Normal forms for discrete-time switched linear systems

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***Abstract*— In this paper, we propose an algebraic approach to the analysis of discrete-time switched linear systems, where we investigate linear relations between words over the matrices of the system. We introduce normal form matrix words as well as a numerical criterion to reduce drastically the number of linear matrix inequalities conditions for checking stability. In particular, we relate exponential stability to quadratic stability of another system. We illustrate our methods with an example.**

I. Introduction

In this article we investigate some properties of discrete-time switched linear systems [1], [2]. Such systems consist of a family of linear systems systems with a rule that orchestrates the switching among them. Discrete-time switched linear systems are a popular model in various control domains. Among such application areas, we may cite networked control systems [3], systems with aperiodic sampling [4], discrete-time delay systems [5], [6], etc., they all can be modeled as switched linear systems. Despite the fact that switched systems have been intensively studied over the last two decades, the stability problem is still a complex open problem [7].

Various methods are available for studying the stability of discrete-time switched linear systems. Roughly speaking, stability criteria have been proposed either based on algebraic methods (such as the ones within the framework of the theory of Lie algebras [8], [9] and the ones based on joint spectral radii [10], [11]) or also on numerical approaches, usually based on Linear Matrix Inequalities and Lyapunov functions. Stability criteria have been proposed by checking the existence of quasi-quadratic [12], [13], parameter dependent [14], path-dependent [15], non-monotonic [16], [17], [18], [19] and composite quadratic [13], [20] Lyapunov functions. The aim of the article is to propose a new method for analyzing the stability of switched linear systems by investigating the structure and algebraic properties of some particular Lyapunov based criteria.

In [16], a necessary and sufficient condition is given for a discrete-time switched linear system to be globally uniformly exponentially stable: it is equivalent to the existence of a positive integer N such that a problem consisting in p^N+1 LMI conditions admits a solution. Hence, we obtain the following method for constructing the quasi-quadratic Lyapunov function: starting with $N=1$, the value of N is increased until we find a solution to the LMI problem, from which the corresponding quasi-quadratic Lyapunov function is derived. As mentioned in [20], an optimization of this method consists in reducing the computational complexity, since the number of LMI conditions grows exponentially when N increases.

In the present paper, starting from the condition proposed in [16], we propose an approach to study discrete-time switched linear systems using linear algebra tools. Two stability criteria are provided. These two methods are based on the observation that the matrices involved in the LMI problem from [16] may satisfy algebraic relations. The first method consists in restricting the LMI conditions to a basis of the linear span of all matrices involved in the initial LMI problem and then, to check if the solution for the basis elements still holds for the other matrices. The second method consists in adding numerical constraints to the LMIs corresponding to basis elements in such a way that the LMI conditions corresponding to the other matrices are automatically fulfilled. The main argument used in this second approach is that convex decomposition of matrices does not bring new LMI constraints. We point out that for both methods, the number of LMI conditions is upper-bounded by the dimension of the matrix space, which does not depend on N.

The paper is organized as follows. In Section II, we fix some notations. Section III is a recollection about discrete-time switched linear systems, stability definition and its characterization in terms of LMIs given in [16]. Section IV contains our results: we present two methods for reducing the number of LMI conditions, the first one using linear algebra and the second one using convex decompositions. In Section V, the second method is illustrated with numerical examples. Section VI containes the proof of an intermediate lemma.

II. Notations

We fix the following notations: let $p,n \in \mathbb{N}\setminus\{0\}$.

- We write $\mathcal{I}_p = \{1,\cdots,\ p\}$.
- Given a set S, we denote by S^* the set of words over S, that is the set of formal concatenation of elements of S. Given $N\in\mathbb{N}$, we denote by $S^{(N)}$ the set of words of S^* of length N.

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- Given a matrix $M \in \mathbb{R}^{n\times p}$, we denote by $M^T \in \mathbb{R}^{p\times n}$ the transpose of M.
- For $x = (x_1 \ldots x_n)^T \in \mathbb{R}^n$, we denote by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ the Euclidean vector norm on $\mathbb{R}^n$.
- Given a square symmetric matrix M, we write $M \succ 0$ or $M \prec 0$ according to M is positive or negative definite, respectively.
- Given $\mathcal{P} = \{\mathbf{p_1}, \cdots, \mathbf{p_k}\} \subset \mathbb{R}^n$, the *convex hull* of $\mathcal{P}$ is the set of elements $\sum_{i=1}^k a_i \mathbf{p_i} \in \mathbb{R}^n$, where a_i are non-negative real numbers satisfying $\sum_{i=1}^k a_i = 1$.
- Given $\lambda \in \mathbb{R}$, we denote by $\operatorname{sign}(\lambda)$ the sign of λ.

III. Preliminaries

Consider n, p, two strictly positive integers, and a set of matrices $\mathbf{A} = \{A_i \in \mathbb{R}^{n\times n} \mid i \in \mathcal{I}_p\}$. Given a word $\omega = i_{k-1}\cdots i_0 \in \mathcal{I}_p^*$ on the alphabet $\mathcal{I}_p$, the corresponding matrix word on $\mathbf{A}$ is denoted by $A_\omega = A_{i_{k-1}} \cdots A_{i_0} \in \mathbf{A}^*$. By convention, if $k = 0$, we let $A_\omega = \mathrm{Id}_n$.

We are interested in the class of discrete-time switched linear systems described by the equation

$$x_{k+1} = A_{\sigma_k} x_k,\ \forall k \in \mathbb{N},\ x_0 \in \mathbb{R}^n, \tag{1}$$

where $x : \mathbb{N} \to \mathbb{R}^n$ represents the system state, $x_0 = x(0)$ the initial condition and $\sigma : \mathbb{N} \to \mathcal{I}_p$, $k \mapsto \sigma_k$, the switching function associated to the system. It is assumed that the switching function is not known. In what follows, $\mathbf{A}$ will denote the set of matrices $\{A_{\sigma_k}\}$ defined by the switching function σ of the system (1).

For $k \in \mathbb{N}$, denote by $\mathcal{S}_k(\mathcal{I}_p) \subset \sigma(\mathbb{N})^*$ the set of k-length words formed by the image of the switching function σ starting with σ_0. This set is also called *the set of k-length switching paths*.

For the initial condition $x_0 \in \mathbb{R}^n$ and an infinite word $\omega = (\sigma_k)_{k=0}^\infty \in \mathcal{S}_\infty(\mathcal{I}_p)$, we consider the flow $(k, x_0) \mapsto \phi_\omega(k, x_0)$ defined by

$$\phi_\omega(k, x_0) = A_\omega(k)\ x_0,$$

where $A_\omega(k) = A_{\sigma_{k-1}} \cdots A_{\sigma_0}$, so that $A_\omega(0) = \mathrm{Id}_n$, is called the *k-step transition matrix* of the discrete-time switched system associated to ω.

For an infinite word $\omega \in \mathcal{S}_\infty(\mathcal{I}_p)$, the solution of (1) associated to the initial condition x_0 is given by the sequence of points $(\phi_\omega(k, x_0))_{k=0}^\infty$.

Definition 3.1: The equilibrium point $x = 0$ of the switched linear system (1) is *globally uniformly exponentially stable (GUES)* if there exist constant $c > 0$ and $0 < \lambda < 1$ such that

$$\|\phi_\omega(\ell, x_0)\|^2 \leq c\lambda^\ell \|x_0\|^2 \tag{2}$$

holds for all initial conditions $x_0 \in \mathbb{R}^n$, all $\ell \in \mathbb{N}$ and all sequences of switching $\omega = (\sigma_k)_{k=0}^\infty \in \mathcal{S}_\infty(\mathcal{I}_p)$.

Proposition 3.2: [16] The following assertions are equivalent:

1) The equilibrium point $x = 0$ of (1) is GUES
2) There exists a positive integer N such that the following LMIs problem admits a solution

$$\exists P = P^T \succ 0 \text{ such that } P \succ A_\omega^T P A_\omega,\ \forall \omega \in \mathcal{I}_p^{(N)}. \tag{3}$$

As an immediate corollary, we obtain the following criterion for checking GUES:

Proposition 3.3: Let N be a positive integer. If the LMIs problem (3) admits a solution, then the equilibrium point $x = 0$ of (1) is GUES.

For a fixed N, the criterion of Proposition 3.3 for checking GUES consists of $p^N + 1$ LMI conditions. In Section IV, we propose an approach based on linear algebra methods for reducing the number of LMI conditions.

IV. Conditions based on normal form matrices

Our approach for stability analysis is based on the following observation: the matrices in $\mathbf{A}^*$ may satisfy algebraic relations of the form

$$\sum_{\text{finite}} \lambda_\omega A_\omega = 0,$$

where the sum is taken over a finite set of words $\omega \in \mathcal{I}_p^*$ and $\lambda_\omega \in \mathbb{R}$. Hence, for such a decomposition, if $\lambda_{\omega_0} \neq 0$ for a word ω_0, then we have

$$A_{\omega_0} = \sum \mu_\omega A_\omega,$$

where $\omega \neq \omega_0$ and $\mu_\omega = -\frac{\lambda_\omega}{\lambda_{\omega_0}}$. In the sequel, we present two methods for removing the LMI condition corresponding to ω_0 in (3).

Given N, let us denote by $V_{\mathbf{A}}^{(N)}$ the subspace of $\mathbb{R}^{n\times n}$ spanned by the matrices A_ω, where $\omega \in \mathcal{I}_p^{(N)}$.

Definition 4.1: Let d_N be the dimension of $V_{\mathbf{A}}^{(N)}$ and let $A_{\omega_1}, \cdots, A_{\omega_{d_N}}$, where $\omega_i \in \mathcal{I}_p^{(N)}$, be a basis of $V_{\mathbf{A}}^{(N)}$. We call the matrices A_{ω_i}, $1 \leq i \leq d_N$, *normal form matrices*.

Next, *non-normal form* matrices are denoted by A_{ω_i}, $d_N + 1 \leq i \leq p^N$.

Since the vector space spanned by normal form matrices is a subspace of the n^2-dimensional space $\mathbb{R}^{n\times n}$, the following lemma yields:

Lemma 4.2: The number d_N of normal form matrices is smaller than n^2.

Remark 4.3: Given a positive integer N, a necessary condition for the existence of a matrix P solution of the LMIs problem (3) is that the following LMIs problem admits a solution

$$\exists P = P^T \succ 0, \text{ such that } P \succ A_{\omega_i}^T P A_{\omega_i},\ \forall 1 \leq i \leq d_N. \tag{4}$$

Remark 4.3 has two consequences. On the one hand, if (4) has no solution, then the criterion of Proposition 3.3 for proving GUES does not hold. On the other hand, if (4) admits a matrix P as a solution such that $P \succ A_{\omega_i}^T P A_{\omega_i}$ for each non-normal form matrix A_{ω_i}, then the criterion of Proposition 3.3 holds.

Moreover, from Lemma 4.2, the number of normal form matrices is upper-bounded by n^2, so that the maximal number of LMI conditions of (4) does not depend on N. Nevertheless, it could happen that the problem for normal form matrices admits a solution which is not valid for (3).

We fix a positive integer N, and we denote as previously normal form matrices and non-normal form matrices by $A_{\omega_1}, \cdots, A_{\omega_{d_N}}$ and $A_{\omega_{d_N+1}}, \cdots, A_{\omega_{p^N}}$, respectively. For each non-normal form matrix A_{ω_ℓ}, let $\lambda_{\ell,1}, \cdots, \lambda_{\ell,d_N}$ be its coordinates in the basis of normal form matrices, so that we have

$$A_{\omega_\ell} = \sum_{k=1}^{d_N} \lambda_{\ell,k} A_{\omega_k}.$$

For $1 \leq i \leq d_N$, we define the real number:

$$m_i = \max_{1 \leq \ell \leq p^N} \left\{ \sum_{k=1}^{d_N} \left|\lambda_{\ell,k}\right| \;\middle|\; \lambda_{\ell,i} \neq 0 \right\}. \tag{5}$$

Notice that $m_i \geq \lambda_{\ell,i}$ for all i, ℓ.

Before stating our main result, we need the following Lemma, for which a proof is given the Appendix:

Lemma 4.4: Let $P = P^T \succ 0$ be such that $P \succ m_i A_{\omega_i}^T P A_{\omega_i}$, for every $1 \leq i \leq d_N$. Then, we have

$$P \succ A_{\omega_\ell}^T P A_{\omega_\ell}, \; \forall 1 \leq \ell \leq p^N. \tag{6}$$

We obtain our main result, stated as follows:

Theorem 4.5: Let N be a positive integer and $A_{\omega_1}, \cdots, A_{\omega_{d_N}}$ be the normal form matrices. For $d_N + 1 \leq \ell \leq p^N$, let $\lambda_{\ell,k}$, $1 \leq k \leq d_N$, be the coefficients of the decomposition of the non-normal form matrix A_{ω_ℓ}. Consider the real number defined in (5):

$$m_i = \max_{1 \leq \ell \leq p^N} \left\{ \sum_{k=1}^{d_N} \left|\lambda_{\ell,k}\right| \;\middle|\; \lambda_{\ell,i} \neq 0 \right\}, \; \forall 1 \leq i \leq d_N.$$

If the LMIs problem

$$\exists P = P^T \succ 0, \; P \succ m_i A_{\omega_i}^T P A_{\omega_i}, \; 1 \leq i \leq d_N, \tag{7}$$

admits a solution, then the equilibrium point $x = 0$ of (1) is GUES.

Proof: Assume that (7) admits a solution. From (6), we have $P \succ A_{\omega_\ell}^T P A_{\omega_\ell}$ for every $1 \leq i \leq p^N$. Hence, P is a solution of (3), so that (1) is GUES from Proposition 3.3. ∎

Remark 4.6: We fix an integer N.

1) The existence of a solution for the LMIs problem (7) does not mean that the system (1) is quadratically stable, excepted if $N = 1$. A counter-example is given in Section V.
2) Theorem 4.5 shows that the stability of (1) is related to the quadratic stability of another discrete-time linear switched system, namely

$$z_{k+1} = A_{\omega_{\gamma_k}} z_k, \; \forall k \in \mathbb{N}, \; z_0 \in \mathbb{R}^n, \tag{8}$$

with the switching function $\gamma : \mathbb{N} \to \mathfrak{I}_{d_N}$. More precisely, when (7) admits a solution, (8) is quadratically stable and an estimation of the delay rate equals to $\sqrt{\delta}$, where

$$\delta = \max_{1 \leq i \leq d_N} \left\{ \frac{1}{m_i} \right\}. \tag{9}$$

V. Examples

In this section, we illustrate Theorem 4.5 with an example adapted from [21], [20]: we consider two matrices

$$A_1^c = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } A_2^c = \begin{pmatrix} -1 & -a \\ \frac{1}{a} & -1 \end{pmatrix},$$

where $a \in \mathbb{R}_{>0}$ and we consider the discrete-time switched linear system (1) with $p = 2$, $A_i = \exp\left(A_i^C T\right)$, $T = 1$ and $i \in \{1, \; 2\}$.

The following table was given in [20, Section V]:

	a=5	a=6	a=7	a=8	#LMI conditions
N=1	✓	-	-	-	2
N=3	✓	✓	-	-	9
N=8	✓	✓	✓	✓	257

The meaning is the following: for every box with a ✓, a matrix P for the corresponding LMIs problem (3) was obtained and for every box with $-$, no matrix was obtained [1].

Moreover, each value of N corresponds to the smallest one for which a matrix P was obtained for a new value of a: in particular, for $a = 7, \; 8$ and $4 \leq N \leq 7$, no matrix were obtained.

Given values $5 \leq a \leq 8$ and $2 \leq N \leq 8$, the number of LMI conditions of (7) is constant equal to 5. In particular, for $a = 8$ and $N = 8$, our method provides a positive definite matrix P with 5 LMI conditions instead of 257.

VI. Appendix

In this section, we prove Lemma 4.4.

For that, we use the following result, which relates convex hulls to LMIs problems:

Lemma 6.1: [22] Let $P = P^T \succ 0$ and $M_i \in \mathbb{R}^{n \times n}$, $1 \leq i \leq k$, such that $P \succ M_i^T P M_i$. If $M \in \mathbb{R}^{n \times n}$ is a matrix in the convex hull of the M_i's, then $P \succ M^T P M$.

Now, we prove Lemma 4.4.

Proof: [Lemma 4.4] For $1 \leq \ell \leq d_N$, (6) is a consequence of the following sequence of inequalities and equality:

$$P \succ m_\ell A_{\omega_\ell}^T P A_{\omega_\ell} \succ \lambda_{\ell,\ell} A_{\omega_\ell}^T P A_{\omega_\ell} = A_{\omega_\ell}^T P A_{\omega_\ell}.$$

[1] That does not mean that the LMIs problem does not admit any solution!

For $d_N+1 \leq \ell \leq d_N$, we let $n_\ell = \sum_{i=1}^{\ell} \mid \lambda_{\ell,i} \mid$ and we consider the following decomposition:

$$\begin{aligned} A_{\omega_\ell} &= \sum_{i=1}^{d_N} \lambda_{\ell,i} A_{\omega_i} \\ &= \sum_{i=1}^{d_N} \frac{\mid \lambda_{\ell,i} \mid}{n_\ell} \left(\operatorname{sign}\left(\lambda_{\ell,i}\right) n_\ell A_{\omega_i} \right). \end{aligned}$$

For $1 \leq i \leq d_N$ such that $\lambda_{\ell,i} \neq 0$, by definition of m_i in (5) and by definition of n_ℓ, we have $n_\ell A_{\omega_i} \preceq m_i A_{\omega_i}$, which implies $P \succ (n_\ell A_{\omega_i})^T P (n_\ell A_{\omega_i}) = \left(\operatorname{sign}\left(\lambda_{\ell,i}\right) n_\ell A_{\omega_i}\right)^T P \left(\operatorname{sign}\left(\lambda_{\ell,i}\right) n_\ell A_{\omega_i}\right)$. Hence, (6) follows from Lemma 6.1. ∎

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