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ABSTRACT. This paper gives a bound on the number of rational points on an absolutely irreducible curve $\mathcal{C}$ lying on a minimal toric surface X. This upper bound improves pre-existing ones if $\mathcal{C}$ has large genus. The strategy consists in finding another curve that intersects $\mathcal{C}$ with good multiplicity at its rational points outside some well-handled closed set. Finding such a curve relies on an extension of K.O. Stöhr and F.J. Voloch's idea for plane curves to the toric framework based on homogenization.

INTRODUCTION

By the Hasse-Weil bound, the number of $\mathbb{F}_q$-points on a smooth, geometrically integral projective curve C defined over $\mathbb{F}_q$ of genus g is bounded from above by $q+1+2g\sqrt{q}$. K.O. Stöhr and F.J. Voloch [SV86] gave an upper bound on the number of $\mathbb{F}_q$-points on an irreducible non-singular projective curve. This bound depends on the Frobenius order-sequence and the genus of the curve. M. Homma and S. J. Kim [HK09] [HK10a] [HK10b] used Stöhr-Voloch theory to prove that a curve on $\mathbb{P}^2$ of degree d without $\mathbb{F}_q$-linear components has at most $(d-1)q+1$ $\mathbb{F}_q$-points, except for a certain curve over $\mathbb{F}_4$. Few years later, M. Homma managed to extend this result on $\mathbb{P}^r$ for $r \geq 3$ [Hom12].

These latter bounds are sharper than Weil's general one for projectively embedded curves for a certain range of parameters [see Figures 3 p16 and 4 p17]. Such bounds are interesting in themselves and also have applications in coding theory, for example the computation of the minimum distance of algebraic geometric codes introduced by V.D.Goppa in 1980. In this paper, we focus on curves embedded in a certain class of surfaces, namely toric smooth surfaces. One can expect that constraining a curve in a specific ambient space and taking advantage of its geometry enables one to enhance the upper bound. We concentrate on irreducible curves, as the reducible case has already been dealt with in the context of Hirzebruch surfaces via coding theory [see [CD13], [CN16], [Nar18]].

More precisely, our strategy is to adapt Stöhr and Voloch's idea [SV86] for plane curves to fit into the toric framework. They bounded the number of $\mathbb{F}_q$-points on a plane curve by computing the number of points whose image under the Frobenius map belongs to their tangent line. They put to good use the nice property of the tangent line to the curve $f=0$ at a point $P=(x_P, y_P)$, namely that it has a global equation $(x-x_P)f_x(P)+(y-y_P)f_y(P)=0$ on the affine plane. Their bound can thus easily be computed as half the intersection number of C and the curve defined by $(x^q-x)f_x+(y^q-y)f_y=0$, since a $\mathbb{F}_q$-point has multiplicity 2 in this

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intersection. To put it another way, the authors displayed a polynomial that vanishes with multiplicity at least 2 at the rational points of the curve. Their bound only depends on the size of the field q and the degree of the curve.

We aim to generalize this idea. Given a curve $\mathcal{C}$ on a toric surface, we want to find an interpolation curve that intersects $\mathcal{C}$ at its rational points with good multiplicity using the same "tangent trick" as K.O. Stöhr and F.J. Voloch. However, adapting their idea on toric surfaces other than $\mathbb{P}^2$ is not trivial since there is no notion of global tangent line of a curve. A naive idea to overcome this issue would be to consider the local tangent line at a point P on a curve $\mathcal{C}$ then take its Zariski closure in the whole surface. Unfortunately "tangents" constructed in this way at two points P_1 and P_2 on $\mathcal{C}$ would not have the pleasant property of always being linearly equivalent.

Happily we can benefit from handy geometric properties of toric varieties. First, toric varieties are endowed with a graded polynomial coordinate ring, named the Cox ring. In the same way that an affine polynomial can be made into a homogeneous one on $\mathbb{P}^n$, there exists a process of homogenization, detailed in Section 1.2, that turns a regular function on the dense torus $\mathbb{T}$ of the toric variety into a polynomial of the Cox ring [see [CLS11] [CD97]].

Moreover, on toric surfaces, a curve can be defined as the zero locus of a polynomial in this Cox ring. In dimension 2, this means that, given the equation of a curve on the dense torus of a toric surface, it is possible to get an equation of a curve on the whole toric surface containing the first one. The degree of the polynomial defining a curve corresponds to its Picard class.

In addition, a toric surface is covered by affine charts (U_σ) isomorphic to $\mathbb{A}^2$ with explicit transition maps. Modifying the regular fonction $g = (x^q - x)f_x + (y^q - y)f_y$ according to these maps, we are able for each toric affine patch U_σ to easily define a curve on the torus that intersects the curve $\mathcal{C} \cap \mathbb{T}^2$ at the set of points in $\mathbb{T}^2$ whose image under the Frobenius map belongs to their tangent. Homogenizing its equation, we thus get a curve on the toric surface, with explicit Picard class in terms of the one of $\mathcal{C}$. Repeating this process on each affine chart, we define as many curves as there are affine charts on the surface whose intersection with $\mathcal{C}$ contains the set of $\mathbb{F}_q$-points of $\mathcal{C}$ outside a well-handled closed set.

Finally the Picard group of a toric variety is well-understood: its generators and relations are completely determined by its fan. Therefore, the intersection number of $\mathcal{C}$ with one of these curves divided by 2 – the lowest intersection multiplicity at a $\mathbb{F}_q$-point of $\mathcal{C}$ – gives an effortlessly computable upper bound, provided that they have no common components. This yields several bounds according to the ambient surface:

Theorem 1. *Let $\mathcal{C}$ be an absolutely irreducible curve on a minimal toric surface X defined over a finite field $\mathbb{F}_q$.*

- *For $X = \mathbb{P}^2$ (Thm 1 [SV86]) and $2 \nmid q$, if $\mathcal{C}$ has degree $d \geq 2$, then*
$$\#\mathcal{C}(\mathbb{F}_q) \leq \frac{1}{2}d(d+q-1),$$
provided that $\mathcal{C}$ has a non flex point.
- *For $X = \mathbb{P}^1 \times \mathbb{P}^1$ (Thm 3), if $\mathcal{C}$ has bidegree $(\alpha, \beta) \in (\mathbb{N}^*)^2$, then*
$$\#\mathcal{C}(\mathbb{F}_q) \leq \alpha\beta + \frac{q}{2}(\alpha+\beta).$$
- *For $X = \mathcal{H}_\eta$ with $\eta \neq 0$ (Thm 4), if $\mathcal{C}$ has bidegree $(\alpha, \beta) \in (\mathbb{N}^*)^2$, then*
$$\#\mathcal{C}(\mathbb{F}_q) \leq \frac{\beta}{2}(2\alpha - \eta\beta - \eta + 1) + \frac{q}{2}(\alpha+\beta).$$

Although the method we develop here can be applied to any toric surface, this paper solely focuses on the projective plane and Hirzeburch surfaces, which are the only minimal rational surfaces – except for $\mathcal{H}_1 \simeq \widetilde{\mathbb{P}^2}$. Also any smooth complete toric surface is obtained by toric blowups from either $\mathbb{P}^2$ or a Hirzebruch surface. It seems that we get a better upper bound using this fact than using the method elaborated here, as illustrated for $\mathcal{H}_1$ in Section 5.

It is worth to note that the bound on $\mathbb{A}^2$ or $\mathbb{P}^2$ requires the curve to have a least one non-flex point on each of its irreducible components whereas such kind of condition is not required on Hirzebruch surfaces, and thus on $\mathcal{H}_1$. On top of that, our method can doubtlessly be extended to higher dimensional toric varieties, notably to adapt F.J. Voloch's idea for surfaces in $\mathbb{P}^3$ [Vol03].

1 Some tools on toric varieties

1.1 Cox ring and characters

General results about toric varieties are compiled here. The reader is invited to read [CLS11] for further details.

Fix an integer $n \in \mathbb{N}^*$. Set $N \simeq \mathbb{Z}^n$ a $\mathbb{Z}$-lattice and $M = \operatorname{Hom}(N, \mathbb{Z})$ its dual lattice. Let $\mathbb{T}^n$ be the *n-dimensional algebraic torus*, then $\mathbb{T}^n(\bar{k}) = (\bar{k}^\times)^n$. A *character* of $\mathbb{T}^n$ is a morhism $\chi : \mathbb{T}^n \to k^\times$ which is a group homomorphism. M is called the *character group*, forms the set of regular functions on $\mathbb{T}^n$ and is isomorphic to $\mathbb{Z}^n$ via the map

$$\left\{ \begin{array}{ccl} \mathbb{Z}^n & \to & M \\ m & \mapsto & \chi^m : (t_1, \dots, t_n) \mapsto t_1^{m_1} \dots t_n^{m_n} \end{array} \right.$$

Let us set the dual pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ which is $\mathbb{Z}$-bilinear. Let $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$ and $M_\mathbb{R} = M \otimes \mathbb{R}$, its dual vector space. The dual pairing extends as a $\mathbb{R}$-bilinear pairing.

Let σ be a *strongly convex rational cone* in $N_\mathbb{R}$, i.e. $\sigma \cap (-\sigma) = \{0\}$ and σ is generated by vectors in N. From now, we assume any cone to be strongly convex rational. For any cone σ, we define its dual cone

$$\sigma^\vee := \{ m \in M_\mathbb{R} \mid \forall\ u \in \sigma,\ \langle m, u \rangle \geq 0 \}$$

and associate to σ the affine toric variety $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$. A fan Σ in N is a finite set of cones in $N_\mathbb{R}$ such that each face of a cone in Σ is also a cone in Σ and the intersection of two cones in Σ is a face of each of both cones. The set of r-dimensional cones in Σ is denoted by $\Sigma(r)$. A 1-dimensional cone is called a ray. Any ray $\rho \in \Sigma(1)$ has a unique minimal generator $u_\rho \in \rho \cap \mathbb{Z}^n$. A n-dimensional cone is said to be *maximal*.

The toric variety X_Σ associated to the fan Σ is defined as the union of the affine toric varieties $(U_\sigma)_{\sigma \in \Sigma}$. If a cone τ is included in another cone σ, the variety U_σ contains U_τ, which means that

$$X_\Sigma = \bigcup_{\sigma \in \Sigma(n)} U_\sigma. \tag{1}$$

Moreover, the torus $\mathbb{T}^n$ is a dense open subset of X_Σ acting on X_Σ. The complement of $\mathbb{T}^n$ in X_σ is well-known. A $\rho \in \Sigma(1)$ corresponds to a codimension 1 orbit under $\mathbb{T}^n$, whose Zariski closure is a $\mathbb{T}^n$-invariant divisor, denoted by D_ρ. Then

$$X_\sigma = \mathbb{T}^n \sqcup \left(\bigcup_{\rho \in \Sigma(1)} D_\rho \right). \tag{2}$$

Assume that the set of minimal generators $\{u_\rho,\ \rho \in \Sigma(1)\}$ spans $\mathbb{R}^n$, i.e. X_Σ has *no torus factors*. Set $\mathrm{Div}_{\mathbb{T}^n}(X)$ the group of $\mathbb{T}^n$-invariant Weil divisors on X_Σ. Then we have a short exact sequence

$$0 \to M \to \mathrm{Div}_{\mathbb{T}^n}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0 \tag{3}$$

where the map $M \to \mathrm{Div}_{\mathbb{T}^n}(X_\Sigma)$ associates to a character χ^m the principal divisor

$$\mathrm{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

In other words, any divisor on X_Σ is linearly equivalent to a $\mathbb{T}^n$-invariant divisor, $\mathbb{Z}$-linear combination of the divisors D_ρ, and the divisors associated to characters are exactly the one linearly equivalent to 0. Thus, the Picard group has rank $\#\Sigma(1) - n$.

A variable x_ρ is associated to each ray $\rho \in \Sigma(1)$. The *Cox Ring* of X_Σ is defined by $S = k\left[x_\rho \mid \rho \in \Sigma(1)\right]$. The function field of X_Σ is $\mathrm{Frac}(S)$. The ring S can be endowed with a graduation, using the short exact sequence

$$0 \longrightarrow M \xrightarrow{a} \mathbb{Z}^{\Sigma(1)} \xrightarrow{b} \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

with $a(m) = \left(\langle m, u_\rho\rangle\right)_{\rho\in\Sigma(1)}$ for $m \in M$ and $b(\alpha) = \left[\sum_\rho \alpha_\rho D_\rho\right]$ for $\alpha = (\alpha_\rho) \in \mathbb{Z}^{\Sigma(1)}$. The *degree* of a monomial $x^\alpha = \prod x_\rho^{\alpha_\rho}$ in S, where $\alpha \in \mathbb{N}^{\Sigma(1)}$, is defined as the Picard class of the divisor $\sum_\rho \alpha_\rho D_\rho$. Then

$$S = \bigoplus_{\beta\in\mathrm{Cl}(X_\Sigma)} S_\beta$$

where S_β is the vector k-space of homogeneous polynomials of degree β. As in projective spaces, we have some Euler relations. For any divisor class $\beta \in \mathrm{Cl}(X_\Sigma)$ and any group homomorphism $\phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{Z})$,

$$\forall F \in S_\beta, \sum_{\rho\in\Sigma(1)} \phi([D_\rho]) x_\rho \frac{\partial F}{\partial x_\rho} = \phi(\beta) F \tag{Eu}$$

Let $D = \sum a_\rho D_\rho$ be a $\mathbb{T}^n$-invariant Weil divisor on X_Σ. Let us set the polytope

$$P_D = \{m \in M_{\mathbb{R}} \mid \forall \rho \in \Sigma(1),\ \langle m, u_\rho\rangle \geq -a_\rho\}.$$

If D and D' are two linearly equivalent divisors, i.e. there exists $m \in M$ such that

$$D' = D + \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho,$$

then P'_D is the translate of P_D by the translation of vector m.

The lattice points of this polytope give a description of the global sections of $\mathcal{O}_{X_\Sigma}(D)$:

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m\in P_D\cap M} k.\chi^m. \tag{4}$$

1.2 HOMOGENIZING A CHARACTER

Let $f \in k[t_1^{\pm1}, \ldots, t_n^{\pm1}]$ be a Laurent polynomial. It defines a regular function on the torus $\mathbb{T}^n$. We would like to give it a meaning on the whole variety X_Σ: we want to find a polynomial F in the Cox ring S of X_Σ such that the variety defined by $F = 0$ is – or at least contains –F the Zariski closure of the affine variety $f = 0$ on $\mathbb{T}^n$.

More practically, we aim to generalize the very natural operation of homogenization in the projective case.

Example 1. *On $\mathbb{P}^2$, the polynomial $f = x^m + x + y$ defines a regular function on $\mathbb{P}^2 \smallsetminus \{Z = 0\}$ for $m \geq 1$. It can be homogenized as $F = X^m + XZ^{m-1} + YZ^{m-1}$ of degree m. It is also possible to homogenize this polynomial as $F' = Z^{d-m}(X^m + XZ^{m-1} + YZ^{m-1})$ of degree d, for any $d \geq m$. However, even if we can homogenize x and y by X and Y, of degree $d \geq 1$, we cannot homogenize the whole polynomial f by a polynomial of degree $d < m$, as it is not possible for x^m.*

As illustrated by Example 1, one have to choose a degree before homogenizing in the projective case. Since the Cox ring is graded by the Picard group, the analogous method in other toric varities will consist in choosing a Picard class.

Definition 1 (Homogenization of a character)**.** *Let $m \in M$ and D a $\mathbb{T}^n$-invariant divisor such that $m \in P_D$. The D-homogenization of the character χ^m is defined by*

$$x^{\langle m,D\rangle} = \prod_\rho x_\rho^{\langle m,u_\rho\rangle + a_\rho}.$$

Remark 1. *The assumption $m \in P_D$ in Definition 1 is analagous to the assumption $m \leq d$ in Example 1.*

It is thus possible to homogenize a Laurent monomial, using the method detailed in [CD97]. To homogenize a Laurent polynomial, we have to find a divisor D such that any character that appears in this polynomial can be D-homogenized. In order to find such a divisor, we use the Newton polytope of the Laurent polynomial. Set $f = \sum c_m \chi^m \in k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. The *Newton polytope* $\Delta(f)$ of f is defined as the convex hull of the set $\{m \in \mathbb{Z}^n,\ c_m \neq 0\}$ in $\mathbb{R}^n$.

Notation 1. *Let $f = \sum c_m \chi^m \in k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. For all $\rho \in \Sigma(1)$, set*

$$a_\rho^f = -\min_{\nu \in \Delta(f)} \langle \nu, u_\rho \rangle \tag{5}$$

and $D_f = \sum a_\rho^f D_\rho$.

On can easily check that the Newton polytope $\Delta(f)$ of the Laurent polynomial $f \in k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is contained in the polytope P_{D_f}. Moreover any divisor $D = \sum b_\rho D_\rho$ such that $\Delta(f) \subset P_D$ satisfies $a_\rho^f \leq b_\rho$ for all $\rho \in \Sigma(1)$.

Definition 2. *Let $D = \sum a_\rho D_\rho$ be a $\mathbb{T}^n$-invariant divisor such that $\Delta(f) \subset P_D$. Then the D-homogenization of f is the polynomial*

$$F = \sum_{m \in \Delta(f)} c_m \prod_{\rho \in \Sigma(1)} x_\rho^{\langle m,u_\rho\rangle + a_\rho^f}$$

of degree $[D]$.

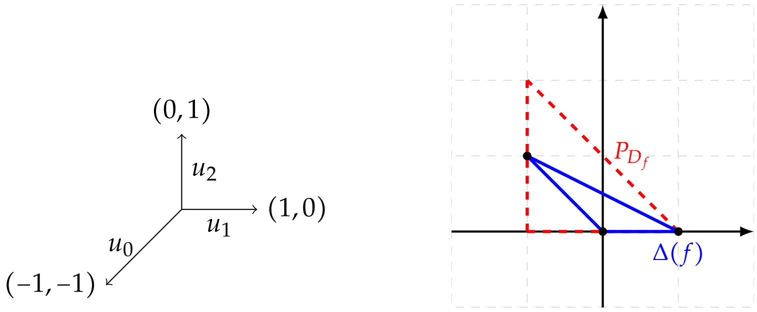
Remark 2. *The D-homogenization of a Laurent polynomial does not depend on the representative of $[D]$.*

Example 2. *See Figure (1a) for the fan of the toric surface $\mathbb{P}^2$. As usual, we denote the variable associated to the ray spanned by u_i by x_i for $i \in \{0,1,2\}$.*

Let us consider the Laurent polynomial $f = t_1 + t_1^{-1} t_2 + 1$. Its Newton polygon $\Delta(f) = \mathrm{Conv}_{\mathbb{R}^2}\{(1,0),(-1,1),(0,0)\}$ is drawn in Figure (1b). In this case

$$-\min_{\nu \in \Delta(f)} \langle \nu, u_i \rangle = \begin{cases} 1 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ 0 & \text{if } i = 2. \end{cases}$$

hence $D_f = D_0 + D_1$, where D_i is the $\mathbb{T}^2$-invariant divisor associated to the ray spanned by u_i. The D_f-homogenization of f is thus $F = x_1^2 + x_0 x_1 + x_0 x_2$, which is the same *polynomial as in Example 1. Two Laurent polynomials equal up to multiplication by a monomial have the same homogenization with respect to two linearly equivalent divisors.*



(A) Fan of $\mathbb{P}^2$ (B) Newton polygon of f (Example 2)

FIGURE 1

2 PRINCIPLE

This section is dedicated to the implementation of the method used later. It essentially relies on Stöhr and Voloch's idea to bound the number of $\mathbb{F}_q$-points on a plane curve [SV86]. Given a plane curve $\mathcal{C}$ of equation $f = 0$, they display an interpolation polynomial h (see (6)) that vanishes at the $\mathbb{F}_q$-points of $\mathcal{C}$ with multiplicity at least 2, as proved in Lemma 1. This enables to give an upper bound for the cardinality of $\mathcal{C}(\mathbb{F}_q)$ by half the intersection number of $\mathcal{C}$ and the curve $\mathcal{D}$ defined by $h = 0$.

Our method aims to adapt this idea on another toric surface X. Given a curve $\mathcal{C}$ on X, we want to find an interpolation curve $\mathcal{D}$ that passes through the $\mathbb{F}_q$-points of $\mathcal{C}$ with mutliplicity at least 2. Since intersection multiplicity is a local property, we shall use the polynomial h in (6) and rewrite it in terms of the coordinates on each affine toric patch of X as displayed in (7). Next, it remains to homogenize this polynomial to get a global equation (9) on the whole surface X. Its results as many interpolation curves as there are affine toric patches.

2.1 STÖHR AND VOLOCH'S INTERPOLATION POLYNOMIAL ON $\mathbb{A}^2$

The following lemma by [SV86] exhibits a good interpolation polynomial of the $\mathbb{F}_q$-points of a given plane curve. Its proof is given here to make this paper comprehensive and understandable.

Lemma 1 ([SV86]). *Let $\mathcal{C}$ be a plane curve defined by a polynomial $f \in \mathbb{F}_q[x,y]$ on $\mathbb{A}^2$. The intersection multiplicity at a $\mathbb{F}_q$-points of $\mathcal{C}$ with the variety defined by $h = 0$, where*

$$h = (x^q - x)f_x + (y^q - y)f_y = 0, \tag{6}$$

is at least 2.

Proof. Take $P \in \mathcal{C}(\mathbb{F}_q)$. First, P is clearly a zero of h. Moreover, the multiplicity of P in $f = h = 0$ is greater than the product of the multiplicities on $f = 0$ and $h = 0$ with equality occurring if and only if the gradients of f and h are not collinear at P [see [Ful89] section 3.3]. The case when P is a singular point of $\mathcal{C}$ is thus straightforward. Otherwise, if P is a simple $\mathbb{F}_q$-point on $\mathcal{C}$, then

$$dh_P = -f_x(P)dx_P - f_y(P)dy_P$$

and $\nabla h(P)$ and $\nabla f(P)$ are collinear, which concludes the proof. □

The polynomial h given in Lemma 1 has the advantage of interpolating $\mathbb{F}_q$-rational points of a given curve, with intersection multiplicity at least 2. We aim to generalize this idea. Given a polynomial F in the Cox ring, we want to display another polynomial G such that the intersection of the curves defined by $F = 0$ and $G = 0$ contains the $\mathbb{F}_q$-points of $F = 0$ and has multiplicity at least 2 at these points.

2.2 Toric framework

Let X be a complete normal toric surface with fan Σ. Let us fix a polynomial $F \in S$ of degree $[D_F] = \left[\sum a_\rho D_\rho\right]$, defining a curve $\mathcal{C} \subset X$. Then

$$F = \sum_{m \in P_{D_F}} c_m \prod_{\rho \in \Sigma(1)} x_\rho^{\langle m, u_\rho \rangle + a_\rho}$$

and we set

$$f = \sum_{m \in P_{D_F}} c_m \chi^m.$$

the equation of $C \cap \mathbb{T}^2$. The toric surface X is covered by as many affines charts (U_σ) as there are maximal cones $\sigma \in \Sigma(2)$ in the fan Σ.

2.2.1 Interpolation polynomial on a toric affine patch

Let us consider a maximal cone $\sigma = \mathrm{Cone}(u_{\rho_1}, u_{\rho_2})$ in Σ. Set A_σ the square matrix created by juxtaposing the column vectors u_{ρ_1} and u_{ρ_2}. Set

$$\Delta_\sigma = \det A_\sigma.$$

Up to exchange ρ_1 and ρ_2, we assume $\Delta_\sigma > 0$. We denote by n_1^σ and n_2^σ the row vectors of $\Delta_\sigma \times A_\sigma^{-1}$, which entries are integers. Then the dual cone of σ is equal to $\sigma^\vee = \mathrm{Cone}(n_1^\sigma, n_2^\sigma)$, since $\left\langle n_i^\sigma, u_{\rho_j} \right\rangle = \Delta_\sigma \delta_{i,j}$ by construction . The affine toric variety U_σ associated to the cone σ corresponds to $\mathrm{Spec}\, k\left[\chi^{n_1^\sigma}, \chi^{n_1^\sigma}\right] \simeq \mathbb{A}^2$.

To adapt Stohr and Voloch's idea and take advantage of Lemma 1, we want to homogenize the polynomial

$$g_\sigma = (\chi^{(q-1)n_1^\sigma} - 1)\chi^{n_1^\sigma} \frac{\partial f}{\partial \chi^{n_1^\sigma}} + (\chi^{(q-1)n_2^\sigma} - 1)\chi^{n_2^\sigma} \frac{\partial f}{\partial \chi^{n_2^\sigma}}. \tag{7}$$

The points of $U_\sigma = \mathrm{Spec}\, k\left[\chi^{n_1^\sigma}, \chi^{n_1^\sigma}\right]$ lying on the curve $\mathcal{C}$ at which g_σ vanishes are exactly the points of $\mathcal{C} \cap U_\sigma$ whose image under the Frobenius map belongs to their tangent line in U_σ.

Remark 3. *For any $m \in M$ and $\lambda \in \mathbb{Z}$, one can write $(\chi^m)^\lambda$ or $\chi^{\lambda m}$ without ambiguity, as λm also belongs to the $\mathbb{Z}$-lattice M.*

2.2.2 Homogenization of the interpolation polynomial

In order to homogenize g_σ, we need to compute its Newton polygon. On this purpose, we shall express g in terms of the coefficients of f, which will enables to write the Newton polygon of g_σ depending on the one of f.

First let rewrite f with respect to $\chi^{n_1^\sigma}$ and $\chi^{n_2^\sigma}$. It is equivalent to find a_1 and a_2 such that $m = a_1 n_1^\sigma + a_2 n_2^\sigma$. Computing the scalar product of m with u_{ρ_1} and u_{ρ_2}, we have $a_i = \frac{1}{\Delta_\sigma}\left\langle m, u_{\rho_i}\right\rangle$ for $i \in \{1,2\}$. Then

$$\chi^m = \left(\chi^{n_1^\sigma}\right)^{\frac{1}{\Delta_\sigma}\left\langle m,u_{\rho_1}\right\rangle}\left(\chi^{n_2^\sigma}\right)^{\frac{1}{\Delta_\sigma}\left\langle m,u_{\rho_2}\right\rangle}$$

and the polynomial f can written

$$f = \sum_{m\in P_D} c_m \left(\chi^{n_1^\sigma}\right)^{\frac{1}{\Delta_\sigma}\left\langle m,u_{\rho_1}\right\rangle}\left(\chi^{n_2^\sigma}\right)^{\frac{1}{\Delta_\sigma}\left\langle m,u_{\rho_2}\right\rangle}.$$

Note that f is not a polynomial with respect to o $\chi^{n_1^\sigma}$ and $\chi^{n_2^\sigma}$ if $\Delta_\sigma \neq 1$, that is to say when the cone is not smooth. Anyway, for $i \in \{1,2\}$, we have

$$\chi^{n_i^\sigma}\frac{\partial f}{\partial \chi^{n_i^\sigma}} = \frac{1}{\Delta_\sigma}\sum c_m \left\langle m, u_{\rho_i}\right\rangle \chi^m, \tag{8}$$

which is a Laurent polynomial even if $\Delta_\sigma \neq 1$.

To determine in which degree we will homogenize the polynomial g_σ, we need to find a divisor E_σ such that the Newton polygon of g_σ is contained in P_{E_σ}. Using (8), we have

$$\Delta_\sigma g_\sigma = \sum c_m \left((\chi^{(q-1)n_1^\sigma} - 1)\left\langle m, u_{\rho_1}\right\rangle + (\chi^{(q-1)n_2^\sigma} - 1)\left\langle m, u_{\rho_2}\right\rangle \right)\chi^m.$$

We can deduce that

$$\Delta(g_\sigma) \subset \operatorname{Conv}\left(\left(\bigcup_{\substack{m\in\Delta(f)\\ \langle m,u_{\rho_1}\rangle \neq 0}} \{m, m + n_1^\sigma(q-1)\} \right) \cup \left(\bigcup_{\substack{m\in\Delta(f)\\ \langle m,u_{\rho_2}\rangle \neq 0}} \{m, m + n_2^\sigma(q-1)\} \right)\right).$$

Set $b_\rho^\sigma = -\min_{m\in\Delta(g_\sigma)}\left\langle m, u_\rho\right\rangle = a_\rho + (q-1)\epsilon_\rho^\sigma$ with

$$\epsilon_\rho^\sigma = -\min\left\{0, \left\langle n_1^\sigma, u_\rho\right\rangle, \left\langle n_2^\sigma, u_\rho\right\rangle\right\} \geq 0$$

and

$$E_\sigma = \sum_{\rho\in\Sigma(1)} b_\rho^\sigma D_\rho.$$

By construction, $\Delta(g_\sigma) \subset P_{E_\sigma}$.

The E_σ-homogenization of $\Delta_\sigma g_\sigma$ is the polynomial $G_\sigma \in S$ given by

$$G_\sigma = \left(\prod_{\rho\in\Sigma(1)} x_\rho^{(q-1)\epsilon_\rho^\sigma}\right)\sum_{j=1}^{2}\left(\prod_{\rho\in\Sigma(1)} x_\rho^{(q-1)\left\langle n_j^\sigma, u_\rho\right\rangle} - 1\right) x_{\rho_j}\frac{\partial F}{\partial x_{\rho_j}}. \tag{9}$$

3 Application to the projective plane: Stöhr and Voloch's bound

Employing the method above on $\mathbb{P}^2$, we recover the dimension 2 case of K.O. Stöhr and F.J. Voloch's general bound [SV86]. The proof of Theorem 1 uses our tools up to (10). From there, the proof, given here for the convenience of the reader, follows Stöhr and Voloch's one in the affine case.

Let us fix F a homogeneous polynomial of degree d. Set $\sigma_0 = \operatorname{Cone}(u_1, u_2)$, $\sigma_1 = \operatorname{Cone}(u_0, u_2)$ and $\sigma_2 = \operatorname{Cone}(u_0, u_1)$ [see Figure 1a].

Let us detail the computation on the cone σ_0. We have $\sigma_0^\vee = \mathrm{Cone}(n_1^0, n_2^0)$ with $n_1^0 = (1,0)$ and $n_2^0 = (0,1)$.

j	$\langle u_i, n_1^0\rangle$	$\langle u_i, n_2^0\rangle$	$\epsilon^0_{\rho_j}$
0	−1	−1	1
1	1	0	0
2	0	1	0

Therefore (9) gives

$$G_{\sigma_0} = x_0^{q-1}\left[\left(x_0^{-(q-1)}x_1^{q-1}-1\right)x_1\frac{\partial F}{\partial x_1}+\left(x_0^{-(q-1)}x_2^{q-1}-1\right)x_2\frac{\partial F}{\partial x_2}\right]$$
$$= \left(x_1^{q-1}-x_0^{q-1}\right)x_1\frac{\partial F}{\partial x_1}+\left(x_2^{q-1}-x_0^{q-1}\right)x_2\frac{\partial F}{\partial x_2},$$

that has degree $d+q-1$. By standard Euler Identity, it can be written as follow:

$$G_{\sigma_0} = x_0^q\frac{\partial F}{\partial x_0}+x_1^q\frac{\partial F}{\partial x_1}+x_2^q\frac{\partial F}{\partial x_2}-x_0^{q-1}dF.$$

One can easily check that for $i \in \{1,2\}$, we also have

$$G_{\sigma_i} = x_0^q\frac{\partial F}{\partial x_0}+x_1^q\frac{\partial F}{\partial x_1}+x_2^q\frac{\partial F}{\partial x_2}-x_i^{q-1}dF.$$

The three polynomials given by (9) are thus all equal modulo F to $G = x_0^q F_{x_0} + x_1^q F_{x_1} + x_2^q F_{x_2}$.

Proposition 1 ([SV86]). *Let $\mathcal{C}$ be an absolutely irreducible curve of degree d in $\mathbb{P}^2$ defined over a finite field with q elements of characteristic different from 2. If there exists at least a non flex point on $\mathcal{C}$, then*

$$\mathcal{C}(\mathbb{F}_q) \leq \frac{1}{2}d(d+q-1).$$

Proof. Let $F \in k[x_0,x_1,x_2]$ be a polynomial defining the curve $\mathcal{C}$. Consider the homogeneous polynomial $G \in k[x_0,x_1,x_2]$ defined by $G = x_0^q F_{x_0} + x_1^q F_{x_1} + x_2^q F_{x_2}$ and let $\mathcal{D}$ the curve defined by $G = 0$.

Let us fix $P \in \mathcal{C}(\mathbb{F}_q)$. The symmetry of G with respect to the indeterminates allows us to assume without loss of generality that $P \notin (x_2 = 0)$. In the affine chart $(x_2 \neq 0)$, the equations of $\mathcal{C}$ and $\mathcal{D}$ are $f(x,y) = 0$ and

$$h(x,y) = (x^q - x)f_x + (y^q - y)f_y + df = 0, \tag{10}$$

where $f(x,y) = F(x,y,1)$. By Lemma 1, the multiplicity of P in $\mathcal{C} \cap \mathcal{D}$ is at least 2. If F does not divide G, then $2\#C(\mathbb{F}_q) \leq \mathcal{C}\cdot\mathcal{D}$, which gives the expected bound.

Let us assume that F divides G. Therefore f divides h. Differentiating the equality $h = 0$ with respect to x and y modulo f, we get

$$-f_x + (x^q - x)f_{xx} + (y^q - y)f_{xy} = 0 \tag{11}$$

$$-f_y + (x^q - x)f_{xy} + (y^q - y)f_{yy} = 0 \tag{12}$$

Replacing f_x and f_y thanks to (11) and (12) in h gives

$$(x^q - x)^2 f_{xx} + 2(x^q - x)(y^q - y)f_{xy} + (y^q - y)^2 f_{yy} = 0 \tag{13}$$

On $\mathcal{C} \cap (f_x \neq 0)$, we have $(x^q - x) = -(y^q - q)\dfrac{f_y}{f_x}$, which gives by subtituting this expression in (13)

$$\frac{(y^q - y)^2}{(f_x)^2}\left[f_{xx}\left(f_y\right)^2 - 2f_{xy}\left(f_x\right)\left(f_y\right) + f_{yy}\left(f_x\right)^2\right] = 0$$

Therefore, $f_{xx}\left(f_y\right)^2 - 2f_{xy}\left(f_x\right)\left(f_y\right) + f_{yy}\left(f_x\right)^2 = 0$ on $\mathcal{C} \cap ((f_x)(y^q - y) \neq 0)$. This implies that f divides $f_{xx}\left(f_y\right)^2 - 2f_{xy}\left(f_x\right)\left(f_y\right) + f_{yy}\left(f_x\right)^2$. By homogenizing, it means that F divides $F_{x_0x_0}(F_{x_1})^2 - 2F_{x_0x_1}F_{x_0}F_{x_1} + F_{x_1x_1}(F_{x_0})^2$. This means exactly that every point is inflectional [see [HK96] Theorem 2.5].

□

4 Application to Hirzebruch surfaces

4.1 Background on Hirzebruch surfaces

Let $\eta \in \mathbb{N}$. The Hirzebruch surface $\mathcal{H}_\eta$ is the toric variety associated to the fan Σ defined by 4 rays $\rho_1, \dots, \rho_4$ respectively spanned by the vectors $u_1 = (1,0)$, $u_2 = (0,1)$, $u_3 = (-1,\eta)$ et $u_4 = (0,-1)$.

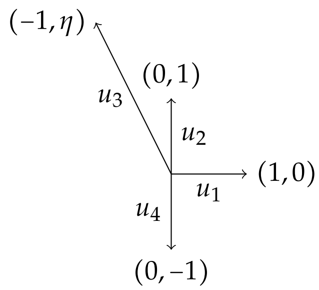


Figure 2. Fan Σ_η

According to the exact short sequence (3), a divisor D is principal if and only if there exists $m = (a,b) \in \mathbb{Z}^2$ such that

$$D = \sum_{i=1}^{4} \langle m, u_i \rangle D_{\rho_i} = a(D_{\rho_1} - D_{\rho_3}) + b(D_{\rho_2} + \eta D_{\rho_3} - D_{\rho_4}).$$

The divisors D_{ρ_1} and D_{ρ_2} thus form a $\mathbb{Z}$-basis of $\mathrm{Pic}(\mathcal{H}_\eta)$, with the intersection pairings

$$D_{\rho_1}^2 = 0, \quad D_{\rho_2}^2 = -\eta, \quad D_{\rho_1} \cdot D_{\rho_2} = 1. \tag{14}$$

A curve $\mathcal{C}$ is said to have *bidegree* (α,β) if $\mathcal{C}$ is linearly equivalent to $\alpha D_{\rho_1} + \beta D_{\rho_2}$. A non-zero polynomial $F \in S$ is said to have *bidegree* (α,β) if it belongs to $S_{[\alpha D_{\rho_1} + \beta D_{\rho_2}]}$, which also means that the curve defined by $F = 0$ has bidegree (α,β).

Notation 2. *The variables of S are chosen to be renamed to coincide with the Notations of Reid* [Rei97]*: $x_{\rho_1} = t_1$, $x_{\rho_2} = x_1$, $x_{\rho_3} = t_2$ and $x_{\rho_4} = x_2$.*

Let us take the group homomorphism $\phi_i : \mathrm{Cl}(X_\Sigma) \to \mathbb{Z}$ such that $\phi_i(D_{\rho_j}) = \delta_{i,j}$ for $(i,j) \in \{1,2\}^2$. Applying generalized Euler relation (Eu) with ϕ_1 and ϕ_2, for $F \in S$ of bidegree $(\alpha,\beta) \in \mathbb{Z}^2$, we have

$$t_1 \frac{\partial F}{\partial t_1} + t_2 \frac{\partial F}{\partial t_2} + \eta x_2 \frac{\partial F}{\partial x_2} = \alpha F \tag{Eu1}$$

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} = \beta F \tag{Eu2}$$

Finally, it is worth pointing out the essential role of Hirzeburch surfaces in the classification of rational surfaces. First, these surfaces for $\eta \neq 2$, together with $\mathbb{P}^2$ are minimal among smooth toric surfaces.

Theorem 2 ([CLS11]). *Every smooth complete toric surface is obtained from either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathcal{H}_\eta$ with $\eta \geq 2$ by a finite sequence of blowups at fixed points of the torus action.*

More generally, it is well-known that these particular surfaces are exactly the minimal rational surfaces.

4.2 Computation of the polynomials G_σ

Let us fix a polynomial $F \in S$ of bidegree (α, β). Set $\sigma_i = \mathrm{Cone}(u_i, u_{i+1})$ for $i \in \{1,2,3\}$ and $\sigma_4 = \mathrm{Cone}(u_4, u_1)$. Let us compute G_{σ_i} for each $i \in \{1,\dots,4\}$. Let us denote G_{σ_i} by G_i to simplify notations. To this end, we have to compute the generating vectors n_1^i and n_2^i of the dual cone $\sigma_i^\vee$ and their scalar product with the vectors u_j in order the determine the value of ϵ_{ρ_j} for $j \in \{1,\dots,4\}$.

- Cone σ_1: $n_1^1 = (1,0)$ and $n_2^1 = (0,1)$.

j	$\langle u_i, n_1^1\rangle$	$\langle u_i, n_2^1\rangle$	$\epsilon_{\rho_j}^1$
1	1	0	0
2	0	1	0
3	-1	η	1
4	0	-1	1

Then
$$\begin{aligned} E_{\sigma_1} &= \alpha D_{\rho_1} + \beta D_{\rho_2} + (q-1)(D_{\rho_3} + D_{\rho_4}) \\ &\sim (\alpha + (q-1)(\eta+1))D_{\rho_1} + (\beta + q - 1)D_{\rho_2} \end{aligned}$$
$$G_1 = \left(t_1^{q-1}x_2^{q-1} - t_2^{q-1}x_2^{q-1}\right) t_1 F_{t_1} + \left(x_1^{q-1}t_2^{(\eta+1)(q-1)} - t_2^{q-1}x_2^{q-1}\right) x_1 F_{x_1}$$

- Cone σ_2: $n_1^2 = (\eta,1)$ and $n_2^2 = (-1,0)$.

j	$\langle u_i, n_1^2\rangle$	$\langle u_i, n_2^2\rangle$	$\epsilon_{\rho_j}^2$
1	η	-1	1
2	1	0	0
3	0	1	0
4	-1	0	1

Then
$$E_{\sigma_2} = (\alpha + q - 1)D_{\rho_1} + \beta D_{\rho_2} + (q-1)D_{\rho_4} \sim E_{\sigma_1}$$
$$G_2 = \left(t_1^{(\eta+1)(q-1)}x_1^{q-1} - t_1^{q-1}x_2^{q-1}\right) x_1 F_{x_1} + \left(t_2^{q-1}x_2^{q-1} - t_1^{q-1}x_2^{q-1}\right) t_2 F_{t_2}$$

- Cone σ_3: $n_1^3 = (-1,0)$ and $n_2^3 = (-\eta,-1)$.

j	$\langle u_i, n_1^3\rangle$	$\langle u_i, n_2^3\rangle$	$\epsilon_{\rho_j}^3$
1	-1	$-\eta$	$\begin{cases} 1 & \text{if } \eta = 0, \\ \eta & \text{if } \eta \geq 1. \end{cases}$
2	0	-1	1
3	1	0	0
4	0	1	0

Then
$$E_{\sigma_3} = \begin{cases} (\alpha + q - 1)D_{\rho_1} + (\beta + q - 1)D_{\rho_2} \sim E_{\sigma_1} & \text{if } \eta = 0, \\ (\alpha + \eta(q-1))D_{\rho_1} + (\beta + q - 1)D_{\rho_2} & \text{if } \eta \geq 1 \end{cases}$$

$$G_3 = \begin{cases} \left(x_1^{q-1}t_2^{q-1} - t_1^{q-1}x_1^{q-1}\right)t_2F_{t_2} + \left(t_1^{q-1}x_2^{q-1} - t_1^{q-1}x_1^{q-1}\right)x_2F_{x_2} & \text{if } \eta = 0 \\ \left(t_1^{(\eta-1)(q-1)}x_1^{q-1}t_2^{q-1} - t_1^{\eta(q-1)}x_1^{q-1}\right)t_2F_{t_2} + \left(x_2^{q-1} - t_1^{\eta(q-1)}x_1^{q-1}\right)x_2F_{x_2} & \text{if } \eta \geq 1 \end{cases}$$

- Cone σ_4: $n_1^4 = (0,-1)$ and $n_2^4 = (1,0)$.

i	$\langle u_i, n_1^4\rangle$	$\langle u_i, n_2^4\rangle$	$\epsilon^4_{\rho_i}$
1	0	1	0
2	-1	0	1
3	$-\eta$	-1	$\begin{cases} 1 & \text{if } \eta = 0, \\ \eta & \text{if } \eta \geq 1. \end{cases}$
4	1	0	0

$$E_{\sigma_4} = \begin{cases} \alpha D_{\rho_1} + (\beta + q - 1)D_{\rho_2} + (q-1)D_{\rho_3} \sim E_{\sigma_1} & \text{if } \eta = 0, \\ \alpha D_{\rho_1} + (\beta + q - 1)D_{\rho_2} + \eta(q-1)D_{\rho_3} \sim E_{\sigma_3} & \text{if } \eta \geq 1 \end{cases}$$

$$G_4 = \begin{cases} \left(t_2^{q-1}x_2^{q-1} - x_1^{q-1}t_2^{q-1}\right)x_2F_{x_2} + \left(t_1^{q-1}x_1^{q-1} - x_1^{q-1}t_2^{q-1}\right)t_1F_{t_1} & \text{if } \eta = 0 \\ \left(x_2^{q-1} - x_1^{q-1}t_2^{\eta(q-1)}\right)x_2F_{x_2} + \left(t_1^{q-1}x_1^{q-1}t_2^{(\eta-1)(q-1)} - x_1^{q-1}t_2^{\eta(q-1)}\right)t_1F_{t_1} & \text{if } \eta \geq 1 \end{cases}$$

In sum we have

$$G_1 = x_2^{q-1}(t_1^{q-1} - t_2^{q-1})t_1F_{t_1} + t_2^{q-1}\left(x_1^{q-1}t_2^{\eta(q-1)} - x_2^{q-1}\right)x_1F_{x_1}$$

$$G_2 = x_2^{q-1}\left(t_2^{q-1} - t_1^{q-1}\right)t_2F_{t_2} + t_1^{q-1}\left(t_1^{\eta(q-1)}x_1^{q-1} - x_2^{q-1}\right)x_1F_{x_1}$$

$$G_3 = \begin{cases} x_1^{q-1}\left(t_2^{q-1} - t_1^{q-1}\right)t_2F_{t_2} + t_1^{q-1}\left(x_2^{q-1} - x_1^{q-1}\right)x_2F_{x_2} & \text{if } \eta = 0 \\ t_1^{(\eta-1)(q-1)}x_1^{q-1}\left(t_2^{q-1} - t_1^{q-1}\right)t_2F_{t_2} + \left(x_2^{q-1} - t_1^{\eta(q-1)}x_1^{q-1}\right)x_2F_{x_2} & \text{if } \eta \geq 1 \end{cases}$$

$$G_4 = \begin{cases} x_1^{q-1}\left(t_1^{q-1} - t_2^{q-1}\right)t_1F_{t_1} + t_2^{q-1}\left(x_2^{q-1} - x_1^{q-1}\right)x_2F_{x_2} & \text{if } \eta = 0 \\ t_2^{(\eta-1)(q-1)}x_1^{q-1}\left(t_1^{q-1} - t_2^{q-1}\right)t_1F_{t_1} + \left(x_2^{q-1} - x_1^{q-1}t_2^{\eta(q-1)}\right)x_2F_{x_2} & \text{if } \eta \geq 1 \end{cases}$$

4.3 Result for $\mathcal{H}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$

Theorem 3. *Let $\mathcal{C}$ be an absolutely irreducible curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(\alpha, \beta) \in (\mathbb{N}^*)^2$ defined over $\mathbb{F}_q$. Then*

$$\#\mathcal{C}(\mathbb{F}_q) \leq \frac{1}{2}\mathcal{C} \cdot \left(\mathcal{C} - \frac{q}{2}K\right) = \alpha\beta + \frac{q}{2}(\alpha + \beta).$$

Proof. Let F be the equation of the curve $\mathcal{C}$. For every $i \in \{1, 2\}$, set

$$H_1 = x_2^{q-1}(t_1^qF_{t_1} + t_2^qF_{t_2}) + t_2^{q-1}(x_1^qF_{x_1} + x_2^qF_{x_2}),$$
$$H_2 = x_2^{q-1}(t_1^qF_{t_1} + t_2^qF_{t_2}) + t_1^{q-1}(x_1^qF_{x_1} + x_2^qF_{x_2}).$$

Note that, using Euler relations (Eu1) and (Eu2), the difference between H_i and G_i is a multiple of F.

First let us prove that there exists $i \in \{1, 2\}$ such that F does not divide H_i. On the contrary, assume that F divides H_1 and H_2. Then F divides

$$H_1 - H_2 = (t_2^{q-1} - t_1^q)(x_1^qF_{x_1} + x_2^qF_{x_2}) = \left(\prod_{\zeta \in \mathbb{F}_q^\times}(t_2 - \zeta t_1)\right)(x_1^qF_{x_1} + x_2^qF_{x_2}).$$

Since F is absolutely irreducible and α and β are larger than 1, this means using (Eu2) that F divides $(x_1^{q-1} - x_2^{q-1})x_1 F_{x_1} = \left(\prod_{\xi \in \mathbb{F}_q}(x_1 - \xi x_2)\right) F_{x_1}$, which is impossible.

Let us assume that F does not divide H_1 and set $\mathcal{D} \subset \mathbb{P}^1 \times \mathbb{P}^1$ the curve defined by $H_1 = 0$. Using Euler relations (Eu1) and (Eu2), we clearly have $\mathcal{C}(\mathbb{F}_q) \subset \mathcal{C} \cap \mathcal{D}$. The calculations and the conclusion are the same if F does not divides H_2.

By Lemma 1, any $P \in \mathcal{C}(\mathbb{F}_q) \setminus (x_2 t_2 = 0)$ the intersection multiplicity of $\mathcal{C}$ and $\mathcal{D}$ at P is at least 2. Indeed, on the affine chart $(t_2 \neq 0) \cap (x_2 \neq 0)$, setting $x = \frac{x_1}{x_2}$ and $t = \frac{t_1}{t_2}$, the curve $\mathcal{D}$ is defined by

$$h(x,y) = (t^q - t)f_t + (x^q - x)f_x,$$

where $f(x,y) = F(1,t,1,x)$.

We thus have

$$\#\left(\mathcal{C}(\mathbb{F}_q) \cap (t_2 x_2 = 0)\right) + 2\#\left(\mathcal{C}(\mathbb{F}_q) \setminus (t_2 x_2 = 0)\right) \leq \mathcal{C} \cdot \mathcal{D}.$$

Note that $K \sim 2(t_2 x_2 = 0)$ and $D \sim C + \frac{q-1}{2}K$. Therefore

$$2\#\mathcal{C}(\mathbb{F}_q) \leq \mathcal{C} \cdot \left(C + \frac{q}{2}K\right).$$

Since $\mathcal{C}$ et $\mathcal{D}$ do not have any common component, we get

$$2\#\mathcal{C}(\mathbb{F}_q) \leq \alpha(\beta + q - 1) + \beta(\alpha + q - 1) + (\alpha + \beta),$$

which establishes the excepted result.

□

Remark 4. *There is no geometrical reason that motivates the rewriting with respect to the canonical divisor K of $\mathcal{H}_0$. This is only possible because the sum of the two "lines" at infinity we consider happens to be equal to half of the canonical divisor. Such phemomenon does not hold on other Hirzeburch surfaces.*

4.4 Result on other Hirzebruch surfaces

As before, our study focuses on irreducible curves. Let us begin with a small observation about the bidegree and the irreducibility.

Lemma 2. *A polynomial of bidegree (α, β) such that $\alpha < \eta\beta$ is divisible by x_1.*

Proof. By definition of the bidegree, any monomial $t_1^{c_1} t_2^{c_2} x_1^{d_1} x_2^{d_2}$ of the polynomial satisfies

$$\begin{cases} c_1 + c_2 + \eta d_2 = \alpha, \\ d_1 + d_2 = \beta. \end{cases}$$

Then $c_1 + c_2 - \eta d_1 < 0$, which implies that $d_1 > 0$. □

This lemma enables us to concentrate on curves of bidegree (α, β) with $\alpha \geq \eta\beta$. Before establishing our upper bound on Hirzeburch surfaces, we need a preliminary result which guarantees that an absolutely irreducible polynomial F does not divide one of the interpolation polynomials given in Subsection 4.2.

Lemma 3. *Let $\eta \in \mathbb{N}^*$. The polynomial $A \in \mathbb{F}_q[t_1, t_2, x_1, x_2]$ defined by*

$$A(t_1, t_1, x_1, x_2) = (1+\eta){x_2}^{q-1} - \sum_{j=0}^{\eta} {t_1}^{(q-1)j} {t_2}^{(q-1)(\eta-j)} {x_1}^{q-1} \tag{15}$$

is a product of factors of bidegree $(1,0)$ and $(0,1)$ if the characteristic of the finite field $\mathbb{F}_q$ divides $\eta + 1$ and absolutely irreducible otherwise.

Proof. Let p be the characteristic of the finite field $\mathbb{F}_q$.

Assume that p divides $\eta+1$. Then $A(t_1,t_1,x_1,x_2) = -f(t_1,t_2)x_1^{q-1}$ with

$$f(t_1,t_2) = \sum_{j=0}^{\eta} t_1^{(q-1)j} t_2^{(q-1)(\eta-j)} = \frac{t_1^{(\eta+1)(q-1)} - t_2^{(\eta+1)(q-1)}}{t_1^{q-1} - t_2^{q-1}}$$

Let $N \in \mathbb{N}^*$ such that $\eta+1 = pN$. Then

$$t_1^{(\eta+1)(q-1)} - t_2^{(\eta+1)(q-1)} = \left(t_1^{N(q-1)} - t_2^{N(q-1)}\right)^p.$$

Take $\zeta \in \overline{\mathbb{F}_q}$ a primitive Nth root of unity. The polynomial f can be written as a product of factors of bidegree $(1,0)$:

$$f(t_1,t_2) = \prod_{\xi\in\mathbb{F}_q^*} \left((t_1-\xi t_2)^{p-1} \prod_{j=1}^{N-1} (t_1 - \zeta^j \xi t_2)^p \right),$$

which proves that A is a product of factors of bidegree $(1,0)$ and $(0,1)$.

Assume that p does not divide $\eta+1$. The polynomial A is irreducible if and only if the polynomial $a \in k[t,x]$ defined by

$$a(t,x) = A(t,1,1,x) = (1+\eta)x^{q-1} - f(t), \text{ with } f(t) = \sum_{j=0}^{\eta} t^{(q-1)j},$$

is irreducible. Since $\gcd(\eta+1,p)=1$, the polynomial f is separable:

$$f(t) = \prod_{\xi\in\mathbb{F}_q^*} \prod_{j=1}^{\eta} (t-\omega^j\xi)$$

where $\omega \in \overline{\mathbb{F}_q}$ is a primitive $(\eta+1)$th root of unity. Eisenstein's criterion applied with any of this linear factor to $a \in k[t][x]$ ensures that a is irreducible.

□

Remark 5. *Using that for any $\xi \in \mathbb{F}_q^*$, $\xi^{q-1}=1$, the number of $\mathbb{F}_q$-points of the curve $\mathcal{C}_A$ defined by $A=0$ is easily computed. The orbit of a rational point of the Hirzebruch surface $\mathcal{H}_\eta$ contains exactly one point of following form: $(a,1,b,1)$, $(a,1,1,0)$, $(1,0,b,1)$ with $(a,b)\in\mathbb{F}_q^2$ and $(1,0,1,0)$. Set p the characteristic of the finite field $\mathbb{F}_q$.*

One can effortless check that the polynomial vanishes at a point of type $(a,1,b,1)$ with $(a,b)\in(\mathbb{F}_q^*)^2$. If p divides $\eta+1$, it is true for $(a,b)\in\mathbb{F}_q^*\times\mathbb{F}_q$.

Concerning points of type $(a,1,1,0)$ with $a\in\mathbb{F}_q$, the polynomial A vanishes at every of them if p divides $\eta+1$. Otherwise, the polynomial A does not vanish at any of these point.

The polynomial is zero at points of type $(1,0,b,1)$ for $b\in\mathbb{F}_q^*$ if and only if p divides η. It is zero at $(1,0,0,1)$ if and only if $p \mid \eta+1$. Finally, the polynomial A never vanishes at $(1,0,1,0)$.

In sum, we have

$$\#\mathcal{C}_A(\mathbb{F}_q) = \begin{cases} q^2 & \text{if } p \mid \eta+1, \\ q(q-1) & \text{if } p \mid \eta, \\ (q-1)^2 & \text{otherwise.} \end{cases}$$

Theorem 4. *Let $\eta\in\mathbb{N}^*$. Let $\mathcal{C}$ be an absolutely irreducible curve of the Hirzebruch surface $\mathcal{H}_\eta$ of bidegree $(\alpha,\beta)\in(\mathbb{N}^*)^2$ defined over the finite field $\mathbb{F}_q$. Then*

$$\#\mathcal{C}(\mathbb{F}_q) \leq \frac{\beta}{2}(2\alpha - \eta\beta - \eta + 1) + \frac{q}{2}(\alpha+\beta).$$

Proof. Let F be an equation of the curve $\mathcal{C}$. We consider the polynomials

$$G_1 = x_2^{q-1}(t_1^{q-1} - t_2^{q-1})t_1 F_{t_1} + t_2^{q-1}\left(x_1^{q-1}t_2^{\eta(q-1)} - x_2^{q-1}\right)x_1 F_{x_1},$$
$$G_2 = x_2^{q-1}\left(t_2^{q-1} - t_1^{q-1}\right)t_2 F_{t_2} + t_1^{q-1}\left(t_1^{\eta(q-1)}x_1^{q-1} - x_2^{q-1}\right)x_1 F_{x_1}.$$

We begin by proving that there exists $i \in \{1,2\}$, such that G_i is not divisible by F. Assume the contrary. Then, using F divides the polynomial

$$\begin{aligned} G_1 - G_2 =& x_2^{q-1}\left(t_1^{q-1} - t_2^{q-1}\right)\left(t_1 F_{t_1} + t_2 F_{t_2}\right) \\ &+ \left[\left(t_2^{(\eta+1)(q-1)} - t_1^{(\eta+1)(q-1)}\right)x_1^{q-1} + \left(t_1^{q-1} - t_2^{q-1}\right)x_2^{q-1}\right]x_1 F_{x_1} \end{aligned}$$

and so, using Euler relations (Eu1) and (Eu2), it also divides

$$x_1 F_{x_1}\left[(1+\eta)\left(t_1^{q-1} - t_2^{q-1}\right)x_2^{q-1} - \left(t_1^{(\eta+1)(q-1)} - t_2^{(\eta+1)(q-1)}\right)x_1^{q-1}\right],$$

which can be factorized as $x_1 F_{x_1}\left(t_1^{q-1} - t_2^{q-1}\right)A(t_1, t_1, x_1, x_2)$ where A is defined in Equation 15.

Since the polynomial F is absolutely irreducible, it is coprime with its derivative F_{x_1}. By Lemma 2, we have $\alpha \geq \eta\beta \geq 1$, which implies F is coprime with x_1 and $\left(t_1^{q-1} - t_2^{q-1}\right)$ of bidegree $(q-1, 0)$. Finally, unless $F = A$, Lemma 3 entails that F does not divide A, which arises a contradiction.

If $F = A$, one can easily verify that the bound we aim to prove is larger than the exact number of points of $\mathcal{C}_A$ given in Remark 5.

Now, let us assume that F does not divide

$$G_1 = x_2^{q-1}(t_1^{q-1} - t_2^{q-1})t_1 F_{t_1} + t_2^{q-1}\left(x_1^{q-1}t_2^{\eta(q-1)} - x_2^{q-1}\right)x_1 F_{x_1}.$$

Set $\mathcal{D} \sim (\alpha + (q-1)(\eta+1))D_{\rho_1} + (\beta + q - 1)D_{\rho_2}$ the curve defined by $G_1 = 0$.

First, let us check that $\mathcal{C}(\mathbb{F}_q) \smallsetminus (t_2 = 0) \subset \mathcal{C} \cap \mathcal{D}$.

Any $\mathbb{F}_q$-point $p = (t_1(p), t_2(p), x_1(p), x_2(p))$ of $\mathcal{C}$ such that $t_2(p) \neq 0$ is obviously a zero of the first term of G_1. It is also clear that it is a zero of the second term if $x_2(p) \neq 0$. If $x_2(p) = 0$ then $x_1 \neq 0$ and, using (Eu2), we can deduce that $F_{x_1}(p) = 0$, which guarantees that the second term also vanishes at p.

Second, let us prove that for any point $p \in \mathcal{C}(\mathbb{F}_q) \smallsetminus (t_2 x_1 = 0)$, the intersection multiplicity of $\mathcal{C}$ and $\mathcal{D}$ at p is at least 2.

On the affine chart $(t_2 \neq 0) \cap (x_1 \neq 0)$, the curve $\mathcal{D}$ is defined by the polynomial

$$g(t, x) = (t^q - t)f_t + (x^q - x)f_x$$

where f is the equation of $\mathcal{C}$ in this affine open set. Using Lemma 1, we thus get

$$\#\left(\mathcal{C}(\mathbb{F}_q) \cap (x_1 = 0)\right) + 2\#\left(\mathcal{C}(\mathbb{F}_q) \smallsetminus (t_2 x_1 = 0)\right) \leq \mathcal{C} \cdot \mathcal{D},$$

which can be written

$$2\#\mathcal{C}(\mathbb{F}_q) \leq \mathcal{C} \cdot (\mathcal{D} + (x_1 = 0) + 2(t_2 = 0)).$$

It remains to compute the right handside. Knowing that $(x_1 = 0) = D_{\rho_2}$ and $(t_2 = 0) = D_{\rho_3} \sim D_{\rho_1}$, we get

$$\begin{aligned} 2\#\mathcal{C}(\mathbb{F}_q) &\leq \left(\alpha D_{\rho_1} + \beta D_{\rho_2}\right) \cdot \left((\alpha + (q-1)(\eta+1) + 2)D_{\rho_1} + (\beta + q)D_{\rho_2}\right) \\ &= \alpha(\beta + q) + \beta(\alpha + (q-1)(\eta+1) + 2) - \eta\beta(\beta + q) \\ &= \beta(2\alpha - \eta\beta - \eta + 1) + q(\alpha + \beta) \end{aligned}$$

□

4.5 Comparison with existing bounds

The linear system associated to the divisor $D = D_{\rho_3} + D_{\rho_4} \sim (\eta+1)D_{\rho_1} + D_{\rho_2}$ is very ample on the surface $\mathcal{H}_\eta$ of dimension $\#P_D \cap \mathbb{Z}^2$ by (4), where

$$P_D = \{(a,b) \in \mathbb{R}^2 \mid 0 \le a \le \eta b + 1 \text{ and } 0 \le b \le 1\}.$$

Then $\#P_D \cap \mathbb{Z}^2 = \eta + 3$. The linear system associated to the divisor $D = \left((\eta+1)D_{\rho_1} + D_{\rho_2}\right)$ is thus very ample on $\mathcal{H}_\eta$ and gives a closed immersion $\varphi_D : \mathcal{H}_\eta \to \mathbb{P}^{\eta+3}$. For $\eta = 0$, this immersion is nothing but the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$.

Let C be a curve of bidegree (α, β) on $\mathcal{H}_\eta$. By the Adjunction formula, we have $2g(C) - 2 = C \cdot (K + C)$, where K is the canonical divisor of $\mathcal{H}_\eta$. Since $K = -\sum_{i=1}^4 D_{\rho_i} \sim -(2+\eta)D_{\rho_1} - 2D_{\rho_2}$, we have

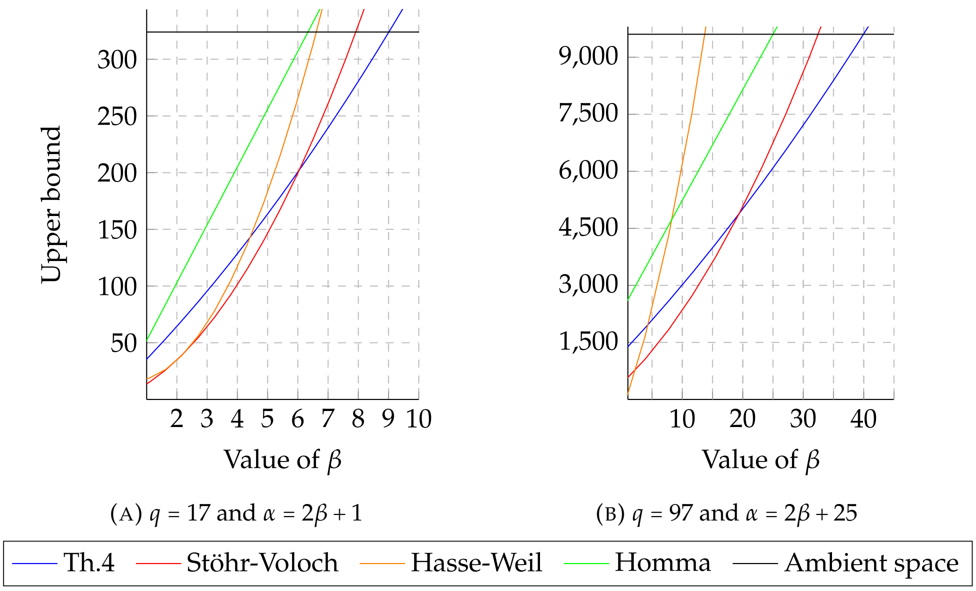
$$\begin{aligned} 2g(C) - 2 &= (\alpha D_{\rho_1} + \beta D_{\rho_2}) \cdot \left((\alpha - 2 - \eta)D_{\rho_1} + (\beta - 2)D_{\rho_2}\right) \\ &= \alpha(\beta - 2) + \beta(\alpha - 2 - \eta) - \eta\beta(\beta - 2) \\ &= 2(\alpha - 1)(\beta - 1) - \eta\beta(\beta - 1) - 2, \end{aligned}$$

which gives $g(C) = (\beta - 1)\left(\alpha - 1 - \frac{\eta\beta}{2}\right)$. Unless $\alpha \le \eta + 1$ and $\beta \le 1$, the curve $\varphi(C)$ does not lie on a hyperplane. Moreover it has degree $C \cdot D = \alpha + \beta$.

If the curve C is Frobenius-classical, K.O. Stöhr and F.J. Voloch [SV86] state that

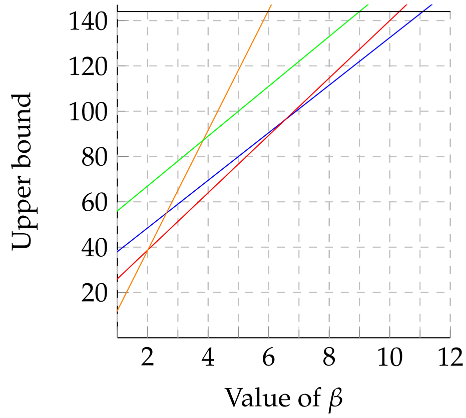
$$\#C(\mathbb{F}_q) \le (\eta + 2)(g - 1) + \frac{q + \eta + 3}{\eta + 3}(\alpha + \beta).$$

A sufficient condition for $\varphi(C)$ to be Frobenius-classical is $\deg(\phi(C)) = \alpha + \beta \le p$ where p is the characteristic of the finite field $\mathbb{F}_q$. If the curve is not Frobenius-classical, the coefficient of the genus g is greater than $\eta + 2$ and the upper bound grows.

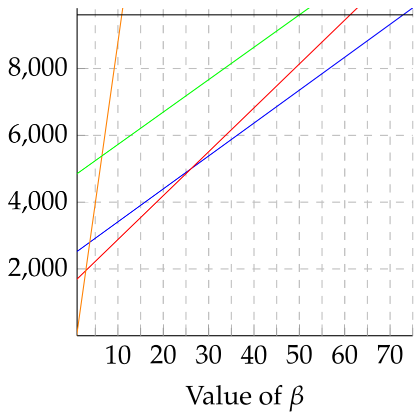


(A) $q = 17$ and $\alpha = 2\beta + 1$ (B) $q = 97$ and $\alpha = 2\beta + 25$

Figure 3. Comparison of bounds on the number of $\mathbb{F}_q$-points on a curve on $\mathcal{H}_2$ of bidegree (α, β)



(A) $q = 11$ and $\alpha = 1$



(B) $q = 97$ and $\alpha = 50$

Th.3 Stöhr-Voloch Hasse-Weil Homma Ambient space

FIGURE 4. Comparison of bounds on the number of $\mathbb{F}_q$-points on a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (α, β)

As displayed in Figures 3 and 4, the upper bounds given by Theorems 3 and 4 are sharper than the pre-existing ones for large bidegrees. It happens that previous bounds turn to be larger than the number of $\mathbb{F}_q$-points of $\mathcal{H}_\eta$, that equals to $(q+1)^2$ and is represented by the horizontal line labelled "Ambient space", whereas our bound is below this number.

5 WHAT'S NEXT?

The present work only studies curves on the projective plane or on a Hirzeburch surface. Although all the needed method to get a similar result on some other toric surfaces is detailed in Section 2, such idea does not seem to be fruitful, due to Theorem 2. The bound obtained from our method applied to a non minimal surface seems to be looser than the one deduced from the bound on the minimal surface it comes from and rough majorizations via multiplicities under blowups.

Let us take the example of the Hirzeburch surface $\mathcal{H}_1$, which is the blowup of $\mathbb{P}^2$. An irreducible curve on $\mathcal{H}_1$ is either the strict transform of an irreducible curve on $\mathbb{P}^2$ or the exceptional divisor D_{ρ_2}. The assumption on α and β forces a curve C to which Theorem 4 applies to be the strict transform of a plane projective curve C_0. More precisely, if C_0 has degree d and multiplicity m at the blown up point $(m < d)$, then C has bidegree $(d, d-m)$.

Therefore, a naive upper bound from Proposition 1 is

$$\#C(\mathbb{F}_q) \leq \frac{d}{2}(d+q-1) + m - 1.$$

Proposition 4 gives $\#C(\mathbb{F}_q) \leq \frac{1}{2}(d^2 - m^2 + 2dq - mq)$. A simple computation shows that the latter quantity is lesser than the first one if $d + q + 2 \leq d - 1$, which never happens. Nevertheless, the bound given by Proposition 4 holds without assumption of the existence of a non-inflectional point.

On the bright side, our method can applied to singular toric surfaces. It also can easily be extended to higher-dimensional varieties. Given an hypersurface of a toric variety, we can compute an interpolation polynomial that vanishes on $\mathbb{F}_q$-points of the hypersurface on each toric affine open set. Our routine can also be adapted to homogenize higher-degree interpolation polynomials, as the ones used by F. Voloch to upperbond the number of $\mathbb{F}_q$-points lying on a surface in $\mathbb{P}^3$ [Vol03].

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