter are much more sensitive to σ. The Dirichlet results agree with those of Gill and Davey (1969). At both Prandtl numbers the loss of the stabilizing effect of wall conduction in changing from Dirichlet to Neumann is evident in the shift of the thermal lobe of each marginal curve to the left: the effect being particularly pronounced for $\sigma = 0.7$.

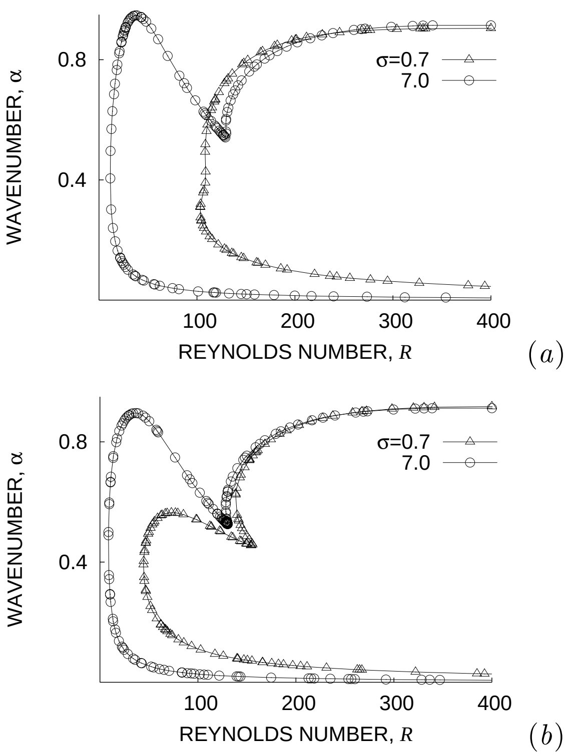

Figure 2: Marginal stability curves for half space with (a) Dirichlet and (b) Neumann thermal boundary condition.

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