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Topological derivatives of leading- and second-order homogenized coefficients in bi-periodic media

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Abstract We derive the topological derivatives of the homogenized coefficients associated to a periodic material, with respect of the small size of a penetrable inhomogeneity introduced in the unit cell that defines such material. In the context of antiplane elasticity, this work extends existing results to (i) time-harmonic wave equation and (ii) second-order homogenized coefficients, whose contribution reflects the dispersive behavior of the material.

Keywords: homogenization, topological derivatives.

Introduction Consider an elastic material occupying a 2D domain and characterized by periodic shear modulus µ and density ρ. The unit cell Y has characteristic length ℓ. Under time-harmonic conditions, the antiplane displacement u satisfies the wave equation:

$$\nabla \cdot (\mu \nabla u) + \omega^2 \rho u = 0$$

For long-wavelength configurations (i.e. ℓ ≪ λ), two-scale periodic homogenization of this equation in terms of ε = ℓ/λ [4] leads to the equation satisfied by the mean field U:

$$\mu^0 : \nabla^2 U + \omega^2 \rho^0 U = -\varepsilon^2 \left[ \mu^2 : \nabla^4 U + \omega^2 \rho^2 : \nabla^2 U \right] + O(\varepsilon^4),$$

where the leading-order and second-order homogenized coefficients (µ^0, ρ^0, µ^2, ρ^2) are constant tensors and ∇^k U stands for the k-th gradient of U.

This study considers a periodic perturbation of this material, whereby a penetrable inhomogeneity B_a, of size a and shape B, characterized by contrasts (Δµ, Δρ) is introduced at point z ∈ Y (Fig. 1). Then, the leading-order expansion coefficients of ⟨µ^0, ρ^0, µ^2, ρ^2⟩ w.r.t. a, namely their topological derivatives, are computed, as in [3] for in-plane elastostatics.

Leading-order coefficients Let ⟨·⟩ = 1/|Y| \int_Y · denote an average on the unit cell. The homogenized density ρ^0 is defined by ρ^0 = ⟨ρ⟩, so that the perturbed coefficient ρ_a^0 and the topological derivative Dρ^0 are exactly given by:

$$\rho_a^0 = \rho^0 + a^2 |Y|^{-1} D\rho^0; \quad D\rho^0 = |B| D\rho.$$  

The homogenized shear modulus µ^0 is defined by µ^0 = ⟨µ(I + ∇P)⟩, where I is the identity tensor, the first cell function P [4] is the Y-periodic and zero-mean vector-valued solution of:

$$\nabla \cdot (\mu(I + ∇P)) = 0 \quad (1)$$

and the superscript -S means symmetrization w.r.t. all index permutations. Consequently, µ^0_a is computed as:

$$\mu_a^0 = \mu^0 + ⟨\mu \nabla p_a⟩ - S  + (\chi_B δ µ(I + ∇P_a))^S$$

where p_a := P_a - P is the perturbation of P.

The analysis of this perturbation is done by re-formulating problem (1) and its perturbed counterpart using domain integral equations [2]. With the help of the adjoint state method, it leads to the following leading-order expansion:

$$\mu_a^0 = \mu^0 + a^2 |Y|^{-1} D\mu^0(z) + o(a^2 |Y|^{-1}), \quad (2)$$

with the topological derivative D\mu^0 given by:

$$D\mu^0(z) = [(I + ∇P) : A · (I + ∇P)^T](z)$$

and A(z) = A(B, µ(z), Δµ) is the polarization tensor [1] associated to shape B and moduli µ(z) and µ(z) + Δµ. Under notational adjustments, this result is similar to [3]. For homogeneous background materials, in which case P = 0, it reduces to D\mu^0 = A as shown by [1].
Second-order coefficients The second-order homogenized density is defined by \( \rho^2 = (\rho Q)^{S} \), where the second cell function \( Q \) is the \( Y \)-periodic, zero-mean, tensor-valued solution of:

\[
\nabla \cdot (\mu(P \otimes I + \nabla Q)) = -\mu(I + \nabla P) + (\rho/\rho^0)\mu^0 \quad (3)
\]

Relying on the same integral equation framework, and with careful analysis of the influence of the source terms involving \( P_a \) when addressing the perturbed cell function \( Q_a \), we show that \( \rho^2_a \) has an expansion of the same form as (2), with its topological derivative \( \mathcal{D}\rho^2 \) given by:

\[
\mathcal{D}\rho^2(z) = \left[ (I + \nabla P) \cdot A \cdot \left( \beta I + \nabla \hat{X}[\beta] \right) \right]^{T} - (P \otimes I + \nabla Q) \cdot A \cdot \nabla \beta
- \left( \mathcal{D}\mu^0 - \left( \mathcal{D}\rho^0/\rho^0 \right)\mu^0 \right) \left( \rho(\beta/\rho^0) \right)
- \mathcal{D}\rho^3 \left( (\beta/\rho^0)\mu^0 - Q \right)^{S}(z). \quad (4)
\]

The above expression features (i) various combinations of the previously computed cell solutions and topological derivatives and (ii) two new adjoint fields \( \beta \) and \( X[\beta] \) defined as the \( (Y \)-periodic, zero-mean) solutions of:

\[
\nabla \cdot (\mu \nabla \beta) = -\rho - \rho^0
\]

and \( \nabla \cdot (\mu(\beta I + \nabla \hat{X}[\beta])) = -\mu \nabla \beta \).

In particular, all the fields involved in (4) solve problems posed on the unperturbed cell.

The second-order homogenized shear modulus is defined by \( \mu^2 = (\mu(Q \otimes I + \nabla R))^S \) in terms of \( Q \) and a third cell function \( R \). Once again, an analysis of the problems satisfied by \( R \) and \( R_a \) is conducted. As a result, \( \mu^2_a \) is found to have an expansion similar to (2), and its topological derivative \( \mathcal{D}\mu^2 \) (not shown here for brevity) is expressed in terms of the cell solutions \( (P, Q, R) \) and the previously determined topological derivatives \( (\mathcal{D}\rho^0, \mathcal{D}\mu^0, \mathcal{D}\rho^2) \).

Perspectives. The obtained expansions of the homogenized coefficients are useful on their own right, e.g. for computing quickly an approximation of the properties of a perturbed periodic material for several trial inhomogeneity locations \( z \) without solving the new cell problems. As an example, an approximation of \( \mu^0 \) is obtained by neglecting the remainder in (2), as illustrated on Fig. 2 for a chessboard-like cell.

![Figure 2: Relative error on shear modulus](image)

However, as already intended in [3], the main usefulness of such expansions occurs for optimizing a periodic structure towards some desirable property. Since they address the time-harmonic case and the second-order homogenized coefficients, our results should notably allow to tune the dispersive properties of the homogenized material, in particular the so-called band-gaps (forbidden frequencies for which no wave propagates through the structure).

References


