Asymptotically ω-Periodic Functions in the Stepanov Sense and Its Application for an Advanced Differential Equation with Piecewise Constant Argument in a Banach Space

William Dimbour, Solym Manou-Abi

ASYMPTOTICALLY ω-PERIODIC FUNCTIONS IN THE STEPANOV SENSE AND ITS APPLICATION FOR AN ADVANCED DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENT IN A BANACH SPACE

WILLIAM DIMBOUR, SOLYM MAWAKI MANOU-ABI

Abstract. In this paper, we give sufficient conditions for the existence and uniqueness of Asymptotically ω-periodic solutions for a nonlinear differential equation with piecewise constant argument in a Banach space via Asymptotically ω-periodic functions in the Stepanov sense. This is done using the Banach fixed point Theorem.

1. Introduction

We are concerned in this paper with the existence of asymptotically ω-periodicity of the following nonlinear differential equation with piecewise constant argument

$$\begin{cases} x'(t) = A(t)x(t) + \sum_{j=0}^{N} A_j(t)x([t+j]) + f(t, x([t]))dt, \\ x(0) = c_0, \end{cases} \tag{1}$$

where $c_0 \in \mathbb{X}$, $[\cdot]$ is the largest integer function, f is a continuous function on $\mathbb{R}^+ \times \mathbb{X}$ and $A(t)$ generates an exponentially stable evolutionnary process in $\mathbb{X}$.

The study of differential equations with piecewise constant argument (EPCA) is an important subject because these equations have the structure of continuous dynamical systems in intervals of unit length. Therefore they combine the properties of both differential and difference equations. There have been many papers studying EPCA, see for instance [11], [12], [13], [14], [15] and the references therein. The study of the existence of asymptotically ω-periodic solutions is one of the most attracting topics in the qualitative theory due to its applications in mathematical biology, control theory, physics. Some concepts generalise asymptotically ω-periodic functions. It is the case of S-asymptotically ω-periodic functions ([4],[5],[6],[9]), S-asymptotically ω-periodic functions in the Stepanov sense ([3],[17]) and asymptotically ω-periodic function in the Stepanov sense ([16]). S-asymptotically ω-periodic

2010 *Mathematics Subject Classification*: 34K05; 34A12; 34A40.

Key words and phrases: Asymptotically ω-periodic functions, differential equations with piecewise constant argument, evolutionnary process.

functions have properties similar to those of periodic functions, but the theory of S-asymptotically ω-periodic functions has the advantage to easily allowing the consideration of initial distortions to periodicity. S-asymptotically ω-periodic functions has been introduced by Henriquez et al. in [5, 8]. In [1], the concept of S-asymptotically ω-periodic in the Stepanov sense was introduced and the application to semilinear first-order abstract differential equations were studied. In [3], the authors show the existence of a functions wich is not S-asymptotically ω-periodic, but wich is S-asymptotically ω-periodic in the Stepanov sense. They study the existence and uniqueness of S-asymptotically ω-periodic of the following differential equation with piecewise constant argument

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x([t])), \\ x(0) = c_0, \end{cases}$$

considering S-asymptotically ω-periodic functions in the Stepanov sense. In [16], Xie and Zhang characterize the asymptotically ω-periodic functions in the Stepanov sense. They apply a criteria obtained to investigate the existence and uniqueness of asymptotically ω-periodic mild solutions to semilinear fractional integro-differential equations with Stepanov asymptotically ω-periodic coefficients.

Recently, N'Guérékata and Valmorin introduced the concept of asymptotically antiperiodic functions and studies their properties in [7]. In this paper, they also studied the existence os asymptotically antiperiodic mild solution of the following semilinear integro-differential equation in a Banach space $\mathbb{X}$

$$u'(t) = Au(t) + \int_{\infty}^{t} a(t-s)Au(s)ds + f(t, Cu(t))$$

where $C : \mathbb{X} \to \mathbb{X}$ is a bounded linear operator, A is a closed linear operator defined in a Bancah space $\mathbb{X}$, and $a \in L^1_{loc}(\mathbb{R}^+)$ is a scalar-valued kernel. In [2], the existence and uniqueness of asymptotically ω-antiperiodic solution for the following nonlinear differential equation with piecewise constant argument

$$\begin{cases} x'(t) = Ax(t) + A_0 x([t]) + f(t, x([t]))dt, \\ x(0) = c_0, \end{cases}$$

is studied, when ω is an integer. Motivated by the work presented in [2], [3] and [16], we investigate the existence of asymptotically ω-periodic solutions for the equation (1), when ω is an integer.

This paper is organized as follows. In Section 2, we recall the concepts of asymptotically ω-periodic functions, asymptotically ω-periodic functions in the Stepanov sense and their basic properties. In Section 3, we present some results showing the existence of function wich are not asymptotically ω-periodic but asymptotically ω-periodic in the Stepanov sense. In section 4, we study the existence and uniqueness of asymptotically ω-periodic solution of the equation (1).

2. Preliminaries

Let $\mathbb{X}$ be a Banach space. The space $BC(\mathbb{R}^+, \mathbb{X})$ of the continuous bounded functions from $\mathbb{R}^+$ into $\mathbb{X}$, endowed with the norm $\|f\|_\infty := \sup_{t\geq 0} \|f(t)\|$, is a Banach space. Set $C_0(\mathbb{R}^+, \mathbb{X}) = \{f \in BC(\mathbb{R}^+, \mathbb{X}) : \lim_{t\to\infty} f(t) = 0\}$ and $P_\omega(\mathbb{R}^+, \mathbb{X}) = \{f \in BC(\mathbb{R}^+, \mathbb{X}) : \ f \text{ is periodic}\}$.

Definition 2.1. *A function $f \in BC(\mathbb{R}^+, \mathbb{X})$ is said to be asymptotically ω-periodic if it can be expressed as $f = g + h$, where $g \in P_\omega(\mathbb{R}^+, \mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$. The collection of such function will be denoted by $AP_\omega(\mathbb{R}^+, \mathbb{X})$.*

Theorem 2.1. [16] *Let $f \in BC(\mathbb{R}^+, \mathbb{X})$ and $\omega > 0$. Then the following statements are equivalent:*

(1) $f \in AP_\omega(\mathbb{R}^+, \mathbb{X})$
(2) $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *uniform on* $\mathbb{R}^+$*;*
(3) $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *uniformly on compact subset of* $\mathbb{R}^+$*;*
(4) $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *is well defined for each* $t \in \mathbb{R}^+$ *and* $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *uniformly on* $[0, \omega]$.

Let $p \in [1, \infty[$. The space $BS^p(\mathbb{R}^+, \mathbb{X})$ of all Stepanov bounded functions, with the exponent p,consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{X}$ such that $f^b \in \mathbb{L}^\infty(\mathbb{R}, L^p([0, 1]; \mathbb{X}))$, where f^b is the Bochner transform of f defined by $f^b(t, s) := f(t+s), t \in \mathbb{R}^+, s \in [0, 1]$. $BS^p(\mathbb{R}^+, X)$ is a Banach space with the norm

$$||f||_{S^p} = ||f^b||_{\mathbb{L}^\infty(\mathbb{R}^+,L^p)} = \sup_{t\in\mathbb{R}^+} \Big(\int_t^{t+1} ||f(\tau)||^p d\tau \Big).$$

It is obvious that $L^p(\mathbb{R}, \mathbb{X}) \subset BS^p(\mathbb{R}, \mathbb{X}) \subset L^p_{loc}(\mathbb{R}, \mathbb{X})$ and $BS^p(\mathbb{R}, \mathbb{X}) \subset BS^q(\mathbb{R}, \mathbb{X})$ for $p \geq q \geq 1$. Define the subspaces of $BS^p(\mathbb{R}^+\mathbb{X})$ by

$$S^pP_\omega(\mathbb{R}^+, X) = \left\{ f \in BS^p(\mathbb{R}^+, \mathbb{X}) : \int_t^{t+1} ||f(s + \omega) - f(s)||^p ds = 0, t \in \mathbb{R}^+ \right\}$$

and

$$BS^p_0(\mathbb{R}^+, \mathbb{X}) = \left\{ f \in BS^p(\mathbb{R}^+, \mathbb{X}) : \lim_{t\to\infty} \int_t^{t+1} ||f(s)||^p ds = 0 \right\}.$$

Definition 2.2. [16] *A function $f \in BS^p(\mathbb{R}^+, \mathbb{X})$ is called asymptotically ω-periodic in the Stepanov sense if it can be expressed as $f = g + h$, where $g \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ and $h \in BS^p_0(\mathbb{R}^+, \mathbb{X})$. The collection of such functions will be denoted by $S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$.*

Definition 2.3. [16] *A function $f \in BS^p(\mathbb{R}^+\times\mathbb{X}, \mathbb{X})$ with $f(t, x) \in L^p_{loc}(\mathbb{R}^+, \mathbb{X})$ for each $x \in \mathbb{X}$ is said to be asymptotically ω-periodic in the Stepanov sense uniformly on bounded sets of $\mathbb{X}$ if there exists a function $g : \mathbb{R}_+ \times \mathbb{X} \to \mathbb{X}$*

with $g(t,x) \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ for each $x \in \mathbb{X}$ such that for every bounded set $K \subset \mathbb{X}$ we have

$$\Big(\int_t^{t+1} ||f(s+n\omega,x)-g(s,x)||^p\Big)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$ uniformly for $x \in K$. The collection of such functions will be denoted by $S^pAP_\omega(\mathbb{R}^+\times\mathbb{X},\mathbb{X})$.

Theorem 2.2. [16] *Let $f \in L^p_{loc}(\mathbb{R}^+,\mathbb{X})$ and $\omega > 0$. Then the following statements are equivalent:*

(1) $f \in S^pAP_\omega(\mathbb{R}^+,\mathbb{X})$
(2) *There exists a function $g \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ such that $\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds \to 0$ as $n \to \infty$ uniformly for $t \in \mathbb{R}^+$;*
(3) *There exists a function $g \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ such that $\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds \to 0$ as $n \to \infty$ pointwise for $t \in \mathbb{R}^+$.*

Lemma 2.3. [16] *Suppose $f \in S^pAP_\omega(\mathbb{R}^+,\mathbb{X})$, $f = g + h$ where $g \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ and $h \in BS^p_0(\mathbb{R}^+,\mathbb{X})$. Let $\omega = n_0 + \theta$, where $n_0 \in \mathbb{N}$ and $\theta \in (0,1)$. then the following statements are true.*

(1) $\int_t^{t+\omega} ||f(s)||ds \leq (n_0+1)||f||_{S^p}$ *for each $t \in \mathbb{R}^+$;*
(2) $\int_t^{t+\omega} ||g(s+m\omega)-g(s)|| = 0$ *for each $t \in \mathbb{R}^+$ and any $m \in \mathbb{N}$;*
(3) $\lim\limits_{n\to\infty}\displaystyle\int_t^{t+\omega} ||h(s+n)||ds = 0$ *uniformly for $t \in \mathbb{R}^+$.*

3. Properties of Asymptotically ω-periodic functions in the Stepanov sense

In this section we study some qualitative properties of Asymptotically ω-periodic functions in the Stepanov sense.

Proposition 3.1. *Let $u \in AP_\omega(\mathbb{R}^+,\mathbb{X})$ where $\omega \in \mathbb{N}^*$. Then the function $t \to u([t+k])$, where $k \in \mathbb{N}$ is Asymptotically ω-periodic in the Stepanov sense but is not Asymptotically ω-periodic.*

Proof. Since $u \in AP_\omega(\mathbb{R}^+,\mathbb{X})$, we can write $u = v + h$, where $v \in P_\omega(\mathbb{R}^+,\mathbb{X})$ and $h \in C_0(\mathbb{R}^+,\mathbb{X})$. We observe that

$$\begin{aligned} v([t+k+\omega]) &= v([t+k]+\omega) \\ &= v([t+k]). \end{aligned}$$

The function $t \to v([t+k])$ is not continuous. Therefore $t \to v([t+k])$ can not be ω-periodic. However, since $s \to v([s+k+\omega])-v([s+k])$ is a step function, we deduce so that $t \to v([t+k]) \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})\backslash P_\omega(\mathbb{R}^+,\mathbb{X})$. Since $h \in C_0(\mathbb{R}^+,\mathbb{X})$ then $\lim\limits_{t\to\infty} h([t+k]) = 0$, but $t \to h([t+k]) \notin C_0(\mathbb{R}^+,\mathbb{X})$ because this function is not continous. However, since the function $t \to h([t+k])$ is a step function, we deduce so that $t \to h([t+k]) \in BS^p_0(\mathbb{R}^+,\mathbb{X})\backslash C_0(\mathbb{R}^+,\mathbb{X})$. $\square$

Example 3.1. *Let the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(t) = g(t) + h(t)$ for each $t \in \mathbb{R}^+$, where $f(t) = \sin(\pi[t])$ and $h(t) = \frac{1}{[t]}$. Then we have $f \in S^pAP_\omega(\mathbb{R}^+, \mathbb{R})\backslash AP_\omega(\mathbb{R}^+, \mathbb{R})$, where $\omega = 2n$ and $n \in \mathbb{N}^*$.*

Theorem 3.2. *Let $\omega \in \mathbb{N}^*$. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a continuous function such that:*

(i) $\forall\ (t, x) \in \mathbb{R} \times \mathbb{X},\ f(t + \omega, x) = f(t, x)$;
(ii) $\exists L_f > 0,\ \forall(t, x) \in \mathbb{R} \times \mathbb{X}$

$$||f(t, x) - f(t, y)|| \leq L_f ||x - y||$$

If $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$, then the function $t \to f(t, u([t]))$ is Asymptotically ω-periodic in the Stepanov sense but is not Asymptotically ω-periodic.

Proof. Since $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$, then $u = v + l$, with $v \in P_\omega(\mathbb{R}^+, \mathbb{X})$ and $l \in C_0(\mathbb{R}^+, \mathbb{X})$. We have $f(t, u([t])) = f(t, v([t])) + h(t)$ where $h(t) = f(t, u([t])) - f(t, v([t]))$ is a piecewise continuous function wich satisfies

$$||h(t)|| \leq L_f ||l([t])||.$$

Since $l \in C_0(\mathbb{R}^+, \mathbb{X})$, then $\lim\limits_{t\to\infty} l([t]) = 0$. We deduce so that $\lim\limits_{t\to\infty} h(t) = 0$. Moreover $f(t+\omega, v([t+\omega])) = f(t, v([t]))$ because $f(t, v([t+\omega])) = f(t, v([t]+\omega))$. Since the function $t \to f(t, v([t]))$ is not continuous on $\mathbb{R}^+$, it can't be ω-periodic. However, since $f(t + \omega, v([t + \omega])) = f(t, v([t]))$ and that the function $t \to f(t, v([t]))$ is piecewise constant on $\mathbb{R}^+$, we deduce so that $t \to f(t, v([t])) \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})\backslash AP_\omega(\mathbb{R}^+, \mathbb{R})$.
Since the function $t \to h(t)$ is not continuous, then $h \notin C_0(\mathbb{R}^+, \mathbb{X})$. We have also $\lim_{t\to\infty} h(t) = 0$. Then $\forall\epsilon^{1/p} > 0, \exists T_\epsilon > 0, t > T_\epsilon \Rightarrow ||h(t)|| < \epsilon^{1/p}$. The function $t \to h(t)$ is a piecewise continuous function and it is measurable on $\mathbb{R}^+$. Then for $t \geq [T_\epsilon] + 1$, we have

$$\begin{aligned} \int_t^{t+1} ||h(s)||^p &\leq \int_t^{t+1} \epsilon\ ds \\ &\leq \epsilon. \end{aligned}$$

This means that $h \in BS_0^p(\mathbb{R}^+, \mathbb{X})\backslash C_0(\mathbb{R}^+, \mathbb{X})$.

□

Lemma 3.3. *Let $\omega \in \mathbb{N}^*$. Assume that $f \in S^pAP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ and assume that f satisfies a Lipschitz condition in $\mathbb{X}$ uniformly in $t \in \mathbb{R}^+$:*

$$||f(t, x) - f(t, y)|| \leq L||x - y||$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant. Let $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$. Then the function $F : \mathbb{R}^+ \to \mathbb{X}$ defined by $F(t) = f(t, u([t]))$ is asymptotically ω-periodic in the Stepanov sense.

Proof. Since $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$, we can write $u = v + l$, where $v \in P_\omega(\mathbb{R}^+, \mathbb{X})$ and $l \in C_0(\mathbb{R}^+, \mathbb{X})$. The function $u([t]) = v([t]) + l([t]) \in S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$ according to the proposition 3.1. In particular, we have $t \to u([t]) \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ and $t \to l([t]) \in BS^p_0(\mathbb{R}^+, \mathbb{X})$. By theorem 2.2, we obtain

$$\Big(\int_t^{t+1} ||u([s+n\omega]) - v([s])||^p ds\Big) \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$.

Denote $K = \overline{\{v([t]) : t \in \mathbb{R}^+\}}$; K is a bounded set. Since f is asymptotically ω-periodic in the Stepanov sense uniformly on bounded sets of $\mathbb{X}$, there exists a function $g : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ with $g(t, x) \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ for each $x \in \mathbb{X}$ such that for every bounded set $K \subset \mathbb{X}$ we have

$$\Big(\int_t^{t+1} ||f(s+n\omega, x) - g(s, x)||^p\Big)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$ uniformly for $x \in K$.

We observe that

$$\begin{aligned}
& \Big(\int_t^{t+1} ||f(s+n\omega, u([s+n\omega])) - g(s, v([s]))||^p ds\Big)^{\frac{1}{p}} \\
\leq\ & \Big(\int_t^{t+1} ||f(s+n\omega, u([s+n\omega])) - f(s+n\omega, v([s]))||^p ds\Big)^{\frac{1}{p}} \\
+\ & \Big(\int_t^{t+1} ||f(s+n\omega, v([s])) - g(s, v([s]))||^p ds\Big)^{\frac{1}{p}} \\
\leq\ & L\Big(\int_t^{t+1} ||u([s+n\omega]) - v([s])||^p ds\Big)^{\frac{1}{p}} \\
+\ & \Big(\int_t^{t+1} ||f(s+n\omega, v([s])) - g(s, v([s]))||^p ds\Big)^{\frac{1}{p}}
\end{aligned}$$

Hence, we deduce so that

$$\Big(\int_t^{t+1} ||f(s+n\omega, u([s+n\omega])) - g(s, v([s]))||^p ds\Big)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$. By Theorem 2.2, we deduce that $F \in S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$.

□

4. Main Results

Definition 4.1. *A solution of (1) on $\mathbb{R}^+$ is a function $x(t)$ that satisfies the conditions:*

(1) *$x(t)$ is continuous on $\mathbb{R}^+$.*
(2) *The derivative $x'(t)$ exists at each point $t \in \mathbb{R}^+$, with possible exception of the points $[t] \in \mathbb{R}^+$ where one-sided derivatives exists.*
(3) *The equation (1) is satisfied on each interval $[n, n+1[$ with $n \in \mathbb{N}$.*

Now we make the following hypthesis:

(H1) : The function $f \in S^p AP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ and satisfies a Lipschitz condition in $\mathbb{X}$ uniformly in $t \in \mathbb{R}^+$:

$$||f(t,x) - f(t,y)|| \leq L||x-y||$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant.

We assume that $A(t)$ generates an evolutionary process $(U(t,s))_{t\geq s}$ in $\mathbb{X}$. Then the function g defined by $g(s) = U(t,s)x(s)$, where x is a solution of (1), is differentiable for $s < t$.

$$\begin{aligned}
\frac{dg(s)}{ds} &= -A(s)U(t,s)x(s) + U(t,s)\frac{d\,x(s)}{ds} \\
&= -A(s)U(t,s)x(s) + U(t,s)A(s)x(s) \\
&+ \sum_{j=0}^{N} U(t,s)\,A_j(s)x([t+s]) + U(t,s)f(s,x([s])) \\
&= \sum_{j=0}^{N} U(t,s)A_j(s)x([t+s]) + U(t,s)f(s,x([s]))
\end{aligned}$$

.

$$\frac{dg(s)}{ds} = \sum_{j=0}^{N} A_j(s)x([t+s]) + U(t,s)f(s,x([s])). \tag{2}$$

The function $x([s])$ is a step function. Therefore $\sum_{j=0}^{N} U(t,s)A_j(s)x([t+s])$ is integrable on $[0,t[$. By **(H1)**, $f(s,x([s]))$ is piecewise continuous. Therefore $f(s,x([s]))$ is integrable on $[0,t]$ where $t \in \mathbb{R}^+$. Integrating (2) on $[0,t]$ we obtain that

$$x(t) - U(t,0)c_0 = \sum_{j=0}^{N} \int_0^t U(t,s)A_j(s)x([s+j])ds + \int_0^t U(t,s)f(s,x([s]))ds.$$

Therefore, we define

Definition 4.2. *We assume* **(H1)** *is satisfied and that $A(t)$ generates an evolutionary process $(U(t,s))_{t\geq s}$ in $\mathbb{X}$. The continuous function x given by*

$$x(t) = U(t,0)c_0 + \sum_{j=0}^{N} \int_0^t U(t,s)A_j(s)x([s+j])ds + \int_0^t U(t,s)f(s,x([s]))ds$$

is called the mild solution of equation (1).

Now we make the following hypothesis.

(H2): $A(t)$ generates an exponentially stable evolutionnary process $(U(t,s))_{t\geq s}$ in $\mathbb{X}$, that is, a two-parameter family of bounded linear operators that satisfies the following conditions:

1. $U(t,t) = I$ for all $t \geq 0$ where I is the identity operator.
2. $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$.
3. The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbb{X}$.
4. $U(t+\omega, s+\omega) = U(t,s)$ for all $t \geq s$ (ω-periodicity).
5. There exist $K > 0$ and $a > 0$ such that $||U(t,s)|| \leq Ke^{-a(t-s)}$ for $t \geq s$.

Theorem 4.1. *We assume that* **(H2)** *is satisfied and that $f \in S^pAP_\omega(\mathbb{R}_+, \mathbb{X})$. Then*

$$(\wedge f)(t) = \int_0^t U(t,s)f(s)ds \in AP_\omega(\mathbb{R}_+, \mathbb{X}), t \in \mathbb{R}^+.$$

Proof. Let $u(t) = \int_0^t U(t,s)f(s)ds$.
For $n \leq t \leq n+1,\ n \in \mathbb{N}$, we observe

$$\begin{aligned}
||u(t)|| &\leq \int_0^t ||U(t,s)f(s)||\, ds \\
&\leq \int_0^n ||U(t,s)f(s)||\, ds + \int_n^t ||U(t,s)f(s)||\, ds \\
&\leq \int_0^n Me^{-a(t-s)}||f(s)||\, ds + \int_n^t Me^{-a(t-s)}||f(s)||\, ds \\
&\leq \int_0^n Me^{-a(n-s)}||f(s)||\, ds + \int_n^t M||f(s)||\, ds \\
&\leq \sum_{k=0}^{n-1} \int_k^{k+1} Me^{-a(n-s)}||f(s)||\, ds + \int_n^t M||f(s)||\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{n-1}\int_k^{k+1} Me^{-a(n-k-1)}||f(s)||\,ds+\int_n^{n+1} M||f(s)||\,ds\\
&\leq \sum_{k=0}^{n-1} Me^{-a(n-k-1)}\int_k^{k+1}||f(s)||\,ds+M\int_n^{n+1}||f(s)||\,ds\\
&\leq \sum_{k=0}^{n-1} Me^{-a(n-k-1)}\Big(\int_k^{k+1}||f(s)||^p\,ds\Big)^{\frac{1}{p}}+M\Big(\int_n^{n+1}||f(s)||^p\,ds\Big)^{\frac{1}{p}}\\
&\leq \sum_{k=0}^{n-1} Me^{-a(n-k-1)}\,||f||_{S^p}+M\,||f||_{S^p}\\
&\leq M\Big(e^{-a(n-1)}+e^{-a(n-2)}+...+e^{-a}+1\Big)||f||_{S^p}+M||f||_{S^p}\\
&\leq M\Big(e^{-a(n-1)}+e^{-a(n-2)}+...+e^{-a}\Big)||f||_{S^p}+2M||f||_{S^p}\\
&\leq M\int_0^{n-1} e^{-at}dt\,||f||_{S^p}+2M||f||_{S^p}\\
&\leq M\int_0^{+\infty} e^{-at}dt\,||f||_{S^p}+2M||f||_{S^p}\\
&\leq \frac{M}{a}||f||_{S^p}+2M||f||_{S^p}.
\end{aligned}
$$

Therefore u is bounded. It is clear that u is continuous for each $t\in\mathbb{R}^+$. Therefore $u\in BC(\mathbb{R}^+,\mathbb{X})$. We observe that

$$
\begin{aligned}
u(t+n\omega) &= \int_0^{t+n\omega} U(t+n\omega,s)f(s)ds\\
&= \int_{-n\omega}^{t} U(t+n\omega,s+n\omega)f(s+n\omega)ds\\
&= \int_{-n\omega}^{t} U(t,s)f(s+n\omega)ds\\
&= \int_{-n\omega}^{0} U(t,s)f(s+n\omega)ds+\int_0^{t} U(t,s)f(s+n\omega)ds\\
&= I_1(t,n)+I_2(t,n).
\end{aligned}
$$

Next we will prove that $I_1(t,n)$ is a cauchy sequence in $\mathbb{X}$ for each $t \in \mathbb{R}^+$.
Let $\epsilon > 0$. For any $p \in \mathbb{N},\ n \in \mathbb{N}$, we observe that

$$\begin{aligned} I_1(t,n+p) - I_1(t,n) &= \int_{-(n+p)\omega}^{0} U(t,s)f(s+(n+p)\omega)ds - \int_{-n\omega}^{0} U(t,s)f(s+n\omega)ds \\ &= \int_{-(n+p)\omega}^{-n\omega} U(t,s)f(s+(n+p)\omega)ds \\ &+ \int_{-n\omega}^{0} U(t,s)\big(f(s+(n+p)\omega) - f(s+n\omega)\big)ds \\ &= I_3(t,n,p) + I_4(t,n,p) \end{aligned}$$

Now we estimate the term $I_3(t,n,p)$.

$$\begin{aligned} ||I_3(t,n,p)|| &\leq \int_{-(n+p)\omega}^{-n\omega} ||U(t,s)||\ ||f(s+(n+p)\omega)||ds \\ &\leq \int_{-(n+p)\omega}^{-n\omega} Me^{-a(t-s)}\ ||f(s+(n+p)\omega)||ds \\ &\leq \int_{0}^{p\omega} Me^{-a(t+s+n\omega)}\ ||f(p\omega - s)||ds \\ &= \sum_{k=0}^{p-1} \int_{k\omega}^{(k+1)\omega} Me^{-a(t+s+n\omega)}\ ||f(p\omega - s)||ds \\ &\leq \sum_{k=0}^{p-1} Me^{-a(t+k\omega+n\omega)} \int_{k\omega}^{(k+1)\omega} ||f(p\omega - s)||ds. \end{aligned}$$

By Lemma 2.3, we deduce so that

$$\begin{aligned} ||I_3(t,n,p)|| &\leq M(n_0+1)||f||_{S^p} \sum_{k=0}^{p-1} e^{-a(t+k\omega+n\omega)} \\ &\leq M(n_0+1)||f||_{S^p}\Big(e^{-a(t+n\omega)} + e^{-a(t+(n+1)\omega)} + ... + e^{-a(t+(n+p-1)\omega)}\Big) \\ &\leq M(n_0+1)||f||_{S^p} \int_{t+(n-1)\omega}^{t+(n+p-1)\omega} e^{-as}ds \\ &\leq M(n_0+1)||f||_{S^p} \int_{t+(n-1)\omega}^{\infty} e^{-as}ds \\ &\leq \frac{M(n_0+1)||f||_{S^p}\ e^{-a(t+(n-1)\omega)}}{a} \\ &\leq \frac{M(n_0+1)||f||_{S^p}\ e^{-a(n-1)\omega}}{a}. \end{aligned}$$

Hence, we deduce that there exists $N_1 \in \mathbb{N}$ such that $||I_3(t,n,p)|| < \epsilon$ when $n \geq N_1$ uniformly for $t \in \mathbb{R}^+$.

For $n \geq N_1$, we observe that

$$\begin{aligned} I_4(t,n,p) &= \int_{-N_1\omega}^{0} U(t,s)\big(f(s+(n+p)\omega) - f(s+n\omega)\big)ds \\ &+ \int_{-n\omega}^{-N_1\omega} U(t,s)\big(f(s+(n+p)\omega) - f(s+n\omega)\big)ds \\ &= I_5(t,n,p) + I_6(t,n,p) \end{aligned}$$

Then we have

$$\begin{aligned} ||I_5(t,n,p)|| &\leq \int_{-N_1\omega}^{0} ||U(t,s)||\, ||f(s+(n+p)\omega) - f(s+n\omega)||ds \\ &\leq \int_{-N_1\omega}^{0} Me^{-a(t-s)}\, ||f(s+(n+p)\omega) - f(s+n\omega)||ds \\ &\leq \int_{0}^{N_1\omega} Me^{-a(t+s)}\, ||f((n+p)\omega - s) - f(n\omega - s)||ds \\ &\leq M\sum_{k=0}^{N_1-1}\int_{k\omega}^{(k+1)\omega} ||f((n+p)\omega - s) - f(n\omega - s)||ds \end{aligned}$$

Since $f \in S^pAP_\omega(\mathbb{R}^+,\mathbb{X})$, it can be expressed as $f = g + h$, where $g \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$, and $h \in BS_0^p(\mathbb{R}^+,\mathbb{X})$. Then we can write

$$\begin{aligned} ||I_5(t,n,p)|| &\leq M\sum_{k=0}^{N_1-1}\Big[\int_{k\omega}^{(k+1)\omega} ||g((n+p)\omega - s) - g(n\omega - s)||ds \\ &+ \int_{k\omega}^{(k+1)\omega} ||h((n+p)\omega - s)||ds + \int_{k\omega}^{(k+1)\omega} h(n\omega - s)||ds\Big] \\ &\leq M\sum_{k=0}^{N_1-1}\Big[\int_{n\omega-(k+1)\omega}^{n\omega-k\omega} ||g(p\omega + s) - g(s)||ds \\ &+ \int_{0}^{\omega} ||h(s+((n+p)-(k+1))\omega)||ds + \int_{0}^{\omega} h(s+(n-(k+1)\omega)||ds\Big]. \end{aligned}$$

By Lemma 2.3(2), we get

$$\begin{aligned} ||I_5(t,n,p)|| &\leq M\sum_{k=0}^{N_1-1}\Big[\int_{0}^{\omega} ||h(s+((n+p)-(k+1))\omega)||ds \\ &+ \int_{0}^{\omega} h(s+(n-(k+1)\omega)||ds\Big]. \end{aligned}$$

By Lemma 2.3(3), we can choose $N_2 \in \mathbb{N}$ such that $N_2 \geq N_1$ and

$$M \sum_{k=0}^{N_1-1} \Big[\int_0^{\omega} ||h(s+((n+p)-(k+1))\omega)||ds + \int_0^{\omega} h(s+(n-(k+1)\omega)||ds \Big] < \epsilon$$

when $n \geq N_2$. Therefore $||I_5(t,n,p)|| < \epsilon$ $(n \geq N_2)$ uniformly for $t \in \mathbb{R}^+$.

Now we estimate the term $I_6(t,n,p)$:

$$\begin{aligned}
||I_6(t,n,p)|| &\leq \int_{-n\omega}^{-N_1\omega} ||U(t,s)||\ ||f(s+(n+p)\omega) - f(s+n\omega)||ds \\
&\leq \int_{-n\omega}^{-N_1\omega} Me^{-a(t-s)}\ ||f(s+(n+p)\omega) - f(s+n\omega)||ds \\
&\leq \int_{N_1\omega}^{n\omega} Me^{-a(t+s)}\ ||f((n+p)\omega - s) - f(n\omega - s)||ds \\
&\leq \sum_{k=N_1}^{n-1} \int_{k\omega}^{(k+1)\omega} Me^{-a(t+s)}\ ||f((n+p)\omega - s) - f(n\omega - s)||ds \\
&\leq \sum_{k=N_1}^{n-1} Me^{-a(t+k\omega)} \int_{k\omega}^{(k+1)\omega} ||f((n+p)\omega - s) - f(n\omega - s)||ds \\
&\leq \sum_{k=N_1}^{n-1} Me^{-a(t+k\omega)} \Big[\int_{k\omega}^{(k+1)\omega} ||f((n+p)\omega - s)||ds + \int_{k\omega}^{(k+1)\omega} ||f(n\omega - s)||ds \Big].
\end{aligned}$$

By Lemma 2.3(1), we obtain

$$\begin{aligned}
||I_6(t,n,p|| &\leq 2M(n_0+1)||f||_{S^p} \sum_{k=N_1}^{n-1} e^{-a(t+k\omega)} \\
&\leq 2M(n_0+1)||f||_{S^p} (e^{-a(t+N_1\omega)} + e^{-a(t+(N_1+1)\omega)} + ... + e^{-a(t+(n-1)\omega)}) \\
&\leq 2M(n_0+1)||f||_{S^p} \int_{t+(N_1-1)\ \omega}^{t+(n-1)\omega} e^{-as} ds \\
&\leq 2M(n_0+1)||f||_{S^p} \int_{t+(N_1-1)\ \omega}^{\infty} e^{-as} ds \\
&\leq 2M(n_0+1)||f||_{S^p} \frac{e^{-a(t+(N_1-1)\omega)}}{a} \\
&\leq 2\epsilon
\end{aligned}$$

uniformly for $t \in \mathbb{R}^+$.

Thus $||I_1(t,n+p)-I_1(t,n)|| \leq ||I_3(t,n,p)||+||I_5(t,n,p)||+||I_6(t,n,p)|| < 4\epsilon$ when $n \geq N_2$. Therefore $I_1(t,n)$ is a cauchy sequence and we denote $\lim_{n\to\infty} I_1(t,n)$ by $F(t)$ for each $t \in \mathbb{R}^+$. We also have that $h(t) = \lim_{n\to\infty} I_1(t,n)$ uniformly for $t \in \mathbb{R}^+$.

Now we consider the term $I_2(t,n)$. Since $f \in S^pAP_\omega(\mathbb{R}^+,\mathbb{X})$, $f = g+h$, where $g \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ and $h \in BS_0^p(\mathbb{R}^+,\mathbb{X})$, by Theorem 2.2(2), we have

$$\lim_{n\to\infty}\Big(\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}} = 0$$

uniformly for $t \in \mathbb{R}^+$. We also have $I_2(t,n)$, $\int_0^t U(t,s)g(s)ds \in BC(\mathbb{R}^+,\mathbb{X})$, wich is like the case of u. For $m \leq t < m+1$, $m \in \mathbb{N}$, we have

$$\begin{aligned}
||I_2(t,n)-\int_0^t U(t,s)g(s)ds|| &\leq \int_0^t ||U(t,s)||\,||f(s+n\omega)-g(s)||ds \\
&\leq \int_0^t Me^{-a(t-s)}\,||f(s+n\omega)-g(s)||ds \\
&\leq \int_0^m Me^{-a(t-s)}\,||f(s+n\omega)-g(s)||ds \\
&+ \int_m^t Me^{-a(t-s)}\,||f(s+n\omega)-g(s)||ds \\
&\leq \sum_{k=0}^{m-1}\int_k^{k+1} Me^{-a(t-s)}\,||f(s+n\omega)-g(s)||ds \\
&+ M\int_m^t ||f(s+n\omega)-g(s)||ds \\
&\leq \sum_{k=0}^{m-1} Me^{-a(t-(k+1))}\int_k^{k+1} ||f(s+n\omega)-g(s)||ds \\
&+ M\int_m^t ||f(s+n\omega)-g(s)||ds \\
&\leq \sum_{k=0}^{m-1} Me^{-a(t-(k+1))}\int_k^{k+1} ||f(s+n\omega)-g(s)||ds \\
&+ M\int_m^{m+1} ||f(s+n\omega)-g(s)||ds.
\end{aligned}$$

By the Holder inequality, we obtain

$$\begin{aligned}
||I_2(t,n) - \int_0^t U(t,s)g(s)ds|| &\leq \sum_{k=0}^{m-1} Me^{-a(t-(k+1))}\Big(\int_k^{k+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}} \\
&\quad + M\Big(\int_m^{m+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}} \\
&\leq M\big(e^{-a(t-1)} + e^{-a(t-2)} + ... + e^{-a(t-m)} + 1\big) \\
&\quad \times \sup_{t\in\mathbb{R}^+}\Big(\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}} \\
&\leq M\Big(\int_{t-2}^{t-1} e^{-as}ds + \int_{t-3}^{t-2} e^{-as}ds + ... + \int_{t-m}^{t-(m-1)} e^{-as}ds \\
&\quad + \int_0^{t-m} e^{-as}ds + 1\Big)\sup_{t\in\mathbb{R}^+}\Big(\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}} \\
&\leq M\Big(\int_{t-m}^{t-1} e^{-as}ds + 2\Big)\sup_{t\in\mathbb{R}^+}\Big(\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}} \\
&\leq M\Big(\int_0^{\infty} e^{-as}ds + 2\Big)\sup_{t\in\mathbb{R}^+}\Big(\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}} \\
&\leq M\Big(\frac{1}{a} + 2\Big)\sup_{t\in\mathbb{R}^+}\Big(\int_t^{t+1} ||f(s+n\omega)-g(s)||^p ds\Big)^{\frac{1}{p}}
\end{aligned}$$

Therefore, it follows that

$$\lim_{n\to\infty} I_2(t,n) = \int_0^t U(t,s)d(s)ds$$

uniformly for $t \in \mathbb{R}^+$.

We deduce so that

$$\lim_{n\to\infty} u(t+n\omega) = \lim_{n\to\infty} I_1(t,n) + \lim_{n\to\infty} I_2(t,n) = F(t) + \int_0^t U(t,s)g(s)ds$$

uniformly for $t \in \mathbb{R}^+$. By Theorem 2.1, we have $u \in AP_\omega(\mathbb{R}^+,\mathbb{X})$. □

Theorem 4.2. *Let $\omega \in \mathbb{N}^*$. We assume that* **(H2)** *is satisfied and that A_j is an asymptotically ω-periodic operator. Then*

$$(\wedge_j\ \phi)(t) = \int_0^t U(t,s)A_j(s)x([s+j])ds$$

maps $AP_\omega(\mathbb{R}^+,\mathbb{X})$ into itself.

Proof. Since $A_j \in AP_\omega(\mathbb{R}^+,\mathbb{X})$, we can write $A_j = u_j + h_j$ where $u_j \in P_\omega(\mathbb{R}^+,\mathbb{X})$ and $h \in C_0(\mathbb{R}^+,\mathbb{X})$. Similarly, since $\phi \in AP_\omega(\mathbb{R}^+,\mathbb{X})$, we can write $\phi([t+j]) = v([t+j]) + l([t+j])$, where $v([t+j+\omega]) = v([t+j])$ for all $t \geq 0$ and $\lim_{t\to\infty} l([t+j]) = 0$. We observe that

$$A_j(t)\phi([t+j]) = u_j(t)v([t+j]) + L(t)$$

where

$$L(t) = u_j(t)l([t+j]) + v([t+j]h_j(t) + h_j(t)l([t+j]).$$

Since $t \to u_j(t)v([t+j])$ is not continuous, this function can't belong to $P_\omega(\mathbb{R}^+, \mathbb{X})$. However, this piecewise continuous function satisfy

$$u_j(t+\omega)v([t+\omega+j]) = u_j(t)v([t+j]).$$

Therefore $t \to u_j(t)v([t+j])$ is ω-periodic in the Stepanov sense. We observe also that $\lim\limits_{t\to\infty} L(t) = 0$ because

$$||u_j(t)l([t+j]) + v([t+j]h_j(t) + h_j(t)l([t+j])||$$
$$\leq ||u_j||_\infty\, ||l([t+j])|| + ||h_j(t)||\, ||v||_\infty + ||h_j||_\infty ||l([t+j])||.$$

We deduce so that we deduce so that $t \to L(t) \in BS_0^p(\mathbb{R}^+, \mathbb{X}) \backslash C_0(\mathbb{R}^+, \mathbb{X})$. Therefore the function $t \to A_j(t)\phi([t+j])$ is asymptotically ω-periodic in the stepanov sense but is not asymptotically ω-periodic. According to the Theorem 4.1 the operator $\wedge_j$ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. □

Theorem 4.3. *Let $\omega \in \mathbb{N}^*$. We assume that the hypothesis* **(H1)** *and* **(H2)** *are satisfied. Then (1) has a unique Asymptotically ω-periodic mild solution provided*

$$\Theta := \frac{M\big(\sum_{j=0}^{N} ||A_j||_\infty + L\big)}{a} < 1.$$

Proof. We define the nonlinear operator Γ by the expression

$$\begin{aligned}(\Gamma\phi)(t) &= U(t,0)c_0 + \sum_{j=0}^{N}\int_0^t U(t,s)A_j(s)\phi([s+j])ds + \int_0^t U(t,s)f(s,\phi([s]))ds \\ &= U(t,0)c_0 + \sum_{j=0}^{N}(\wedge_j\,\phi)(t) + (\wedge^*\phi)(t)\end{aligned}$$

where

$$(\wedge_j\,\phi)(t) = \int_0^t U(t,s)A_j(s)\phi([s+j])ds$$

and

$$(\wedge^*\phi)(t) = \int_0^t U(t,s)f(s,\phi([s])).$$

According to the hypothesis **(H.2)**, we have

$$||U(t,0)|| \leq Me^{-at}$$

Therefore $\lim\limits_{t\to\infty} ||U(t,0)|| = 0$.

According to the Lemma 3.3 the function $t \to f(t, u(\big[t\big]))$ belongs to $S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$. According to the Theorem 4.1 the operator $\wedge^*$ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.

According to the Theorem 4.2 the operators $\wedge_j$ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. Therefore the operator Γ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.
We have

$$\begin{aligned}
||(\Gamma\phi)(t) - \Gamma\psi)(t)|| &= \left|\left|\sum_{j=0}^{N}\int_0^t U(t,s)A_j(s)\big(\phi([s+j]) - \psi([s+j])\big)ds\right|\right| \\
&+ \left|\left|\int_0^t U(t,s)\big(f(s,\phi([s])) - f(s,\psi([s]))\big)ds\right|\right| \\
&\leq \sum_{j=0}^{N}\int_0^t ||U(t,s)||\,||A_j(s)||\,||\phi([s+j]) - \psi([s+j])||ds \\
&+ \int_0^t ||U(t,s)||\,||f(s,\phi([s])) - f(s,\psi([s]))||ds \\
&\leq \sum_{j=0}^{N}\int_0^t ||U(t,s)||\,||A_j||_\infty\,||\phi([s+j]) - \psi([s+j])||ds \\
&+ L\int_0^t ||U(t,s)||\,||\phi([s]) - \psi([s])||ds \\
&\leq \sum_{j=0}^{N}||A_j||_\infty M\int_0^t e^{-a(t-s)}\,||\phi([s+j]) - \psi([s+j])||ds \\
&+ LM\int_0^t e^{-a(t-s)}\,||\phi([s]) - \psi([s])||ds \\
&\leq \sum_{j=0}^{N}||A_j||_\infty M\int_0^t e^{-a(t-s)}\,||\phi - \psi||_\infty ds \\
&+ LM\int_0^t e^{-a(t-s)}\,||\phi - \psi||_\infty ds \\
&\leq \sum_{j=0}^{N}||A_j||_\infty M\frac{1-e^{-at}}{a}||\phi - \psi||_\infty + LM\frac{1-e^{-at}}{a}||\phi - \psi||_\infty \\
&\leq \frac{M\big(\sum_{j=0}^{N}||A_j||_\infty + L\big)}{a}||\phi - \psi||_\infty.
\end{aligned}$$

Hence we have :

$$||\Gamma\phi - \Gamma\psi||_\infty \leq \frac{M\big(\sum_{j=0}^{N}||A_j||_\infty + L\big)}{a}||\phi - \psi||_\infty$$

which proves that Γ is a contraction and we conclude that Γ has a unique fixed point in SAP_ω. The proof is complete.

□

Example 4.1. *Consider the following heat equation with Dirichlet conditions:*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + q(t,x)u(t,x) + \alpha u([t],x) + f(t,u([t],x)), \\ u(t,0) = u(t,\pi) = 0, t \in \mathbb{R}^+, \\ u(0,x) = c_0, \end{cases} \tag{3}$$

where $c_0 \in L^2[0,\pi]$, $q \in \mathcal{C}(\mathbb{R}^+ \times [0,\pi], \mathbb{R})$, $q(t+\omega,x) = q(t,x)$ *for* $\omega \in \mathbb{N}$, *and there exists* $\gamma_0 > 0$ *such that* $q(t,x) \leq -\gamma_0$. *The function* $f \in S^pAP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ *and satisfies a Lipschitz condition in* $\mathbb{X}$ *uniformly in* $t \in \mathbb{R}^+$*:*

$$||f(t,x) - f(t,y)|| \leq L||x-y||$$

for all $x, y \in \mathbb{X}$ *and* $t \in \mathbb{R}^+$, *where* L *is a positive constant.*
Let $\mathbb{X} = L^2[0,\pi]$ *be endowed with it's natural topology. Define*

$$\mathcal{D}(A) = \{u \in L^2[0,\pi] \; such\, that \; u'' \in L^2[0,\pi] \\ and\, u(0) = u(\pi) = 0\} \\ Au = u'' \; for\, all\, u \in \mathcal{D}(A).$$

Let $\phi_n(t) = \sqrt{\frac{2}{\pi}}\sin(nt)$ *for all* $n \in \mathbb{N}$. ϕ_n *are eigenfunctions of the operator* $(A, \mathcal{D}(A))$ *with eigenvalues* $\lambda_n = -n^2$. A *is the infinitesimal generator of a semi-group* $T(t)$ *of the form*

$$T(t)\phi = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \phi, \phi_n \rangle \phi_n, \; \forall \phi \in L^2[0,\pi],$$

and

$$||T(t)|| \leq e^{-t}, \; for\, t \geq 0,$$

(see [10],[18]*).*
Now define $A(t)$ *by:*

$$\begin{cases} \mathcal{D}(A(t)) = \mathcal{D}(A) \\ A(t) = A + q(t,x). \end{cases}$$

Note that $A(t)$ *generates an evolutionnary process* $U(t,s)$ *of the form*

$$U(t,s) = T(t-s)e^{\int_s^t q(t,x)dx}.$$

Since $q(t,x) \leq -\gamma_0$, *we have*

$$||U(t,s)|| \leq e^{-(1+\gamma_0)(t-s)}.$$

Since $q(t+\omega,x) = q(t,x)$, *we conclude that* $U(t,s)$ *is a* ω*-periodic evolutionnary process exponentially stable.*
The equation (3) is of the form

$$\begin{cases} x'(t) = A(t)x(t) + A_0(t)x([t]) + f(t,x([t])), \\ x(0) = c_0. \end{cases}$$

By Theorem 4.3, we claim that

Theorem 4.4. *If* $L + |\alpha| < 1 + \gamma_0$ *then the equation (3) admits a unique mild solution* $u(t) \in AP_\omega(\mathbb{R}^+, \mathbb{X})$.

Acknowledgment

The authors are grateful to the referee for valuable comments and suggestions.

References

[1] C. Cuevas, J. de Souza. *Existence of S-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay. Nonlinear Analysos, 72, 2010.*

[2] W. Dimbour, V. Valmorin. *Asymptotically antiperiodic solutions for a nonlinear differential equation with piecewise constant argument in a Banach space.* Applied Mathematics, 7 (2016).

[3] W. Dimbour, S. Manou-Abi. *S-asymptotically ω-periodic solution for a nonlinear differential equation with piecewise constant argument via S-asymptotically ω-periodic functions in the Stepanov sense.* To appear

[4] W. Dimbour, G.Mophou and G.M. N'Guérékata. *S asymptotically ω-periodic solution for partial differential equations with finite delay. Electron.J.Differ.Equa. 2011, 1-12, 2011.*

[5] H. R. Henríquez, M. Pierri and P. Táboas. *On S asymptotically ω-periodic function on Banach spaces and applications. J. Math. Anal Appl. 343, 1119-1130, 2008.*

[6] H. R. Henríquez, M. Pierri and P. Táboas. *Existence of S-asymptotically ω-periodic solutions for abstract neutral equations. Bull. Aust. Math.Soc. 78, 365-382, 2008.*

[7] G.M. N'Guérékata, V. Valmorin. *Antiperiodic solutions of semilinear integrodifferential equations in Banach spaces.* Applied Mathematics and Computation, 218, 2012.

[8] S. Nicola, M. Pierri. *A note on S-asymptotically periodic functions.* Nonlinear Anaklysis, Real World Application, 10 (2009).

[9] M. Pierri. *On S-Asymptotically ω-periodic functions and applications. Nonlinear-Anal.75, 651-661, 2012.*

[10] Rong-Hua, He. *Stepanov-like pseudo-almost automorphic mild solutions for some abstract differential equations., Advances in Fixed Point Theory, 2(3),258-272, 2012*

[11] J.Wiener, *A Second-Order delay differential equation with multiple Periodic solutions*, Journal of Mathematical Analysis and Appliction, 229 (1999) 6596-676.

[12] J.Wiener, L.Debnath *Boundary Value Problems for the diffusion equation with piecewise continuous time delay*, Internat.J.Math.and Math.Sci., Vol.20 (1997) 187-195.

[13] J.Wiener, L.Debnath *A survey of partial differential equations with piecewise continuous arguments*, Internat.J.Math.and Math.Sci., Vol.18, No.2 (1995) 209-228.

[14] J.Wiener, V.Lakshmikantham, *Excitability of a second-order delay differential equation*, Nonlinear Analysis, Vol.38 (1999) 1-11.

[15] J.Wiener, *Generalized solutions of functional differential equations*, World Scientific (1999).

[16] R. Xie and C. Zhang. *Criteria of asymptotic ω-periodicity and their applications in a class of fractional differential equations. Advances in Difference Equations, 2015.*

[17] Z.Xia. *Asymptotically periodic of semilinear fractional integro-differential equations. Advances in Difference Equations, 1-19, 2014 .*

[18] Z.Xia. *Weighted pseudo asymptotically periodic mild solutions of evolutions equations. Acta Mathematica Sinica, 31(8), 1215-1232, 2015.*

William Dimbour, UMR Espace-Dev, Université de Guyane, Campus de Troubiran 97300 Cayenne Guyane (FWI)
E-mail address: `William.dimbour@espe-guyane.fr`

Solym mawaki manou-Abi, Centre Universitaire de Mayotte, Dpartement Sciences et Technologies Route Nationale 3 BP 53 - 97660 DEMBENI
E-mail address: solym-mawaki.manou-abi@univ-mayotte.fr