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KLEIN-GORDON-MAXWELL-PROCA TYPE SYSTEMS IN THE ELECTRO-MAGNETO-STATIC CASE - THE HIGH DIMENSIONAL CASE

EMMANUEL HEBEY AND PIERRE-DAMIEN THIZY

ABSTRACT. We investigate Klein-Gordon-Maxwell-Proca type systems in the context of closed n -dimensional manifolds with $n \geq 4$. We prove existence of solutions and compactness of the system both in the subcritical and in the critical case.

1. INTRODUCTION

Let (M, g) be a smooth closed (connected) Riemannian manifold of dimension $n \geq 4$. We study systems like

$$\begin{cases} \Delta_g u + \Phi(x, v, A)u = u^{p-1} \\ \Delta_g v + (b + q^2 u^2)v = qu^2 \\ \Delta_g A + bA = q(\nabla S - qA)u^2, \end{cases} \quad (1.1)$$

where $q > 0$ and ω are real numbers, a, b, S are smooth functions such that $b \geq 0$ in M , $p \in (2, 2^*]$ and $2^* = \frac{2n}{n-2}$. The unknowns in (1.1) are two positive functions $u, v > 0$ and a 1-form A . The function Φ in (1.1) is given by

$$\Phi(x, v, A) = a - \omega^2(qv - 1)^2 + |\nabla S - qA|^2. \quad (1.2)$$

In the above, Δ_g denotes the Laplace-Beltrami operator $\Delta_g = -\operatorname{div}_g \nabla$ when acting on functions, and the Hodge-de Rham Laplacian $\Delta_g = \delta d + d\delta$ when acting on 1-forms, where d stands for the differential and δ for the codifferential. The 3-dimensional case of (1.1) has been studied recently by the authors in [21]. Systems of equations like (1.1) are derived from the full KGMP system when we look for solutions of such systems in the form $\Psi(x, t) = u(x, t)e^{iS(x, t)}$ with u depending only on x and S in the splitted form $S(x, t) = S(x) - \omega t$ (see Section 2 below). Such types of solutions were introduced in the very nice paper by Benci and Fortunato [2] for the Klein-Gordon-Maxwell equations in $\mathbb{R}^3$ (see also D'Avenia, Mederski and Pomponio [7] for an analogue of [2] in the Riemannian setting). A special choice of S in these papers gives rise to vortex solutions of the system. In the whole paper, the subscript R for functional spaces (e.g. L_R^p , H_R^1 , $W_R^{k,p}$, $C_R^{k,\theta}$ etc.) means that we refer to a space of real valued functions, while the subscript V (e.g. L_V^p , H_V^1 , $W_V^{k,p}$, $C_V^{k,\theta}$ etc.) means that we refer to a space of 1-forms. It is well known that 2^* is the critical Sobolev exponent for the embeddings of H_R^1 (H_V^1 respectively) into Lebesgue spaces. Our main result establishes the existence of a smooth nontrivial solution to (1.1) and also the compactness in the C^2 -topology of the set of solutions of (1.1). It reads as follows.

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Theorem 1.1. *Let $n \geq 4$ be an integer, let (M, g) be a smooth closed n -manifold, let $q > 0$ and ω be real numbers, and let a, b, S be smooth functions such that $b \geq 0$ in M and such that $Rc_g + bg > 0$ in M , in the sense of the bilinear forms, where Rc_g is the Ricci curvature of g . Let $p \in (2, 2^*]$, where $2^* = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embeddings of H_R^1 into Lebesgue spaces. We assume that $\Delta_g + f$ is coercive, where $f = a + |\nabla S|^2 - \omega^2$ if $b \neq 0$, and $f = a + |\nabla S|^2$ otherwise. When $p = 2^*$ we also assume either that*

$$\begin{cases} \max_M \left(a + |\nabla S|^2 - \frac{S_g}{6} \right) < \omega^2 & \text{if } b \neq 0, \\ \max_M \left(a + |\nabla S|^2 - \frac{S_g}{6} \right) < 0 & \text{otherwise,} \end{cases} \quad (1.3)$$

when $n = 4$, or that

$$\max_M \left(a - \frac{(n-2)S_g}{4(n-1)} \right) < 0, \quad (1.4)$$

when $n \geq 5$, where S_g is the scalar curvature of g . Then, (1.1) possesses a smooth solution (u, v, A) such that $u > 0$ and $v > 0$ in M . Moreover, the set $\mathcal{S}$ consisting of the smooth solutions (u, v, A) of (1.1) with $u \geq 0$ and $v \geq 0$ in M is compact in the C^2 -topology.

We actually prove in Sections 5 and 6 the stability of the system. This is a stronger statement than the sole compactness of the set of solutions of (1.1). Concerning existence, the leading equation in (1.1) is the first equation. It is subcritical when $p < 2^*$ and critical when $p = 2^*$. From the variational viewpoint, the other equations are critical when $n = 4$ and supercritical when $n \geq 5$ (while subcritical when $n = 3$). They are purely supercritical when $n > 6$ (as $2 > 2^* - 1$ in these dimensions, independently of the fact that v might be bounded or not). Also the vector structure in the third equation of (1.1) brings terms with indefinite sign in the energy. A weak variational setting has been proposed in [18, 30] to address the question of the existence of a solution to (1.1). In [21], the $n = 3$ case was handled. We overcome the difficulty in the present paper by using Leray-Schauder degree theory. Even if this approach drops a little the variational flavor of the problem, our rather general procedure has its own interest for such autoinductive critical systems. As a remark, the situation $a < 0$ corresponds to an imaginary mass (see for instance [13]).

The paper is organized as follows. The building of the equations is discussed in Section 2. Preliminary lemmas and notations are given in Sections 3 and 4. Theorem 1.1 in the subcritical case $p \in (2, 2^*)$ is proved in Section 5. A general stability analysis of (1.1) in the critical asymptotic $p \rightarrow 2^*$, ($p \leq 2^*$) is developed in Section 6, which implies the compactness of the set of the solutions of (1.1) claimed in Theorem 1.1. As a by product of Sections 5 and 6, we eventually get in Section 7 the existence of solutions for $p = 2^*$, thus concluding the proof of Theorem 1.1.

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2. THE BUILDING OF THE EQUATIONS

The physics purpose of Klein-Gordon-Maxwell-Proca systems is that they provide a model for the interaction between a charged relativistic matter scalar field and the electromagnetic field that it generates. In other words, the electromagnetic field is both generated by and drives the particle field. To be more precise the particle field interacts with the external field via the minimum coupling rule in a nonlinear Klein-Gordon equation. Formally, that means we replace time and space derivatives in the nonlinear Klein-Gordon total functional by gauge covariant derivatives such as

$$\partial_t \rightarrow \partial_t + iq\varphi \text{ and } \nabla \rightarrow \nabla - iqA ,$$

where A and φ are gauge potentials which represent the electromagnetic field generated by the particle. In this theory they are governed by the Maxwell-Proca Lagrangian.

Let us assume for a while that the manifold is orientable. Then, from the variational viewpoint, we consider the two Lagrangian densities $\mathcal{L}_{NKG}$ and $\mathcal{L}_{MP}$ given by

$$\begin{aligned} \mathcal{L}_{NKG}(\psi, \varphi, A) &= \frac{1}{2} \left| \left(\frac{\partial}{\partial t} + iq\varphi \right) \psi \right|^2 - \frac{1}{2} |(\nabla - iqA)\psi|^2 - \frac{m_0^2}{2} |\psi|^2 + \frac{1}{p} |\psi|^p \text{ and} \\ \mathcal{L}_{MP}(\varphi, A) &= \frac{1}{2} \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 - \frac{1}{2} |\nabla \times A|^2 + \frac{m_1^2}{2} |\varphi|^2 - \frac{m_1^2}{2} |A|^2 , \end{aligned}$$

where the curl operator $\nabla \times$ is given by $\nabla \times = \star d$, $\star$ is the Hodge dual, and d is the standard differential operator on forms. Basically $\mathcal{L}_{NKG}$ is like 1/2 of the square of the time derivatives minus 1/2 of the square of the space derivative after the gauge change of derivatives, plus the potential term and the nonlinear term. This is the nonlinear Klein-Gordon part of the energy. Then $\mathcal{L}_{MP}$ is only a functional of φ and A . It mixes space and time derivatives of A , and space derivatives of φ . This is the Maxwell-Proca part of the energy. One can note here that $|\varphi|^2 - |A|^2$ in this Maxwell-Proca part of the energy is nothing but the Lorentz norm of the external vector field (φ, A) . And Proca comes here because we are giving a mass m_1 to the field. This means again that we are in a massive version of the more classical Klein-Gordon-Maxwell equations. In this model ψ represents the matter field, m_0 is its mass, q is its electric charge, (A, φ) is the gauge potential which represents the electromagnetic vector field generated by ψ , and m_1 is the mass of this vector field. Let $\mathcal{S}$ be the total action functional for ψ , φ , and A given by

$$\mathcal{S}(\psi, \varphi, A) = \int \int (\mathcal{L}_{NKG} + \mathcal{L}_{MP}) dv_g dt$$

Writing ψ in the polar form

$$\psi(x, t) = u(x, t) e^{iS(x, t)} ,$$

where $u \geq 0$ and $u, S : M \times \mathbb{R} \rightarrow \mathbb{R}$ are real valued, the total action functional rewrites as a functional of the four variables u, S, φ, A . Taking the variation of the total energy with respect to these four variables we get four equations which are

written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left(\left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla \cdot \left((\nabla S - qA) u^2 \right) = 0 \\ -\nabla \cdot \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 \varphi + q \left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 A = q (\nabla S - qA) u^2 , \end{cases} \quad (KGMP)$$

where $\overline{\Delta}_g = \delta d$ is half the Hodge-de Rham Laplacian acting on forms, and δ (as well as $\nabla \cdot$) represents the codifferential. This is the system which we refer to as the nonlinear Klein-Gordon-Maxwell-Proca system. Letting $m_1 = 0$, we face the more traditional nonlinear Klein-Gordon-Maxwell system.

Assume for the rest of this section that $n = 3$, and define the electric field E , the magnetic induction H , the charge density ρ and the current density J by the following equations:

$$\begin{aligned} E &= - \left(\frac{\partial A}{\partial t} + \nabla \varphi \right) , \quad H = \nabla \times A , \\ \rho &= - \left(\frac{\partial S}{\partial t} + q\varphi \right) q u^2 , \quad J = (\nabla S - qA) q u^2 . \end{aligned}$$

Then the two last equations in the $(KGMP)$ -system give rise to the first pair of the Maxwell-Proca equations with respect to a matter distribution whose charge and current density are respectively ρ and J . On the other hand, because of the definitions of E and H , this is classical in physics, we also get for free that the second pair of the Maxwell-Proca equations holds true. In other words we get the full Maxwell-Proca equations from the two last equations in the system with this change of variables and the two last equations in the system can be rewritten as the following system of Maxwell-Proca equations:

$$\begin{aligned} \nabla \cdot E &= \rho - m_1^2 \varphi , \\ \nabla \times H - \frac{\partial E}{\partial t} &= J - m_1^2 A , \\ \nabla \times E + \frac{\partial H}{\partial t} &= 0 , \\ \nabla \cdot H &= 0 . \end{aligned} \quad (2.1)$$

The first equation in the system is the nonlinear Klein-Gordon matter equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \frac{\rho^2 - |J|^2}{q^2 u^3} . \quad (2.2)$$

And the second equation is the charge continuity equation which, in turn, is equivalent to the Lorentz condition

$$\nabla \cdot A + \frac{\partial \varphi}{\partial t} = 0 . \quad (2.3)$$

In other words, the nonlinear $(KGMP)$ -system is equivalent to this system (2.1)-(2.3) of six equations. More on the system of Maxwell-Proca equations can be found in the papers by Goldhaber and Nieto [16, 17] and Gillies, Luo and Tu [26].

As a remark, there is this breaking of the gauge invariance in this construction. The equivalence between the charge continuity equation and the Lorentz condition uses in a fundamental way the condition $m_1 \neq 0$. Taking the derivation of the first

Maxwell equation in (2.1) with respect to time, and the divergence of the second equation in (2.1), we get that

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot J &= \nabla \cdot \frac{\partial E}{\partial t} + m_1^2 \frac{\partial \varphi}{\partial t} + \nabla \cdot (\nabla \times H) - \nabla \cdot \frac{\partial E}{\partial t} + m_1^2 \nabla \cdot A \\ &= m_1^2 \left(\nabla \cdot A + \frac{\partial \varphi}{\partial t} \right) \end{aligned}$$

since $\nabla \cdot (\nabla \times H) = \delta(\star d)H$, $\delta = \star^{-1}d\star$ in Λ^1 , $\star\star = \text{Id}$ in Λ^2 , and $d^2 = 0$ so that $\nabla \cdot (\nabla \times H) = 0$. In other words, the condition $m_1 \neq 0$ breaks the gauge invariance and enforces the Lorentz gauge.

Let us assume now that we are in the static case of the equations. In other words we assume that A and φ depend on the sole spatial variables. And we look for solutions of the system which are like

$$\Psi(x, t) = u(x)e^{i(S(x) - \omega t)}.$$

These solutions for such kind of systems were introduced in a very nice work by Benci and Fortunato [2] with the specific choice of S given by

$$S(x) = l \text{Im} \ln(x_1 + ix_2),$$

where l is an integer. In doing so Benci and Fortunato produce existence in $\mathbb{R}^3$ of a solution which they refer to as a spinning Q -ball. This idea was discussed in the specific case of the KGMP-system by D'Avenia, Mederski and Pomponio [7]. We slightly differ from this approach here by considering that S is any given but smooth function in M , and hence a free parameter in the system. When we plug Ψ in the system, and we assume that A and φ do not depend on t , we do get the following static system of equations

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + \left((q\varphi - \omega)^2 - |\nabla S - qA|^2 \right) u \\ \nabla \cdot ((\nabla S - qA) u^2) = 0 \\ \Delta_g \varphi + m_1^2 \varphi + q(q\varphi - \omega) u^2 = 0 \\ \overline{\Delta}_g A + m_1^2 A = q(\nabla S - qA) u^2. \end{cases} \quad (2.4)$$

In the limit case where $m_1 = 0$, the second equation is automatically satisfied since the divergence of $\overline{\Delta}_g$ is automatically zero. When $m_1 \neq 0$, thanks to the fourth equation in this system, still since $\delta \overline{\Delta}_g = 0$, the second equation in the system can be omitted and replaced by the Coulomb gauge equation $\delta A = 0$. Then the system can be rewritten as

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + \left((q\varphi - \omega)^2 - |\nabla S - qA|^2 \right) u \\ \Delta_g \varphi + m_1^2 \varphi + q(q\varphi - \omega) u^2 = 0 \\ \overline{\Delta}_g A + m_1^2 A = q(\nabla S - qA) u^2 \\ \delta A = 0 \text{ (2nd eqt in original system)}. \end{cases} \quad (2.5)$$

As one can check, there holds that $\overline{\Delta}_g A = \Delta_g A$ when we do have the Coulomb gauge equation $\delta A = 0$, where $\Delta_g = d\delta + \delta d$ is the usual Hodge-de Rham Laplacian on forms. Letting $\varphi = \omega v$, and if we replace m_0^2 by a positive function a , and m_1^2 by a positive function b , the system (2.5) reduces to (1.1) when we forget about the last equation (the gauge condition). However, as a remark, the compactness (resp. stability) part in Theorem 1.1 (resp. in Theorems 5.1 and 6.1) applies with

no restriction to (2.5) as the set of solutions of (2.5) is a closed subset in the C^2 -topology of the set of solutions of (1.1).

3. AUXILIARY MAPS

We derive tools in this section that will help us controlling the two last equations in (1.1).

Lemma 3.1. *Let $q > 0$ be a real number and let b, S be smooth functions such that $b \geq 0$ in M and such that $Rc_g + bg > 0$ in the sense of the bilinear forms in M . Then, for all $u \in C_R^0$, there exists a unique $\phi_{b,S}(u) \in H_V^1 \cap C_V^1$ such that*

$$\Delta_g \phi_{b,S}(u) + b \phi_{b,S}(u) = q(\nabla S - q \phi_{b,S}(u))u^2. \quad (3.1)$$

Moreover, for all $k \geq 0$, $\phi_{b,S}$ restricts in a continuous map from C_R^k to C_V^{k+1} and we have that

$$q|\phi_{b,S}(u)| \leq \max_M |\nabla S| \text{ in } M, \quad (3.2)$$

for all u .

Proof of Lemma 3.1. We split the proof into different parts.

(1) Proof of existence. Let $u \in C_R^0$ be given. If $u \nabla S = 0$, then $\phi_{b,S}(u) = 0$ is solution of (3.1). We may thus assume that $u \nabla S$ is not identically zero. Let $I : H_V^1 \rightarrow \mathbb{R}$ be the functional given by

$$I(\phi) = \int_M |d\phi|^2 dv_g + \int_M |\delta\phi|^2 dv_g + \int_M (b + q^2 u^2) |\phi|^2 dv_g$$

and let $\mathcal{H} = \{\phi \in H_V^1 \text{ s.t. } q \int_M (\nabla S, \phi) u^2 dv_g = 1\}$. Clearly $\mathcal{H} \neq \emptyset$ as $\lambda \nabla S \in \mathcal{H}$ for some suitable $\lambda \in \mathbb{R}$. Let

$$\mu = \inf_{\phi \in \mathcal{H}} I(\phi).$$

Let $(\phi_\alpha)_\alpha$ be a smooth minimizing sequence for μ . By the Weitzenböck formula, for any 1-form ϕ ,

$$(\Delta_g \phi, \phi) = \frac{1}{2} \Delta_g |\phi|^2 + |\nabla \phi|^2 + Rc_g(\phi^\sharp, \phi^\sharp), \quad (3.3)$$

where $\phi^\sharp$ is the vector field we get from ϕ by the musical isomorphism. Then, by (3.3) and the Stokes formula,

$$\begin{aligned} I(\phi_\alpha) &= \int_M (\Delta_g \phi_\alpha, \phi_\alpha) dv_g + \int_M (b + q^2 u^2) |\phi_\alpha|^2 dv_g \\ &= \int_M |\nabla \phi_\alpha|^2 dv_g + \int_M (Rc_g + bg) (\phi_\alpha^\sharp, \phi_\alpha^\sharp) dv_g + q^2 \int_M u^2 |\phi_\alpha|^2 dv_g, \end{aligned}$$

and since $I(\phi_\alpha) \rightarrow \mu$ as $\alpha \rightarrow +\infty$ and $Rc_g + bg > 0$, we get that $\mu \geq 0$ and that the sequence $(\phi_\alpha)_\alpha$ is bounded in H_V^1 . Up to passing to a subsequence, by the reflexivity of H_V^1 , we can assume that $\phi_\alpha \rightharpoonup \phi$ in H_V^1 as $\alpha \rightarrow +\infty$, and by Rellich-Kondrakov we can assume that $\phi_\alpha \rightarrow \phi$ in L_V^2 for some $\phi \in H_V^1$. Then $\phi \in \mathcal{H}$. The bilinear form in H_V^1 given by

$$\mathcal{B}(A, B) = \int_M (dA, dB) dv_g + \int_M (\delta A)(\delta B) dv_g$$

is nonnegative. If $\mathcal{Q}$ is the quadratic form associated to $\mathcal{B}$, then (same proof as for the Cauchy-Schwarz inequality),

$$|\mathcal{B}(A, B)| \leq \sqrt{\mathcal{Q}(A)} \sqrt{\mathcal{Q}(B)}$$

for all $A, B \in H_V^1$. By the weak convergence $\phi_\alpha \rightharpoonup \phi$ in H_V^1 we then get that

$$\mathcal{Q}(\phi) = \lim_{\alpha \rightarrow +\infty} \mathcal{B}(\phi_\alpha, \phi) \leq \liminf_{\alpha \rightarrow +\infty} \sqrt{\mathcal{Q}(\phi_\alpha)} \sqrt{\mathcal{Q}(\phi)},$$

and thus that

$$\mathcal{Q}(\phi) \leq \liminf_{\alpha \rightarrow +\infty} \mathcal{Q}(\phi_\alpha).$$

Then, since $\phi_\alpha \rightarrow \phi$ in L_V^2 , $I(\phi) \leq \mu$ and in particular, $I(\phi) = \mu$. By differentiating we get that for any $B \in H_V^1$,

$$\mathcal{B}(\phi, B) + \int_M (b + q^2 u^2) (\phi, B) dv_g = q\mu \int_M (\nabla S, B) u^2 dv_g.$$

There holds that $\phi \not\equiv 0$ since $\phi \in \mathcal{H}$. In particular, $\mu > 0$. The 1-form $\tilde{\phi} = \frac{1}{\mu} \phi$ solves (3.1) in H_V^1 . By elliptic regularity, noting that $u^2 \phi \in L_V^2$, we get that $\tilde{\phi} \in H_V^2$. Bootstrapping, $\tilde{\phi} \in H_V^{2,k}$ for all $k \geq 1$ and we then get that $\tilde{\phi} \in C_V^1$. Clearly it follows from standard regularity that $\tilde{\phi} \in C_V^{k+1}$ as soon as $u \in C_R^k$.

(2) Proof of uniqueness. If ϕ_1 and ϕ_2 both solve (3.1), then

$$\Delta_g(\phi_2 - \phi_1) + (b + q^2 u^2) (\phi_2 - \phi_1) = 0$$

and contracting by $\phi_2 - \phi_1$, since $\text{Rc}_g + bg > 0$ in M and by the Weitzenböck formula again, we get that $\phi_2 \equiv \phi_1$.

(3) Proof of the continuity of the map $\phi_{b,S} : C_R^k \rightarrow C_V^{k+1}$. Fix $k \in \mathbb{N}$ and $u \in C_R^k$. Let $\mathcal{B}_u = \mathcal{B}_u(1)$ be the ball of center u and radius 1 in C_R^k . For $h \in \mathcal{B}_u$, let

$$\phi_h = \phi_{b,S}(u + h) - \phi_{b,S}(u).$$

Then,

$$\Delta_g \phi_h + (b + q^2 u^2) \phi_h = (q \nabla S - q^2 \phi_{b,S}(u + h)) (2u + h) h. \quad (3.4)$$

Anticipating on the proof of (3.2), there holds that $q \nabla S - q^2 \phi_{b,S}(u + h)$ is bounded in L_V^∞ (independently of h). Then, by (3.4), the Weitzenböck formula, and since $\text{Rc}_g + bg > 0$, we get that

$$\|\phi_h\|_{H_V^1} \leq C \|h\|_{C_R^0} \quad (3.5)$$

for all $h \in \mathcal{B}_u$, where $C > 0$ is independent of h . There also holds that

$$\Delta_g \phi_h + (b + q^2 (u + h)^2) \phi_h = (q \nabla S - q^2 \phi_{b,S}(u)) (2u + h) h$$

and thus that

$$\Delta_g \phi_h = Fh + G\phi_h, \quad (3.6)$$

where $F \in C_V^k$ and $G \in C_R^k$ are bounded with respect to $h \in \mathcal{B}_u$. By (local) elliptic estimates (e.g. see Lemma 1.7.9 in Biquard [3]), we get with (3.6) that for $p \leq k$,

$$\begin{aligned} \|\phi_h\|_{H_V^{2+p,2}} &\leq C \|\Delta_g \phi_h\|_{H_V^{p,2}} + C \|\phi_h\|_{L_V^2} \\ &\leq C \|h\|_{H_R^{p,2}} + C \|\phi_h\|_{H_V^{p,2}}. \end{aligned} \quad (3.7)$$

By (3.5) and (3.7), having p running from $p = 1$ to k , we get that

$$\|\phi_h\|_{H_V^{2+k,2}} \leq C \|h\|_{C_R^k}, \quad (3.8)$$

where $C > 0$ does not depend on h . By the Weitzenböck formula, for any 1-form A ,

$$\Delta_g A = \bar{\Delta}_g A + Rc_g(A) ,$$

where $\bar{\Delta}_g$ is the rough Laplacian and $Rc_g(A)$ is the 1-form with coordinates $Rc_g(A)_i = g^{\alpha\beta} R_{i\alpha} A_\beta$. In particular, in local coordinates, for any $i = 1, \dots, n$,

$$-(\Delta_g A)_i = g^{\mu\nu} \partial_{\mu\nu}^2 A_i + F_i^{\mu\nu} \partial_\mu A_\nu + G_i^\mu A_\mu , \quad (3.9)$$

where the $F_i^{\mu\nu}$'s and G_i^μ 's are smooth functions in the chart. By (3.6), (3.8) with $k = 0$, (3.9) and elliptic regularity as in Gilbarg-Trudinger [15], we get by induction and using Sobolev that for any $p \geq 1$,

$$\begin{aligned} \|\phi_h\|_{H_V^{2,p}} &\leq C \|\phi_h\|_{H_V^{1,p}} + C \|h\|_{C_R^0} \\ &\leq C_p \|h\|_{C_R^0} , \end{aligned} \quad (3.10)$$

where $C, C_p > 0$ do not depend on h . For $p \gg 1$ sufficiently large, there holds that $\|\phi_h\|_{C_V^1} \leq C \|\phi_h\|_{H_V^{2,p}}$. This proves the continuity of the map $\phi_{b,S} : C_R^0 \rightarrow C_V^1$. Higher regularity goes in a similar way.

(4) Proof of (3.2). Let $\Lambda = \max_M (|\phi_{b,S}(u)| |\nabla S|)$. If $\Lambda = 0$ then $(\phi_{b,S}(u), \nabla S) = 0$ by the Cauchy-Schwartz inequality, and by the Weitzenböck formula (3.3), contracting (3.1) by $\phi_{b,S}(u)$, we get that $\phi_{b,S}(u) \equiv 0$ since $Rc_g + bg > 0$ in M . We assume now that $\Lambda > 0$. By (3.1) and (3.3), we have that

$$\begin{aligned} &\frac{1}{2} \Delta_g |\phi_{b,S}(u)|^2 + (b + q^2 u^2) |\phi_{b,S}(u)|^2 \\ &= q (\phi_{b,S}(u), \nabla S) u^2 - |\nabla \phi_{b,S}(u)|^2 - Rc_g (\phi_{b,S}(u)^\sharp, \phi_{b,S}(u)^\sharp) . \end{aligned} \quad (3.11)$$

In particular, by (3.11),

$$\begin{aligned} &\frac{1}{2} \Delta_g \left(\frac{1}{q} - \frac{|\phi_{b,S}(u)|^2}{\Lambda} \right) + (b + q^2 u^2) \left(\frac{1}{q} - \frac{|\phi_{b,S}(u)|^2}{\Lambda} \right) \\ &= \frac{b}{q} + q u^2 \left(1 - \frac{1}{\Lambda} (\phi_{b,S}(u), \nabla S) \right) + \frac{|\nabla \phi_{b,S}(u)|^2}{\Lambda} \\ &\quad + \frac{1}{\Lambda} Rc_g (\phi_{b,S}(u)^\sharp, \phi_{b,S}(u)^\sharp) . \end{aligned} \quad (3.12)$$

Since $Rc_g + bg > 0$ in M there exists $\varepsilon_0 > 0$ such that $Rc_g \geq -(b - \varepsilon_0)g$ in M . In particular

$$Rc_g (\phi_{b,S}(u)^\sharp, \phi_{b,S}(u)^\sharp) \geq -(b - \varepsilon_0) |\phi_{b,S}(u)|^2$$

in M . Then, by (3.12),

$$\begin{aligned} &\frac{1}{2} \Delta_g \left(\frac{1}{q} - \frac{|\phi_{b,S}(u)|^2}{\Lambda} \right) + (\varepsilon_0 + q^2 u^2) \left(\frac{1}{q} - \frac{|\phi_{b,S}(u)|^2}{\Lambda} \right) \\ &\geq \frac{\varepsilon_0}{q} + q u^2 \left(1 - \frac{1}{\Lambda} (\phi_{b,S}(u), \nabla S) \right) + \frac{|\nabla \phi_{b,S}(u)|^2}{\Lambda} , \end{aligned} \quad (3.13)$$

and since $|\frac{1}{\Lambda} (\phi_{b,S}(u), \nabla S)| \leq 1$, we get from (3.13) that

$$\frac{1}{2} \Delta_g \left(\frac{1}{q} - \frac{|\phi_{b,S}(u)|^2}{\Lambda} \right) + (\varepsilon_0 + q^2 u^2) \left(\frac{1}{q} - \frac{|\phi_{b,S}(u)|^2}{\Lambda} \right) \geq 0 \quad (3.14)$$

in M . By the weak maximum principle it follows that $\frac{1}{q} - \frac{|\phi_{b,S}(u)|^2}{\Lambda} \geq 0$ and thus that $|\phi_{b,S}(u)| \leq \frac{1}{q} \max |\nabla S|$ in M . This proves (3.2).

The proof of Lemma 3.1 follows from (1)-(4) above. $\square$

Another lemma we need now concerns the v -part of the equation. The idea here goes back to Benci and Fortunato [1].

Lemma 3.2. *Let $q > 0$ be a real number and let b be a smooth function such that $b \geq 0$ in M . For all $u \in C_R^0$, there exists a unique $\psi_b(u) \in H_R^1 \cap C_R^1$ such that*

$$\Delta_g \psi_b(u) + (b + q^2 u^2) \psi_b(u) = q u^2 \quad (3.15)$$

as long as $b \not\equiv 0$ (or $u \not\equiv 0$). Define $\psi_b(u) \equiv \frac{1}{q}$ if $b \equiv 0$. Then, for all $k \geq 0$, ψ_b restricts in a continuous map from C_R^k to C_R^{k+1} and we have that

$$0 \leq q \psi_b(u) \leq 1 \text{ in } M \quad (3.16)$$

for all u . At last, if $u \not\equiv 0$, then $\psi_b(u) > 0$ in M .

Proof of Lemma 3.2. Here again we split the proof into different parts.

(1) Proof of existence and uniqueness. Let $u \in C_R^0$ be given. Suppose $b + q^2 u^2 \not\equiv 0$. Then existence easily follows from standard variational arguments. Concerning uniqueness it suffices to prove that if $u \in C_R^0$ and $\psi \in H_R^1$ are such that

$$\Delta_g \psi + (b + q^2 u^2) \psi = 0 \quad (3.17)$$

in M , then $\psi \equiv 0$. We multiply (3.17) by ψ and integrate. Then $\nabla \psi \equiv 0$ so that ψ is a constant, and we get that $\psi \equiv 0$ as soon as $b + q^2 u^2 \not\equiv 0$. In particular, since $b \geq 0$, we get that $\psi \equiv 0$ if $b \not\equiv 0$ or if $b \equiv 0$ and $u \not\equiv 0$. This proves uniqueness. By the maximum principle, $\psi_b(u) > 0$ in M if $u \not\equiv 0$. By regularity theory, $\psi_b(u) \in C_R^{k+1}$ if $u \in C_R^k$.

(2) Proof of (3.16). Let $u \in C_R^0$ and $\kappa = \frac{1}{q} - \psi_b(u)$. It is clear from the maximum principle that $\psi_b(u) \geq 0$. Noting that

$$\Delta_g \kappa + (b + q^2 u^2) \kappa = \frac{b}{q}$$

it also follows from the maximum principle that $\kappa \geq 0$. Then $\psi_b(u) \leq \frac{1}{q}$, and this proves (3.16).

(3) Proof of the continuity of the map $\psi_b : C_R^k \rightarrow C_R^{k+1}$. We may assume that $b \not\equiv 0$ since ψ_0 is constant. Fix $k \in \mathbb{N}$ and $u \in C_R^k$. Let $\mathcal{B}_u = \mathcal{B}_u(1)$ be the ball of center u and radius 1 in C_R^k . For $h \in \mathcal{B}_u$, let

$$\psi_h = \psi_b(u + h) - \psi_b(u) \text{ .}$$

Then,

$$\Delta_g \psi_h + (b + q^2 u^2) \psi_h = q(1 - q \psi_b(u + h))(h^2 + 2uh) \text{ .} \quad (3.18)$$

Since $b \geq 0$ and $b \not\equiv 0$, the operator $\Delta_g + b$ is coercive in the sense that there exists $\varepsilon > 0$ such that

$$\int_M |\nabla u|^2 dv_g + \int_M b u^2 dv_g \geq \varepsilon \int_M (|\nabla u|^2 + u^2) dv_g$$

for all $u \in H_R^1$ (this is easily proved by contradiction, using the reflexivity of H_R^1 , the compactness of the embedding $H_R^1 \subset L_R^2$ and restricting the inequality to the unit sphere in H_R^1). Then, multiplying (3.18) by ψ_h and integrating over M , using (3.16), we get that

$$\|\psi_h\|_{H_R^1} \leq C \|h\|_{C_R^0} \quad (3.19)$$

for all $h \in \mathcal{B}_u$, where $C > 0$ is independent of h . There also holds that

$$\Delta_g \psi_h + (b + q^2(u + h)^2) \psi_h = q(1 - q\psi_b(u))(h + 2u)h$$

and thus that

$$\Delta_g \psi_h = Fh + G\psi_h, \quad (3.20)$$

where $F, G \in C_R^k$ are bounded with respect to h . We conclude to the continuity of ψ_b using (3.19), (3.20) and local estimates as in Gilbarg and Trudinger [15].

The proof of Lemma 3.2 follows from (1)-(3) above. $\square$

Klein-Gordon-Maxwell-Proca systems have been investigated in Clapp, Ghimenti and Micheletti [6], d'Avenia, Medreski and Pomponio [7], Druet and Hebey [10], Druet, Hebey and Vtois [12], Hebey and Truong [22], Hebey and Thizy [21], Hebey and Wei [23] and Thizy [30].

4. BASIC CONVERGENCE

For the reader's convenience we state and prove the following basic convergence result in this section.

Lemma 4.1. *Let $n \geq 4$ be an integer, let (M, g) be a smooth closed n -manifold, let $q > 0$ and ω be real numbers, and let a, b, S be smooth functions such that $b \geq 0$ in M and such that $Rc_g + bg > 0$ in M , in the sense of bilinear forms. Let $p \in (2, 2^*)$ and $\theta \in (0, 1)$. Let $(a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon$ and $(S_\varepsilon)_\varepsilon$ be sequences of smooth functions such that $a_\varepsilon \rightarrow a, b_\varepsilon \rightarrow b$ in $C_R^{0, \theta}$ and $S_\varepsilon \rightarrow S$ in $C_R^{1, \theta}$ as $\varepsilon \rightarrow 0$, and such that $b_\varepsilon \geq 0$ in M for all $0 < \varepsilon \ll 1$. Let $(\omega_\varepsilon)_\varepsilon$ and $(p_\varepsilon)_\varepsilon$ be sequences of real numbers such that $p_\varepsilon \rightarrow p$ in $(2, 2^*)$ and $\omega_\varepsilon \rightarrow \omega$ in $\mathbb{R}$ as $\varepsilon \rightarrow 0$. Let $(u_\varepsilon, v_\varepsilon, A_\varepsilon)_\varepsilon$ be a sequence such that*

$$\begin{cases} \Delta_g u_\varepsilon + \Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon) u_\varepsilon = u_\varepsilon^{p_\varepsilon - 1} \\ \Delta_g v_\varepsilon + (b_\varepsilon + q^2 u_\varepsilon^2) v_\varepsilon = q u_\varepsilon^2 \\ \Delta_g A_\varepsilon + b_\varepsilon A_\varepsilon = q(\nabla S_\varepsilon - q A_\varepsilon) u_\varepsilon^2, \end{cases} \quad (4.1)$$

for all ε , where

$$\Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon) = a_\varepsilon - \omega_\varepsilon^2(qv_\varepsilon - 1)^2 + |\nabla S_\varepsilon - qA_\varepsilon|^2 \quad (4.2)$$

and $u_\varepsilon > 0$ in M . Assume $\|u_\varepsilon\|_{L_R^\infty} = O(1)$. Then, up to passing to a subsequence, $u_\varepsilon \rightarrow u, v_\varepsilon \rightarrow v$ in C_R^2 and $A_\varepsilon \rightarrow A$ in C_V^2 as $\varepsilon \rightarrow 0$, where $u, v \in C_R^2$ and $A \in C_V^2$ solve (1.1).

Proof of Lemma 4.1. By Lemma 3.1, $A_\varepsilon = \phi_{b_\varepsilon, S_\varepsilon}(u_\varepsilon)$ for all ε , while by Lemma 3.2, $v_\varepsilon = \psi_{b_\varepsilon}(u_\varepsilon)$ for all ε . By assumption, by (3.2) and (3.16), thanks to the convergences we have on the a_ε 's and S_ε 's, we then get that the sequences $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon$ and $(\Phi_\varepsilon)_\varepsilon$ are bounded in L_R^∞ and that the sequence $(A_\varepsilon)_\varepsilon$ is bounded in L_V^∞ . By the Calderon-Zygmund inequality and the third equation in (4.1), $(A_\varepsilon)_\varepsilon$ is actually bounded in H_V^2 . By the third equation in (4.1), by (3.9) and the regularity theory for functions (see Gilbarg-Trudinger [15]) we then get that the $(A_\varepsilon)_i$'s are bounded in $H_R^{2, 2^*}$, and by bootstrapping we then get that they are bounded in $C_R^{1, \theta}$. By the two first equations in (4.1), and again by standard regularity theory, the u_ε 's and the v_ε 's are then bounded in $C_R^{2, \theta}$. Still by standard regularity theory we then obtain from the third equation in (4.1) and from (3.9) that the $(A_\varepsilon)_i$'s are actually bounded in $C_R^{2, \theta}$. This ends the proof of Lemma 4.1. $\square$

As a remark on Lemma 4.1, the following result holds true.

Lemma 4.2. *Under the same assumptions than in Lemma 4.1, if we assume either that $b \not\equiv 0$ or that $b_\varepsilon \equiv 0$ for all ε , and if we also assume that $\Delta_g + f$ is coercive, where $f = a + |\nabla S|^2 - \omega^2$ if $b \not\equiv 0$, and $f = a + |\nabla S|^2$ otherwise, then there holds that $u > 0$ and $v > 0$. There also holds that $A \not\equiv 0$ if S is not a constant.*

Proof of Lemma 4.2. By Lemma 4.1, up to passing to a subsequence, we get that

$$u_\varepsilon \rightarrow u, \quad v_\varepsilon \rightarrow v, \quad A_\varepsilon \rightarrow A$$

in C_R^2 and C_V^2 , $\Phi_\varepsilon \rightarrow \Phi$ in $C_R^{0,\theta}$, where $\Phi = a - \omega^2(qv - 1)^2 + |\nabla S - qA|^2$ if $b \not\equiv 0$, and

$$\Delta_g u + \Phi u = u^{2^*-1}$$

in M . By the maximum principle, either $u \equiv 0$ or $u > 0$. If $u > 0$ then we are done since we also have that

$$\begin{cases} \Delta_g v + (b + q^2 u^2)v = qu^2 \\ \Delta_g A + bA = q(\nabla S - qA)u^2. \end{cases}$$

Suppose by contradiction that $u \equiv 0$. Then $\Phi = f$, where $f = a + |\nabla S|^2 - \omega^2$ if $b \not\equiv 0$ or $f = a + |\nabla S|^2$ otherwise as there are no nontrivial harmonic 1-forms when $\text{Rc}_g > 0$. By our assumption, $\Delta_g + f$ is coercive. Let $\delta > 0$ be such that $\Delta_g + f_\delta$ is still coercive, where $f_\delta = f - \delta$. For all $\varepsilon \ll 1$, we can write

$$\begin{aligned} \int_M (|\nabla u_\varepsilon|^2 + f_\delta u_\varepsilon^2) dv_g &\leq \int_M (|\nabla u_\varepsilon|^2 + \Phi_\varepsilon u_\varepsilon^2) dv_g \\ &= \int_M u_\varepsilon^{p_\varepsilon} dv_g \\ &\leq C \left(\int_M (|\nabla u_\varepsilon|^2 + f_\delta u_\varepsilon^2) dv_g \right)^{p_\varepsilon/2}. \end{aligned}$$

In particular, we get that

$$\int_M (|\nabla u|^2 + f_\delta u^2) dv_g > 0$$

and this is the contradiction we were looking for. This proves the lemma. $\square$

Another useful complement to Lemma 4.1 is given by the following result.

Lemma 4.3. *Under the same assumptions than in Lemma 4.1, and if we assume that $|\omega| < \min_M \sqrt{a}$, then there holds that $u > 0$ and $v > 0$. There also holds that $A \not\equiv 0$ if S is not a constant.*

Proof of Lemma 4.3. Since $|\omega| < \min_M \sqrt{a}$, the operator $\Delta_g + a - \omega^2$ is coercive. Then, for $\delta_0 > 0$ sufficiently small, $\Delta_g + (a - \omega^2 - \delta_0)$ is still coercive. Since $u_\varepsilon > 0$ in M it follows from Lemma 3.2 that $0 \leq v_\varepsilon \leq \frac{1}{q}$ for all $\varepsilon \ll 1$. In particular, by

(4.1) and the Sobolev inequality, for any $\varepsilon \ll 1$ sufficiently small,

$$\begin{aligned}
& \int_M (|\nabla u_\varepsilon|^2 + (a - \omega^2 - \delta_0) u_\varepsilon^2) dv_g \\
& \leq \int_M |\nabla u_\varepsilon|^2 dv_g + \int_M \Phi_\varepsilon u_\varepsilon^2 dv_g \\
& = \int_M u_\varepsilon^{p_\varepsilon} dv_g \\
& \leq C \left(\int_M (|\nabla u_\varepsilon|^2 + (a - \omega^2 - \delta_0) u_\varepsilon^2) dv_g \right)^{p_\varepsilon/2}
\end{aligned}$$

for some $C > 0$ independent of ε . This implies that $u > 0$. Passing to the limit we get from (4.1) that

$$\begin{cases} \Delta_g v + (b + q^2 u^2) v = q u^2 \\ \Delta_g A + b A = q (\nabla S - q A) u^2, \end{cases}$$

and it follows that $v > 0$. Also $A \neq 0$ if $\nabla S \neq 0$. The lemma follows. $\square$

5. PROOF OF THEOREM 1.1 IN THE SUBCRITICAL CASE $p \in (2, 2^*)$

Let $n \geq 4$ be an integer, let (M, g) be a smooth closed n -manifold, let $q > 0$ and ω be real numbers, and let a, b, S be smooth functions such that $b \geq 0$ in M and such that $Rc_g + bg > 0$ in M in the sense of the bilinear forms, where Rc_g is the Ricci curvature of g . Let $p \in (2, 2^*)$ be given. We assume that $\Delta_g + f$ is coercive, where $f = a + |\nabla S|^2 - \omega^2$ if $b \neq 0$, and $f = a + |\nabla S|^2$ otherwise. First we prove the following key lemma.

Lemma 5.1. *Let $\varepsilon_0 > 0$ be given. Let $h_t : M \times \mathbb{R} \rightarrow \mathbb{R}$ be given by*

$$h_t(\cdot, u) = (1 - t)\varepsilon_0 + t\Phi(\cdot, \psi_b(u), \phi_{b,S}(u)) \quad (5.1)$$

for all $u \in C_R^0$ and all $t \in [0, 1]$, where Φ is as in (1.2), $\phi_{b,S}$ is as in Lemma 3.1 and ψ_b is as in Lemma 3.2. Then,

$$\alpha_0 := \inf_{t \in [0, 1]} \inf_{u \in \mathcal{S}_t} \min_{x \in M} u > 0, \quad (5.2)$$

where $\mathcal{S}_t \subset H_R^{2,q} \cap C_R^1$, $q \gg 1$, is the set of the smooth solutions of

$$\Delta_g u + h_t(\cdot, u)u = u^{p-1} \quad (5.3)$$

which are such that $u > 0$ in M .

Proof of Lemma 5.1. First, by Lemmas 3.1 and 3.2, there exists $C_1 > 0$ such that

$$h_t(\cdot, u) \leq C_1 \quad (5.4)$$

in M for all $u \in C_R^0$ and all $t \in [0, 1]$. Then, as a corollary of Gidas and Spruck [14], there exists $C_2 > 1$ such that

$$u \leq C_2 \quad (5.5)$$

in M for all $t \in [0, 1]$ and for all $u \in \mathcal{S}_t$. It is clear that $\alpha_0 \geq 0$ in (5.2). From now on we assume by contradiction that $\alpha_0 = 0$. Then there exists a sequence $(t_\varepsilon)_\varepsilon$ of numbers in $[0, 1]$, $u_\varepsilon \in \mathcal{S}_{t_\varepsilon}$ and $x_\varepsilon \in M$ for all ε , such that $u_\varepsilon(x_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. By (5.3) with $u = u_\varepsilon$ and $t = t_\varepsilon$, by (5.4), (5.5) and by the Harnack inequality (e.g.

see Gilbarg and Trudinger [15], Theorem 8.20), we then get that $\|u_\varepsilon\|_{C_R^0} \rightarrow 0$ as $\varepsilon \rightarrow 0$. It follows that $\|u_\varepsilon\|_{H_R^1} \rightarrow 0$ as $\varepsilon \rightarrow 0$ by noting that by (5.3) and the above,

$$\Delta_g u_\varepsilon + u_\varepsilon = f_\varepsilon$$

in M , where $\|f_\varepsilon\|_{C_R^0} \rightarrow 0$ as $\varepsilon \rightarrow 0$. By the continuity of $\phi_{b,S}$ and ψ_b , see Lemmas 3.1 and 3.2, and since $\|u_\varepsilon\|_{C_R^0} \rightarrow 0$ as $\varepsilon \rightarrow 0$, we can write that

$$\psi_b(u_\varepsilon) \rightarrow \begin{cases} 0 & \text{if } b \neq 0 \\ \frac{1}{q} & \text{if } b \equiv 0 \end{cases}$$

in C_R^0 , and that $\phi_{b,S}(u_\varepsilon) \rightarrow 0$ in C_V^0 as $\varepsilon \rightarrow 0$. However, multiplying (5.3) by u_ε , integrating by parts, using the coercivity of $\Delta_g + f$, where $f = a + |\nabla S|^2 - \omega^2$ if $b \neq 0$, and $f = a + |\nabla S|^2$ otherwise, using the Sobolev inequality and since $p > 2$, we get from the above that

$$\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{H_R^1} > 0.$$

This is the contradiction we look for. Lemma 5.1 is proved. $\square$

A rather standard lemma, following Lin, Ni and Takagi [25], is Lemma 5.2 below. We refer to Brézis-Li [4] and Hebey [19] for the critical version of this lemma. We skip its proof as it is rather standard.

Lemma 5.2. *Let $p \in (2, 2^*)$. There exists $\delta_0 > 0$ such that the equation*

$$\Delta_g u + \varepsilon u = u^{p-1} \quad (5.6)$$

with $u > 0$ in M , admits the sole solution $u \equiv \varepsilon^{1/(p-2)}$ for all $\varepsilon \in (0, \delta_0)$.

Now we are in position to prove Theorem 1.1 in the subcritical case.

Proof of Theorem 1.1 in the subcritical case. We fix $\varepsilon_0 \in (0, \delta_0)$, where δ_0 is as in Lemma 5.2. Let L be the compact operator given by

$$L = (\Delta_g + 1)^{-1} \quad (5.7)$$

in C_R^0 . By Lemmas 3.1 and 3.2, and elliptic theory, given $t \in [0, 1]$ and h_t as in (5.1), the elements in $\mathcal{S}_t$, where $\mathcal{S}_t$ is as in Lemma 5.1, are precisely the solutions $u \in C_R^0$ of

$$T_t(u) := u - L(u^{p-1} - (h_t(\cdot, u) - 1)u) = 0 \quad (5.8)$$

with $u > 0$ in M . Up to reducing $\varepsilon_0 > 0$, the operator $DT_0\left(\varepsilon_0^{1/(p-2)}\right)$ given by

$$\begin{aligned} DT_0\left(\varepsilon_0^{1/(p-2)}\right)(\varphi) &= \varphi - L(((p-2)\varepsilon_0 + 1)\varphi) \\ &= L(\Delta\varphi - (p-2)\varepsilon_0\varphi) \end{aligned} \quad (5.9)$$

possesses $-(p-2)\varepsilon_0$ as a unique simple negative eigenvalue since its eigenvalues are the $\lambda_i - (p-2)\varepsilon_0$, where the λ_i 's are the eigenvalues of Δ_g . By the Leray-Schauder degree theory and our above remarks, we can deduce that

$$\deg(T_0, \Omega, 0) = -1, \quad (5.10)$$

where Ω is given by

$$\Omega = \left\{ u \in C_R^0 \text{ s.t. } \frac{1}{2}\alpha_0 < u < 2C_2 \right\} \quad (5.11)$$

where α_0 is as in (5.2) and C_2 is as in (5.5). Moreover, by the homotopy invariance of the Leray-Schauder degree, using in a crucial way Lemma 5.1, we get in particular that T_t does not vanishes in $\partial\Omega$ and conclude that

$$\deg(T_t, \Omega, 0) = -1 \quad (5.12)$$

for all $t \in [0, 1]$. Since $\deg(T_1, \Omega, 0) \neq 0$, (5.3) admits at least one positive solution $u \in \Omega$ for $t = 1$. By Lemmas 3.1 and 3.2 with elliptic theory, u turns out to be smooth; moreover $\psi_b(u) > 0$ in M and $(u, \psi_b(u), \phi_{b,S}(u))$ is the smooth nontrivial solution of (1.1) we look for in Theorem 1.1. Also any solution is in Ω . By Lemmas 3.1 and 3.2 with elliptic theory, we then get that $\mathcal{S}_1$ is compact in C_R^2 , so that $\mathcal{S}$ in Theorem 1.1 is also compact in the C^2 -topology. This concludes the proof of Theorem 1.1 when $p \in (2, 2^*)$. $\square$

As a remark we can prove more than the compactness of the set of solutions and actually prove the stability of the equation. This is what we state in the following theorem.

Theorem 5.1. *Let $n \geq 4$ be an integer, let (M, g) be a smooth closed n -manifold, let $q > 0$ and ω be real numbers, and let a, b, S be smooth functions such that $b \geq 0$ in M and such that $Rc_g + bg > 0$ in M , in the sense of bilinear forms. Let $p \in (2, 2^*)$ and $\theta \in (0, 1)$. Let $(a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon$ and $(S_\varepsilon)_\varepsilon$ be sequences of smooth functions such that $a_\varepsilon \rightarrow a, b_\varepsilon \rightarrow b$ in $C_R^{0,\theta}$ and $S_\varepsilon \rightarrow S$ in $C_R^{1,\theta}$ as $\varepsilon \rightarrow 0$, and such that $b_\varepsilon \geq 0$ in M for all $0 < \varepsilon \ll 1$. Let $(\omega_\varepsilon)_\varepsilon$ and $(p_\varepsilon)_\varepsilon$ be sequences of real numbers such that $p_\varepsilon \rightarrow p$ in $(2, 2^*)$ and $\omega_\varepsilon \rightarrow \omega$ in $\mathbb{R}$ as $\varepsilon \rightarrow 0$. Then, for any sequence $(u_\varepsilon, v_\varepsilon, A_\varepsilon)_\varepsilon$ such that*

$$\begin{cases} \Delta_g u_\varepsilon + \Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon) u_\varepsilon = u_\varepsilon^{p_\varepsilon - 1} \\ \Delta_g v_\varepsilon + (b_\varepsilon + q^2 u_\varepsilon^2) v_\varepsilon = q u_\varepsilon^2 \\ \Delta_g A_\varepsilon + b_\varepsilon A_\varepsilon = q(\nabla S_\varepsilon - q A_\varepsilon) u_\varepsilon^2, \end{cases} \quad (5.13)$$

for all ε , where

$$\Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon) = a_\varepsilon - \omega_\varepsilon^2 (q v_\varepsilon - 1)^2 + |\nabla S_\varepsilon - q A_\varepsilon|^2 \quad (5.14)$$

and $u_\varepsilon > 0$ in M , there holds that, up to a subsequence, $u_\varepsilon \rightarrow u, v_\varepsilon \rightarrow v$ in C_R^2 and $A_\varepsilon \rightarrow A$ in C_V^2 as $\varepsilon \rightarrow 0$, where $u, v \in C_R^2$ and $A \in C_V^2$ solve (1.1).

Proof of Theorem 5.1. By Lemma 3.1, $A_\varepsilon = \phi_{b_\varepsilon, S_\varepsilon}(u_\varepsilon)$ for all ε , while by Lemma 3.2, $v_\varepsilon = \psi_{b_\varepsilon}(u_\varepsilon)$ for all ε . In particular, by (3.2) and (3.16), thanks to the convergences we have on the a_ε 's and S_ε 's, we get that $|\Phi_\varepsilon| \leq C$ for all ε , where $\Phi_\varepsilon = \Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon)$ and $C > 0$ is independent of ε . By Lemma 4.1 it suffices to prove that $(u_\varepsilon)_\varepsilon$ is bounded in L_R^∞ . At this point we assume by contradiction that

$$\max_M u_\varepsilon \rightarrow +\infty \quad (5.15)$$

as $\varepsilon \rightarrow 0$. Let $x_\varepsilon \in M$ and $\mu_\varepsilon > 0$ be such that

$$u_\varepsilon(x_\varepsilon) = \max_M u_\varepsilon = \mu_\varepsilon^{-2/(p_\varepsilon - 2)}.$$

By (5.15), $\mu_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Define $\tilde{u}_\varepsilon$ by

$$\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{2}{p_\varepsilon - 2}} u_\varepsilon(\exp_{x_\varepsilon}(\mu_\varepsilon x))$$

and g_ε by $g_\varepsilon(x) = (\exp_{x_\varepsilon}^* g)(\mu_\varepsilon x)$ for $x \in B_0(\delta\mu_\varepsilon^{-1})$, where $\delta > 0$ is small. Since $\mu_\varepsilon \rightarrow 0$, we get that $g_\varepsilon \rightarrow \xi$ in $C_{loc}^2(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$, where ξ is the Euclidean metric. Moreover, by (5.13),

$$\Delta_{g_\varepsilon} \tilde{u}_\varepsilon + \mu_\varepsilon^2 \hat{\Phi}_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon^{p_\varepsilon-1} \quad (5.16)$$

for all ε , where $\hat{\Phi}_\varepsilon$ is given by

$$\hat{\Phi}_\varepsilon(x) = \Phi_\varepsilon(\exp_{x_\varepsilon}(\mu_\varepsilon x)) .$$

In addition, $\tilde{u}_\varepsilon(0) = 1$ and $0 \leq \tilde{u}_\varepsilon \leq 1$. By (5.16) and standard elliptic theory arguments, we can write that, after passing to a subsequence, $\tilde{u}_\varepsilon \rightarrow u$ in $C_{loc}^{1,\theta}(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$, where u is such that $u(0) = 1$ and $0 \leq u \leq 1$. Then

$$\Delta_\xi u = u^{p-1} \quad (5.17)$$

in $\mathbb{R}^n$, where Δ_ξ is the Euclidean Laplacian and, since $2 < p < 2^*$, we get a contradiction with the Liouville result of Gidas and Spruck [14]. This ends the proof of Theorem 5.1. $\square$

6. STABILITY IN THE CRITICAL CASE $p = 2^*$

We prove the following stability result in this section.

Theorem 6.1. *Let $n \geq 4$ be an integer, let (M, g) be a smooth closed n -manifold, let $q > 0$ and ω be real numbers, and let a, b, S be smooth functions such that $b \geq 0$ in M and such that $Rc_g + bg > 0$ in M , in the sense of bilinear forms. Let $\theta \in (0, 1)$ be given. Let $(a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon$ and $(S_\varepsilon)_\varepsilon$ be sequences of smooth functions such that $a_\varepsilon \rightarrow a, b_\varepsilon \rightarrow b$ in $C_R^{0,\theta}$ and $S_\varepsilon \rightarrow S$ in C_R^3 as $\varepsilon \rightarrow 0$, and such that $b_\varepsilon \geq 0$ in M for all $0 < \varepsilon \ll 1$. We assume either (1.3) if $n = 4$, or (1.4) if $n \geq 5$. Let $(\omega_\varepsilon)_\varepsilon$ and $(p_\varepsilon)_\varepsilon$ be sequences of real numbers such that $p_\varepsilon \rightarrow 2^*$ in $(2, 2^*]$ and $\omega_\varepsilon \rightarrow \omega$ in $\mathbb{R}$ as $\varepsilon \rightarrow 0$. Then, for any sequence $(u_\varepsilon, v_\varepsilon, A_\varepsilon)_\varepsilon$ such that*

$$\begin{cases} \Delta_g u_\varepsilon + \Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon) u_\varepsilon = u_\varepsilon^{p_\varepsilon-1} \\ \Delta_g v_\varepsilon + (b_\varepsilon + q^2 u_\varepsilon^2) v_\varepsilon = q u_\varepsilon^2 \\ \Delta_g A_\varepsilon + b_\varepsilon A_\varepsilon = q(\nabla S_\varepsilon - q A_\varepsilon) u_\varepsilon^2, \end{cases} \quad (6.1)$$

for all ε , where

$$\Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon) = a_\varepsilon - \omega_\varepsilon^2(qv_\varepsilon - 1)^2 + |\nabla S_\varepsilon - qA_\varepsilon|^2 \quad (6.2)$$

and $u_\varepsilon > 0$ in M , there holds that, up to a subsequence, $u_\varepsilon \rightarrow u, v_\varepsilon \rightarrow v$ in C_R^2 and $A_\varepsilon \rightarrow A$ in C_V^2 as $\varepsilon \rightarrow 0$, where $u, v \in C_R^2$ and $A \in C_V^2$ solve (1.1) in the critical case $p = 2^*$.

We start here the proof of Theorem 6.1. Let $n \geq 4$ be an integer, let (M, g) be a smooth closed n -manifold, let $q > 0$ and ω be real numbers, and let a, b, S be smooth functions such that $b \geq 0$ in M and such that

$$Rc_g + bg > 0 \quad (6.3)$$

in M in the sense of the bilinear forms. Let $\theta \in (0, 1)$ be given. Let $(a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon$ and $(S_\varepsilon)_\varepsilon$ be sequences of smooth functions such that

$$a_\varepsilon \rightarrow a, \quad b_\varepsilon \rightarrow b \text{ in } C_R^{0,\theta} \text{ and } S_\varepsilon \rightarrow S \text{ in } C_R^3 \quad (6.4)$$

as $\varepsilon \rightarrow 0$, and such that $b_\varepsilon \geq 0$ in M for all $0 < \varepsilon \ll 1$. Let $(\omega_\varepsilon)_\varepsilon$ and $(p_\varepsilon)_\varepsilon$ be sequences of real numbers such that $p_\varepsilon \rightarrow 2^*$ in $(2, 2^*]$ and $\omega_\varepsilon \rightarrow \omega$ in $\mathbb{R}$ as $\varepsilon \rightarrow 0$. Let $(u_\varepsilon, v_\varepsilon, A_\varepsilon)$ be smooth, such that (6.1) holds true and such that $u_\varepsilon > 0$ in M

for all ε . Observe in particular that since $u_\varepsilon > 0$ in M , we have that $v_\varepsilon = \psi_{b_\varepsilon}(u_\varepsilon)$ even if $b_\varepsilon \equiv 0$, so that we can use (3.16). Observe also that, by (6.3) and (6.4), we have $Rc_g + b_\varepsilon g > 0$ in M in the sense of the bilinear forms, so that we get $A_\varepsilon = \phi_{b_\varepsilon, S_\varepsilon}(u_\varepsilon)$ for all ε small. Then, by Lemmas 3.1 and 3.2, and (6.4), we clearly have that

$$\limsup_{\varepsilon \rightarrow 0} \|\Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon)\|_{L_R^\infty} \leq \|a\|_{L_R^\infty} + 4\|\nabla S\|_{L_V^\infty}^2 + \omega^2, \quad (6.5)$$

where $\Phi_\varepsilon(x, v_\varepsilon, A_\varepsilon)$ is as in (6.2). By Lemma 4.1 the convergence of $(u_\varepsilon, v_\varepsilon, A_\varepsilon)_\varepsilon$ in the C^2 -topology at the end of Theorem 6.1 holds true as soon as $\|u_\varepsilon\|_{L_R^\infty} = O(1)$. We proceed by contradiction and assume from now on that, up to a subsequence,

$$\lim_{\varepsilon \rightarrow 0} \max_M u_\varepsilon = +\infty. \quad (6.6)$$

In order to get Theorem 6.1, our goal is to prove eventually that our assumptions (1.3)-(1.4) allow to rule out (6.6). Following the global strategy in Druet, Hebey and Vétois [11], which goes back to Coda-Marques [27], Druet [8, 9], Li-Zhu [24] and Schoen [29] (See Hebey [19] for a presentation in book form), we start by analyzing the local blow-up behavior of the u_ε 's in a model case. In the sequel, $i_g > 0$ denotes the injectivity radius of (M, g) . We let $(x_\varepsilon)_\varepsilon$ be a sequence of points in M , $(\rho_\varepsilon)_\varepsilon$ be a sequence of positive real numbers such that $7\rho_\varepsilon < i_g$. Up to a subsequence, we have that

$$\lim_{\varepsilon \rightarrow 0} x_\varepsilon = x_0, \quad (6.7)$$

for some $x_0 \in M$. We assume that there exists $C > 0$ such that

$$\begin{cases} d_g(x_\varepsilon, \cdot)^{\frac{2}{p_\varepsilon-2}} u_\varepsilon \leq C \text{ in } B_{x_\varepsilon}(7\rho_\varepsilon) \text{ and all } \varepsilon, \\ \nabla u_\varepsilon(x_\varepsilon) = 0, \\ \lim_{\varepsilon \rightarrow 0} \rho_\varepsilon^{\frac{2}{p_\varepsilon-2}} \sup_{B_{x_\varepsilon}(6\rho_\varepsilon)} u_\varepsilon = +\infty, \end{cases} \quad (6.8)$$

where $d_g(\cdot, \cdot)$ denotes the Riemannian distance on (M, g) , and where $B_x(r) \subset M$ denotes the open geodesic ball of center $x \in M$ and radius $r > 0$. Let $\mu_\varepsilon > 0$ be given by

$$\mu_\varepsilon = u_\varepsilon(x_\varepsilon)^{-\frac{p_\varepsilon-2}{2}}. \quad (6.9)$$

Focusing on the first equation in (6.1), and using (6.5) and (6.8), as a by product of the classification in Caffarelli, Gidas and Spruck [5], we get as in [9, 20] that $\mu_\varepsilon \rightarrow 0$, that $\rho_\varepsilon/\mu_\varepsilon \rightarrow +\infty$ and that

$$\mu_\varepsilon^{\frac{2}{p_\varepsilon-2}} u_\varepsilon(\exp_{x_\varepsilon}(\mu_\varepsilon \cdot)) \rightarrow u_0 \quad (6.10)$$

in $C_{loc}^1(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$, where u_0 is given in $\mathbb{R}^n$ by

$$u_0(y) = \left(1 + \frac{|y|^2}{n(n-2)}\right)^{-\frac{n-2}{2}}. \quad (6.11)$$

We derive below several useful lemmas and aim to get precise asymptotics for $\Phi_\varepsilon := \Phi_\varepsilon(\cdot, v_\varepsilon, A_\varepsilon)$. The first lemma we prove basically only uses the estimate (6.10) on the u_ε 's.

Lemma 6.1 (Control of $(\Phi_\varepsilon)_\varepsilon$ for $n \geq 5$). *Let n , (M, g) , $q > 0$, b , S , θ be given as in Theorem 6.1, satisfying in particular $b \geq 0$ in M and (6.3). Let $(b_\varepsilon)_\varepsilon$ and $(S_\varepsilon)_\varepsilon$ be sequences of smooth functions such that (6.4) holds true and such that $b_\varepsilon \geq 0$ in M for all ε . Let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth positive functions in M . Let $(x_\varepsilon)_\varepsilon$ be a sequence of points in M such that $\mu_\varepsilon > 0$ given by (6.9) satisfies*

$\mu_\varepsilon = o(1)$ as $\varepsilon \rightarrow 0$. We assume that $n \geq 5$ and that (6.10) holds true. Then, up to a subsequence, for all given $p \in [1, +\infty)$, we have that

$$|q\phi_{b_\varepsilon, S_\varepsilon}(u_\varepsilon) - \nabla S_\varepsilon|(\exp_{x_\varepsilon}(\mu_\varepsilon \cdot)) \rightarrow 0 \text{ in } L_{loc}^p(\mathbb{R}^n), \quad (6.12)$$

$$\psi_{b_\varepsilon}(u_\varepsilon)(\exp_{x_\varepsilon}(\mu_\varepsilon \cdot)) \rightarrow \frac{1}{q} \text{ in } L_{loc}^p(\mathbb{R}^n), \quad (6.13)$$

as $\varepsilon \rightarrow 0$, where $\phi_{b_\varepsilon, S_\varepsilon}$ and ψ_{b_ε} are given by Lemmas 3.1 and 3.2 respectively.

Proof of Lemma 6.1. Even if $(b_\varepsilon)_\varepsilon$ is not necessarily here a constant sequence, i.e. such that $b_\varepsilon = b$ for all ε , (6.13) may be easily obtained by following the lines of [30, Lemma 2.1] and the end of the argument below. We focus now on the slightly less obvious proof of (6.12). For all ε small, we set $A_\varepsilon = \phi_{b_\varepsilon, S_\varepsilon}(u_\varepsilon)$ and let $\tilde{A}_\varepsilon$ be given by

$$\tilde{A}_\varepsilon = \frac{\nabla S_\varepsilon}{q} - A_\varepsilon. \quad (6.14)$$

Then, we get from (3.1) that

$$\Delta_g \tilde{A}_\varepsilon + q^2 u_\varepsilon^2 \tilde{A}_\varepsilon = \frac{\nabla(\Delta_g S_\varepsilon)}{q} + b_\varepsilon A_\varepsilon \quad (6.15)$$

in M . According to the Weitzenböck formula, for all smooth 1-form A in M , we have that

$$(\Delta_g A, A) = \frac{1}{2} \Delta_g |A|^2 + |\nabla A|^2 + Rc_g(A^\sharp, A^\sharp), \quad (6.16)$$

where $\sharp$ is the musical isomorphism. Then, we get from (6.15), dropping the non-negative critical term $|\nabla \tilde{A}_\varepsilon|^2$, that

$$\frac{1}{2} \Delta_g |\tilde{A}_\varepsilon|^2 + q^2 u_\varepsilon^2 |\tilde{A}_\varepsilon|^2 \leq -Rc_g(\tilde{A}_\varepsilon^\sharp, \tilde{A}_\varepsilon^\sharp) + b_\varepsilon(A_\varepsilon, \tilde{A}_\varepsilon) + \frac{1}{q}(\nabla(\Delta_g S_\varepsilon), \tilde{A}_\varepsilon) \quad (6.17)$$

in M . Let $f_\varepsilon \geq 0$ be given by $f_\varepsilon(y) = |\tilde{A}_\varepsilon|(\exp_{x_\varepsilon}(\mu_\varepsilon y))^2$. Let $R > 0$ and

$$\delta_R := \left(1 + \frac{|2R|^2}{n(n-2)}\right)^{-\frac{n-2}{2}} > 0,$$

be fixed. It follows from (3.2), (6.4), (6.10) and (6.17) that there exists $C > 0$ such that

$$\frac{1}{2} \Delta_{g_\varepsilon} f_\varepsilon + (q\delta_R)^2 \mu_\varepsilon^{2-\frac{4}{p_\varepsilon-2}} f_\varepsilon \leq C \mu_\varepsilon^2 \quad (6.18)$$

in $B_0(R) \subset \mathbb{R}^n$ for all ε small, where $g_\varepsilon := (\exp_{x_\varepsilon}^* g)(\mu_\varepsilon \cdot) \rightarrow \xi$, the Euclidean metric as $\varepsilon \rightarrow 0$. Let us control f_ε with a Dirichlet and a quasi-harmonic term, more precisely, we can write $f_\varepsilon \leq f_{\varepsilon, H} + f_{\varepsilon, D}$, where $f_{\varepsilon, H}$ satisfies

$$\begin{cases} \frac{1}{2} \Delta_{g_\varepsilon} f_{\varepsilon, H} + (q\delta_R)^2 \mu_\varepsilon^{2-\frac{4}{p_\varepsilon-2}} f_{\varepsilon, H} = 0 \text{ in } B_0(R), \\ f_{\varepsilon, H} = \Lambda \text{ on } \partial B_0(R), \end{cases} \quad (6.19)$$

where $\Lambda > 0$ is chosen such that

$$\Lambda > \frac{4\|\nabla S\|_{L_V^\infty}^2}{q^2},$$

using (3.2) and (6.4) with the maximum principle, and where $f_{\varepsilon, D}$ satisfies

$$\begin{cases} \frac{1}{2} \Delta_{g_\varepsilon} f_{\varepsilon, D} + (q\delta_R)^2 \mu_\varepsilon^{2-\frac{4}{p_\varepsilon-2}} f_{\varepsilon, D} = C \mu_\varepsilon^2 \text{ in } B_0(R), \\ f_{\varepsilon, D} = 0 \text{ on } \partial B_0(R), \end{cases} \quad (6.20)$$

where C is as in (6.18). By the maximum principle,

$$0 \leq f_{\varepsilon, H} \leq \Lambda \quad (6.21)$$

in $B_0(R)$. Also we get from (6.19) that for any smooth function φ with compact support in $B_0(R)$,

$$\int_{B_0(R)} \left(\Delta_{g_\varepsilon} \varphi + (q\delta_R)^2 \mu_\varepsilon^{2-\frac{4}{p_\varepsilon-2}} \varphi \right) f_{\varepsilon, H} dv_{g_\varepsilon} = 0$$

and then we get with (6.21) that

$$\mu_\varepsilon^{2-\frac{4}{p_\varepsilon-2}} \int_{B_0(R)} \varphi f_{\varepsilon, H} dv_{g_\varepsilon} = O(1) .$$

Since $n \geq 5$ there holds that $\mu_\varepsilon^{2-\frac{4}{p_\varepsilon-2}} \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Letting φ be nonnegative and such that $\varphi = 1$ in $B_0(R/2)$, since $f_{\varepsilon, H} \geq 0$, we get that

$$\int_{B_0(R/2)} f_{\varepsilon, H} dv_{g_\varepsilon} \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Then, by (6.21), it follows that $f_{\varepsilon, H} \rightarrow 0$ in any $L^p(B_0(R/2))$ as $\varepsilon \rightarrow 0$. By (6.20) and the maximum principle,

$$0 \leq f_{\varepsilon, D} \leq \frac{C}{(q\delta_R)^2} \mu_\varepsilon^{\frac{4}{p_\varepsilon-2}} .$$

Then it follows that $f_{\varepsilon, D} \rightarrow 0$ in $L^p(B_0(R))$ as $\varepsilon \rightarrow 0$, and thus that $f_\varepsilon \rightarrow 0$ in $L^p(B_0(R/2))$ as $\varepsilon \rightarrow 0$. This concludes the proof of (6.12) since $R > 0$ is arbitrary. $\square$

For $r \in (0, i_g)$ and $x \in M$, $|\partial B_x(r)|_g$ denotes the volume of the $\partial B_x(r)$ with respect to the metric induced by g . We let φ_ε be the smooth function given in $(0, 2\rho_\varepsilon)$ by

$$\varphi_\varepsilon(r) = \frac{1}{|\partial B_{x_\varepsilon}(r)|_g} \int_{\partial B_{x_\varepsilon}(r)} u_\varepsilon d\sigma_g . \quad (6.22)$$

Set $\Lambda_0 = 2\sqrt{n(n-2)}$ and define $r_\varepsilon \in [\Lambda_0\mu_\varepsilon, \rho_\varepsilon]$ by

$$r_\varepsilon = \sup \left\{ r \in [\Lambda_0\mu_\varepsilon, \rho_\varepsilon] \text{ s.t. } \frac{d}{ds} \left(s^{\frac{2}{p_\varepsilon-2}} \varphi_\varepsilon(s) \right) \leq 0 \text{ for all } s \text{ in } [\Lambda_0\mu_\varepsilon, r] \right\} . \quad (6.23)$$

By (6.10), we get that

$$\mu_\varepsilon = o(r_\varepsilon) \quad (6.24)$$

as $\varepsilon \rightarrow 0$. Moreover, (6.23) implies that

$$\frac{d}{ds} \left(s^{\frac{2}{p_\varepsilon-2}} \varphi_\varepsilon(s) \right) \Big|_{s=r_\varepsilon} = 0 \quad (6.25)$$

if $r_\varepsilon < \rho_\varepsilon$. Let B_ε be given in M by

$$B_\varepsilon(x) = \frac{1}{\mu_\varepsilon^{\frac{2}{p_\varepsilon-2}}} \left(\frac{1}{1 + \frac{d_g(x_\varepsilon, x)^2}{\mu_\varepsilon^2 n(n-2)}} \right)^{\frac{n-2}{2}} . \quad (6.26)$$

At last, using only the information in (6.5) to control the Φ_ε , following closely the lines of [20, Lemmas 9.1-9.3], for all $R > 1$ such that $Rr_\varepsilon \leq 6\rho_\varepsilon$ for all ε , there

exists $C > 0$ and a sequence $(\delta_\varepsilon)_\varepsilon$ of positive real numbers converging to 0 such that

$$|\nabla u_\varepsilon| \leq C \mu_\varepsilon^{n-2-\frac{2}{p_\varepsilon-2}} d_g(x_\varepsilon, \cdot)^{1-n}, \quad (6.27)$$

and

$$|u_\varepsilon - B_\varepsilon| \leq C \mu_\varepsilon^{n-2-\frac{2}{p_\varepsilon-2}} (r_\varepsilon^{2-n} + d_g(x_\varepsilon, \cdot)^{3-n}) + \delta_\varepsilon B_\varepsilon, \quad (6.28)$$

in $B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus \{0\}$, for all ε . In the same spirit, keeping our notations, in dimension $n = 4$, there also holds that the following lemma holds true.

Lemma 6.2. *Assume $n = 4$. Then there exist $C > 0$ and a sequence $(\delta_\varepsilon)_\varepsilon$ of positive real numbers converging to 0 such that*

$$|\nabla u_\varepsilon - B_\varepsilon^\nabla| \leq C \mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}} (\Theta_\varepsilon r_\varepsilon^{-3} + d_g(x_\varepsilon, \cdot)^{-2} + \delta_\varepsilon d_g(x_\varepsilon, \cdot)^{-3}) \quad (6.29)$$

in $B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(\mu_\varepsilon)$, for all ε , where B_ε^∇ is given in $B_{x_\varepsilon}(i_g)$ by

$$\mu_\varepsilon^{\frac{p_\varepsilon}{p_\varepsilon-2}} B_\varepsilon^\nabla(\exp_{x_\varepsilon}(\mu_\varepsilon \cdot)) = \nabla u_0$$

for u_0 as in (6.11), and where $(\Theta_\varepsilon)_\varepsilon$ is a sequence of positive real numbers which satisfies that $\Theta_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$ and $\Theta_\varepsilon = o\left(\ln\left(\frac{r_\varepsilon}{\mu_\varepsilon}\right)\right)$.

Proof of Lemma 6.2. Assume that $n = 4$. Fix $(\Theta_\varepsilon)_\varepsilon$ as in the lemma. By a diagonal argument and (6.10) we get that there exists a sequence $(R_\varepsilon)_\varepsilon$ of positive real numbers converging to $+\infty$ such that, up to passing to a subsequence,

$$\lim_{\varepsilon \rightarrow 0} \left\| \mu_\varepsilon^{\frac{2}{p_\varepsilon-2}} u_\varepsilon(\exp_{x_\varepsilon}(\mu_\varepsilon \cdot)) - u_0 \right\|_{C_R^1(B_0(R_\varepsilon))} = 0. \quad (6.30)$$

Let $(y_\varepsilon)_\varepsilon \in B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(R_\varepsilon \mu_\varepsilon)$, where by $(y_\varepsilon)_\varepsilon \in B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(R_\varepsilon \mu_\varepsilon)$ we mean that $y_\varepsilon \in B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(R_\varepsilon \mu_\varepsilon)$ for all ε . We assume that we also have that $d_g(x_\varepsilon, y_\varepsilon) \leq \Theta_\varepsilon^{-1/3} r_\varepsilon$ for all ε . In particular, $d_g(x_\varepsilon, y_\varepsilon) = o(1)$. Set $\tilde{u}_\varepsilon = u_\varepsilon(\exp_{x_\varepsilon}(\cdot))$ and $\tilde{B}_\varepsilon = B_\varepsilon(\exp_{x_\varepsilon}(\cdot))$. By a standard abuse of notations, if ξ is the Euclidean metric, we may write in the exponential chart at x_ε that

$$\begin{aligned} \Delta_\xi \tilde{u}_\varepsilon + O(|\cdot| |\nabla \tilde{u}_\varepsilon|) + O(\tilde{u}_\varepsilon) &= \tilde{u}_\varepsilon^{p_\varepsilon-1}, \\ \Delta_\xi \tilde{B}_\varepsilon &= \tilde{B}_\varepsilon^{p_\varepsilon-1}, \end{aligned} \quad (6.31)$$

uniformly in $B_{x_\varepsilon}(i_g/2)$, for all ε . The first equation in (6.31) comes from (6.1), and uses the expansion of the Δ_g in normal coordinates and (6.5). Independently, we get from (6.27) and (6.28) that

$$|\tilde{u}_\varepsilon - \tilde{B}_\varepsilon| + |\cdot| |\nabla \tilde{u}_\varepsilon| + \tilde{u}_\varepsilon = O(\tilde{B}_\varepsilon), \quad (6.32)$$

uniformly in $B_0(Rr_\varepsilon/2) \setminus B_0(\mu_\varepsilon)$ and for all ε . Then, since $p_\varepsilon \rightarrow 2^*$, using the basic inequality

$$(1+t)^{p_\varepsilon-1} = O(t^{p_\varepsilon-2}|t-1| + |t-1|^{p_\varepsilon-1}), \quad (6.33)$$

uniformly in $t \geq 0$ and for all ε , we get from (6.31)-(6.32) that

$$|\Delta_\xi(\tilde{u}_\varepsilon - \tilde{B}_\varepsilon)| = O(\tilde{B}_\varepsilon + \tilde{B}_\varepsilon^{p_\varepsilon-2} |\tilde{u}_\varepsilon - \tilde{B}_\varepsilon|), \quad (6.34)$$

uniformly in $B_0(Rr_\varepsilon/2) \setminus B_0(\mu_\varepsilon)$ and for all ε . Setting now $d_\varepsilon = d_g(x_\varepsilon, y_\varepsilon)$, $\hat{u}_\varepsilon = \tilde{u}_\varepsilon(d_\varepsilon \cdot)$, $\hat{B}_\varepsilon = \tilde{B}_\varepsilon(d_\varepsilon \cdot)$ and $D = B_0(2) \setminus B_0(1/2)$, we get from (6.30) and (6.34) that

$$\begin{aligned} & \|\Delta_\xi(\hat{u}_\varepsilon - \hat{B}_\varepsilon)\|_{L^\infty(D)} \\ &= d_\varepsilon^2 \left(O \left(\mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}} d_\varepsilon^{-2} \right) + O \left(\mu_\varepsilon^{2p_\varepsilon-6} d_\varepsilon^{-2p_\varepsilon+4} \|\hat{u}_\varepsilon - \hat{B}_\varepsilon\|_{L^\infty(D)} \right) \right), \\ &= d_\varepsilon O \left(\mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}} d_\varepsilon^{-2} \right) + O \left(\left(\frac{1}{R_\varepsilon} \right)^{2p_\varepsilon-6} \|\hat{u}_\varepsilon - \hat{B}_\varepsilon\|_{L^\infty(D)} \right), \end{aligned} \quad (6.35)$$

for all ε . By standard elliptic theory, we have that

$$\|\nabla(\hat{u}_\varepsilon - \hat{B}_\varepsilon)\|_{L^\infty(\partial B_0(1))} = O \left(\|\hat{u}_\varepsilon - \hat{B}_\varepsilon\|_{L^\infty(D)} + \|\Delta_\xi(\hat{u}_\varepsilon - \hat{B}_\varepsilon)\|_{L^\infty(D)} \right),$$

and we conclude that there exists $C > 0$ and a sequence $(\delta_\varepsilon)_\varepsilon$ of positive real numbers converging to 0 such that

$$\begin{aligned} & |\nabla u_\varepsilon(y_\varepsilon) - B_\varepsilon^\nabla(y_\varepsilon)| \\ & \leq C \mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}} (r_\varepsilon^{-2} d_g(x_\varepsilon, y_\varepsilon)^{-1} + d_g(x_\varepsilon, y_\varepsilon)^{-2} + \delta_\varepsilon d_g(x_\varepsilon, y_\varepsilon)^{-3}) \\ & \leq C \mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}} (d_g(x_\varepsilon, y_\varepsilon)^{-2} + \delta_\varepsilon d_g(x_\varepsilon, y_\varepsilon)^{-3} + \Theta_\varepsilon^{-2/3} d_g(x_\varepsilon, y_\varepsilon)^{-3}) \end{aligned} \quad (6.36)$$

for all sequences $(y_\varepsilon)_\varepsilon \in B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(R_\varepsilon \mu_\varepsilon)$ such that $d_g(x_\varepsilon, y_\varepsilon) \leq \Theta_\varepsilon^{-1/3} r_\varepsilon$ for all ε . On the other hand, by (6.27), there exists $C > 0$ such that

$$|\nabla u_\varepsilon(y_\varepsilon) - B_\varepsilon^\nabla(y_\varepsilon)| \leq |\nabla u_\varepsilon(y_\varepsilon)| + |B_\varepsilon^\nabla(y_\varepsilon)| \leq C \mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}} \Theta_\varepsilon r_\varepsilon^{-3}$$

for all sequences $(y_\varepsilon)_\varepsilon \in B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(R_\varepsilon \mu_\varepsilon)$ such that $d_g(x_\varepsilon, y_\varepsilon) \geq \Theta_\varepsilon^{-1/3} r_\varepsilon$ for all ε . Let now $(y_\varepsilon)_\varepsilon$ be a sequence of points in the set $B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(\mu_\varepsilon)$, i.e. $y_\varepsilon \in B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(\mu_\varepsilon)$ for all ε , such that $(y_\varepsilon)_\varepsilon \notin B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(R_\varepsilon \mu_\varepsilon)$. If we assume that $y_\varepsilon \in B_{x_\varepsilon}(R_\varepsilon \mu_\varepsilon)$ for all ε , then by (6.24) and (6.30)

$$\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon^{\frac{p_\varepsilon}{p_\varepsilon-2}} |\nabla u_\varepsilon(y_\varepsilon) - B_\varepsilon^\nabla(y_\varepsilon)| = 0.$$

From the above, and by contradiction, we clearly get that there exist $C > 0$ and a sequence $(\delta_\varepsilon)_\varepsilon$ of positive real numbers converging to 0 such that, up to passing to a subsequence, (6.29) holds true in $B_{x_\varepsilon}(\frac{Rr_\varepsilon}{2}) \setminus B_{x_\varepsilon}(\mu_\varepsilon)$, for all ε . This ends the proof of Lemma 6.2. $\square$

The following lemma is the complement of Lemma 6.1 when the dimension $n = 4$.

Lemma 6.3 (Control of $(\Phi_\varepsilon)_\varepsilon$ for $n = 4$). *Assume that $n = 4$. Let (M, g) , $q > 0$, a, b, S, θ, ω be given as in Theorem 6.1, satisfying in particular $b \geq 0$ in M and (6.3). Let $(a_\varepsilon)_\varepsilon$, $(b_\varepsilon)_\varepsilon$ and $(S_\varepsilon)_\varepsilon$ be sequences of smooth functions such that (6.4) holds true and such that $b_\varepsilon \geq 0$ in M for all ε . Let $(\omega_\varepsilon)_\varepsilon$ and $(p_\varepsilon)_\varepsilon$ be sequences of real numbers such that $p_\varepsilon \rightarrow 2^*$ in $(2, 2^*]$ and $\omega_\varepsilon \rightarrow \omega$ in $\mathbb{R}$ as $\varepsilon \rightarrow 0$. Let $(u_\varepsilon, v_\varepsilon, A_\varepsilon)_\varepsilon$ be a sequence of smooth maps solving (6.1) such that $u_\varepsilon > 0$ in M for all ε . We assume that $(x_\varepsilon)_\varepsilon$ is a sequence of points in M such that $\mu_\varepsilon > 0$ given by (6.9) satisfies $\mu_\varepsilon = o(1)$ as $\varepsilon \rightarrow 0$. We assume (6.10) and that (6.28) holds true in $B_{x_\varepsilon}(r_\varepsilon) \setminus \{0\}$, where the r_ε 's are numbers in $(0, i_g/2)$ satisfying*

$$\mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}} = o(r_\varepsilon^2). \quad (6.37)$$

Let F_ε be given by

$$F_\varepsilon(x) = \mu_\varepsilon^{2 - \frac{4}{p_\varepsilon - 2}} \frac{\ln \left(2 + \frac{d_g(x_\varepsilon, \cdot)}{\mu_\varepsilon} \right)}{1 + \frac{d_g(x_\varepsilon, \cdot)^2}{\mu_\varepsilon^2}}. \quad (6.38)$$

Then we can split v_ε according to $v_\varepsilon = v_\varepsilon^D + v_\varepsilon^H$ in $B_{x_\varepsilon}(r_\varepsilon)$, where, up to a subsequence, v_ε^H satisfies

$$\lim_{\varepsilon \rightarrow 0} v_\varepsilon^H(\exp_{x_\varepsilon}(r_\varepsilon \cdot)) = v_0, \quad (6.39)$$

in $C_{loc}^1(B_0(1))$ for some C^1 -function v_0 valued in $[0, 1/q]$, and where v_ε^D satisfies for all ε that

$$0 \leq v_\varepsilon^D \leq CF_\varepsilon \quad (6.40)$$

in $B_{x_\varepsilon}(r_\varepsilon)$, for some $C > 0$ independent of ε . Moreover, there exist $C > 0$ and a sequence $(\delta_\varepsilon)_\varepsilon$ of positive real numbers converging to 0 such that

$$|A_\varepsilon|^2 \leq CF_\varepsilon + \delta_\varepsilon \quad (6.41)$$

in $B_{x_\varepsilon}(r_\varepsilon)$ (whether $b \equiv 0$ or not), and such that

$$v_\varepsilon \leq CF_\varepsilon + \delta_\varepsilon, \quad (6.42)$$

in $B_{x_\varepsilon}(r_\varepsilon)$ if $b \not\equiv 0$, for all ε .

Proof of Lemma 6.3. First we aim to prove the global control (6.47) below of v_ε and $|A_\varepsilon|^2$ by u_ε . We get from the third equation of (6.1) and from (6.16) that

$$\frac{1}{2} \Delta_g |A_\varepsilon|^2 + (Rc_g + b_\varepsilon g)(A_\varepsilon^\sharp, A_\varepsilon^\sharp) \leq q(A_\varepsilon, \nabla S_\varepsilon) u_\varepsilon^2,$$

in M . Thus, since $Rc_g + b_\varepsilon g > 0$, and by (3.2), there exists a smooth function $\eta_0 > 0$ in M and $C > 0$ such that

$$\Delta_g |A_\varepsilon|^2 + \eta_0 |A_\varepsilon|^2 \leq C u_\varepsilon^2 \quad (6.43)$$

in M for all ε small, thanks to (6.4). Now since v_ε is nonnegative, by the second equation of (6.1) and by (6.4), if $b \not\equiv 0$, there exists a smooth function $\eta_0 \geq 0$ in M , $\eta_0 \not\equiv 0$, such that

$$\Delta_g v_\varepsilon + \eta_0 v_\varepsilon \leq q u_\varepsilon^2 \quad (6.44)$$

in M for all ε small. Since (6.5) holds true and by the first equation of (6.1), we may also find a constant $C_0 > 0$ such that

$$\Delta_g u_\varepsilon + C_0 u_\varepsilon \geq u_\varepsilon^{p_\varepsilon - 1} + u_\varepsilon \quad (6.45)$$

in M for all ε small. By Robert [28], given any smooth function c in M such that the operator $\Delta_g + c$ is coercive in M , there exists $C > 1$ such that the Green's function $G_c > 0$ of $\Delta_g + c$ satisfies

$$\frac{1}{C d_g(x, y)^2} \leq G_c(x, y) \leq \frac{C}{d_g(x, y)^2}, \quad (6.46)$$

for all $x, y \in M$ such that $x \neq y$. Then, by (6.43)-(6.46), by the Green's representation formula, there exists $C > 0$ such that

$$C u_\varepsilon \geq \begin{cases} |A_\varepsilon|^2 & \text{in any case,} \\ v_\varepsilon & \text{if } b \not\equiv 0, \end{cases} \quad (6.47)$$

in M for all ε , and we used here that $t^2 \leq t^{p_\varepsilon-1} + t$ for all $t \geq 0$ and all $0 < \varepsilon \ll 1$. Now, we turn to the proof of (6.39)-(6.40). We write $v_\varepsilon = v_\varepsilon^D + v_\varepsilon^H$, where v_ε^D and v_ε^H are given by

$$\begin{cases} \Delta_g v_\varepsilon^D + b_\varepsilon v_\varepsilon^D = q u_\varepsilon^2 (1 - q v_\varepsilon) \text{ in } B_{x_\varepsilon}(r_\varepsilon), \\ v_\varepsilon^D = 0 \text{ on } \partial B_{x_\varepsilon}(r_\varepsilon), \end{cases} \quad (6.48)$$

and by

$$\begin{cases} \Delta_g v_\varepsilon^H + b_\varepsilon v_\varepsilon^H = 0 \text{ in } B_{x_\varepsilon}(r_\varepsilon), \\ v_\varepsilon^H = v_\varepsilon \text{ on } \partial B_{x_\varepsilon}(r_\varepsilon). \end{cases} \quad (6.49)$$

Let $G_\varepsilon > 0$ be the Green's function of Δ_g in $B_{x_\varepsilon}(r_\varepsilon)$ with zero Dirichlet boundary condition. By construction and the maximum principle, there exists $C > 0$ such that

$$G_\varepsilon(x, y) \leq C d_g(x, y)^{-2},$$

for all ε and all $x, y \in B_{x_\varepsilon}(r_\varepsilon)$, $x \neq y$. Then, since $b_\varepsilon \geq 0$ and by (3.16), $\Delta_g v_\varepsilon^D$ is smaller than $q u_\varepsilon^2$, and we get from the Green's representation formula, (6.10), (6.28) and Giraud's type computations that (6.40) holds true. The lower bound in (6.40) holds true by the maximum principle, since b_ε and the RHS of (6.48) are both nonnegative. Independently, we set

$$\hat{v}_\varepsilon = v_\varepsilon^H(\exp_{x_\varepsilon}(r_\varepsilon \cdot)), \quad \hat{b}_\varepsilon = b_\varepsilon(\exp_{x_\varepsilon}(r_\varepsilon \cdot)) \quad \text{and} \quad \hat{g}_\varepsilon = (\exp_{x_\varepsilon}^* g)(r_\varepsilon \cdot)$$

in the Euclidean ball $B_0(2) \subset \mathbb{R}^4$. Whether $r_\varepsilon = o(1)$ or not, up to a subsequence, we may assume that $\hat{g}_\varepsilon \rightarrow \hat{g}_0$ in $C^2(B_0(2))$ as $\varepsilon \rightarrow 0$, where $\hat{g}_0$ is some C^2 -metric in $B_0(2)$, and that $\hat{v}_\varepsilon$ satisfies

$$\begin{cases} \Delta_{\hat{g}_\varepsilon} \hat{v}_\varepsilon + r_\varepsilon^2 \hat{b}_\varepsilon \hat{v}_\varepsilon = 0 \text{ in } B_0(1), \\ 0 \leq q \hat{v}_\varepsilon \leq 1 \text{ on } \partial B_0(1), \end{cases} \quad (6.50)$$

for all ε , using (3.16). By standard elliptic theory and the maximum principle using $b_\varepsilon \geq 0$, we get from (6.4), (6.50) that $\hat{v}_\varepsilon \rightarrow v_0$ in $C_{loc}^1(B_0(1))$, up to a subsequence, where v_0 is valued in $[0, 1/q]$. In other words, (6.39) is proved. Now we focus on (6.42) and then assume that $b \not\equiv 0$. By (6.40), it is sufficient to prove that

$$\|v_\varepsilon^H\|_{L_R^\infty(B_{x_\varepsilon}(r_\varepsilon))} = o(1). \quad (6.51)$$

By (6.28) and (6.37),

$$\|u_\varepsilon\|_{L^\infty(\partial B_{x_\varepsilon}(r_\varepsilon))} = o(1). \quad (6.52)$$

Then, by the maximum principle with $b_\varepsilon \geq 0$, (6.47) and (6.49), we get that (6.51) holds true, which concludes the proof of (6.42). In order to conclude the proof of Lemma 6.3, it remains to prove (6.41). The first estimate in (6.47) holds true whatever $b \equiv 0$ or not. We split $|A_\varepsilon|^2$ into $|A_\varepsilon|^2 = A_\varepsilon^H + A_\varepsilon^D$ where A_ε^D is given by

$$\begin{cases} \Delta_g A_\varepsilon^D = \Delta_g |A_\varepsilon|^2 \text{ in } B_{x_\varepsilon}(r_\varepsilon), \\ A_\varepsilon^D = 0 \text{ on } \partial B_{x_\varepsilon}(r_\varepsilon), \end{cases}$$

and $A_\varepsilon^H \geq 0$ is given by

$$\begin{cases} \Delta_g A_\varepsilon^H = 0 \text{ in } B_{x_\varepsilon}(r_\varepsilon), \\ A_\varepsilon^H = |A_\varepsilon|^2 \text{ on } \partial B_{x_\varepsilon}(r_\varepsilon). \end{cases}$$

By the maximum principle, (6.47) and (6.52), $\|A_\varepsilon^H\|_{L^\infty(B_{x_\varepsilon}(r_\varepsilon))} = o(1)$. By the Green's representation formula, using (6.43) and that G_ε, η_0 are nonnegative, we get from Giraud's type computations as above that there exists $C > 0$ such that

$$A_\varepsilon^D \leq CF_\varepsilon \text{ in } B_{x_\varepsilon}(r_\varepsilon),$$

for all ε . This concludes the proof of (6.41) and that of Lemma 6.3. $\square$

Let X_ε be the 1-form given for x in a neighborhood of x_ε by

$$X_\varepsilon(x) = \left(1 - \frac{1}{6(n-1)} Rc_g^\sharp(\nabla d_\varepsilon(x), \nabla d_\varepsilon(x))\right) \nabla d_\varepsilon(x) \quad (6.53)$$

where $d_\varepsilon = \frac{1}{2}d_g(x_\varepsilon, \cdot)^2$. By the Pohozaev identity as in Druet, Hebey and Vétois [11] (see also Hebey [19] and Hebey and Thizy [20]):

$$T_{0,\varepsilon} = T_{1,\varepsilon} + T_{2,\varepsilon} + R_{1,\varepsilon} + R_{2,\varepsilon}, \quad (6.54)$$

where $T_{1,\varepsilon}$ is given by

$$\begin{aligned} T_{1,\varepsilon} &= \frac{n-2}{2n} \int_{\partial B_{x_\varepsilon}(r_\varepsilon)} (\operatorname{div}_g X_\varepsilon) \partial_\nu u_\varepsilon u_\varepsilon d\sigma_g \\ &\quad - \int_{\partial B_{x_\varepsilon}(r_\varepsilon)} \left(\frac{1}{2} X_\varepsilon(\nu) |\nabla u_\varepsilon|^2 - \partial_\nu u_\varepsilon X_\varepsilon(\nabla u_\varepsilon^\sharp) \right) d\sigma_g, \end{aligned} \quad (6.55)$$

$T_{2,\varepsilon}$ is given by

$$T_{2,\varepsilon} = \left(\frac{1}{2^\star} - \frac{1}{p_\varepsilon} \right) \int_{B_{x_\varepsilon}(r_\varepsilon)} (\operatorname{div}_g X_\varepsilon) u_\varepsilon^{p_\varepsilon} dv_g, \quad (6.56)$$

$R_{1,\varepsilon}$ is given by

$$R_{1,\varepsilon} = - \int_{B_{x_\varepsilon}(r_\varepsilon)} \left(\nabla X_\varepsilon - \frac{1}{n} (\operatorname{div}_g X_\varepsilon) g \right)^\sharp (\nabla u_\varepsilon, \nabla u_\varepsilon) dv_g,$$

$R_{2,\varepsilon}$ is given by

$$R_{2,\varepsilon} = \frac{1}{p_\varepsilon} \int_{\partial B_{x_\varepsilon}(r_\varepsilon)} X_\varepsilon(\nu) u_\varepsilon^{p_\varepsilon} d\sigma_g - \frac{n-2}{4n} \int_{\partial B_{x_\varepsilon}(r_\varepsilon)} \partial_\nu (\operatorname{div}_g X_\varepsilon) u_\varepsilon^2 d\sigma_g, \quad (6.57)$$

and $T_{0,\varepsilon}$, involving Φ_ε , is given by

$$\begin{aligned} T_{0,\varepsilon} &= \int_{B_{x_\varepsilon}(r_\varepsilon)} \Phi_\varepsilon \left(u_\varepsilon X_\varepsilon(\nabla u_\varepsilon^\sharp) + \frac{n-2}{2n} (\operatorname{div}_g X_\varepsilon) u_\varepsilon^2 \right) dv_g \\ &\quad + \frac{n-2}{4n} \int_{B_{x_\varepsilon}(r_\varepsilon)} (\Delta_g (\operatorname{div}_g X_\varepsilon)) u_\varepsilon^2 dv_g. \end{aligned} \quad (6.58)$$

Concerning the notations in (6.54), the $T_{i,\varepsilon}$'s terms may matter, while the $R_{i,\varepsilon}$'s terms will eventually be only remainder terms. As in Hebey and Thizy [20, Lemma 9.4] we get from (6.27)-(6.28), using (6.5), that

$$R_{1,\varepsilon} = \begin{cases} o\left(\mu_\varepsilon^{\frac{4-\frac{4}{p_\varepsilon-2}}{\frac{4}{p_\varepsilon-2}}} \ln \frac{1}{\mu_\varepsilon}\right) + o\left(\mu_\varepsilon^{\frac{4-\frac{4}{p_\varepsilon-2}}{\frac{4}{p_\varepsilon-2}}} r_\varepsilon^{-2}\right) & \text{if } n = 4, \\ o\left(\mu_\varepsilon^{\frac{n-\frac{4}{p_\varepsilon-2}}{\frac{4}{p_\varepsilon-2}}}\right) + o\left(\mu_\varepsilon^{\frac{2n-4-\frac{4}{p_\varepsilon-2}}{\frac{4}{p_\varepsilon-2}}} r_\varepsilon^{2-n}\right) & \text{if } n \geq 5, \end{cases} \quad (6.59)$$

for all ε . Still following Hebey and Thizy [20, Lemma 9.4] we also get the estimate

$$R_{2,\varepsilon} = \mu_\varepsilon^{\frac{2n-4-\frac{4}{p_\varepsilon-2}}{\frac{4}{p_\varepsilon-2}}} r_\varepsilon^{2-n} (o(1) + O(r_\varepsilon^2)), \quad (6.60)$$

using (6.24). At last, still from (6.10) and (6.28), we get that

$$\begin{aligned} T_{2,\varepsilon} &\leq -n \frac{2^\star - p_\varepsilon}{(2^\star)^2} \left(\int_{\mathbb{R}^n} u_0^{2^\star} + o(1) \right) \mu_\varepsilon^{4\left(\frac{1}{2^\star-2} - \frac{1}{p_\varepsilon-2}\right)}, \\ &\leq -\bar{\delta}(2^\star - p_\varepsilon), \end{aligned} \quad (6.61)$$

for all ε small, for some given constant $\bar{\delta} > 0$. Now, we consider $T_{0,\varepsilon}$. By (6.5), and Lemmas 6.1 and 6.3, we are in position to get the estimates gathered in the following lemma.

Lemma 6.4. *Up to passing to a subsequence, there holds that*

$$T_{0,\varepsilon} = O\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right), \quad (6.62)$$

if $n = 4$, and that

$$T_{0,\varepsilon} = \left(\int_{\mathbb{R}^n} u_0^2 dy \right) \left(\frac{n-2}{4(n-1)} S_g(x_0) - a(x_0) \right) \mu_\varepsilon^{n-\frac{4}{p_\varepsilon-2}} + o\left(\mu_\varepsilon^{n-\frac{4}{p_\varepsilon-2}}\right), \quad (6.63)$$

if $n \geq 5$. Moreover, if $n = 4$ and (6.37) holds true, we also have that

$$\begin{aligned} T_{0,\varepsilon} &= 64\omega_3 \left(\frac{S_g(x_0)}{6} - h(x_0) \right) \mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{r_\varepsilon}{\mu_\varepsilon} + o\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right) \\ &\quad + O\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2} + \frac{4(p_\varepsilon-4)}{p_\varepsilon-2}}\right), \end{aligned} \quad (6.64)$$

where u_0 is as in (6.11), x_0 is as in (6.7), μ_ε is as in (6.9), r_ε is as in (6.23), ω_n is the volume of the unit n -sphere in $\mathbb{R}^{n+1}$ and h in (6.64) is given by

$$h = \begin{cases} a + |\nabla S|^2 - \omega^2 & \text{if } b \neq 0, \\ a + |\nabla S|^2 - \omega^2(1 - qv_0(0))^2 & \text{if } b \equiv 0, \end{cases} \quad (6.65)$$

where v_0 is as in (6.39).

Proof of Lemma 6.4. By (6.10), (6.27) and (6.28), we have that

$$\int_{B_{x_\varepsilon}(r_\varepsilon)} (u_\varepsilon^2 + d_g(x_\varepsilon, \cdot) |\nabla u_\varepsilon| u_\varepsilon) dv_g = \begin{cases} O\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right) & \text{if } n = 4, \\ O\left(\mu_\varepsilon^{n-\frac{4}{p_\varepsilon-2}}\right) & \text{if } n \geq 5. \end{cases} \quad (6.66)$$

Writing that $\Delta_g(\operatorname{div}_g X_\varepsilon) = \frac{n}{n-1} S_g(x_\varepsilon) + O(d_g(x_\varepsilon, \cdot))$ in M for all ε , we get from (6.10) and (6.28) that

$$\begin{aligned} &\frac{n-2}{4n} \int_{B_{x_\varepsilon}(r_\varepsilon)} (\Delta_g(\operatorname{div}_g X_\varepsilon)) u_\varepsilon^2 dv_g \\ &= \begin{cases} \frac{32\omega_3}{3} S_g(x_0) \mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{r_\varepsilon}{\mu_\varepsilon} + o\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right) & \text{if } n = 4, \\ \left(\int_{\mathbb{R}^n} u_0^2 dy \right) \left(\frac{n-2}{4(n-1)} S_g(x_0) \right) \mu_\varepsilon^{n-\frac{4}{p_\varepsilon-2}} + o\left(\mu_\varepsilon^{n-\frac{4}{p_\varepsilon-2}}\right) & \text{if } n \geq 5. \end{cases} \end{aligned} \quad (6.67)$$

Then, if $n = 4$, (6.62) follows from rough computations with (6.5), (6.10), (6.27), (6.28), and the estimates

$$\operatorname{div}_g X_\varepsilon = n + O(d_g(x_\varepsilon, \cdot)) \quad \text{and} \quad |X_\varepsilon| = O(d_g(x_\varepsilon, \cdot)) \quad (6.68)$$

in M for all ε . Independently, we observe that, if $n \geq 5$,

$$\int_{\mathbb{R}^n} \left(u_0 X(\nabla u_0^\sharp) + \frac{n-2}{2} u_0^2 \right) dy = - \int_{\mathbb{R}^n} u_0^2 dy, \quad (6.69)$$

integrating by parts, where $X(\nabla u_0^\sharp)(x) = x^i (\nabla u_0)_i(x)$. Then, we get from (3.2), (3.16), (6.4), (6.10), (6.27), (6.28), Lemma 6.1, (6.67) and the dominated convergence theorem that (6.63) holds true. Thus, from now on, we focus on the remaining case, namely we assume that $n = 4$ and (6.37) holds true. First, for F_ε as in (6.38), we observe that

$$\int_{B_{x_\varepsilon}(r_\varepsilon)} F_\varepsilon^i(u_\varepsilon^2 + d_g(x_\varepsilon, \cdot) |\nabla u_\varepsilon| u_\varepsilon) dy = O \left(\mu_\varepsilon^{4 - \frac{4}{p_\varepsilon - 2} + \frac{4(p_\varepsilon - 4)}{p_\varepsilon - 2}} \right), \quad (6.70)$$

for all $i \in \{1/2, 1, 2\}$, by (6.10), (6.27) and (6.28). Consider first the case where $b \neq 0$. Then (6.41)-(6.42) hold true. We currently use here and in the sequel that, if $(z_\varepsilon)_\varepsilon$ is some sequence of points and $(f_\varepsilon)_\varepsilon$ is some sequence in C_R^1 converging in C_R^1 to some $f \in C_R^1$, then $f_\varepsilon = f_\varepsilon(z_\varepsilon) + O(d_g(z_\varepsilon, \cdot))$. We have that

$$\begin{aligned} & \int_{B_{x_\varepsilon}(r_\varepsilon)} (a_\varepsilon - \omega_\varepsilon^2 + |\nabla S_\varepsilon|^2) \left(u_\varepsilon X_\varepsilon(\nabla u_\varepsilon^\sharp) + \frac{n-2}{2n} (\operatorname{div}_g X_\varepsilon) u_\varepsilon^2 \right) dv_g \\ &= -64\omega_3 (a - \omega^2 + |\nabla S|^2)(x_0) \mu_\varepsilon^{4 - \frac{4}{p_\varepsilon - 2}} \ln \frac{r_\varepsilon}{\mu_\varepsilon} + o \left(\mu_\varepsilon^{4 - \frac{4}{p_\varepsilon - 2}} \ln \frac{1}{\mu_\varepsilon} \right), \end{aligned} \quad (6.71)$$

by (6.4), (6.10), (6.28), (6.29) and (6.66). Then, we get (6.64) in the case $b \neq 0$, by (6.41)-(6.42), and by (6.67), (6.70), (6.71). Consider now the case $b \equiv 0$. Then, (6.39), and (6.40)-(6.41) hold true. Computing as in (6.71), we get that

$$\begin{aligned} & \int_{B_{x_\varepsilon}(r_\varepsilon/2)} (a_\varepsilon - \omega_\varepsilon^2 (1 - qv_\varepsilon^H)^2 + |\nabla S_\varepsilon|^2) \left(u_\varepsilon X_\varepsilon(\nabla u_\varepsilon^\sharp) + \frac{n-2}{2n} (\operatorname{div}_g X_\varepsilon) u_\varepsilon^2 \right) dv_g \\ &= -64\omega_3 (a - \omega^2 (1 - qv_0(0))^2 + |\nabla S|^2)(x_0) \mu_\varepsilon^{4 - \frac{4}{p_\varepsilon - 2}} \ln \frac{r_\varepsilon}{\mu_\varepsilon} \\ &+ o \left(\mu_\varepsilon^{4 - \frac{4}{p_\varepsilon - 2}} \ln \frac{1}{\mu_\varepsilon} \right), \end{aligned} \quad (6.72)$$

using also (6.24) and (6.39). We also have that

$$\int_{B_{x_\varepsilon}(r_\varepsilon) \setminus B_{x_\varepsilon}(r_\varepsilon/2)} (u_\varepsilon^2 + d_g(x_\varepsilon, \cdot) |\nabla u_\varepsilon| u_\varepsilon) dv_g = O \left(\mu_\varepsilon^{4 - \frac{4}{p_\varepsilon - 2}} \right), \quad (6.73)$$

by (6.27) and (6.28). By (6.67), using (6.4), (6.40), (6.41), (6.70), (6.72) to compute in $B_{x_\varepsilon}(r_\varepsilon/2)$ the terms involving Φ_ε , and using only (6.5) and (6.73) to estimate these terms in $B_{x_\varepsilon}(r_\varepsilon) \setminus B_{x_\varepsilon}(r_\varepsilon/2)$, we get (6.64) also for $b \equiv 0$, which concludes the proof of Lemma 6.4. $\square$

We are in position to state the following key proposition, and we assume from now on (1.3)-(1.4) as in Theorem 6.1.

Proposition 6.1. *Assume that (1.3) holds true if $n = 4$ or that (1.4) holds true if $n \geq 5$. Then, up to passing to a subsequence,*

$$r_\varepsilon = o(1), \quad (6.74)$$

$r_\varepsilon = \rho_\varepsilon$ and

$$r_\varepsilon^{n-2} \mu_\varepsilon^{\frac{2}{p_\varepsilon-2}-n+2} u_\varepsilon(\exp_{x_\varepsilon}(r_\varepsilon \cdot)) \rightarrow \frac{\alpha_n}{| \cdot |^{n-2}} + \mathcal{H} \quad (6.75)$$

in $C_{loc}^1(B_0(2) \setminus \{0\})$ as $\varepsilon \rightarrow 0$, where $\alpha_n = (n(n-2))^{\frac{n-2}{2}}$, where ρ_ε is as in (6.8), μ_ε as in (6.9), r_ε as in (6.23), and $\mathcal{H}$ is a harmonic function in $B_0(2)$ such that

$$\mathcal{H}(0) \leq 0. \quad (6.76)$$

Proof of Proposition 6.1. By (6.27) and (6.28), we get first the estimate

$$T_{1,\varepsilon} = O\left(\mu_\varepsilon^{2n-4-\frac{4}{p_\varepsilon-2}} r_\varepsilon^{2-n}\right), \quad (6.77)$$

where $T_{1,\varepsilon}$ is as in (6.55). If $n = 4$, by plugging in (6.54) the estimates in (6.59), (6.60), (6.61), (6.62) and (6.77), we get that

$$4 - p_\varepsilon = O\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right) + O\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} r_\varepsilon^{-2}\right), \quad (6.78)$$

so that, if (6.37) holds true, we clearly have that

$$0 \leq 4 - p_\varepsilon \leq \mu_\varepsilon^{2-\frac{2}{p_\varepsilon-2}}.$$

In particular, we have that

$$\mu_\varepsilon^{\frac{4(p_\varepsilon-4)}{p_\varepsilon-2}} = o\left(\ln \frac{1}{\mu_\varepsilon}\right). \quad (6.79)$$

We prove now (6.74). Since (6.74) is obvious if (6.37) is not satisfied, we assume that (6.37) holds true in order to conclude the proof of (6.74). Mimicking the argument to get (6.78), but using now (6.79), (6.63) and (6.64) instead of (6.62), using the sign of $T_{2,\varepsilon}$ we get in (6.61) and our assumptions (1.3)-(1.4), we get that

$$\begin{aligned} \mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{r_\varepsilon}{\mu_\varepsilon} &= O\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} r_\varepsilon^{-2}\right) + o\left(\mu_\varepsilon^{4-\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right) \text{ if } n = 4, \\ \mu_\varepsilon^{n-\frac{4}{p_\varepsilon-2}} &= O\left(\mu_\varepsilon^{2n-4-\frac{4}{p_\varepsilon-2}} r_\varepsilon^{2-n}\right) + o\left(\mu_\varepsilon^{n-\frac{4}{p_\varepsilon-2}}\right) \text{ if } n \geq 5. \end{aligned} \quad (6.80)$$

By (6.80), (6.74) is clearly satisfied also when (6.37) holds true. This proves (6.74). Assuming (6.74), the proof of (6.75) is by now rather standard (see for instance Hebey and Thizy [20, Lemma 9.6]). It uses in particular (6.5), the first equation in (6.1) and the Bôcher's theorem about nonnegative harmonic functions. The value of α_n is then obtained by integrating the first equation of (6.1) in $B_{x_\varepsilon}(r_\varepsilon)$, and by using again (6.10) and (6.28). At last, if we assume (6.76), then we get $r_\varepsilon = \rho_\varepsilon$ from (6.25) (see for instance Hebey and Truong [22]). Thus, it remains to prove the key inequality (6.76). Since we have now (6.74) and (6.75), we may improve (6.77) and get

$$T_{1,\varepsilon} = -(\beta_n \mathcal{H}(0) + o(1)) r_\varepsilon^{2-n} \mu_\varepsilon^{2n-4-\frac{4}{p_\varepsilon-2}}, \quad (6.81)$$

where $\beta_n = \omega_{n-1} n^{n-2} (n-2)^n / 2 > 0$. Assume either that $n \geq 5$, or that $n = 4$ and (6.37) holds true. Then, using now (6.81) instead of (6.77), we may resume the arguments used to get (6.80) from (6.54), and obtain that

$$-(\beta_n \mathcal{H}(0) + o(1)) r_\varepsilon^{2-n} \mu_\varepsilon^{2n-4-\frac{4}{p_\varepsilon-2}} > 0 \quad (6.82)$$

for all ε small, and then that (6.76) is true. In order to conclude the proof of (6.76) and thus that of Proposition 6.1, we assume that we are in the remaining case:

namely $n = 4$ and (6.37) is not true. We assume by contradiction that (6.76) is not satisfied, in other words that $\mathcal{H}(0)$ is positive. Starting from (6.54), resuming the estimates to get (6.82) from (6.54), but using now (6.62) instead of (6.64), we get that

$$(\beta_n \mathcal{H}(0) + o(1)) r_\varepsilon^{2-n} \mu_\varepsilon^{\frac{4}{p_\varepsilon-2}} = O\left(\mu_\varepsilon^{\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right) + T_{2,\varepsilon}. \quad (6.83)$$

Assuming that

$$T_{2,\varepsilon} = O\left(\mu_\varepsilon^{\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon}\right),$$

(6.83) with $\mathcal{H}(0) > 0$ enforces (6.37) to be satisfied. Since we assumed that (6.37) is not satisfied there must be the case that

$$\mu_\varepsilon^{\frac{4}{p_\varepsilon-2}} \ln \frac{1}{\mu_\varepsilon} = o(|T_{2,\varepsilon}|).$$

Since $T_{2,\varepsilon}$ has a sign, which is given by (6.61), the RHS of (6.83) is negative, and this contradicts $\mathcal{H}(0) > 0$. This concludes the proof of (6.76). Proposition 6.1 is proved. $\square$

At that stage, we can conclude the proof of Theorem 6.1 following that of Hebey and Thizy [20, Theorem 8.1].

Proof of Theorem 6.1. The goal is to prove that (6.6) cannot hold true if one assumes (1.3)-(1.4). We will use for this Proposition 6.1 where (1.3)-(1.4) are crucially assumed. As in Druet [9] and Druet, Hebey and Vtois [11] (see also Hebey [19]), there exist $C > 0$ such that there exists $N_\varepsilon \in \mathbb{N}^*$, N_ε critical points of u_ε denoted by $x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}$ and such that

$$d_g(x_{i,\varepsilon}, x_{j,\varepsilon})^{\frac{2}{p_\varepsilon-2}} u_\varepsilon \geq 1 \quad (6.84)$$

for all $i, j \in \{1, \dots, N_\varepsilon\}$, $i \neq j$, and such that

$$\left(\min_{i \in \{1, \dots, N_\varepsilon\}} d_g(x_{i,\varepsilon}, \cdot)\right)^{\frac{2}{p_\varepsilon-2}} u_\varepsilon \leq C \text{ in } M, \quad (6.85)$$

for all ε . We let $d_\varepsilon > 0$ be given by

$$d_\varepsilon = \begin{cases} \frac{i_g}{4} & \text{if } N_\varepsilon = 1, \\ \min_{i,j \in \{1, \dots, N_\varepsilon\}, i \neq j} d_g(x_{i,\varepsilon}, x_{j,\varepsilon}) & \text{otherwise,} \end{cases} \quad (6.86)$$

where i_g is the injectivity radius. In case $N_\varepsilon \geq 2$, we reorder the $x_{i,\varepsilon}$'s such that

$$d_\varepsilon = d_g(x_{1,\varepsilon}, x_{2,\varepsilon}) \leq d_g(x_{1,\varepsilon}, x_{3,\varepsilon}) \leq \dots \leq d_g(x_{1,\varepsilon}, x_{N_\varepsilon,\varepsilon}). \quad (6.87)$$

Our first claim is that, up to passing to a subsequence,

$$d_\varepsilon \rightarrow d \quad (6.88)$$

as $\varepsilon \rightarrow 0$ for some $d > 0$. We prove (6.88) by contradiction and assume that $d_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then, $N_\varepsilon \geq 2$ for $\varepsilon > 0$ small. We set for $x \in B_0(\hat{\delta}d_\varepsilon^{-1})$, $0 < \hat{\delta} < \frac{1}{2}$ fixed,

$$\begin{aligned}\hat{u}_\varepsilon(x) &= d_\varepsilon^{\frac{2}{p_\varepsilon-2}} u_\varepsilon \left(\exp_{x_{1,\varepsilon}}(d_\varepsilon x) \right), \\ \hat{\Phi}_\varepsilon(x) &= \Phi_\varepsilon \left(\exp_{x_{1,\varepsilon}}(d_\varepsilon x) \right), \text{ and} \\ \hat{g}_\varepsilon(x) &= \left(\exp_{x_{1,\varepsilon}}^* g \right) (d_\varepsilon x).\end{aligned}$$

Since $d_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, we get that $\hat{g}_\varepsilon \rightarrow \xi$ in $C_{loc}^2(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$, where ξ is the Euclidean metric. Thanks to (6.1), we have that

$$\Delta_{\hat{g}_\varepsilon} \hat{u}_\varepsilon + d_\varepsilon^2 \hat{\Phi}_\varepsilon \hat{u}_\varepsilon = \hat{u}_\varepsilon^{p_\varepsilon-1} \quad (6.89)$$

in $B_0(\hat{\delta}d_\varepsilon^{-1})$. For any $R > 0$, we let $1 \leq N_{R,\varepsilon} \leq N_\varepsilon$ be such that

$$\begin{aligned}d_g(x_{1,\varepsilon}, x_{i,\varepsilon}) &\leq R d_\varepsilon \text{ for } 1 \leq i \leq N_{R,\varepsilon} \text{ and} \\ d_g(x_{1,\varepsilon}, x_{i,\varepsilon}) &> R d_\varepsilon \text{ for } N_{R,\varepsilon} + 1 \leq i \leq N_\varepsilon.\end{aligned}$$

Such an $N_{R,\varepsilon}$ does exist thanks to (6.87). We also have $N_{R,\varepsilon} \geq 2$ for $R > 1$ and that the sequence $(N_{R,\varepsilon})_\varepsilon$ is bounded for all R by (6.86). Given $R > 0$, there holds that

$$\begin{aligned}\text{either } \hat{u}_\varepsilon(\hat{x}_{i,\varepsilon}) &= O(1) \text{ for all } 1 \leq i \leq N_{R,\varepsilon}, \\ \text{or } \hat{u}_\varepsilon(\hat{x}_{i,\varepsilon}) &\rightarrow +\infty \text{ as } \varepsilon \rightarrow 0 \text{ for all } 1 \leq i \leq N_{R,\varepsilon},\end{aligned} \quad (6.90)$$

where

$$\hat{x}_{i,\varepsilon} = \frac{1}{d_\varepsilon} \exp_{x_{1,\varepsilon}}^{-1}(x_{i,\varepsilon}). \quad (6.91)$$

At this point we split the proof into the study of two cases. In the first case, we assume that there exist $R > 0$ and $1 \leq i \leq N_{R,\varepsilon}$ such that $\hat{u}_\varepsilon(\hat{x}_{i,\varepsilon}) = O(1)$. Then, by (6.90), $\hat{u}_\varepsilon(\hat{x}_{j,\varepsilon}) = O(1)$ for all $1 \leq j \leq N_{R,\varepsilon}$ and all $R > 0$. Noting that the two first equations in (6.8) are satisfied by $x_\varepsilon = x_{j,\varepsilon}$ and $\rho_\varepsilon = \frac{1}{8}d_\varepsilon$, it follows from (6.10) that the sequence $(\hat{u}_\varepsilon)_\varepsilon$ is uniformly bounded in the balls $B_{\hat{x}_{j,\varepsilon}}(\frac{1}{2})$ and, adding (6.85), in $B_0(R/2)$. Thus, by (6.89) and elliptic theory, up to passing to a subsequence, we get that the $\hat{u}_\varepsilon$'s converge in $C_{loc}^1(\mathbb{R}^n)$, as $\varepsilon \rightarrow 0$, to some nonnegative $\hat{u}$ which solves (5.17) with $p = 2^*$ and which satisfies that $\hat{u} \neq 0$ by (6.84). Moreover, still up to passing to a subsequence, $\hat{u}$ has at least two critical points, namely 0 and $\hat{x}_2 \in S^{n-1}$, the limit of $(\hat{x}_{2,\varepsilon})_\varepsilon$. By the classification result of the nonnegative solutions of (5.17) with $p = 2^*$, see Caffarelli, Gidas and Spruck [5], this is not possible and we are left with the second case of our study, where we assume that there exist $R > 0$ and $1 \leq i \leq N_{R,\varepsilon}$ such that $\hat{u}_\varepsilon(\hat{x}_{i,\varepsilon}) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Then, by (6.90), $\hat{u}_\varepsilon(\hat{x}_{j,\varepsilon}) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$ for all $1 \leq j \leq N_{R,\varepsilon}$ and all $R > 0$. The assumptions (6.8) are satisfied by $x_\alpha = x_{1,\varepsilon}$ and $\rho_\varepsilon = \frac{1}{8}d_\varepsilon$. Let

$$\hat{w}_\varepsilon = \hat{u}_\alpha(0)^{\frac{(p_\varepsilon-2)(n-2)}{2}-1} \hat{u}_\varepsilon.$$

By (6.89)

$$\Delta_{\hat{g}_\varepsilon} \hat{w}_\varepsilon + d_\varepsilon^2 \hat{\Phi}_\varepsilon \hat{w}_\varepsilon = \frac{1}{\hat{u}_\alpha(0)^{(p_\varepsilon-2)(\frac{(p_\varepsilon-2)(n-2)}{2}-1)}} \hat{w}_\varepsilon^{p_\varepsilon-1}. \quad (6.92)$$

Up to passing to a subsequence, we let $\hat{x}_i$ be the limit of $(\hat{x}_{i,\varepsilon})_\varepsilon$ in (6.91) and $I = \{1, \dots, \lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} N_{R,\varepsilon}\}$. We pick a compact set $K \subset \mathbb{R}^n \setminus \{\hat{x}_i, i \in I\}$.

The sequence $(\hat{u}_\varepsilon)_\varepsilon$ is bounded in K by (6.85). Hence by (6.92), we get that

$$\Delta_{\hat{g}_\varepsilon} \hat{w}_\varepsilon = F_\varepsilon \hat{w}_\varepsilon$$

in K , where $(F_\varepsilon)_\varepsilon$ is bounded in L^∞ , and by the Harnack inequality

$$\sup_K \hat{w}_\varepsilon \leq \tilde{C} \inf_K \hat{w}_\alpha \quad (6.93)$$

for some positive constant $\tilde{C}$ independent of ε . Adding Proposition 6.1, we get that, up to a subsequence, $\hat{w}_\varepsilon \rightarrow \hat{G}$ in $C_{loc}^1(\mathbb{R}^n \setminus \{\hat{x}_i\}_{i \in I})$, where, for any $R > 0$,

$$\begin{aligned} \hat{G}(x) &= \sum_{i=1}^{\hat{N}_R} \frac{\Lambda_i}{|x - \hat{x}_i|^{n-2}} + \hat{\mathcal{H}}_R(x) = \frac{\Lambda_1}{|x|^{n-2}} + X(x), \\ X(x) &= \sum_{i=2}^{\hat{N}_R} \frac{\Lambda_i}{|x - \hat{x}_i|^{n-2}} + \hat{\mathcal{H}}_R(x) \end{aligned} \quad (6.94)$$

in $B_0(R) \setminus \{\hat{x}_i, i \in I\}$, and where $2 \leq \hat{N}_R \leq N_{2R}$ is such that $|\hat{x}_{\hat{N}_R}| \leq R$ and $|\hat{x}_{\hat{N}_R+1}| > R$, and N_{2R} is such that $N_{2R,\varepsilon} \rightarrow N_{2R}$ as $\varepsilon \rightarrow 0$. The Λ_i 's in (6.94) are positive and $\hat{\mathcal{H}}_R$ is a harmonic function in $B_0(R)$. In order to see that $\Lambda_i \neq 0$ for $i \geq 2$, we can apply Proposition 6.1 around $x_{i,\varepsilon}$ as we did around $x_{1,\varepsilon}$, and use the Harnack inequality again to get that the ratio of $\hat{u}_\varepsilon(\hat{x}_{i,\varepsilon})$ over $\hat{u}_\varepsilon(0)$ tends to a positive limit. We have that $\hat{G} \geq 0$. Hence, by the maximum principle, we get that $X(0) \geq -\Lambda_1 R^{2-n}$ for all $R > 1$, so that $X(0) \geq 0$. Then, by Proposition 6.1, we get that $X(0) = 0$. By the maximum principle,

$$X(0) \geq \Lambda_2 - \frac{\Lambda_1}{R^{n-2}} - \frac{\Lambda_2}{(R-1)^{n-2}}.$$

Choosing $R \gg 1$ sufficiently large, we get that $X(0) > 0$ and this is a contradiction with $X(0) = 0$. This proves (6.88). We are now in position to conclude the proof of Theorem 6.1, mixing the results of (6.88) and of Proposition 6.1. By (6.88), M being compact, $(N_\varepsilon)_\varepsilon$ is a bounded sequence. Up to passing to a subsequence, we can assume that $N_\varepsilon = N$ for all ε and some $N \in \mathbb{N}^*$. Let $(x_\varepsilon)_\varepsilon$ be a sequence of maximal points of u_ε . By (6.6) and (6.85), we get that $d_g(x_\varepsilon, x_{i,\varepsilon}) \rightarrow 0$ as $\varepsilon \rightarrow 0$ for some i , maybe after passing to a subsequence. Then, by (6.88), noting that

$$d_g(x_\varepsilon, x) u_\varepsilon(x)^{\frac{p_\varepsilon-2}{2}} \leq d_g(x_\varepsilon, x_{i,\varepsilon}) u_\varepsilon(x_\varepsilon)^{\frac{p_\varepsilon-2}{2}} + d_g(x_{i,\varepsilon}, x) u_\varepsilon(x)^{\frac{p_\varepsilon-2}{2}}$$

for all x , we get with (6.85) that (6.8) holds true with the x_ε 's and $\rho_\varepsilon = \delta$, for some $\delta > 0$. But this contradicts Proposition 6.1 for which $\rho_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$. In particular, assumption (6.6) cannot hold true. Theorem 6.1 is proved. $\square$

7. PROOF OF THEOREM 1.1 IN THE CRITICAL CASE $p = 2^*$

Let $n \geq 4$ be an integer, let (M, g) be a smooth closed n -manifold, let $q > 0$ and ω be real numbers, and let a, b, S be smooth functions such that $b \geq 0$ in M and such that $Rc_g + bg > 0$ in M , in the sense of the bilinear forms, where Rc_g is the Ricci curvature of g . We assume that $\Delta_g + f$ is coercive, where $f = a + |\nabla S|^2 - \omega^2$ if $b \neq 0$, and $f = a + |\nabla S|^2$ otherwise. We also assume that (1.3)-(1.4) hold true. The compactness part in Theorem 1.1 is an easy consequence of Theorem 6.1. It remains to prove the existence of a nontrivial solution. Let $(p_\varepsilon)_\varepsilon$ be a sequence of subcritical exponents in $(2, 2^*)$ converging to 2^* , and let $(u_\varepsilon, v_\varepsilon, A_\varepsilon)_\varepsilon$, $u_\varepsilon > 0$, be a

sequence of solutions of (6.1) given by Section 5. Here we apply Theorem 1.1 with a, b, S fixed and $p = p_\varepsilon$. By Theorem 6.1, up to passing to a subsequence, we get that $(u_\varepsilon)_\varepsilon$ is bounded in L^∞_R . Then we conclude thanks to Lemma 4.1 and Lemma 4.2. This ends the proof of Theorem 1.1 in the critical case.

REFERENCES

- [1] Benci, V., and Fortunato, D., Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations, *Rev. Math. Phys.*, 14, 409–420, 2002.
- [2] ———, Spinning Q -balls for the Klein-Gordon-Maxwell equations, *Commun. Math. Phys.*, 295, 639–668, 2010.
- [3] Biquard, O., Polycopié on Differential Geometry and Global Analysis, *Unpublished Notes*, 2008.
- [4] Brézis, H., and Li, Y.Y., Some nonlinear elliptic equations have only constant solutions, *J. Partial Differential Equations*, 19, 2006, 208–217.
- [5] Caffarelli, L. A., Gidas, B., and Spruck, J., Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.*, 42, 271–297, 1989.
- [6] Clapp, M., Ghimenti, M., and Micheletti, A.M., Semiclassical states for a static supercritical Klein-Gordon-Maxwell-Proca system on a closed Riemannian manifold, *Commun. Contemp. Math.*, 18, 2016.
- [7] d’Avenia, P., Medreski, J., and Pomponio, P., Vortex ground states for Klein-Gordon-Maxwell-Proca type systems, *J. Math. Phys.*, 58, pages 041503, 2017.
- [8] Druet, O., From one bubble to several bubbles: The low-dimensional case, *J. Differential Geom.*, 63, 399–473, 2003.
- [9] ———, Compactness for Yamabe metrics in low dimensions, *Internat. Math. Res. Notices*, 23, 1143–1191, 2004.
- [10] Druet, O., and Hebey, E., Existence and a priori bounds for electrostatic Klein-Gordon-Maxwell systems in fully inhomogeneous spaces, *Commun. Contemp. Math.*, 12, 831–869 2010.
- [11] Druet, O., Hebey, E., and Véttois, J., Bounded stability for strongly coupled critical elliptic systems below the geometric threshold of the conformal Laplacian, *J. Funct. Anal.*, 258, 999–059, 2010.
- [12] ———, Static Klein-Gordon-Maxwell-Proca systems in 4-dimensional closed manifolds. II, *J. reine angew. Math.*, 713, 149–179, 2016.
- [13] Epstein, H., and Moschella, U., de Sitter tachyons and related topics, *Comm. Math. Phys.*, 336(1):3811430, 2015.
- [14] Gidas, B., and Spruck, J., A priori bounds for positive solutions of nonlinear elliptic equations, *Comm. Part. Diff. Eq.*, 6, 883–901, 1981.
- [15] Gilbarg, G., and Trudinger, N.S., *Elliptic partial differential equations of second order*, Second edition, Grundlehren der Mathematischen Wissenschaften, 224, Springer, Berlin–New York, 1983.
- [16] Goldhaber, A.S., and Nieto, M.M., Terrestrial and Extraterrestrial limits on the photon mass, *Rev. Mod. Phys.*, 43, 277–296, 1971.
- [17] ———, Photon and Graviton mass limits, *Rev. Mod. Phys.*, 82, 939–979, 2010.
- [18] Hebey, E., Solitary waves in critical abelian gauge theories, *Discrete Contin. Dyn. Syst.*, 32(5):174771761, 2012.
- [19] ———, *Compactness and stability for nonlinear elliptic equations*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2014.
- [20] Hebey, E., and Thizy, P.D., Stationary Kirchhoff systems in closed high dimensional manifolds, *Commun. Contemp. Math.*, 18(2):1550028, 53, 2016.
- [21] ———, Klein-Gordon-Maxwell-Proca Type Systems in the Electro-Magneto-Static Case, *J. Part. Diff. Eq.*, 31, 119–58, 2018.
- [22] Hebey, E., and Truong, T.T., Static Klein-Gordon-Maxwell-Proca systems in 4-dimensional closed manifolds, *J. reine angew. Math.*, 667, 221–248, 2012.
- [23] Hebey, E., and Wei, J., Resonant states for the static Klein-Gordon-Maxwell-Proca system, *Math. Res. Lett.*, 19, 953–967, 2012.

- [24] Li, Y.Y., and Zhu, M., Yamabe type equations on three dimensional Riemannian manifolds, *Commun. Contemp. Math.*, 1, 1–50, 1999.
- [25] Lin, C.S., Ni, W.M., and Takagi. I., Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations*, 72(1):127, 1988.
- [26] Luo, J., Gillies, G.T., and Tu, L.C., The mass of the photon, *Rep. Prog. Phys.*, 68, 77–130, 2005.
- [27] Marques, F.C., A priori estimates for the Yamabe problem in the non-locally conformally flat case, *J. Differential Geom.*, 71, 2005, 315–346.
- [28] Robert, F., Existence et asymptotiques optimales des fonctions de Green des opérateurs elliptiques d’ordre deux, *Unpublished Notes*, 2009.
- [29] Schoen, R.M., *Lecture notes from courses at Stanford*, written by D.Pollack, preprint, 1988.
- [30] Thizy, P.D., Klein-Gordon-Maxwell equations in high dimensions, *Commun. Pure Appl. Anal.*, 14, 1097–1125, 2015.

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