Normal forms of matrix words for stability analysis of discrete-time switched linear systems

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***Abstract*— In this paper, we propose a new method for investigating the stability of discrete-time switched linear systems by means of linear algebra techniques. Exponential stability of such systems is equivalent to the existence of solutions of linear matrix inequalities indexed by matrix words (*i.e.* products of matrices of the sub-systems). Our method consists in using the link between this characterization of exponential stability and linear dependency among matrix words. In particular, we introduce a criterion to reduce drastically the number of linear matrix inequalities, by removing redundant ones. This is achieved by eliminating matrix words that depend on the others. From this criterion, we also relate exponential stability to quadratic stability of another switched system. An example is given to illustrate our methods.**

I. Introduction

In this article, we investigate some properties of discrete-time switched linear systems [1], [2]. Such systems consist of a family of linear systems with a rule that orchestrates the switching among them. Discrete-time switched linear systems are a popular model in various control domains. Among the application areas, we may cite networked control systems [3], systems with aperiodic sampling [4], discrete-time delay systems [5], [6], etc., they all can be modeled as switched linear systems. Despite the fact that switched systems have been intensively studied over the last two decades, the stability problem is still a complex open problem [7].

Various methods are available for studying the stability of discrete-time switched linear systems. Roughly speaking, there are stability criteria that have been proposed based on algebraic methods, such as the ones within the framework of the theory of Lie algebras [8], [9] or the ones based on joint spectral radii [10], [11]. Numerical approaches, usually based on Linear Matrix Inequalities (LMI) and Lyapunov functions were also given in different works. In this direction, stability criteria have been proposed by checking the existence of quasi-quadratic [12], [13], parameter dependent [14], path-dependent [15], non-monotonic [16], [17], [18], [19], [20], with an augmented state vector [21], composite quadratic [13], [22], [23] Lyapunov functions or using a Gaussian elimination procedure [24]. The aim of the present article is to propose a new method to analyze the stability of switched linear systems by investigating the structural and algebraic properties of some particular Lyapunov based criteria.

A remarkable result, that can be found in [17], provides a necessary and sufficient condition for a discrete-time switched linear system to be globally uniformly exponentially stable: it is equivalent to the existence of a positive integer N such that a particular problem consisting in p^N+1 LMI conditions admits a solution. From this condition, it follows the construction of the quasi-quadratic Lyapunov function: starting with $N=1$, the value of N is increased until a solution to the LMI problem can be found. From that, the corresponding quasi-quadratic Lyapunov function is derived. As mentioned in [22], a significant improvement of this method would be to reduce the computational complexity, since the number of LMI conditions grows exponentially when N increases.

In this work, starting with the condition proposed in [17], we use tools from linear algebra to introduce two sufficient conditions for proving stability of study discrete-time switched linear systems. Two criteria are provided. These two criteria are based on the remark that the matrices involved in the LMI problem from [17] may satisfy algebraic relations. So our first method consists in restricting the LMI conditions to a smaller problem, by reducing it to a basis of the linear span of all matrices involved in the initial LMI problem. Next, we must check if the LMI solution for the basis elements still holds for the remaining original matrices. The second method is also based on the basis: it consists in adding numerical constraints to the LMIs corresponding to basis elements. Thanks to this criterion, the remaining LMI conditions corresponding to the other matrices are automatically fulfilled. The main argument used in this second approach is that the convex decomposition of matrices does not bring new LMI constraints. Moreover, it should be stressed that for both proposed methods, the number of LMI conditions is upper-bounded by the dimension of the matrix space, which does not depend on N.

The paper is organized as follows. In Section II, some useful notions are given. In Section III, we recall notions about discrete-time switched linear systems, stability definition and its characterization in terms of LMIs given in [17]. Section IV contains our results: we present two

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novel methods for reducing the number of LMI conditions, the first one using linear algebra and the second one using convex decomposition. In Section V, the second method is illustrated with a numerical example. Section VI contains the proof of an intermediate lemma.

II. Preliminaries

Definition 2.1: Given an arbitrary set S, a *word* w over S is a finite formal concatenation of elements of S:

$$\omega = s_1 s_2 s_3 \dots s_N$$

where $s_j \in S$, for all $j \in \{1, \cdots, N\}$.

The set S is called an *alphabet*. We denote by S^* the set of words over S and by $S^{(N)}$ the set of words of S^* of length N for a given $N \in \mathbb{N}$, so that $S^{(0)}$ is reduced to the *empty word* and $S^{(1)} = S$.

Example 2.2: As an illustration, consider $S = \{a, b\}$. Then

$$\begin{aligned} S^{(2)} &= \{aa, ab, ba, bb\}, \\ S^{(3)} &= \{aaa, aab, aba, abb, baa, bab, bba, bbb\}. \end{aligned}$$

The notion of words can be naturally applied to the set of matrices, as in the following definition.

Definition 2.3: Let p, $n \in \mathbb{N} \setminus \{0\}$. Set $\mathcal{I}_p := \{1, \cdots, p\}$ and consider a set of matrices $\mathcal{A} = \{A_i \in \mathbb{R}^{n \times n} \mid i \in \mathcal{I}_p\}$. Given a word $\omega = i_{k-1} \cdots i_0 \in \mathcal{I}_p^*$ on the alphabet $\mathcal{I}_p$, the corresponding matrix word on $\mathcal{A}$ is denoted by

$$A_\omega = A_{i_{k-1}} \cdots A_{i_0} \in \mathcal{A}^*$$

where the concatenation of matrices stands for the matrix product. By convention, if $k = 0$, we let $A_\omega = \mathrm{Id}_n$ where Id_n denotes the identity matrix of order n.

Example 2.4: If $p = 2$, then $\mathcal{I}_2 = \{1, 2\}$. Consider $\mathcal{A} = \{A_1, A_2\}$ with

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The set of words of length 2 in $\mathcal{I}_2$ is $\mathcal{I}_2^{(2)} = \{11, 12, 21, 22\}$. So there are four matrix words of length 2 corresponding to the words in $\mathcal{I}_2^{(2)}$, they are:

$$\begin{aligned} A_{11} &= A_1 A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\ A_{12} &= A_1 A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\ A_{21} &= A_2 A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\ A_{22} &= A_2 A_2 = \mathrm{Id}_2 . \end{aligned}$$

III. Problem Statement

In this Section, a stability problem for a class of discrete-time switched linear systems is presented. Let n and p be two strictly positive integers and a set $\mathcal{A}$ of p matrices A_1, A_2, …, A_p. Consider the following switched linear system:

$$x_{k+1} = A_{\sigma_k} x_k, \ \forall k \in \mathbb{N}, \ x_0 \in \mathbb{R}^n, \tag{1}$$

where $x : \mathbb{N} \to \mathbb{R}^n$ represents the system state, $x_0 = x(0)$ the initial condition and $\sigma : \mathbb{N} \to \mathcal{I}_p$, $k \mapsto \sigma_k$ the switching function associated to the system. It is assumed that the switching function is not known.

For $k \in \mathbb{N}$, denote by $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}(\mathcal{I}_p) \subset \sigma(\mathbb{N})^*$ the set of k-length words formed by the image of the switching function σ starting with σ_0. So, an element of $\mathcal{S}^{(k)}$ is a word of the form $\sigma_{i_{k-1}} \dots \sigma_{i_1} \sigma_0$ with $\sigma_{i_j} = \sigma(i_j)$, $i_j \in \mathcal{I}_p$. For example, with the notation in Example 2.4 and a switching law σ defined by $\sigma(i) = 1$ if i is odd and 2 if i is even, for all $i \in \mathbb{N} \setminus \{0\}$, $\sigma_0 = 1$, the elements of $\mathcal{S}^{(2)}$ are 11, 21. This set is also called *the set of k-length switching paths*.

For the initial condition $x_0 \in \mathbb{R}^n$ and an infinite word $\omega = (\sigma_k)_{k=0}^\infty \in \mathcal{S}^\infty$, we consider the flow $(k, x_0) \mapsto \phi_\omega(k, x_0)$ defined by

$$\phi_\omega(k, x_0) = A_\omega(k) \ x_0,$$

where $A_\omega(k) = A_{\sigma_{k-1}} \cdots A_{\sigma_0} \in \mathbf{A}^* = \mathcal{A}^*$, so that $A_\omega(0) = \mathrm{Id}_n$, is called the *k-step transition matrix* of the discrete-time switched system associated to ω.

For an infinite word $\omega \in \mathcal{S}^\infty$, the solution of (1) associated to the initial condition x_0 is given by the sequence of points $(\phi_\omega(k, x_0))_{k=0}^\infty$.

Next, some notations are set and classical definition recalled. So, given a matrix $M \in \mathbb{R}^{n \times p}$, $M^\top \in \mathbb{R}^{p \times n}$ denotes the transpose of M. For $x = (x_1 \dots x_n)^\top \in \mathbb{R}^n$, we denote by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ the Euclidean vector norm on $\mathbb{R}^n$.

Definition 3.1: The equilibrium point $x = 0$ of the switched linear system (1) is *globally uniformly exponentially stable (GUES)* if there exists a constant $c > 0$ and $0 < \lambda < 1$ such that

$$\|\phi_\omega(\ell, x_0)\|^2 \leq c \lambda^\ell \|x_0\|^2 \tag{2}$$

holds for all initial conditions $x_0 \in \mathbb{R}^n$, all $\ell \in \mathbb{N}$ and all sequences of switching $\omega = (\sigma_k)_{k=0}^\infty \in \mathcal{S}^\infty$.

The following well-known stability criteria can be found for instance in [17]. Note that given a square symmetric matrix M, we write $M \succ 0$ or $M \prec 0$ accordingly if M is positive or negative definite.

Proposition 3.2: The following assertions are equivalent:

1) The equilibrium point $x = 0$ of (1) is GUES
2) There exists a positive integer N and a matrix $P = P^T \succ 0$ such that

$$P \succ A_\omega^T P A_\omega, \ \forall \omega \in \mathcal{S}^{(N)}. \tag{3}$$

As an immediate corollary, we obtain the following criterion for checking the GUES property:

Corollary 3.3: Let N be a positive integer. If the LMI problem (3) admits a solution, then the equilibrium point $x=0$ of (1) is GUES.

It is a straightforward conclusion that for a fixed N, the LMI criterion of Proposition 3.3 for checking the GUES property consists of p^N+1 LMI conditions. In Section IV, a method to simplify Proposition 3.3 is proposed. This new approach is based on linear algebra concepts and it aims to drastically reduce the number of LMI conditions.

IV. Conditions based on normal form of matrix words

In this section, a new algebraic method for analyzing the GUES property is presented. The main idea is to take into consideration linear dependency between matrix words, that is the elements of $\mathcal{A}^*$. As an illustration of this approach, we show in Theorem 4.5 that the number of LMI conditions in (3) may be reduced and become independent from the length of matrix words.

For that, matrix words $A_\omega \in \mathcal{A}^*$ are realized as elements of the vector space $\mathbb{R}^{n\times n}$. Since any vector space has a basis, it may exist a matrix word $A_{\omega_0} \in \mathcal{A}^*$ which is a linear combination of other matrix words of the same length, that is

$$A_{\omega_0} = \sum \mu_\omega A_\omega, \tag{4}$$

for scalars $\mu_\omega \in \mathbb{R}$. So, our method consists in rewriting the conditions (3) without taking into account the redundant inequalities corresponding to the words such as in (4). Moreover, we wish to consider only LMIs associated to matrix words A_ω that form a basis of the subspace of $\mathbb{R}^{n\times n}$ generated by all matrix words of the same length in $\mathcal{A}^*$ (those for which no equality of the form (4) holds). This is possible when the parameters μ_ω in (4) satisfy particular numerical constraints as we shall see in Theorem 4.5.

Given N, let us denote by $V_{\mathcal{A}}^{(N)}$ the subspace of $\mathbb{R}^{n\times n}$ spanned by the matrices A_ω, where $\omega \in \mathcal{I}_p^{(N)}$.

Definition 4.1: Let d_N be the dimension of $V_{\mathcal{A}}^{(N)}$ and let $A_{\omega_1},\cdots,A_{\omega_{d_N}} \in \mathcal{A}^*$, be a basis of $V_{\mathcal{A}}^{(N)}$ composed of matrix words, that is $\omega_i \in \mathcal{I}_p^{(N)}$. The matrices A_{ω_i} $(1 \le i \le d_N)$ will be called *normal form* matrices.

Next, *non-normal form* matrices are denoted by A_{ω_i}, for all $d_N+1 \le i \le p^N$.

Since the vector space spanned by normal form matrices is a subspace of the n^2-dimensional space $\mathbb{R}^{n\times n}$, the following lemma holds immediately:

Lemma 4.2: The number d_N of normal form matrices is smaller than n^2.

Remark 4.3: Given a positive integer N, a necessary condition for the existence of a matrix P solution of the LMIs problem (3) is that the following LMI problem admits a solution:

$$\text{there exists } P=P^T \succ 0, \text{ such that } P \succ A_{\omega_i}^T P A_{\omega_i},\ \forall 1 \le i \le d_N. \tag{5}$$

Remark 4.3 has two important consequences. On the one hand, if (5) has no solution, then the criterion of Proposition 3.3 proving the GUES property does not hold. On the other hand, if (5) admits a matrix P as a solution such that $P \succ A_{\omega_i}^T P A_{\omega_i}$ for each non-normal form matrix A_{ω_i}, then the criterion of Proposition 3.3 holds.

Moreover, from Lemma 4.2, the number of normal form matrices is upper-bounded by n^2, so that the maximal number of LMI conditions of (5) does not depend on N. However, a solution of (5) is not necessarily a solution of (3). Our objective is to introduce a LMI problem with the same number of conditions than (5) and for which any solution is automatically a solution of (3).

We fix a positive integer N, and we denote as in Definition 4.1, normal form matrices and non-normal form matrices by $A_{\omega_1},\cdots,A_{\omega_{d_N}}$ and $A_{\omega_{d_N+1}},\cdots,A_{\omega_{p^N}}$, respectively. For each non-normal form matrix A_{ω_ℓ}, let $\lambda_{\ell,1},\cdots,\lambda_{\ell,d_N}$ be its coordinates in the basis of normal form matrices, so that we have

$$A_{\omega_\ell} = \sum_{k=1}^{d_N} \lambda_{\ell,k} A_{\omega_k}. \tag{6}$$

For $1 \le i \le d_N$, we define the real number:

$$m_i = \max_{1\le \ell \le p^N} \left\{ \sum_{k=1}^{d_N} \left|\lambda_{\ell,k}\right| \;\middle|\; \lambda_{\ell,i} \ne 0 \right\}. \tag{7}$$

Notice that $m_i \ge \lambda_{\ell,i}$ for all i,ℓ.

Before stating our main result, we need the following Lemma, for which a proof is given the Appendix:

Lemma 4.4: Let $P=P^T \succ 0$ be such that $P \succ m_i A_{\omega_i}^T P A_{\omega_i}$, for every $1 \le i \le d_N$. Then, we have

$$P \succ A_{\omega_\ell}^T P A_{\omega_\ell},\ \forall 1 \le \ell \le p^N. \tag{8}$$

We obtain our main result, stated as follows:

Theorem 4.5: Consider the system (1). Let N be a positive integer and $A_{\omega_1},\cdots,A_{\omega_{d_N}}$ be the normal form matrices. For $d_N+1 \le \ell \le p^N$, let $\lambda_{\ell,k}$ $(1 \le k \le d_N)$ be the coefficients of the decomposition of the non-normal form matrix A_{ω_ℓ} (see (6)). Consider the real number defined in (7):

$$m_i = \max_{1\le \ell \le p^N} \left\{ \sum_{k=1}^{d_N} \left|\lambda_{\ell,k}\right| \;\middle|\; \lambda_{\ell,i} \ne 0 \right\},\ \forall 1 \le i \le d_N.$$

If the LMI problem

$$\exists P=P^T \succ 0,\ P \succ m_i A_{\omega_i}^T P A_{\omega_i},\ 1 \le i \le d_N, \tag{9}$$

admits a solution, then the equilibrium point $x=0$ of (1) is GUES.

Proof: Assume that (9) admits a solution. From (8), we have $P \succ A_{\omega_\ell}^T P A_{\omega_\ell}$ for every $1 \leq i \leq p^N$. Hence, P is a solution of (3), so that (1) is GUES from Proposition 3.3. ■

Remark 4.6: We fix an integer N.

1) The existence of a solution for the LMI problem (9) does not mean that the system (1) is quadratically stable, excepted if $N = 1$. A counter-example is given in Section V.
2) Theorem 4.5 shows that the stability of (1) is related to the quadratic stability of another discrete-time linear switched system, namely

$$z_{k+1} = A_{\omega_{\gamma_k}} z_k, \ \forall k \in \mathbb{N}, \ z_0 \in \mathbb{R}^n, \tag{10}$$

with the switching function $\gamma : \mathbb{N} \to \mathcal{I}_{d_N}$. More precisely, when (9) admits a solution, (10) is quadratically stable and an estimation of the decay rate equals to $\sqrt{\delta}$, where

$$\delta = \max_{1 \leq i \leq d_N} \left\{ \frac{1}{m_i} \right\}. \tag{11}$$

V. Examples

In this section, we illustrate Theorem 4.5 with an example adapted from [25], [22]: we consider two matrices

$$A_1^c = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad A_2^c = \begin{pmatrix} -1 & -a \\ \frac{1}{a} & -1 \end{pmatrix},$$

where $a \in \mathbb{R}_{>0}$ and we consider the discrete-time switched linear system (1) with $p = 2$ and $A_i = \exp(A_i^c T)$ with $T = 1$ for $i \in \mathcal{I}_2 = \{1, \ 2\}$. So $\mathbf{A} = \{A_1, A_2\}$.

The following table was given in [22, Section V]:

	a=5	a=6	a=7	a=8	#LMI conditions
N=1	✓	-	-	-	2
N=3	✓	✓	-	-	9
N=8	✓	✓	✓	✓	257

The meaning is the following: for every box with a ✓, a matrix P for the corresponding LMI problem (3) was obtained and for every box with $-$, no matrix was obtained [1].

Moreover, each value of N corresponds to the smallest one for which a matrix P was obtained for a new value of a: in particular, for $a = 7, \ 8$ and $4 \leq N \leq 7$, no matrix were obtained.

The method presented in the previous section consists in extracting a basis of the linear span of A_ω, for a fixed length of words. Given values $5 \leq a \leq 8$ and $2 \leq N \leq 8$, the matrices $A_{\omega 11}, A_{\omega 12}, A_{\omega 21}, A_{\omega 22}$, where the length of ω is $N-2$, form a basis of $\mathbb{R}^{2\times 2}$ since they are linearly independent [2]. Hence, the criterion of Theorem 4.5 requires to solve the 5 LMI conditions of (9).

[1] That does not mean that the LMI problem does not admit any solution!

[2] It was checked by a case analysis with Maple 2019.

In particular, for $a = 8$ and $N = 8$, our method provides a positive definite matrix P with 5 LMI conditions instead of 257. Moreover, for these particular choices of a and N, the values of the elements $m_1, \ldots, m_4$ (in Theorem 4.5) are:

$$m_1 = m_2 = m_3 = m_4 \sim 423.$$

VI. Appendix

In this section, we prove Lemma 4.4. For that, we use the following result, which relates convex hulls to LMI problems.

Leu us recall that given $\mathcal{P} = \{\mathbf{p_1}, \cdots, \mathbf{p_k}\} \subset \mathbb{R}^n$, the *convex hull* of $\mathcal{P}$ is the set of elements $\sum_{i=1}^k a_i \mathbf{p_i} \in \mathbb{R}^n$, where a_i are non-negative real numbers satisfying $\sum_{i=1}^k a_i = 1$.

Lemma 6.1: [26] Let $P = P^T \succ 0$ and $M_i \in \mathbb{R}^{n\times n}$, $1 \leq i \leq k$, such that $P \succ M_i^T P M_i$. If $M \in \mathbb{R}^{n\times n}$ is a matrix in the convex hull of the M_i's, then $P \succ M^T P M$.

Given $\lambda \in \mathbb{R}$, we denote by $\operatorname{sign}(\lambda)$ the sign of λ. Now, we prove Lemma 4.4.

Proof: [Lemma 4.4] For each non-normal form matrix A_{ω_ℓ}, recall that $\lambda_{\ell,1}, \cdots, \lambda_{\ell,d_N}$ denote its coordinates in the basis of normal form matrices $A_{\omega_1}, \cdots, A_{\omega_{d_N}}$. For $1 \leq \ell \leq d_N$, (8) is a consequence of the following sequence of inequalities and equality:

$$P \succ m_\ell A_{\omega_\ell}^T P A_{\omega_\ell} \succ \lambda_{\ell,\ell} A_{\omega_\ell}^T P A_{\omega_\ell} = A_{\omega_\ell}^T P A_{\omega_\ell}.$$

For $d_N + 1 \leq \ell \leq p^N$, we let $n_\ell = \sum_{i=1}^{d_N} |\lambda_{\ell,i}|$ and we consider the following decomposition:

$$\begin{aligned} A_{\omega_\ell} &= \sum_{i=1}^{d_N} \lambda_{\ell,i} A_{\omega_i} \\ &= \sum_{i=1}^{d_N} \frac{|\lambda_{\ell,i}|}{n_\ell} \left(\operatorname{sign}(\lambda_{\ell,i}) n_\ell A_{\omega_i} \right). \end{aligned}$$

For $1 \leq i \leq d_N$ such that $\lambda_{\ell,i} \neq 0$, by definition of m_i in (7) and by definition of n_ℓ, we have $n_\ell A_{\omega_i} \preceq m_i A_{\omega_i}$, which implies $P \succ (n_\ell A_{\omega_i})^T P (n_\ell A_{\omega_i}) = \left(\operatorname{sign}(\lambda_{\ell,i}) n_\ell A_{\omega_i}\right)^T P \left(\operatorname{sign}(\lambda_{\ell,i}) n_\ell A_{\omega_i}\right)$. Hence, (8) follows from Lemma 6.1. ■

Conclusion

In this paper, we have presented a stability analysis for discrete-time switched linear systems based on linear dependency between matrix words. In particular, we illustrated how this approach may be used to reduced the complexity of LMI systems for checking stability by adding numerical constraints to these LMI. This work admits at least the following two possible extensions. The first one consists in finding better numerical constraints for the LMI problem in order to have a more efficient criterion. The second one is to extend our approach by allowing relations between matrix words of different lengths. In particular, we could use the theory of noncommutative Gröbner [27] bases for constructing bases of the linear span of matrix words.

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