SHARP SPECTRAL MULTIPLIERS WITHOUT SEMIGROUP FRAMEWORK AND APPLICATION TO RANDOM WALKS

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Abstract. In this paper we prove spectral multiplier theorems for abstract self-adjoint operators on spaces of homogeneous type. We have two main objectives. The first one is to work outside the semigroup context. In contrast to previous works on this subject, we do not make any assumption on the semigroup. The second objective is to consider polynomial off-diagonal decay instead of exponential one. Our approach and results lead to new applications to several operators such as differential operators, pseudo-differential operators as well as Markov chains. In our general context we introduce a restriction type estimates à la Stein-Tomas. This allows us to obtain sharp spectral multiplier theorems and hence sharp Bochner-Riesz summability results. Finally, we consider the random walk on the integer lattice $\mathbb{Z}^n$ and prove sharp Bochner-Riesz summability results similar to those known for the standard Laplacian on $\mathbb{R}^n$.

Contents

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1. Introduction

Let (X, d, μ) be a metric measure space, i.e. X is a metric space with distance function d and μ is a nonnegative, Borel, doubling measure on X. Let A be a self-adjoint operator acting on $L^2(X, \mu)$. By the spectral theorem one has

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda),$$

where $dE(\lambda)$ is the spectral resolution of the operator A. Then for any bounded measurable function $F\colon \mathbb{R} \to \mathbb{R}$ one can define operator

$$F(A) = \int_{-\infty}^{\infty} F(\lambda) dE(\lambda).$$

It is a standard fact that the operator $F(A)$ is bounded on L^2 with norm bounded by the L^∞ norm of the function F.

The theory of spectral multipliers consists of finding minimal regularity conditions on F (e.g. existence of a finite number of derivatives of F in a certain space) which ensure that the operator $F(A)$ can be extended to a bounded operator on $L^p(X, \mu)$ for some range of exponents $p \neq 2$. Spectral multipliers results are modeled on Fourier multiplier results described in fundamental works of Mikhlin [32] and Hörmander [25]. The initial motivation for spectral multipliers comes from the problem of convergence of Fourier series or more generally of eigenfunction expansion for differential operators. One of the most famous spectral multipliers is the Bochner-Riesz mean

$$\sigma_{R,\alpha}(A) := \left(1 - \frac{A}{R}\right)_+^\alpha.$$

When α is large, the function $\sigma_{R,\alpha}$ is smooth. The problem is then to prove boundedness on $L^p(X)$ (uniformly w.r.t. the parameter R) for small values of α. This is the reason why, for general function F with compact support, we study $\sup_{t>0} \|F(tA)\|_{p\to p} \leq C < \infty$. The constant C depends on F and measures the (minimal) smoothness required on the function.

In recent years, spectral multipliers have been studied by many authors in different contexts, including differential or pseudo-differential operators on manifolds, sub-Laplacians on Lie groups, Markov chains as well as operators in abstract settings. We refer the reader to [1, 2, 4, 5, 14, 16, 19, 20, 21, 22, 25, 27, 28, 29, 32, 34] and references therein. We mention in particular the recent paper [11] where sharp spectral multiplier results as well as end-point estimates for Bochner-Riesz means are proved. A restriction type estimate was introduced there in an abstract setting which turns out to be equivalent to the classical Stein-Tomas restriction estimate in the case of the Euclidean Laplacian. Also it is proved there (see also [4]) that in an abstract setting, dispersive or Strichartz estimates for the Schrödinger equation imply sharp spectral multiplier results.

1.1. **The main results.** There are two main objectives of the present paper. First, in contrast to the previous papers on spectral multipliers where usually decay assumptions are made on the heat kernel or the semigroup, we do not make directly such assumptions and work outside the semigroup framework. The second objective is to replace the usual exponentiel decay of the heat kernel by a polynomial one. All of this is motivated by applications to new settings and examples which were not covered by previous works. In addition most of spectral multipliers proved before can be included in our framework.

In order to state explicitly our contributions we first recall that (X, d, μ) satisfies the doubling property (see Chapter 3, [15]) if there exists a constant $C > 0$ such that

$$V(x, 2r) \leq CV(x, r) \quad \forall\, r > 0,\ x \in X. \tag{1.1}$$

Note that the doubling property implies the following strong homogeneity property,

$$V(x, \lambda r) \leq C\lambda^n V(x, r) \tag{1.2}$$

for some $C, n > 0$ uniformly for all $\lambda \geq 1$ and $x \in X$. In Euclidean space with Lebesgue measure, the parameter n corresponds to the dimension of the space, but in our more abstract setting, the optimal n need not even be an integer.

Let $\tau > 0$ be a fixed positive parameter and suppose that A is a bounded self-adjoint operator on $L^2(X, \mu)$ which satisfies the following polynomial off-diagonal decay

$$\|P_{B(x,\tau)} V_\tau^{\sigma_p} A P_{B(y,\tau)}\|_{p\to 2} < C\left(1 + \frac{d(x, y)}{\tau}\right)^{-n-a} \qquad \forall x, y \in X \tag{1.3}$$

with $\sigma_p = 1/p - 1/2$ and $P_{B(x,\tau)}$ is the projection on the open ball $B(x, \tau)$. We prove that if $a > [n/2] + 1$ and F is a bounded Borel function such that $F \in H^s(\mathbb{R})$ for some $s > n(1/p - 1/2) + 1/2$, then

$$\|F(A)A\|_{p\to p} \leq C\|F\|_{H^s}.$$

Note that the operator A which we discuss here cannot be, in a natural way, considered as a part of a semigroups framework. See Theorem 3.1 below for more additional information. In the particular case where $A = e^{-\tau L}$ for some non-negative (unbounded) self-adjoint operator L with constant in (1.3) independent of τ, we obtain for $s > n(1/p - 1/2) + 1/2$

$$\sup_{\tau>0} \|F(\tau L)\|_{p\to p} \leq C\|F\|_{H^s}. \tag{1.4}$$

As mentioned previously, this latter property implies Bochner-Riesz L^p-summability with index $\alpha > n(1/p - 1/2)$. See Corollary 3.2.

Some significant spectral multiplier results for operators satisfying polynomial estimates were considered by Hebisch in [22] and indirectly also in [26, 27] by Jensen and Nakamura. Our results are inspired by ideas initiated in [17, 22, 26, 27, 31].

Following [11] we introduce the following restriction type estimate

$$\left\|F(A)AV_\tau^{\sigma_p}\right\|_{p\to 2} \leq C\|F\|_q, \qquad \sigma_p = \frac{1}{p} - \frac{1}{2}. \tag{1.5}$$

We then prove a sharper spectral multiplier result under this condition. Namely,

$$\|AF(A)\|_{p\to p} \leq C_p\|F\|_{W^{s,q}} \tag{1.6}$$

for $F \in W^{s,q}(\mathbb{R})$ for some

$$s > n\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{n}{4[a]}.$$

We refer to Theorem 4.1 for the precise statement. We prove several other results such as bounds for $e^{itA}A$ on L^p for p as in (1.3). The proofs are very much based on Littlewood-Paley type theory, commutator estimates and amalgam spaces [10, 17, 26].

Our result can be applied to many examples. Obviously, if the A has an exponential decay (e.g. a Gaussian upper bound) then it satisfies the previous polynomial off-diagonal decay. Hence our results apply to a wide class of elliptic operators on Euclidean domains or on Riemannian manifolds. They also apply in cases where the heat kernel has polynomial decay. This is the case for example for fractional powers of elliptic or Schrödinger operators. In the last section we discuss applications to Markov chains. We also study spectral multipliers (and hence Bochner-Riesz means) for random walk on $\mathbb{Z}^n$. To be more precise, we consider

$$Af(\mathbf{d}) := \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=\pm 1} f(\mathbf{d}+j\mathbf{e}_i)$$

where $\mathbf{e}_i = (0,\dots,1,\dots 0)$. We prove for appropriate function F

$$\sup_{t>1}\|F(t(I-A)\|_{p\to p} \le C\|F\|_{H^s}$$

for any $s > n|1/p - 1/2|$. If in addition, if supp $F \subset [0, 1/n]$ and

$$\sup_{1/n>t>0}\|\eta F(t\cdot)\|_{H^s} < \infty$$

for some $s > n|1/p-1/2|$ and η is a non-trivial $C_c^\infty(0,\infty)$ function, then $F(I-A)$ is bounded on L^p if $1 < p < (2n+2)/(n+3)$ and weak type $(1,1)$ if $p = 1$. This result is similar to the sharp spectral multiplier theorem for the standard Laplacian on $\mathbb{R}^n$. Here again the operator A cannot be, in a natural way, included in a semigroups framework.

1.2. **Notations and assumptions.** In the sequel we always assume that the considered ambient space is a separable metric measure space (X, d, μ) with metric d and Borel measure μ. We denote by $B(x,r) := \{y \in X,\ d(x,y) < r\}$ the open ball with centre $x \in X$ and radius $r > 0$. We often use B instead of $B(x,r)$. Given $\lambda > 0$, we write λB for the λ-dilated ball which is the ball with the same centre as B and radius λr. For $x, \in X$ and $r > 0$ we set $V(x,r) := \mu(B(x,r))$ the volume of $B(x,r)$. We set

$$V_r(x) := V(x,r), \quad r > 0, \quad x \in X. \tag{1.7}$$

We will often write $V(x)$ in place of $V_1(x)$.

For $1 \le p \le +\infty$, we denote by $\|f\|_p$ the norm of $f \in L^p(X)$, $\langle .,. \rangle$ the scalar product of $L^2(X)$, and if T is a bounded linear operator from $L^p(X)$ to $L^q(X)$ we write $\|T\|_{p\to q}$ for its corresponding operator norm. For a given $p \in [1,2)$ we define

$$\sigma_p = \frac{1}{p} - \frac{1}{2}. \tag{1.8}$$

Given a subset $E \subseteq X$, we denote by χ_E the characteristic function of E and set $P_E f(x) = \chi_E(x) f(x)$. For every $x \in X$ and $r > 0$.

Throughout this paper we always assume that the space X is of homogeneous type in the sense that it satisfies the classical doubling property (1.2) with some constants C and n independent of $x \in X$ and $\lambda \geq 1$. In the Euclidean space with Lebesgue measure, n is the dimension of the space. In our results critical index is always expressed in terms of homogeneous dimension n.
Note also that there exists c and $D, 0 \leq D \leq n$ so that

$$V(y,r) \leq c\Big(1 + \frac{d(x,y)}{r}\Big)^D V(x,r) \tag{1.9}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, the property (1.9) with $D = n$ is a direct consequence of triangle inequality of the metric d and the strong homogeneity property. In the cases of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, D can be chosen to be 0.

2. Preliminary results

In this this section we give some elementary results which will be used later.

2.1. A criterion for L^p-L^q boundedness for linear operators.

We start with a countable partitions of X. For every $r > 0$, we choose a sequence $(x_i)_{i=1}^\infty \in X$ such that $d(x_i, x_j) > r/4$ for $i \neq j$ and $\sup_{x\in X} \inf_i d(x, x_i) \leq r/4$. Such sequence exists because X is separable. Set

$$D = \bigcup_{i\in\mathbb{N}} B(x_i, r/4).$$

Then define $Q_i(r)$ by the formula

$$Q_i(r) = B(x_i, r/4) \bigcup \left[B\left(x_i, r/2\right) \setminus \left(\bigcup_{j<i} B\left(x_j, r/2\right) \setminus D \right) \right], \tag{2.1}$$

so that $(Q_i(r))_i$ is a countable partition of X ($Q_i(r) \cap Q_j(r) = \emptyset$ if $i \neq j$). Note that $Q_i(r) \subset B_i = B(x_i, r)$ and there exists a uniform constant $C > 0$ depending only on the doubling constants in (1.2) such that $\mu(Q_i(r)) \geq C\mu(B_i)$.

We have the following Schur-test for the norm $\|T\|_{p\to q}$ of a given linear operator T.

Lemma 2.1. *Let T be a linear operator and $1 \leq p \leq q \leq \infty$. For every $r > 0$,*

$$\|T\|_{p\to q} \leq \sup_j \sum_i \left\| P_{Q_i(r)} T P_{Q_j(r)} \right\|_{p\to q} + \sup_i \sum_j \left\| P_{Q_i(r)} T P_{Q_j(r)} \right\|_{p\to q},$$

where $(Q_i(r))_i$ is a countable partition of X.

Proof. The proof is inspired by [20]. Given a function $f \in L^p(X)$, we have

$$\begin{aligned}
\|Tf\|_q^q &= \Big\| \sum_{i,j} P_{Q_i(r)} T P_{Q_j(r)} f \Big\|_q^q \\
&= \sum_i \Big\| \sum_j P_{Q_i(r)} T P_{Q_j(r)} f \Big\|_q^q \\
&\leq \sum_i \left(\sum_j \|P_{Q_i(r)} T P_{Q_j(r)}\|_{p\to q} \|P_{Q_j(r)} f\|_p \right)^q
\end{aligned}$$

$$=: \quad \Big\| \sum_j a_{ij} c_j \Big\|_{\ell^q}^q,$$

where $a_{ij} = \|P_{Q_i(r)} T P_{Q_j(r)}\|_{p\to q}$ and $c_j = \|P_{Q_j(r)} f\|_p$.

Next note that, for all $1 \le p \le \infty$,

$$\sum_k \left(\sum_l |a_{lk} c_l| \right)^p \le \left(\max \left\{ \sup_l \sum_k |a_{lk}|, \ \sup_k \sum_l |a_{lk}| \right\} \right)^p \sum_n |c_n|^p, \tag{2.2}$$

with the obvious meaning for $p = \infty$, where c_{lk}, a_l are sequences of real or complex numbers. Indeed, for $p = 1$ or $p = \infty$, (2.2) is easy to obtain. Then we obtain (2.2) for all $1 \le p \le \infty$ by interpolation. Observe that

$$\|Tf\|_q \le \|(a_{ij})\|_{\ell^p \to \ell^q} \|c_j\|_{\ell^p} \le \|(a_{ij})\|_{\ell^p \to \ell^p} \|f\|_p.$$

The lemma follows from (2.2) and the above inequality. □

2.2. **Operators with kernels satisfying off-diagonal polynomial decays.** For a given function $W : X \to \mathbb{R}$, we denote by M_W the multiplication operator by W, that is

$$(M_W f)(x) = W(x) f(x).$$

In the sequel, we will identify the operator M_W with the function W. This means that, if T is a linear operator, we will denote by $W_1 T W_2$ the operators $M_{W_1} T M_{W_2}$. In other words,

$$W_1 T W_2 f(x) := W_1(x) T(W_2 f)(x).$$

Following [7, 8, 9], we introduce the following estimates which are interpreted as polynomial off-diagonal estimates.

Definition 2.2. *Let A be a self-adjoint operator on $L^2(X)$ and $\tau > 0$ be a constant. For $1 \le p < 2$ and $a > 0$, we say that A satisfies the property ($\mathrm{PVE}^a_{p,2}(\tau)$) if there exists a constant $C > 0$ such that for all $x, y \in X$,*

$$\|P_{B(x,\tau)} V_\tau^{\sigma_p} A P_{B(y,\tau)}\|_{p\to 2} < C \left(1 + \frac{d(x,y)}{\tau} \right)^{-n-a} \tag{$\mathrm{PVE}^a_{p,2}(\tau)$}$$

with $\sigma_p = 1/p - 1/2$.

By Hölder's inequality and duality, the condition ($\mathrm{PVE}^a_{p,2}(\tau)$) implies that

$$\|P_{B(x,\tau)} A P_{B(y,\tau)}\|_{p\to p} + \|P_{B(x,\tau)} A P_{B(y,\tau)}\|_{p'\to p'} \le C \left(1 + \frac{d(x,y)}{\tau} \right)^{-n-a}. \tag{2.3}$$

By interpolation,

$$\|P_{B(x,\tau)} A P_{B(y,\tau)}\|_{2\to 2} \le C \left(1 + \frac{d(x,y)}{\tau} \right)^{-n-a}. \tag{2.4}$$

Remark 2.3. *Suppose that ($\mathrm{PVE}^a_{p,2}(\tau)$) holds for some $p < 2$. Then ($\mathrm{PVE}^{\mathrm{a}}_{\mathrm{q},2}$) holds for every $q \in [p, 2]$. This can be shown by applying complex interpolation to the family*

$$F(z) = P_{B(x,\tau)} V_\tau^{z\sigma_p} A P_{B(y,\tau)}.$$

For $z = 1 + is$ we use ($\mathrm{PVE}^a_{p,2}(\tau)$) and for $z = is$ we use (2.4).

In the sequel, for a given $\tau > 0$ we fix a countable partition $\{Q_\ell(\tau)\}_{\ell=1}^\infty$ of X and and a sequence $(x_\ell)_{\ell=1}^\infty \in X$ as in Section 2.1. First, we have the following result.

Proposition 2.4. *Let $1 \le p \le 2$, and A be a self-adjoint operator on $L^2(X)$. Assume that condition ($\mathrm{PVE}^a_{p,2}(\tau)$) holds for some $\tau > 0$ and $a > 0$. There exists a constant $C > 0$ independent of $\tau > 0$ such that*

$$\left\| V_\tau^{\sigma_p} A \right\|_{p\to 2} \le C. \tag{2.5}$$

As a consequence, the operator A is a bounded operator on $L^p(X)$, and $\|A\|_{p\to p} + \|A\|_{2\to 2} \le C$.

Proof. By Lemma 2.1 and condition ($\mathrm{PVE}^a_{p,2}(\tau)$), one is lead to estimate

$$\sup_j \sum_i \left(1 + \frac{d(x_i,x_j)}{\tau}\right)^{-n-a} \le C < \infty. \tag{2.6}$$

Note that for every $k \ge 1$,

$$\begin{aligned} \sup_{j,\tau} \#\{i : 2^k\tau \le d(x_i,x_j) < 2^{k+1}\tau\} &\le \sup_{j,\tau} \sup_{\{i: 2^k\tau \le d(x_i,x_j) < 2^{k+1}\tau\}} \frac{V(x_i, 2^{k+3}\tau)}{V(x_i,\tau)} \\ &\le C 2^{kn} < \infty. \end{aligned} \tag{2.7}$$

This implies that for every $j \ge 1$ and $k \ge 1$,

$$\sum_{i:\, 2^k\tau \le d(x_i,x_j) < 2^{k+1}\tau} \left(1 + \frac{d(x_i,x_j)}{\tau}\right)^{-n-a} \le C(1+2^k)^{-n-a} 2^{kn} \le C 2^{-ka}$$

and we sum over k to get (2.6).

The boundedness of the operator A on L^p is proved in the same way by applying (2.3). □

Note that when the operator A has integral kernel $K_A(x,y)$ satisfying the following pointwise estimate

$$|K_A(x,y)| \le C V(x,\tau)^{-1} \left(1 + \frac{d(x,y)}{\tau}\right)^{-n-a}$$

for some $\tau > 0$ and all $x, y \in X$, then A satisfies the property ($\mathrm{PVE}^a_{p,2}(\tau)$) with $p = 1$. Conversely, we have the following result.

Proposition 2.5. *Suppose that $a > D$ where D is the constant in (1.9). If the operator A satisfies the property ($\mathrm{PVE}^a_{p,2}(\tau)$) for some $\tau > 0$ and $p = 1$, then the operator A^2 has integral kernel $K_{A^2}(x,y)$ satisfying the following pointwise estimate: For any $\epsilon > 0$, there exists a constant $C > 0$ independent of τ such that*

$$|K_{A^2}(x,y)| \le C V(x,\tau)^{-1} \left(1 + \frac{d(x,y)}{\tau}\right)^{-(a-D-\epsilon)} \tag{2.8}$$

for all $x, y \in X$.

Proof. For every $x, y \in X$ and $\tau > 0$, we write

$$P_{B(x,\tau)}A^2P_{B(y,\tau)} = P_{B(x,\tau)}AP_{B(x,\tau)}AP_{B(y,\tau)} + \sum_{\ell=0}^{\infty}\sum_{i:2^\ell\tau\le d(x_i,x)<2^{\ell+1}\tau} P_{B(x,\tau)}AP_{B(x_i,\tau)}AP_{B(y,\tau)}.$$

From the property ($\mathrm{PVE}^a_{p,2}(\tau)$) with $p = 1$ we get

$$\begin{aligned}\left\|P_{B(x,\tau)}A^2P_{B(y,\tau)}\right\|_{L^1\to L^\infty} &\le CV(x,\tau)^{-1}\left(1+\frac{d(x,y)}{\tau}\right)^{-n-a}\\ &\quad + C\sum_{\ell=0}^{\infty}\sum_{i:2^\ell\tau\le d(x_i,x)<2^{\ell+1}\tau} V(x_i,\tau)^{-1}\left(1+\frac{d(x,x_i)}{\tau}\right)^{-n-a}\left(1+\frac{d(x_i,y)}{\tau}\right)^{-n-a}.\end{aligned}$$

This, in combination with the fact that $V(x_i,\tau)^{-1} \le C2^{\ell D}V(x,\tau)^{-1}$ for every $\ell \ge 0$ and $2^\ell\tau \le d(x_i, x) < 2^{\ell+1}\tau$, and the property (2.7), implies that

$$\left\|P_{B(x,\tau)}A^2P_{B(y,r)}\right\|_{L^1\to L^\infty} \le CV(x,\tau)^{-1}\left(1+\frac{d(x,y)}{\tau}\right)^{-(a-D-\epsilon)}$$

for any $\epsilon > 0$. Hence it follows that (2.8) holds. This completes the proof of Proposition 2.5. □

Finally, we mention that if A is the semigroup e^{-tL} generated by (minus) a non-negative self-adjoint operator L, then the condition $\mathrm{PVE}^a_{p,2}(t^{1/m})$ holds for some $m \ge 1$ if the corresponding heat kernel $K_t(x, y)$ has a polynomial decay

$$|K_t(x,y)| \le CV(x,t^{1/m})^{-1}\left(1+\frac{d(x,y)}{t^{1/m}}\right)^{-n-a}. \tag{2.9}$$

it is known that the heat kernel satisfies a Gaussian upper bound, for a wide class of differential operators of order $m \in 2\mathbb{N}$ on Euclidean domains or Riemannian manifolds (see for example [18]). In this case (2.9) holds with any arbitrary $a > 0$.

3. Spectral multipliers via polynomial off-diagonal decay kernels

In this section we prove spectral multiplier results corresponding to compactly supported functions in the abstract setting of self-adjoint operators on homogeneous spaces. Recall that we assume that (X, d, μ) is a metric measure space satisfying the doubling property and n is the homogeneous dimension from condition (1.2). We use the standard notation $H^s(\mathbb{R})$ for the Sobolev space $\|F\|_{H^s} = \|(I - d^2/dx^2)^{s/2}F\|_2$.

Theorem 3.1. *Suppose that A is a bounded self-adjoint operator on $L^2(X)$ which satisfies condition ($\mathrm{PVE}^a_{p,2}(\tau)$) for some $\tau > 0$, $1 \le p < 2$ and $a > [n/2] + 1$. If F is a bounded Borel function such that $F \in H^s(\mathbb{R})$ for some $s > n(1/p - 1/2) + 1/2$, then there exists constant $C > 0$ such that*

$$\|F(A)A\|_{p\to p} \le C\|F\|_{H^s}.$$

The constant C above does not depend on the choice of $\tau > 0$. In addition, if we assume that A is a nonnegative self-adjoint operator and $\operatorname{supp} F \subset [-1, 1]$*, then there exists a constant $C > 0$ which is* *also independent of τ such that*

$$\|F(-\log A)\|_{p\to p} \le C\|F\|_{H^s}.$$

The proof of Theorem 3.1 will be given at the end of this section after a series of preparatory results. The following statement is a direct consequence of Theorem 3.1.

Corollary 3.2. *Suppose that L is a non-negative self-adjoint operator on $L^2(X)$ that for a given $\tau > 0$ the semigroup operator $e^{-\tau L}$ satisfies condition ($\mathrm{PVE}^a_{p,2}(\tau)$) for some $1 \le p < 2$ and $a > [n/2] + 1$. If F is a bounded Borel function such that $\operatorname{supp} F \subset [-1, 1]$ and $F \in H^s(\mathbb{R})$ for some $s > n(1/p - 1/2) + 1/2$, then there exists constant $C > 0$ such that*

$$\|F(\tau L)\|_{p\to p} \le C\|F\|_{H^s}.$$

As a consequence if operators $e^{-\tau L}$ satisfies condition ($\mathrm{PVE}^a_{p,2}(\tau)$) with constant independent of τ then

$$\sup_{\tau>0} \|F(\tau L)\|_{p\to p} \le C\|F\|_{H^s}$$

for the same range of exponents p and s.

Proof. We apply Theorem 3.1 to the operator $A = e^{-\tau L}$ for all $\tau > 0$. Then the theorem follows from the fact that the constants in statement of Theorem 3.1 are independent of τ. □

Remark 3.3. *It is possible to obtain a version of Theorem 3.1 under the weaker assumption $a > 0$. However, this requires a different approach which we do not discuss here. Related results can be found in [22] and [29]. We will consider this more general case in a forthcoming paper [12].*

3.1. **Preparatory results.** The following result plays a essential role in the proof of Theorem 3.1.

Theorem 3.4. *Suppose that A is a bounded self-adjoint operator on $L^2(X)$ and satisfies condition ($\mathrm{PVE}^a_{p,2}(\tau)$) for $1 \le p < 2$, $a > [n/2] + 1$ and for some $\tau > 0$. Then there exists a constant $C > 0$ such that for all $t \in \mathbb{R}$,*

$$\|e^{itA}A\|_{p\to p} \le C(1 + |t|)^{n\sigma_p} \tag{3.1}$$

where $\sigma_p = 1/p - 1/2$.

Remark 3.5. *It is interesting to compare the above statement with Theorem 1.1 of [17]. Note that in Theorem 3.4 we assume condition ($\mathrm{PVE}^a_{p,2}(\tau)$) for one fixed exponent τ but conclusion is valid for all $t \in \mathbb{R}$.*

In order to prove Theorem 3.4 we need two technical lemmas. First, we state the following known formula for the commutator of a Lipschitz function and an operator T on $L^2(X)$ on metric measure space X. Recall our notation convention $\eta T = M_\eta T$.

Lemma 3.6. *Let T be a self-ajoint operator on $L^2(X)$. Assume that for some $\eta \in \mathrm{Lip}(X)$, the commutator $[\eta, T]$, defined by $[\eta, T]f = M_\eta T f - T M_\eta f$, satisfies that for all $f \in \mathrm{Dom}(T)$, $\eta f \in \mathrm{Dom}(T)$ and*

$$\|[\eta, T]f\|_2 \le C\|f\|_2,$$

where $\mathrm{Dom}(T)$ denotes the domain of T . Then the following formula holds:

$$[\eta, e^{itT}]f = it\int_0^1 e^{istT}[\eta, T]e^{i(1-s)tT} f ds \quad \forall t \in \mathbb{R}, \forall f \in L^2(X).$$

Proof. The proof of Lemma 3.6 follows by integration by parts. See for example, [31, Lemma 3.5]. □

Next we recall some useful results concerning amalgam spaces [10, 17, 26]. For a given $\tau > 0$, we recall that $\{Q_\ell(\tau)\}_{\ell=1}^\infty$ is a countable partition of X as in Section 2.1. For $1 \le p, q \le \infty$ and a non-negative number $\tau > 0$, consider a two-scale Lebesgue space $X_\tau^{p,q}$ of measurable functions on X equipped with the norm

$$\|f\|_{X_\tau^{p,q}} := \left(\sum_{\ell=1}^\infty \|P_{Q_\ell(\tau)} f\|_q^p\right)^{1/p}. \tag{3.2}$$

(with obvious changes if $q = \infty$).

Notice that when $q = p$ these spaces are just the Lebesgue spaces $X_\tau^{p,p} = L^p$ for every $\tau > 0$ and p. Note also that for $1 \le p_1 < p_2 \le \infty$, we have that $X_\tau^{p_1,q} \subseteq X_\tau^{p_2,q}$ with

$$\|f\|_{X_\tau^{p_2,q}} \le \|f\|_{X_\tau^{p_1,q}}.$$

The following result gives a criterion for a linear operator A to be bounded on $X_\tau^{p,2}$, $1 \le p \le 2$. We define a family of functions $\{\eta_\ell(x)\}_\ell$ by

$$\eta_\ell(x) := \frac{d(x, x_\ell)}{\tau}, \quad \ell = 1, 2, \cdots \tag{3.3}$$

i.e., the distance function between x_ℓ and x (up to a constant).

Lemma 3.7. *Let $\tau > 0$ and $\{\eta_\ell(x)\}_{\ell=1}^\infty$ be as above. For a bounded operator T on $L^2(X)$, the multi-commutator $\mathrm{ad}_\ell^k(T) : L^2(X) \to L^2(X)$ is defined inductively by*

$$\mathrm{ad}_\ell^0(T) = T, \quad \mathrm{ad}_\ell^k(T) = \mathrm{ad}_\ell^{k-1}(\eta_\ell T - T\eta_\ell), \quad \ell \ge 1,\ k \ge 1. \tag{3.4}$$

Suppose that there exists a constant $M > 1$ such that for all $1 \le \ell < \infty$,

$$\|\mathrm{ad}_\ell^k(T)\|_{2\to 2} \le CM^k, \quad 0 \le k \le [n/2] + 1.$$

Then for $1 \le p \le 2$,

$$\|V_\tau^{\sigma_p} T V_\tau^{-\sigma_p} f\|_{X_\tau^{p,2}} \le CM^{n\sigma_p}\|f\|_{X_\tau^{p,2}} \quad \text{with } \sigma_p = (1/p - 1/2)$$

for some constant $C > 0$ depending on n, p and $\|T\|_{2\to 2}$.

Proof. We prove this lemma by using the complex interpolation method. Let **S** be the closed strip $0 \leq \mathrm{Re} z \leq 1$ in the complex plane. For every $z \in \mathbf{S}$, we consider the analytic family of operators:

$$T_z := V_r^{z/2}\, T\, V_\tau^{-z/2}.$$

Consider $z = 1 + iy$, $y \in \mathbb{R}$ and $p = 1$. Let $N \geq 1$ be a constant large enough to be chosen later. Let $\{Q_\ell(\tau)\}_\ell$ be a countable partition of X in Section 2.1. By definition of $X_\tau^{1,2}$,

$$\begin{aligned} \|T_{1+iy}f\|_{X_\tau^{1,2}} &= \sum_{j=1}^\infty \|P_{Q_j(\tau)} V_\tau^{\frac{1+iy}{2}} T V_\tau^{-\frac{1+iy}{2}} f\|_2 \\ &\leq \sum_{\ell=1}^\infty \sum_{j=1}^\infty \|P_{Q_j(\tau)} V_\tau^{\frac{1+iy}{2}} T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2 \\ &= \left(\sum_\ell \sum_{j:\, d(x_\ell, x_j) > N\tau} + \sum_\ell \sum_{j:\, d(x_\ell, x_j) \leq N\tau} \right) \|P_{Q_j(\tau)} V_\tau^{\frac{1+iy}{2}} T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2 \\ &=: I + II. \end{aligned} \tag{3.5}$$

By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} II &\leq C \sum_{\ell=1}^\infty \left(\sum_{j:\, d(x_\ell, x_j) \leq Nr} V_\tau(x_j) \right)^{1/2} \left(\sum_{j:\, d(x_\ell, x_j) \leq N\tau} \|P_{Q_j(\tau)} T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2^2 \right)^{1/2} \\ &\leq C \sum_\ell V(x_\ell, N\tau)^{1/2} \|T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2 \\ &\leq C N^{n/2} \|T\|_{2\to 2} \sum_\ell \mu(Q_\ell(\tau))^{1/2} \|V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2 \\ &\leq C N^{n/2} \|T\|_{2\to 2} \|f\|_{X_\tau^{1,2}}. \end{aligned} \tag{3.6}$$

Now we estimate the term I. Let $\kappa = [n/2] + 1$. We apply the Cauchy-Schwarz inequality again to obtain

$$\begin{aligned} I &\leq C \sum_{\ell=1}^\infty \left(\sum_{j:\, d(x_\ell, x_j) > N\tau} \mu(Q_j(\tau)) \eta_\ell(x_j)^{-2\kappa} \right)^{1/2} \left(\sum_{j: d(x_\ell, x_j) > N\tau} \eta_\ell(x_j)^{2\kappa} \|P_{Q_j(\tau)} T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2^2 \right)^{1/2} \\ &\leq C \sum_\ell \left(\sum_{k=0}^\infty \sum_{j:\, 2^k N\tau < d(x_\ell, x_j) \leq 2^{k+1} N\tau} \mu(Q_j(\tau)) \eta_\ell(x_j)^{-2\kappa} \right)^{1/2} \|\eta_\ell^\kappa T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2 \\ &\leq C \sum_\ell \left(\sum_{k=0}^\infty (2^k N)^n \mu(Q_\ell(\tau)) (2^k N)^{-2\kappa} \right)^{1/2} \|\eta_\ell^\kappa T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2 \\ &\leq C \sum_\ell N^{-\kappa + n/2} \mu(Q_\ell(\tau))^{1/2} \|\eta_\ell^\kappa T V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f\|_2. \end{aligned} \tag{3.7}$$

To continue we define we let $\Gamma(\kappa, 0) = 1$ for $\kappa \geq 1$, and $\Gamma(\kappa, m)$ defined inductively by $\Gamma(\kappa, m+1) = \sum_{\ell=m}^{\kappa-1} \Gamma(\ell, m)$ for $1 \leq m \leq \kappa - 1$. Applying the following known formula for commutators (see

Lemma 3.1, [27]):

$$\eta_\ell^\kappa T = \sum_{m=0}^{\kappa} \Gamma(\kappa, m)\mathrm{ad}_\ell^m(T)\eta_\ell^{\kappa-m},$$

we obtain

$$\begin{aligned} \|\eta_\ell^\kappa T(1+\eta_\ell)^{-\kappa}\|_{2\to 2} &\leq C\sum_{m=0}^{\kappa} \|\mathrm{ad}_\ell^m(T)\eta_\ell^{\kappa-m}(1+\eta_\ell)^{-\kappa}\|_{2\to 2} \\ &\leq C\sum_{m=0}^{\kappa} \|\mathrm{ad}_\ell^m(T)\|_{2\to 2} \leq CM^\kappa. \end{aligned} \tag{3.8}$$

This implies

$$\begin{aligned} I &\leq CN^{-\kappa+n/2}M^\kappa \sum_\ell \mu(Q_\ell(\tau))^{1/2}\|V_\tau^{-\frac{1+iy}{2}}(1+\eta_\ell)^\kappa P_{Q_\ell(\tau)}f\|_2 \\ &\leq CN^{-\kappa+n/2}M^\kappa \sum_\ell \|P_{Q_\ell(\tau)}f\|_2 \leq CN^{-\kappa+n/2}M^\kappa\|f\|_{X_\tau^{1,2}}. \end{aligned}$$

Next, set $N = M\|T\|_{2\to 2}^{-1/\kappa}$. Then above estimates of I and II give

$$\begin{aligned} \|T_{1+iy}f\|_{X_\tau^{1,2}} &\leq C\left(N^{-\kappa+n/2}M^\kappa + N^{n/2}\|T\|_{2\to 2}\right)\|f\|_{X_\tau^{1,2}} \\ &\leq CM^{n/2}\|T\|_{2\to 2}^{1-1/\kappa}\|f\|_{X_\tau^{1,2}}. \end{aligned} \tag{3.9}$$

On the other hand, we have that for $z = iy$, $y \in \mathbb{R}$,

$$\|T_{iy}\|_{X_\tau^{2,2}\to X_\tau^{2,2}} = \|T_{iy}\|_{2\to 2} \leq \|T\|_{2\to 2} \tag{3.10}$$

From estimates (3.9) and (3.10), we apply the complex interpolation method to obtain

$$\|V_\tau^{\sigma_p}TV_\tau^{-\sigma_p}f\|_{X_\tau^{p,2}} = \|T_{\frac{2}{p}-1}f\|_{X_\tau^{p,2}} \leq CM^{n\sigma_p}\|f\|_{X_\tau^{p,2}}, \quad \sigma_p = (1/p - 1/2)$$

for some constant $C > 0$ depending on n, p and $\|T\|_{2\to 2}$. This finishes the proof of Lemma 3.7. □

Now we apply Lemmas 3.6 and 3.7 to prove Theorem 3.4.

Prooff Theorem 3.4. The proof is inspired by Theorem 1.3 of [27] and Theorem 1.1 of [17]. Note that

$$\|e^{itA}A\|_{p\to p} \leq \|V_\tau^{-\sigma_p}\|_{X_\tau^{p,2}\to L^p}\|V_\tau^{\sigma_p}e^{itA}V_\tau^{-\sigma_p}\|_{X_\tau^{p,2}\to X_\tau^{p,2}}\|V_\tau^{\sigma_p}A\|_{L^p\to X_\tau^{p,2}}.$$

First, we have that $\|V_\tau^{-\sigma_p}\|_{X_\tau^{p,2}\to L^p} \leq C$ by definition of $X_\tau^{p,2}$ and Hölder's inequality. To estimate the term $\|V_\tau^{\sigma_p}A\|_{L^p\to X_\tau^{p,2}}$, we recall that $(\{Q_i(\tau)\}_{i=1}^\infty)$ is a countable partition of X as in Section 2.1 and note that

$$\begin{aligned} \|V_\tau^{\sigma_p}Af\|_{X_\tau^{p,2}} &= \left(\sum_{j=1}^\infty \|P_{Q_j(\tau)}V_\tau^{\sigma_p}Af\|_2^p\right)^{1/p} \\ &\leq \left(\sum_j\left[\sum_\ell \|P_{Q_j(\tau)}V_\tau^{\sigma_p}AP_{Q_\ell(\tau)}\|_{p\to 2}\|P_{Q_\ell(\tau)}f\|_p\right]^p\right)^{1/p}. \end{aligned}$$

For every j,ℓ, we set $a_{j\ell} = \|P_{Q_j(\tau)}V_\tau^{\sigma_p}AP_{Q_\ell(\tau)}\|_{p\to 2}$ and $b_\ell = \|P_{Q_\ell(\tau)}f\|_p$. It follows by interpolation that

$$\begin{aligned}\|V_\tau^{\sigma_p}Af\|_{X_\tau^{p,2}} &\leq \Big\|\sum_\ell a_{j\ell}b_\ell\Big\|_{\ell^p}\\ &\leq \big\|(a_{j\ell})\big\|_{\ell^p\to\ell^p}\|b_\ell\|_{\ell^p}\\ &\leq \big\|(a_{j\ell})\big\|^{\theta}_{\ell^1\to\ell^1}\big\|(a_{j\ell})\big\|^{1-\theta}_{\ell^\infty\to\ell^\infty}\|f\|_p.\end{aligned}$$

Therefore, by condition ($\mathrm{PVE}^a_{p,2}(\tau)$) we have

$$\|(a_{j\ell})\|_{\ell^1\to\ell^1} = \sup_\ell\sum_j\big\|P_{Q_j(\tau)}V_\tau^{\sigma_p}AP_{Q_\ell(\tau)}\big\|_{p\to 2}\leq C$$

and

$$\|(a_{j\ell})\|_{\ell^\infty\to\ell^\infty} = \sup_j\sum_\ell\big\|P_{Q_j(\tau)}V_\tau^{\sigma_p}AP_{Q_\ell(\tau)}\big\|_{p\to 2}\leq C.$$

Thus

$$\big\|V_\tau^{\sigma_p}A\big\|_{L^p\to X_\tau^{p,2}}\leq C. \tag{3.11}$$

Next we show that

$$\big\|V_\tau^{\sigma_p}e^{itA}V_\tau^{-\sigma_p}\big\|_{X_\tau^{p,2}\to X_\tau^{p,2}}\leq C(1+|t|)^{n\sigma_p}. \tag{3.12}$$

By Lemma 3.7, it suffices to show for every $m\in\mathbb{N}$,

$$\|\mathrm{ad}^k_m(e^{itA})\|_{2\to 2}\leq C(1+|t|)^k,\quad 0\leq k\leq [n/2]+1. \tag{3.13}$$

Note that A is a bounded operator on $L^2(X)$. Recall that (cf. Lemma 3.6) for all $m\in\mathbb{N}$:

$$\mathrm{ad}^1_m(e^{itA})f = it\int_0^1 e^{istA}\mathrm{ad}^1_m(A)e^{i(1-s)tA}f\,ds,\quad \forall t\in\mathbb{R},\quad f\in L^2(X).$$

Repeatedly, we are reduced to prove that for every $m\in\mathbb{N}$, there exists a constant $C>0$ independent of m such that,

$$\|\mathrm{ad}^k_m(A)\|_{2\to 2}\leq C,\quad 0\leq k\leq [n/2]+1. \tag{3.14}$$

Fix $m\in\mathbb{N}$. By Lemma 2.1, it suffices to show

$$\sup_j\sum_\ell\|P_{Q_\ell(\tau)}\mathrm{ad}^k_m(A)P_{Q_j(\tau)}\|_{2\to 2}\leq C,\quad 0\leq k\leq [n/2]+1 \tag{3.15}$$

for some constant $C>0$ independent of m.

To show (3.15) we note that

$$\begin{aligned}P_{Q_\ell(\tau)}\mathrm{ad}^k_m(A)P_{Q_j(\tau)} &= \sum_{\alpha+\beta+\gamma=k}\frac{k!}{\alpha!\beta!\gamma!}[\eta_m(x_\ell)-\eta_m(x_j)]^\beta[\eta_m(\cdot)-\eta_m(x_\ell)]^\alpha\times\\ &\qquad\times P_{Q_\ell(\tau)}AP_{Q_j(\tau)}[\eta_m(x_j)-\eta_m(\cdot)]^\gamma.\end{aligned}$$

Observe that:

- $|\eta_m(x_\ell)-\eta_m(x_j)|\leq d(x_\ell,x_j)/\tau$;
- $|\eta_m(x)-\eta_m(x_\ell)|\chi_{Q_\ell(\tau)}\leq d(x,x_\ell)\chi_{Q_\ell(\tau)}/\tau\leq 1$;

• $|\eta_m(x_j) - \eta_m(y)|\chi_{Q_j(\tau)} \le d(y, x_j)\chi_{Q_j(\tau)}/\tau \le 1$.

These, together with estimate ($\mathrm{PVE}^a_{p,2}(\tau)$) and $a > [n/2] + 1 \ge k$, yield

$$\sum_\ell \|P_{Q_\ell(\tau)} \mathrm{ad}^k_m(A) P_{Q_j(\tau)}\|_{2\to 2} \le C \sum_\ell \left(1 + \frac{d(x_\ell, x_j)}{\tau}\right)^k \|P_{Q_\ell(\tau)} A P_{Q_j(\tau)}\|_{2\to 2} \le C$$

for some constant $C > 0$ independent of j. Hence, (3.15) is proved. This, in combination with estimates (3.13) and (3.14), implies (3.12). All together, we obtain that $\|e^{itA}A\|_{p\to p} \le C(1 + |t|)^{n\sigma_p}$ where $\sigma_p = (1/p - 1/2)$. The proof of Theorem 3.4 is complete. □

3.2. **Proof of Theorem 3.1.** We apply Lemma 3.4 to see that

$$\begin{aligned}
\|F(A)A\|_{p\to p} &\le \int_{\mathbb{R}} |\widehat{F}(\xi)| \|e^{i\xi A}A\|_{p\to p}\, d\xi \\
&\le C \int_{\mathbb{R}} |\widehat{F}(\xi)|(1 + |\xi|)^{n\sigma_p}\, d\xi \\
&\le C\|F\|_{H^s} \left(\int_{\mathbb{R}} (1 + |\xi|)^{2(n\sigma_p - s)}\, d\xi\right)^{1/2} \\
&\le C\|F\|_{H^s}.
\end{aligned}$$

If we also assume that $A \ge 0$ and $\mathrm{supp}\, F \subset [-1, 1]$, then we may consider $G(\lambda) := F(-\log\lambda)\lambda^{-1}$ so that $F(-\log A) = G(A)A$. Therefore,

$$\|F(-\log A)\|_{p\to p} \le \|G(A)A\|_{p\to p} \le C\|G\|_{H^s}.$$

Since $\mathrm{supp}\, F \subset [-1, 1]$ we have $\|G\|_{H^s} \le C\|F\|_{H^s}$ and we obtain $\|F(-\log A)\|_{p\to p} \le C\|F\|_{H^s}$. The proof of Theorem 3.1 is complete. □

4. Sharp spectral multiplier results via restriction type estimates

The aim of this section is to obtain sharp L^p boundedness of spectral multipliers from restriction type estimates. We consider the metric measure space (X, d, μ) with satisfies the doubling condition (1.2) with the homogeneous dimension n. Let $q \in [2, \infty]$. Recall that $V_r = V(x, r) = \mu(B(x, r))$. We say that A satisfies the *restriction type condition* if there exist interval $[b, e]$ for some $-\infty < b < e < \infty$ and $\tau > 0$ such that for all Borel functions F with $\mathrm{supp}\, F \subset [b, e]$,

$$(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2}(\tau)) \qquad \left\|F(A)AV_\tau^{\sigma_p}\right\|_{p\to 2} \le C\|F\|_q, \qquad \sigma_p = \frac{1}{p} - \frac{1}{2}.$$

The above conditions originates and in fact is a version of the classical Stein-Tomas restriction estimates. For more detailed discussion and the rationale of formulation of the above condition we referee readers to [11, 34]) for a related definition).

The following statement is our main result in this section.

Theorem 4.1. *Let $1 \leq p < 2 \leq q \leq \infty$. Let A be a bounded self-adjoint operator on $L^2(X)$ satisfying the property ($\mathrm{PVE}^a_{p,2}(\tau)$) for some $a > [n/2] + 1$ and $\tau > 0$. Suppose also that A satisfies the property ($\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2}(\tau)$) on some interval $[R_1, R_2]$ for $-\infty < R_1 < R_2 < \infty$. Let F be a bounded Borel function such that $\operatorname{supp} F \subset [R_1 + \gamma, R_2 - \gamma]$ for some $\gamma > 0$ and $F \in W^{s,q}(\mathbb{R})$ for some*

$$s > n\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{n}{4[a]}.$$

Then $AF(A)$ is bounded on $L^p(X)$ and

$$\|AF(A)\|_{p\to p} \leq C_p \|F\|_{W^{s,q}}. \tag{4.1}$$

Remark 4.2. *One can formulate a version of Corollary 3.2 corresponding to Theorem 4.1. See also the statement of Theorem 5.3.*

Remark 4.3. *Note that if A satisfies ($\mathrm{PVE}^a_{p,2}(\tau)$) for some $a > [n/2] + 1$ and $\tau > 0$ then A satisfies ($\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2}(\tau)$) with $q = \infty$. Indeed, by Proposition 2.4, we have*

$$\begin{aligned} \left\| F(A) A V_\tau^{\sigma_p} \right\|_{p\to 2} &\leq \|F(A)\|_{2\to 2} \left\| A V_\tau^{\sigma_p} \right\|_{p\to 2} \\ &\leq C\|F\|_\infty. \end{aligned}$$

As a consequence of Theorem 4.1 we obtain under the sole assumption ($\mathrm{PVE}^a_{p,2}(\tau)$)

$$\|AF(A)\|_{p\to p} \leq C_p \|F\|_{W^{s,\infty}} \tag{4.2}$$

for every $F \in W^{s,\infty}$ and some $s > n\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{n}{4[a]}$.

Before we start the proof of Theorem 4.1, we need some preliminary result. For a given $r > 0$, we recall that $(Q_i(r))_i$ is a countable partition of X. For a bounded operator T and a given $r > 0$, we decompose the operator T into the on-diagonal part $[T]_{<r}$ and the off-diagonal part $[T]_{>r}$ as follows:

$$[T]_{<r} := \sum_i \sum_{j:d(x_i,x_j)\leq 5r} P_{Q_j(r)} T P_{Q_i(r)} \tag{4.3}$$

and

$$[T]_{>r} := \sum_i \sum_{j:d(x_i,x_j)>5r} P_{Q_j(r)} T P_{Q_i(r)}. \tag{4.4}$$

For the on-diagonal part $[T]_{<r}$, we have the following result.

Lemma 4.4. *Assume that T is a bounded operator from $L^p(X)$ to $L^q(X)$ ($p \leq q$). Then the on-diagonal part $[T]_{<r}$ is bounded on from $L^p(X)$ to $L^q(X)$ and there exists a constant $C = C(n) > 0$ independent of r such that*

$$\|[T]_{<r}\|_{p\to q} \leq C\|T\|_{p\to q}.$$

Proof. Note that

$$\left\| \sum_i \sum_{j:d(x_i,x_j)\leq 5r} P_{Q_j(r)} T P_{Q_i(r)} f \right\|_q^q = \sum_j \left\| \sum_{i:d(x_i,x_j)\leq 5r} P_{Q_j(r)} T P_{Q_i(r)} f \right\|_q^q$$

$$
\begin{aligned}
&\le C\sum_j \sum_{i:d(x_i,x_j)\le 5r} \left\|P_{Q_j(r)}TP_{Q_i(r)}f\right\|_q^q\\
&\le C\sum_i \sum_{j:d(x_i,x_j)\le 5r} \|T\|_{p\to q}^q\|P_{Q_i(r)}f\|_p^q\\
&\le C\|T\|_{p\to q}^q\sum_i \|P_{Q_i(r)}f\|_p^q\\
&\le C\|T\|_{p\to q}^q\|f\|_p^q.
\end{aligned}
$$

This proves Lemma 4.4. □

Proof of Theorem 4.1. Let $\phi\in C_c^\infty(\mathbb{R})$ be a function such that $\operatorname{supp}\phi\subseteq\{\xi: 1/4\le|\xi|\le 1\}$ and $\sum_{\ell\in\mathbb{Z}}\phi(2^{-\ell}\lambda)=1$ for all $\lambda>0$. Set $\phi_0(\lambda)=1-\sum_{\ell=1}^\infty\phi(2^{-\ell}\lambda)$. By the Fourier inversion formula, we can write

$$AF(A)=\sum_{\ell=0}^\infty AF^{(\ell)}(A), \tag{4.5}$$

where

$$F^{(0)}(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi_0(s)\hat{F}(s)e^{is\lambda}\,ds \tag{4.6}$$

and

$$F^{(\ell)}(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi(2^{-\ell}s)\hat{F}(s)e^{is\lambda}\,ds. \tag{4.7}$$

We now estimate L^p-L^p norm of the operator $AF^{(\ell)}(A), \ell\ge 0$. Let $N\ge 1$ be a constant to be chosen later. For every $\ell\ge 0$, we follow (4.3) and (4.4) to write

$$AF^{(\ell)}(A)=:[AF^{(\ell)}(A)]_{<N\tau}+[AF^{(\ell)}(A)]_{>N\tau}. \tag{4.8}$$

Estimate of the term $[AF^{(\ell)}(A)]_{<N\tau}$. Let $\psi\in C_c^\infty(\mathbb{R})$ be a function such that $\psi(\lambda)=1$ for $\lambda\in[R_1+\gamma/2, R_2-\gamma/2]$ and $\operatorname{supp}\psi\subset[R_1,R_2]$. We write

$$[AF^{(\ell)}(A)]_{<N\tau}=[\psi(A)AF^{(\ell)}(A)]_{<N\tau}+[(1-\psi(A))AF^{(\ell)}(A)]_{<N\tau}.$$

Observe that

$$\left\|[\psi(A)AF^{(\ell)}(A)]_{<N\tau}\right\|_p^p \le C\sum_j\sum_{\lambda:d(x_j,x_\lambda)\le 5N\tau}\|P_{Q_\lambda(N\tau)}A\psi(A)F^{(\ell)}(A)P_{Q_j(N\tau)}f\|_p^p.$$

We use the Hölder inequality to obtain

$$
\begin{aligned}
\|P_{Q_\lambda(N\tau)}A\psi(A)F^{(\ell)}(A)P_{Q_j(N\tau)}f\|_p &\le \mu(Q_\lambda(N\tau))^{\sigma_p}\|P_{Q_\lambda(N\tau)}A\psi(A)F^{(\ell)}(A)P_{Q_j(N\tau)}f\|_2\\
&\le C\mu(Q_j(N\tau))^{\sigma_p}\|F^{(\ell)}\|_q\|V_\tau^{1/2-1/p}P_{Q_j(N\tau)}f\|_p.
\end{aligned}
$$

Note that in the last inequality we used the condition $d(x_j,x_\lambda)\le 5N\tau$ and the fact that $\operatorname{supp}\psi F^{(\ell)}\subset[R_1,R_2]$, and it follows from the property ($\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2}(\tau)$) that $\|A\psi(A)F^{(\ell)}(A)V_\tau^{\sigma_p}\|_{p\to 2}\le\|F^{(\ell)}\|_q$. Hence,

$$\left\|[\psi(A)AF^{(\ell)}(A)]_{<N\tau}\right\|_p^p \le CN^{np\sigma_p}\|F^{(\ell)}\|_q^p\sum_j\mu(Q_j(\tau))^{p\sigma_p}\|V_\tau^{-\sigma_p}P_{Q_j(N\tau)}f\|_p^p$$

$$
\begin{aligned}
&\le CN^{np\sigma_p}\|F^{(\ell)}\|_q^p \sum_j \|P_{Q_j(N\tau)}f\|_p^p \\
&\le CN^{np\sigma_p}\|F^{(\ell)}\|_q^p\|f\|_p^p. \qquad (4.9)
\end{aligned}
$$

On the other hand, we apply Lemma 4.4 and Theorem 3.1 to obtain that for some $s > n/2 + 1$ and every large number $M > 0$,

$$
\begin{aligned}
\|[(1-\psi(A))AF^{(\ell)}(A)]_{<N\tau}\|_{p\to p} &\le C\|(1-\psi(A))AF^{(\ell)}(A)\|_{p\to p} \\
&\le C\|(1-\psi(\cdot))F^{(\ell)}(\cdot)\|_{H^s} \le C_{\sigma,M}2^{-M\ell}\|F\|_q.
\end{aligned}
$$

This, together with (4.9), yields

$$
\|[AF^{(\ell)}(A)]_{<N\tau}f\|_p^p \le CN^{np\sigma_p}\Big[\|F^{(\ell)}\|_q^p + 2^{-Mp\ell}\|F\|_q^p\Big]\|f\|_p^p. \tag{4.10}
$$

<u>Estimate of the term $[AF^{(\ell)}(A)]_{\ge N\tau}$.</u> From (4.6) and (4.7), we have that

$$
[AF^{(\ell)}(A)]_{>N\tau} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(2^{-\ell}t)\hat{F}(t)[Ae^{itA}]_{>N\tau}\, dt. \tag{4.11}
$$

To estimate the term $[Ae^{itA}]_{>N\tau}$, we write

$$
\begin{aligned}
Ae^{itA} &= e^{itA}[A]_{>N\tau/2} + [e^{itA}]_{>N\tau/2}[A]_{<N\tau/2} + [e^{itA}]_{<N\tau/2}[A]_{<N\tau/2} \\
&=: T_1 + T_2 + T_3.
\end{aligned}
$$

It is easy to see that

$$
\sum_i \sum_{j:d(x_i,x_j)>5N\tau} P_{Q_j(N\tau)}T_3P_{Q_i(N\tau)} = 0,
$$

and so

$$
[Ae^{itA}]_{>N\tau} =: [T_1]_{>N\tau} + [T_2]_{>N\tau}.
$$

By Lemma 4.4 with $r = N\tau$, we obtain

$$
\begin{aligned}
\|[Ae^{itA}]_{>N\tau}\|_{p\to p} &\le \|T_1 - [T_1]_{<N\tau}\|_{p\to p} + \|T_2 - [T_2]_{<N\tau}\|_{p\to p} \\
&\le (1+C)\|T_1\|_{p\to p} + (1+C)\|T_2\|_{p\to p}, \qquad (4.12)
\end{aligned}
$$

and hence we have to estimate $\|T_i\|_{p\to p}$, $i = 1, 2$.

Let us estimate the term $\|T_1\|_{p\to p}$. We write

$$
\begin{aligned}
\|T_1\|_{p\to p} &= \|e^{itA}[A]_{>N\tau/2}\|_{p\to p} \\
&\le \left\|V_\tau^{-\sigma_p}\right\|_{X_\tau^{p,2}\to L^p}\left\|V_\tau^{\sigma_p}e^{itA}V_\tau^{-\sigma_p}\right\|_{X_\tau^{p,2}\to X_\tau^{p,2}}\left\|V_\tau^{\sigma_p}[A]_{>N\tau/2}\right\|_{L^p\to X_\tau^{p,2}}. \qquad (4.13)
\end{aligned}
$$

We have that $\|V_\tau^{-\sigma_p}\|_{X_\tau^{p,2}\to L^p} \le C$ by definition of $X^{p,2}$ and Hölder's inequality. Also it follows from (3.12) that

$$
\left\|V_\tau^{\sigma_p}e^{itA}V_\tau^{-\sigma_p}\right\|_{X_\tau^{p,2}\to X_\tau^{p,2}} \le C(1+|t|)^{n\sigma_p}. \tag{4.14}
$$

To handle the term $\left\|V_\tau^{\sigma_p}[A]_{>N\tau/2}\right\|_{L^p\to X_\tau^{p,2}}$, we note that

$$
\left\|V_\tau^{\sigma_p}[A]_{>N\tau/2}f\right\|_{X_\tau^{p,2}} = \left(\sum_{j=1}^{\infty}\|P_{Q_j(\tau)}V_\tau^{\sigma_p}[A]_{>N\tau/2}f\|_2^p\right)^{1/p}
$$

$$\leq \left(\sum_j \left[\sum_{\ell: d(x_\ell, x_j) > N\tau/2} \left\| P_{Q_j(\tau)} V_\tau^{\sigma_p} A P_{Q_\ell(\tau)} \right\|_{p\to 2} \left\| P_{Q_\ell(\tau)} f \right\|_p \right]^p \right)^{1/p}.$$

For every j, ℓ, we set $a_{j\ell} = \|P_{Q_j(\tau)} V_\tau^{\sigma_p} A P_{Q_\ell(\tau)}\|_{p\to 2}$ and $b_\ell = \|P_{Q_\ell(\tau)} f\|_p$. This gives

$$\begin{aligned}
\|V_\tau^{\sigma_p}[A]_{>N\tau/2} f\|_{X_\tau^{p,2}} &\leq \Big\| \sum_\ell a_{j\ell} b_\ell \Big\|_{\ell^p} \\
&\leq \left\| (a_{j\ell}) \right\|_{\ell^p \to \ell^p} \|b_\ell\|_{\ell^p} \\
&\leq \left\| (a_{j\ell}) \right\|_{\ell^1 \to \ell^1}^{\theta} \left\| (a_{j\ell}) \right\|_{\ell^\infty \to \ell^\infty}^{1-\theta} \|f\|_p
\end{aligned}$$

by interpolation. It then follows from condition ($\mathrm{PVE}^a_{p,2}(\tau)$) that

$$\|(a_{j\ell})\|_{\ell^1\to\ell^1} = \sup_\ell \sum_{j: d(x_\ell, x_j) > N\tau/2} \left\| P_{Q_j(\tau)} V_\tau^{\sigma_p} A P_{Q_\ell(\tau)} \right\|_{p\to 2} \leq C N^{-a}$$

and

$$\|(a_{j\ell})\|_{\ell^\infty\to\ell^\infty} = \sup_j \sum_{\ell: d(x_\ell, x_j) > N\tau/2} \left\| P_{Q_j(\tau)} V_\tau^{\sigma_p} A P_{Q_\ell(\tau)} \right\|_{p\to 2} \leq C N^{-a},$$

and so

$$\left\| V_\tau^{\sigma_p} [A]_{>N\tau/2} \right\|_{L^p \to X_\tau^{p,2}} \leq C N^{-a}.$$

This, in combination with (4.14) and (4.13), shows that

$$\|T_1\|_{p\to p} = \|e^{itA}[A]_{>N\tau/2}\|_{p\to p} \leq C N^{-a}(1+t)^{n\sigma_p}. \tag{4.15}$$

Next we estimate the term $\|T_2\|_{p\to p}$. We write

$$\begin{aligned}
\|T_2\|_{p\to p} &= \|[e^{itA}]_{>N\tau/2}[A]_{<N\tau/2}\|_{p\to p} \\
&\leq \left\| V_\tau^{-\sigma_p} \right\|_{X_\tau^{p,2}\to L^p} \left\| V_\tau^{\sigma_p} [e^{itA}]_{>N\tau/2} V_\tau^{-\sigma_p} \right\|_{X_\tau^{p,2}\to X_\tau^{p,2}} \left\| V_\tau^{\sigma_p} [A]_{<N\tau/2} \right\|_{L^p \to X_\tau^{p,2}}.
\end{aligned} \tag{4.16}$$

We have that $\|V_\tau^{-\sigma_p}\|_{X_\tau^{p,2}\to L^p} \leq C$ by definition of $X^{p,2}$ and Hölder's inequality. Also, we follow a similar argument as in (3.11) to obtain

$$\|V_\tau^{\sigma_p}[A]_{<N\tau/2}\|_{L^p\to X_\tau^{p,2}} \leq C. \tag{4.17}$$

In order to deal with the term $\|V_\tau^{\sigma_p}[e^{itA}]_{>N\tau/2} V_\tau^{-\sigma_p}\|_{X_\tau^{p,2}\to X_\tau^{p,2}}$, we apply the complex interpolation method. To do it, we let $\mathbf{S}$ denote the closed strip $0 \leq \mathrm{Re} z \leq 1$ in the complex plane. For $z \in \mathbf{S}$, we consider an analytic family of operators:

$$T_z = V_\tau^{\frac{z}{2}} [e^{itA}]_{>N\tau/2} V_\tau^{-\frac{z}{2}}.$$

Case 1: $z = 1 + iy$, $y \in \mathbb{R}$ and $p = 1$. In this case, we observe that

$$\begin{aligned}
\|T_{1+iy} f\|_{X_\tau^{1,2}} &= \sum_{j=1}^\infty \left\| P_{Q_j(\tau)} V_\tau^{\frac{1+iy}{2}} [e^{itA}]_{>N\tau/2} V_\tau^{-\frac{1+iy}{2}} f \right\|_2 \\
&\leq \sum_{\ell=1}^\infty \sum_{j=1}^\infty \left\| P_{Q_j(\tau)} V_\tau^{\frac{1+iy}{2}} [e^{itA}]_{>N\tau/2} V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f \right\|_2
\end{aligned}$$

$$\leq \sum_{\ell=1}^{\infty} \sum_{j:\, d(x_\ell,x_j)>N\tau/2} \left\| P_{Q_j(\tau)} V_\tau^{\frac{1+iy}{2}} e^{itA} V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f \right\|_2. \tag{4.18}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\sum_{j:\, d(x_\ell,x_j)>N\tau/2} \left\| P_{Q_j(\tau)} V_\tau^{\frac{1+iy}{2}} e^{itA} V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f \right\|_2 \\
&\quad \leq \left(\sum_{j:\, d(x_\ell,x_j)>N\tau/2} \mu(Q_j(\tau)) \eta_\ell(x_j)^{-2[a]} \right)^{1/2} \left(\sum_{d(x_\ell,x_j)>N\tau/2} \eta_\ell(x_j)^{2[a]} \left\| P_{Q_j(\tau)} e^{itA} V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f \right\|_2^2 \right)^{1/2},
\end{aligned}$$

where $\eta_\ell(x) := d(x, x_\ell)/\tau$. Since

$$\begin{aligned}
\sum_{j:\, d(x_\ell,x_j)>N\tau/2} \mu(Q_j(\tau)) \eta_\ell(x_j)^{-2[a]} &\leq C \sum_{i=0}^{\infty} \sum_{j:\, 2^i N\tau < d(x_\ell,x_j) \leq 2^{i+1} N\tau} \mu(Q_j(\tau)) \eta_\ell(x_j)^{-2[a]} \\
&\leq C \sum_{i=0}^{\infty} (2^i N)^n \mu(Q_\ell(\tau)) (2^i N)^{-2[a]} \\
&\leq C N^{-2[a]+n} \mu(Q_\ell(\tau))
\end{aligned}$$

and

$$\sum_{j:\, d(x_\ell,x_j)>N\tau/2} \eta_\ell(x_j)^{2[a]} \left\| P_{Q_j(\tau)} e^{itA} V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f \right\|_2^2 \leq C \left\| \eta_\ell^{[a]} e^{itA} V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f \right\|_2^2,$$

we have

$$\|T_{1+iy} f\|_{X_\tau^{1,2}} \leq C \sum_{\ell=1}^{\infty} N^{-[a]+n/2} \mu(Q_\ell(\tau))^{1/2} \left\| \eta_\ell^{[a]} e^{itA} V_\tau^{-\frac{1+iy}{2}} P_{Q_\ell(\tau)} f \right\|_2$$

Following an argument as in (3.8) and (3.13) we obtain

$$\|\eta_\ell^{[a]} e^{itA} (1+\eta_\ell)^{-[a]}\|_{2\to 2} \leq C(1+t)^{[a]}.$$

From this, it follows that

$$\begin{aligned}
\|T_{1+iy} f\|_{X_\tau^{1,2}} &\leq C N^{-[a]+n/2} (1+t)^{[a]} \sum_i \mu(Q_\ell(\tau))^{1/2} \left\| V_\tau^{-\frac{1+iy}{2}} (1+\eta_\ell)^{[a]} P_{Q_\ell(\tau)} f \right\|_2 \\
&\leq C N^{-[a]+n/2} (1+t)^{[a]} \sum_i \|P_{Q_\ell(\tau)} f\|_2 \\
&\leq C N^{-[a]+n/2} (1+t)^{[a]} \|f\|_{X_\tau^{1,2}}.
\end{aligned}$$

Case 2: $z = iy$, $y \in \mathbb{R}$ and $p = 2$. In this case, we note that

$$\left\| T_{iy} \right\|_{X_\tau^{2,2} \to X_\tau^{2,2}} = \|T_{iy}\|_{2\to 2} \leq \|[e^{itA}]_{>N\tau/2}\|_{2\to 2} \leq (1+C)\|e^{itA}\|_{2\to 2} \leq 1 + C.$$

From **Cases 1** and **2**, we apply the complex interpolation method to obtain

$$\left\| V_\tau^{\sigma_p} [e^{itA}]_{>N\tau/2} V_\tau^{-\sigma_p} f \right\|_{X_\tau^{p,2}} = \left\| T_{\frac{2}{p}-1} f \right\|_{X_\tau^{p,2}} \leq C_p \left(N^{-[a]+n/2} (1+t)^{[a]} \right)^\theta \|f\|_{X_\tau^{p,2}}$$

where $\theta = 2/p - 1$. This implies

$$\|T_2\|_{p\to p} \leq C N^{(-[a]+n/2)(2/p-1)} (1+t)^{[a](2/p-1)}$$

Substituting this and estimate (4.15) back into (4.12), yields

$$\|[Ae^{itA}]_{>N\tau}\|_{p\to p} \le C\|T_1\|_{p\to p} + C\|T_2\|_{p\to p} \le CN^{(-[a]+n/2)(2/p-1)}(1+t)^{[a](2/p-1)}.$$

This, in combination with (4.11) and (4.10), shows

$$\begin{aligned}\|AF^{\ell}(A)\|_{p\to p} &\le CN^{n(1/p-1/2)}[\|F^{(\ell)}(\lambda)\|_q + 2^{-M\ell}\|F\|_q] \\ &+ CN^{(-[a]+n/2)(2/p-1)}\int_{2^\ell}^{2^{\ell+1}} |\phi(2^{-\ell}t)||\hat{F}(t)|(1+t)^{[a](2/p-1)}\,dt.\end{aligned}$$

We take $N = 2^{\ell(1+p/(2[a](2-p)))}$ to obtain

$$\|AF^{\ell}(A)\|_{p\to p} \le C2^{\ell n(1/p-1/2)(1+p/(2[a](2-p)))}\left(\|F^{(\ell)}(\lambda)\|_q + 2^{-M\ell}\|F\|_q + \|\phi_\ell\widehat{F}\|_2\right).$$

After summation in ℓ, we obtain

$$\|AF(A)\|_{p\to p} \le C\|F\|_{W^{s,q}}$$

as in (4.1), whenever

$$s > n\left(\frac{1}{p}-\frac{1}{2}\right)\left(1+\frac{p}{2[a](2-p)}\right) = n\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{n}{4[a]}.$$

The proof of Theorem 4.1 is complete. □

5. Applications

As an illustration of our results we shall discuss some examples. Our main results, Theorems 3.1 and 4.1. can be applied to all examples which are discussed in [19], [11] and [34].

5.1. **Symmetric Markov chains.** In [1] Alexopoulos considers bounded symmetric Markov operator A on a homogeneous space X whose powers A^k have kernels $A^k(x,y)$ satisfying the following Gaussian estimates

$$0 \le A^k(x,y) \le \frac{C}{V(x,k^{1/2})}\exp\left(-\frac{d(x,y)^2}{k}\right) \tag{5.1}$$

for all $k \in \mathbb{N}$. The operator $I-A$ is symmetric and satisfies for every $f \in L^2(X)$,

$$\langle (I-A)f,\ f\rangle = \frac{1}{2}\iint (f(x)-f(y))^2 A(x,y)d\mu(x)d\mu(y) \ge 0.$$

Thus, $I-A$ is positive. In addition, $\|(I-A)f\|_2 \le \|f\|_2 + \|Af\|_2 \le 2\|f\|_2$. Hence, $I-A$ admits the spectral decomposition (see for example, [30]) which allows to write

$$I-A = \int_0^2 (1-\lambda)dE_A(\lambda).$$

Let F be a bounded Borel measurable function. Then by the spectral theorem we can define the operator

$$F(I-A) = \int_0^2 F(1-\lambda)dE_A(\lambda).$$

Note that $I - A$ is bounded on $L^2(X)$ and $\|F(I-A)\|_{2\to 2} \le \|F\|_\infty$. In [1] Alexopoulos obtained the following spectral multiplier type result. In the sequel, let us consider a function $0 \le \eta \in C^\infty(\mathbb{R})$ and let us assume that $\eta(t) = 1$ for $t \in [1,2]$ and that $\eta(t) = 0$ for $t \notin [1/2, 4]$.

Theorem 5.1. *Assume that F is a bounded Borel function with $\operatorname{supp} F \subset [0, 1/2]$ and that*

$$\sup_{0<t\le 1} \|\eta(\cdot)F(t\cdot)\|_{W^{n/2+\epsilon,\infty}} < \infty$$

for some $\epsilon > 0$. Then under above assumption on the operator A, the spectral multiplier $F(I-A)$ extends to a bounded operator on L^p for $1 \le p \le \infty$.

Our approach allows us to prove a version of Alexopoulos' result under the weaker assumption of polynomial decay rather than the exponential one. We start with the following statement.

Theorem 5.2. *Let $1 \le p < 2 \le q \le \infty$. Suppose that (X, d, μ) satisfies the doubling condition with the doubling exponent n from (1.2). Assume next that A is a bounded self-adjoint operator and there exists $k \in \mathbb{N}$ such that the kernel of the operator A^k exists and satisfies the following estimate*

$$|A^k(x,y)| \le C \frac{1}{\max(V(x,k^{1/2}), V(y,k^{1/2}))} \left(1 + \frac{d(x,y)^2}{k}\right)^{-N} \tag{5.2}$$

for some $N > n + [n/2] + 1$. Then for every $1 \le p \le \infty$,

$$\|F(A^k)A^k\|_{p\to p} \le C \min\left(\|F\|_{W^{s,\infty}}, \|F\|_{H^{s'}}\right) \tag{5.3}$$

for any $s > n\sigma_p + \frac{n}{4[a]}$ and any $s' > n\sigma_p + \frac{1}{2}$.

If in addition the restriction type bounds ($\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2}(\tau)$) with $\tau = k^{1/2}$ are valid on the interval (R_1, R_2) for some $-\infty < R_1 < R_2 < \infty$ and supp$F \subset (R_1 + \gamma, R_2 - \gamma)$ for some $\gamma > 0$ then

$$\|F(A^k)A^k\|_{p\to p} \le C\|F\|_{W^{s,q}} \tag{5.4}$$

$s > n\sigma_p + \frac{n}{4[a]}$.

Proof. It follows from (5.2) that the operator A^k satisfies the property ($\mathrm{PVE}^a_{p,2}(\tau)$) with $a = N - n$ and $\tau = k^{1/2}$. The theorem follows from Theorems 3.1, 4.1 and Remark 4.3. □

The following result is a direct consequence of the above theorem.

Theorem 5.3. *Let $1 \le p < 2 \le q \le \infty$. Suppose that (X, d, μ) satisfies the doubling condition with the doubling exponent n from (1.2) and that $N > n + [n/2] + 1$. Assume next that A is a bounded self-adjoint operator and that for all $k \in \mathbb{N}$ the kernel of the operator A^k exists and satisfies the following estimate*

$$|A^k(x,y)| \le C \frac{1}{\max(V(x,k^{1/2}), V(y,k^{1/2}))} \left(1 + \frac{d(x,y)^2}{k}\right)^{-N} \tag{5.5}$$

with the constant C independent of k. In addition we assume that the restriction type bounds ($\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2}(\tau)$) with $\tau = k^{1/2}$ are valid on the interval $(R_1, R_2) \subset [-1, 1]$ for all $k \in \mathbb{N}$. Then for function F with supp $F \subset [1/4, 1/2]$

$$\|F(k(I-A))\|_{p\to p} \le C\|F\|_{W^{s,q}} \tag{5.6}$$

for any $s > n\sigma_p + \frac{n}{4[a]}$.

Proof. Note that by (5.5) we have that $\|A^k\|_{2\to 2} \le C < \infty$ for some constant C independent of k. It follows that the spectrum of A is contained in the interval $[-1, 1]$. For a given $k \in \mathbb{N}$, we define a function G as

$$G(\lambda) = \frac{F(k(1-\lambda^{1/k}))}{\lambda}.$$

and so $G(\lambda^k)\lambda^k = F(k(1-\lambda))$. It follows from Theorem 5.2 that for $s > n\sigma_p + \frac{n}{4[a]}$,

$$\|G(A^k)A^k\|_{p\to p} \le C\|G\|_{W^{s,q}},$$

which yields

$$\|F(k(I-A))\|_{p\to p} = \|G(A^k)A^k\|_{p\to p} \le C\|G\|_{W^{s,q}} = C\left\|\frac{F(k(1-\lambda^{1/k}))}{\lambda}\right\|_{W^{s,q}}.$$

Note that $\operatorname{supp} F \subset [1/4, 1/2]$, we have

$$\|F(k(I-A))\|_{p\to p} \le C\left\|\frac{F(k(1-\lambda^{1/k}))}{\lambda}\right\|_{W^{s,q}} \le C\|F\|_{W^{s,q}}.$$

This completes the proof of Theorem 5.3. □

Remark 5.4. *1) Note that we do not assume that the operator A is Markovian.*
2) Using similar technique as in [34] *one can obtain the singular integral version of Theorem 5.2 stated in Theorem 5.1. We do not discuss the details here.*

5.2. **Random walk on $\mathbb{Z}^n$.** In this section, we consider random walk on the n-dimensional integer lattice $\mathbb{Z}^n$. Define the operator A acting on $l^2(\mathbb{Z}^n)$ by the formula

$$Af(\mathbf{d}) = \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=\pm 1} f(\mathbf{d}+j\mathbf{e}_i)$$

where $\mathbf{e}_i = (0, \dots, 1, \dots 0)$ and 1 is positioned on the i-coordinate. The aim of this section is to prove the following result. Recall that η is the auxiliary nonzero compactly supported function $\eta \in C_c^\infty[1/2, 4]$ as in Theorem 5.1.

Theorem 5.5. *Let* $1 < p < \infty$. *Let A be the random walk on the integer lattice defined above. Suppose that* $\operatorname{supp} F \subset (0, 1/n)$. *Then*

$$\sup_{t>1} \|F(t(I-A))\|_{p\to p} \le C\|F\|_{H^s} \tag{5.7}$$

for any $s > n|1/p - 1/2|$.

Next we assume that a bounded Borel function $F : \mathbb{R}_+ \to \mathbb{C}$ *satisfies* $\operatorname{supp} F \subset [0, 1/n]$ *and*

$$\sup_{1/n>t>0} \|\eta F(t\cdot)\|_{H^s} < \infty \tag{5.8}$$

for some $s > n|1/p - 1/2|$. *Then the operator* $F(I-A)$ *is bounded on* L^p *if* $1 < p < (2n+2)/(n+3)$ *and weak type* $(1, 1)$ *if* $p = 1$.

The proof of Theorem 5.5 is given at the end of this section and it is based on the following restriction type estimate.

Proposition 5.6. *Let A be defined as above and $dE(\lambda)$ be the spectral measure of A. Then for $\lambda \in [1-1/n, 1]$*

$$\|dE(\lambda)\|_{p\to p'} \le C(1-\lambda)^{n(1/p-1/p')/2-1} \tag{5.9}$$

for $1 < p < (2n+2)/(n+3)$.

The proof of Proposition 5.6 is based on the following result due to Bak and Seeger [3, Theorem 1.1].

Lemma 5.7. *Consider a probability measure μ on $\mathbb{R}^n$. Assume that for positive constants $0 < a < n$, $0 < b \le a/2$, $M_i \ge 1, i = 1,2$, μ satisfies*

$$\sup_{r_B\le 1} \frac{\mu(B(x_B, r_B))}{r_B^a} \le M_1 \tag{5.10}$$

where the supremum is taken over all balls with radius ≤ 1 and

$$\sup_{|\xi|\ge 1} |\xi|^b |\widehat{d\mu}| \le M_2. \tag{5.11}$$

Let $p_0 = \frac{2(n-a+b)}{2(n-a)+b}$. Then

$$\int \left|\widehat{f}\right|^2 d\mu \le C M_1^{\frac{b}{n-a+b}} M_2^{\frac{n-a}{n-a+b}} \|f\|^2_{L^{p_0,2}(\mathbb{R}^n)}, \tag{5.12}$$

where $L^{p_0,2}$ is the Lorentz space.

Proof of Proposition 5.6. Let $\mathbf{T}^n$ be the n-dimensional torus (note that n is equal to the homogeneous dimension of $\mathbf{T}^n$). For any function $f \in l^2(\mathbb{Z}^n)$, one can define the Fourier series $\mathcal{F}f \colon \mathbf{T}^n \to \mathbb{C}$ of f by

$$\mathcal{F}f(\theta) = \sum_{\mathbf{d}\in\mathbb{Z}^n} f(\mathbf{d}) e^{i\langle \mathbf{d},\theta\rangle}.$$

Then the inverse Fourier series $\mathcal{F}^{-1}f \colon \mathbb{Z}^n \to \mathbb{C}$ is defined by

$$\mathcal{F}^{-1}f(\mathbf{d}) = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(\theta) e^{-i\langle \mathbf{d},\theta\rangle}.$$

Define the convolution of $f, g \in L^2(\mathbb{Z}^n)$ by

$$f * g(\mathbf{d}) = \sum_{\mathbf{d}_1\in\mathbb{Z}^n} f(\mathbf{d}-\mathbf{d}_1) g(\mathbf{d}_1).$$

Note that

$$\mathcal{F}(Af)(\theta) = \left(\frac{1}{n}\sum_{j=1}^n \cos\theta_j\right)\mathcal{F}f(\theta) = (G(\theta))\mathcal{F}f(\theta),$$

where

$$G(\theta) = \frac{1}{n}\sum_{j=1}^n \cos\theta_j.$$

Hence for any continuous function F

$$
\begin{aligned}
\int_{-1}^{1} F(\lambda) dE(\lambda) f = F(A) f(\mathbf{d}) &= \mathcal{F}^{-1}\left(F(G(\theta))\widehat{f}(\theta)\right) \\
&= \left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}}\int_{\sigma_\lambda} F(\lambda) e^{i\langle \mathbf{d},\theta\rangle}\frac{1}{|\nabla G|}\, d\sigma_\lambda(\theta) d\lambda\right) * f \\
&= \int_{\mathbb{R}} F(\lambda)\,(\Gamma_\lambda * f)\, d\lambda,
\end{aligned}
$$

where σ_λ is the level set defined by the formula

$$\sigma_\lambda = \{\theta\colon \ \frac{1}{n}\sum_{j=1}^{n}\cos\theta_j = \lambda\} \subset \mathbf{T}^n$$

and

$$\Gamma_\lambda(\mathbf{d}) = \frac{1}{(2\pi)^n}\int_{\sigma_\lambda} e^{i\langle \mathbf{d},\theta\rangle}\frac{1}{|\nabla G|}\, d\sigma_\lambda(\theta)$$

for all $\lambda \in [-1,1]$. Thus

$$dE(\lambda) f = f * \Gamma_\lambda.$$

For the range $\lambda \in (1-1/n, 1)$ considered in the proposition, changing variable yields

$$\Gamma_\lambda(\mathbf{d}) = \frac{(1-\lambda)^{(n-1)/2}}{(2\pi i)^n}\int_{\sigma'_\lambda} e^{i\langle \mathbf{d},\,(1-\lambda)^{1/2}\theta\rangle}\frac{1}{|\nabla G|((1-\lambda)^{1/2}\theta)}\, d\widetilde{\sigma}_\lambda(\theta),$$

where

$$\widetilde{\sigma}_\lambda = \{\theta\colon \ \frac{1}{n}\sum_{j=1}^{n}\cos((1-\lambda)^{1/2}\theta_j) = \lambda\} \subset \mathbf{T}^n.$$

Next we define a probability measure μ_λ on the surface $\widetilde{\sigma}_\lambda$ by the formula

$$d\mu_\lambda = \frac{1}{|\nabla G|((1-\lambda)^{1/2}\theta) N(\lambda)}\, d\widetilde{\sigma}_\lambda(\theta),$$

where

$$N(\lambda) = \int_{\widetilde{\sigma}_\lambda}\frac{1}{|\nabla G|((1-\lambda)^{1/2}\theta)}\, d\widetilde{\sigma}_\lambda(\theta).$$

For $\lambda \in (1-1/n, 1)$ we define the restriction type operator $R_\lambda\colon\ \ell^1(\mathbb{Z}^n) \to L^2(\widetilde{\sigma}_\lambda, \mu_\lambda)$ by

$$(R_\lambda f)(\theta) = \mathcal{F} f(\theta(1-\lambda)^{1/2}) = \sum_{\mathbf{d}\in\mathbb{Z}^\mathbf{n}} f(\mathbf{d}) e^{i\langle \mathbf{d},\,(1-\lambda)^{1/2}\theta\rangle}, \qquad \theta \in \widetilde{\sigma}_\lambda.$$

Then the dual operator R_λ^* is given by

$$(R_\lambda^* f)(\mathbf{d}) = \int_{\widetilde{\sigma}_\lambda} e^{-i(1-\lambda)^{1/2}\langle \mathbf{d},\theta\rangle}\frac{f(\theta)}{|\nabla G|((1-\lambda)^{1/2}\theta) N(\lambda)}\, d\widetilde{\sigma}_\lambda(\theta).$$

Hence

$$dE(\lambda) f = \frac{N(\lambda)((1-\lambda)^{1/2})^{n-1}}{(2\pi)^n} R_\lambda^* R_\lambda f.$$

Following the standard approach on the Euclidean space we study boundedness of the operator R^*_λ acting from $L^2(\widetilde{\sigma}_\lambda, \mu_\lambda)$ to $\ell^p(\mathbb{Z}^n)$. Next we define the operator $\tilde{R}^*_\lambda \colon L^2(\widetilde{\sigma}_\lambda, \mu_\lambda) \to L^\infty(\mathbb{R}^n)$ by

$$(\tilde{R}^*_\lambda f)(\xi) = \int_{\widetilde{\sigma}_\lambda} e^{-i\langle \xi, (1-\lambda)^{1/2}\theta\rangle} d\mu_\lambda(\theta).$$

By the Plancherel-Pólya inequality (cf. [36, Section 1.3.3])

$$\|(R^*_\lambda f)(\mathbf{d})\|_{L^p(\mathbb{Z}^n)} \sim \|R^*_\lambda f(\xi)\|_{L^p(\mathbb{R}^n)} \tag{5.13}$$

for all $1 \le p \le \infty$. Hence, it suffices to study $R^*_\lambda f(\xi), \xi \in \mathbb{R}^n$. Set $\tilde{G}(\theta) = \frac{1}{n}\sum_{j=1}^n \cos((1-\lambda)^{1/2}\theta_j)$. Denote $H(\tilde{G})$ the Hessian corresponding to $\tilde{G}$. Then the Gaussian curvature for an implicitly defined surface corresponding to the equation $\tilde{G}(\theta) = \lambda$ is given by the following formula

$$\begin{aligned} K &= -\begin{vmatrix} H(\tilde{G}) & \nabla\tilde{G}^T \\ \nabla\tilde{G} & 0 \end{vmatrix} |\nabla\tilde{G}|^{-(n+1)} \\ &= \frac{(-1)^{n+1}((1-\lambda)^{1/2})^{n-1}\prod_{j=1}^n \cos((1-\lambda)^{1/2}\theta_j)\left(\sum_{j=1}^n \sin(\theta_j(1-\lambda)^{1/2})\tan((1-\lambda)^{1/2}\theta_j)\right)}{\left(\sum_{j=1}^n \sin^2((1-\lambda)^{1/2}\theta_j)\right)^{(n+1)/2}}. \end{aligned}$$

Note that if $\lambda \in (1-1/n, 1)$, then $\cos((1-\lambda)^{1/2}\theta_j) > 0$ for all j and

$$\cos((1-\lambda)^{1/2}\theta_j) \ge 1 - (1-\lambda)n.$$

Indeed, otherwise

$$\frac{1}{n}\sum_{j=1}^n \cos((1-\lambda)^{1/2}\theta_j) < \frac{n-1+1-(1-\lambda)n}{n} = \lambda$$

which contradicts $\theta \in \widetilde{\sigma}_\lambda$. It follows that for every $n \ge 2$ there exists a positive constant $C_n > 0$ which does not depend on λ and θ such that

$$|K| \ge C_n.$$

There exists also a constant $C > 0$ such that for all $\lambda \in (1-1/n, 1)$

$$(1-\lambda)^{1/2} \ge |\nabla G|((1-\lambda)^{1/2}\theta) = \frac{1}{n}\left(\sum_{j=1}^n \sin^2((1-\lambda)^{1/2}\theta_j)\right)^{1/2} \ge C(1-\lambda)^{1/2}$$

so $N(\lambda) \sim (1-\lambda)^{-1/2}$. Then from Stein [35, Page 360, Section 5.7 of Chapter VIII], we know that

$$|\widehat{d\mu_\lambda}| \le C(1+|\xi|)^{(1-n)/2}, \tag{5.14}$$

where C just depends on n and does not depend on λ and θ.

Now, it is not difficult to check that surfaces $\widetilde{\sigma}_\lambda$ and measures μ_λ satisfy assumptions of Lemma 5.7. The required exponent for (5.10) is equal to $a = n-1$. In addition $d\mu_\lambda$ satisfies (5.11) with $b = (n-1)/2$ uniformly in $\lambda \in (1-1/n, 1)$. Hence by Lemma 5.7

$$\int_{\widetilde{\sigma}_\lambda} \left|\widehat{f}\right|^2 d\mu_\lambda \le C\|f\|^2_{L^p(\mathbb{R}^n)}$$

Hence

$$\|\tilde{R}_\lambda^* f(\xi)\|_{L^{p'}(\mathbb{R}^n)} \le C(1-\lambda)^{-\frac{n}{2p'}}\|f\|_{L^2(\widetilde{\sigma}_\lambda,\mu_\lambda)}.$$

Thus by the Plancherel-Pólya inequality (5.13)

$$\|R_\lambda^* f(\mathbf{d})\|_{L^{p'}(\mathbb{Z}^n)} \le C(1-\lambda)^{-\frac{n}{2p'}}\|f\|_{L^2(\widetilde{\sigma}_\lambda,\mu_\lambda)}$$

for $1 \le p \le \frac{2n+2}{n+3}$. By duality

$$\|dE(\lambda)\|_{p\to p'} \le \frac{N(\lambda)(1-\lambda)^{\frac{n-1}{2}}}{(2\pi)^n}\|R_\lambda^* R_\lambda\|_{p\to p'} \le C(1-\lambda)^{n(1/p-1/p')/2-1},$$

for $1 \le p \le \frac{2n+2}{n+3}$ and $\lambda \in (1-1/n, 1)$. This completes the proof of Proposition 5.6. □

Now we are able to conclude the proof of Theorem 5.5.

Proof of Theorem 5.5. For every $k \in \mathbb{N}$, we denote $A^k(\mathbf{d}_1,\mathbf{d}_2)$ the kernel of A^k for $k \in \mathbb{Z}$. Note that $V(x,k) \sim k^n$. It is well-known (see e.g. [23]) that $A^k(\mathbf{d}_1,\mathbf{d}_2)$ satisfies the following Gaussian type upper estimate:

$$A^k(\mathbf{d}_1,\mathbf{d}_2) \le Ck^{-n/2}\exp\left(-\frac{|\mathbf{d}_1-\mathbf{d}_2|^2}{ck}\right). \tag{5.15}$$

Next we verify that the operators A^k satisfy condition $(\mathrm{ST}^2_{\mathrm{p},2}(\tau))$ with $\tau = k^{1/2}$ uniformly for all $k \in \mathbb{N}$ for all bounded Borel functions F such that supp $F \subset (1-1/n, 1)$. By T^*T argument and Proposition 5.6,

$$\begin{aligned}\left\|F(A^k)A^k\right\|^2_{p\to 2} = \left\||F|^2(A^k)A^{2k}\right\|_{p\to p'} &\le \int_{[(n-1)/n]^{1/k}}^1 |F|^2(\lambda^k)\lambda^{2k}\left\|dE_A(\lambda)\right\|_{p\to p'}\,d\lambda\\ &\le C\int_{(n-1)/n}^1 |F|^2(\lambda)\lambda^2(1-\lambda^{1/k})^{n(1/p-1/p')/2-1}\,d\lambda^{1/k}\\ &\le C\left(\frac{1}{k}\right)^{n(1/p-1/p')/2}\|F\|_2,\end{aligned}$$

as desired.

Now (5.7) follows from Theorem 5.3. The L^p boundedness of $F(I-A)$ for functions F satisfying condition (5.8) follows from [34, Theorem 3.3]. This completes the proof of Theorem 5.5. □

Remark 5.8. *For $n = 2$ it is enough to assume that* supp $F \subset (0, 1)$.

Remark 5.9. *There is another approach to the proof of Theorem 5.5 via transference type statements on equivalence of L^p boundedness of the Fourier integral and the Fourier series multipliers under suitable condition on the multiplier support (cf. [13]).*

5.3. **Fractional Schrödinger operators.** Let $n \geq 1$ and V, W be locally integrable non-negative functions on $\mathbb{R}^n$.
Consider the fractional Schrödinger operator with a potentials V and W:

$$L = (-\Delta + W)^\alpha + V(x), \quad \alpha \in (0, 1].$$

The particular case $\alpha = \frac{1}{2}$ is often referred to as the relativistic Schrödinger operator. The operator L is self-adjoint as an operator associated with a well defined closed quadratic form. By the classical subordination formula together with the Feynman-Kac formula it follows that the semigroup kernel $p_t(x, y)$ associated to e^{-tL} satisfies the estimate

$$0 \leq p_t(x, y) \leq Ct^{-\frac{n}{2\alpha}} \Big(1 + t^{-\frac{1}{2\alpha}}|x - y|\Big)^{-(n+2\alpha)}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. See page 195 of [33]. Hence, estimates ($\mathrm{PVE}^a_{p,2}(\tau)$) hold for $p = 1$ and $a = 2\alpha$. If $n = 1$ and $\alpha > \frac{1}{2}$ then we can apply Corollary 3.2 and obtain a spectral multiplier result for L.

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