A simple way to explain undecidability

Stavros Tripakis

A simple way to explain undecidability

Stavros Tripakis
Northeastern University

March 3, 2019

1 Introduction

The undecidability of the halting problem for Turing machines is a cornerstone result in computer science [3]. Many students are exposed to rigorous proofs of this result in textbooks on computability theory, such as [1, 2]. These proofs rely on knowing what a Turing machine is and how it operates. This is of course necessary for a rigorous exposition of the undecidability result.

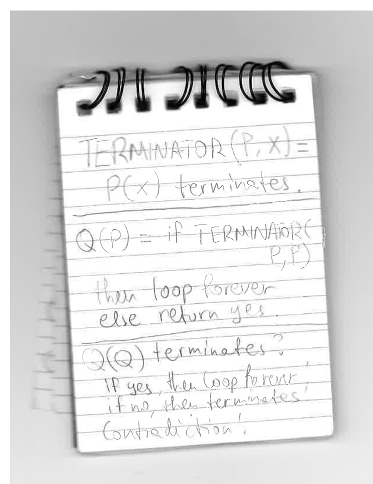
Here's a way to explain undecidability without having to explain Turing machines. This is by no means a rigorous proof, but I have found it a useful way to introduce students to the undecidability concept.

2 Undecidability without Turing machines

The argument is as follows. Suppose there exists a program, call it TERMINATOR, which can decide termination of other programs. TERMINATOR takes as input a program P and an input x and returns YES if P terminates on x, and NO if P does not terminate on x. TERMINATOR always terminates and gives a YES or NO answer.

Now build a new program Q as follows:

```
Q(P) := if (TERMINATOR(P,P) = YES)
          then loop forever
          else return YES.
```



Q takes as input a program P and runs TERMINATOR on P, with the input x also set to P. If TERMINATOR returns YES, then Q goes into an infinite loop. Otherwise, Q returns YES (the actual returned value is in fact not important).

Assuming that program TERMINATOR exists, Q is also a valid program (which calls TERMINATOR as a subroutine). So Q can be given as an input to itself, and we can ask: does $Q(Q)$ terminate? There are two cases:

- Either TERMINATOR(Q, Q) returns YES, which means that $Q(Q)$ terminates. But in that case, Q takes the `then` branch and loops forever, which means it does not terminate!

- Or TERMINATOR(Q, Q) returns NO, which means that $Q(Q)$ does not terminate. But in that case, Q takes the `else` branch and returns a value, which means that it does terminate!

In both cases, we reach a contradiction. Therefore, TERMINATOR cannot exist.

This "proof" is both simple and short: it fits in one page of a small note pad, as the picture above shows.

References

[1] H. Lewis and C. Papadimitriou. *Elements of the Theory of Computation, 2/e.* Prentice-Hall, 1997.

[2] Michael Sipser. *Introduction to the theory of computation.* 1997.

[3] A. M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 1936.