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Homogenization of frame lattices leading to second gradient models coupling classical strain and strain-gradient terms

Houssam Abdoul-Anziz[1], Cédric Bellis[2] and Pierre Seppecher[1]

Abstract

We determine in the framework of static linear elasticity the homogenized behavior of three-dimensional periodic structures made of welded elastic bars. It has been shown that such structures can be modeled as discrete systems of nodes linked by extensional, flexural/torsional interactions corresponding to frame lattices and that the corresponding homogenized models can be strain-gradient models, i.e., models whose effective elastic energy involves components of the first and the second gradients of the displacement field. However, in the existing models, there is no coupling between the classical strain and the strain-gradient terms in the expression of the effective energy. In the present article, under some assumptions on the positions of the nodes of the unit cell, we show that classical strain and strain-gradient strain terms can be coupled. In order to illustrate this coupling we compute the homogenized energy of a particular structure which we call *asymmetrical pantographic structure*.

Keywords

Periodic homogenization, second gradient, strain-gradient, Gamma-convergence

Mathematics Subject Classification: 35B27, 35J30.

Contents

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Introduction

On the one hand, macroscopic behavior of elastic composite materials has been widely studied both in the mechanics as well as in the mathematics literature. From the mathematical point of view, the homogenization of periodic elastic media with moderate contrast is well founded, see, for instance [10,35]. It consists in taking into account the fact that the size ℓ of the unit cell is much smaller than the characteristic size L of the whole sample of the material and in passing to the limit when the ratio $\varepsilon := \ell/L$ tends to zero. This approach called *asymptotic homogenization* has been widely used to study conduction or elasticity problems in static or dynamic cases [17,28,30,35]. The effective behavior of the periodic elastic medium is characterized explicitly in terms of a local minimization problem set in a rescaled cell (see formula (3.6) of [6]).

On the other hand, strain-gradient (or second gradient) materials are expected to have an exotic behavior [24,36]. They are obtained in the literature by using different methods: heuristic homogenization techniques [12,13,25] and variational methods [4,9,15,31]. In articles [4,9,15,31] the second gradient energies obtained involve only the components of the skew-symmetric part of the gradient of the displacement: the limit models obtained enter the framework of the so-called couple-stress models.

Recently, in the the static linear elasticity setting, Abdoul-Anziz and Seppecher [2] have provided, using tools of Γ-convergence, a homogenization result leading to more general strain-gradient energies. They have obtained the term $(\partial^2 u_1/\partial x_1^2)^2$ in the expression of the limit energy. The presence of this second gradient extensional energy ensures that the limit model do not enters the framework of couple-stress models. It has also a very interesting mechanical meaning: it means that a dilatation imposed in a part of the structure tends to spread on the whole structure.

In the article [2], one considers periodic structures made of a single high-rigidity linear elastic material and voids. It is shown that if the thickness of the bars in the unit cell is of order ε^2 (ε being the ratio between the characteristic length ℓ and the macroscopic size L), the considered structures can be reduced to discrete systems corresponding to frame lattices. It is then shown that the homogenization of these frame lattices can lead to strain-gradient models. It can also lead to generalized continua, that are models enriched with extra kinematic variables. The results of the article [2] are extended to dimension 3 in [3] where the algorithm for making explicit the homogenized energy is described precisely and several examples of generalized continua and second gradient models obtained by homogenization of frame lattices are given.

It is important to note that the positions of the nodes of the lattices studied in [2,3] were fixed in the rescaled periodic cell. It is this seemingly natural assumption that we reconsider in the present article. By doing this we are able to generalize the results of [2,3] and to get effective energies in which strain-gradient and classical strain terms are coupled.

Here, the whole study is placed in the framework of static linear elasticity. It is organized as follows. In a first section we explain the effect of strain-gradient terms on the equilibrium. In particular we show on a simple one dimensional example the astonishing effect of coupling strain and strain-gradient terms. Then we describe the particular structure we are interested in: it consists in modifying the pantographic structure which is known to lead to strain-gradient effects and which has been widely studied in order to make it slightly asymmetrical. This structure does not enter in the general framework described in [2], so that we adapt the latter in the section entitled "Statement of the homogenization problem" where most of the needed notation is also fixed. In this section an asymptotic expansion sheds light on the way strain-gradient appears and specially how it can be coupled to classical strain. Next section is devoted to the main result: the energy identified through the previous formal expansion is proved to be the Γ-limit of the initial elastic energy. Then we come back to the asymmetrical pantographic structure and check that it fulfills our goal: its effective behavior is indeed of second gradient type with a coupling of classical strain and strain-gradient terms.

Strain-gradient models in elasticity

Strain-gradient models describe materials whose energy depends not only on the strain but also on its gradient. Equivalently, elastic energy density is a function of the strain $\boldsymbol{e}(\boldsymbol{u})$, that is the symmetric part of the gradient of the displacement field $\boldsymbol{u}$, and of the strain-gradient $\nabla \boldsymbol{e}(\boldsymbol{u})$. Clearly the components of the strain-gradient can be computed from the components of the second gradient $\nabla\nabla\boldsymbol{u}$ of the displacement field. As the inverse is also true, strain-gradient materials can be equivalently called second gradient materials.

In classical linear elasticity, the elastic energy density is a non-negative quadratic form*

$$Q(\boldsymbol{e}(u)) = \frac{1}{2}\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{A} : \boldsymbol{e}(\boldsymbol{u}) \tag{1}$$

of the strain where $\boldsymbol{A}$ is the rigidity tensor (symmetric fourth-order tensor: $A_{ijkl} = A_{klij} = A_{jikl}$) and the stress tensor is its differential $\sigma = \boldsymbol{A} : \boldsymbol{e}(\boldsymbol{u})$. In linear strain-gradient elasticity the elastic energy density is a non negative quadratic form of the strain and its gradient $Q(\boldsymbol{e}(u), \nabla\boldsymbol{e}(u)) = \frac{1}{2}\boldsymbol{e}(\boldsymbol{u}) : (\boldsymbol{A} : \boldsymbol{e}(\boldsymbol{u})) + \boldsymbol{e}(\boldsymbol{u}) : (\boldsymbol{B} \,\vdots\, \nabla\boldsymbol{e}(\boldsymbol{u})) + \frac{1}{2}\nabla\boldsymbol{e}(\boldsymbol{u}) \,\vdots\, (\boldsymbol{C} \,\vdots\, \nabla\boldsymbol{e}(\boldsymbol{u}))$ where $\boldsymbol{C}$ is a symmetric sixth-order tensor ($C_{ijklmn} = C_{lmnijk} = C_{jiklmn}$) and $\boldsymbol{B}$ is a fifth-order tensor (with the symmetry $B_{ijklm} = B_{jiklm} = B_{ijlkm}$).

Equilibrium equations under the action of some external body force $\boldsymbol{f}$ are easily recovered from the minimization of the total energy

$$W(\boldsymbol{u}) := \int_\Omega \Big(Q(\boldsymbol{e}(\boldsymbol{u}), \nabla\boldsymbol{e}(\boldsymbol{u})) - \boldsymbol{f}\cdot\boldsymbol{u} \Big)\, d\boldsymbol{x}.$$

This is a lower semi-continuous (for instance for the L^2 topology) functional over the displacement field $\boldsymbol{u}$ and under suitable coercivity assumptions, the existence of a minimizer is ensured. The variational formulation of the minimization problem reads

$$\forall \boldsymbol{v}, \quad \int_\Omega \left(\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{A} : \boldsymbol{e}(\boldsymbol{v}) + \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{B} \,\vdots\, \nabla\boldsymbol{e}(\boldsymbol{v}) + \boldsymbol{e}(\boldsymbol{v}) : \boldsymbol{B} \,\vdots\, \nabla\boldsymbol{e}(\boldsymbol{u}) + \nabla\boldsymbol{e}(\boldsymbol{u}) \,\vdots\, \boldsymbol{C} \,\vdots\, \nabla\boldsymbol{e}(\boldsymbol{v}) - \boldsymbol{f}\cdot\boldsymbol{v}\right)\, d\boldsymbol{x} = 0.$$

Let us define the third and second-order tensors

$$\boldsymbol{H} := \boldsymbol{B}^T : \boldsymbol{e}(\boldsymbol{u}) + \boldsymbol{C} \,\vdots\, \nabla\boldsymbol{e}(\boldsymbol{u}) \quad \text{and} \quad \boldsymbol{\sigma} := \boldsymbol{A} : \boldsymbol{e}(\boldsymbol{u}) + \boldsymbol{B} \,\vdots\, \nabla\boldsymbol{e}(\boldsymbol{u}) - \mathrm{div}(\boldsymbol{H}), \tag{2}$$

where $\boldsymbol{B}^T$ stands for the transposed tensor: $(\boldsymbol{B}^T)_{ijklm} := (\boldsymbol{B})_{klmij}$. Integrating by parts, the Euler equation of the minimization problem takes, on the interior of the domain, the usual form

$$\mathrm{div}(\boldsymbol{\sigma}) + \boldsymbol{f} = 0 \tag{3}$$

of equilibrium equation. The fundamental difference with classical elasticity is that, now, the stress tensor $\boldsymbol{\sigma}$ depends on the second and third-order partial derivatives of $\boldsymbol{u}$. The boundary conditions are also fundamentally different from these in classical elasticity: the boundary terms which appear when integrating by parts give the natural free boundary conditions associated with the model. In case of a smooth domain, they read

$$(\boldsymbol{H}\cdot\boldsymbol{n})\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{\sigma}\cdot\boldsymbol{n} + \mathrm{div}_{//}(\boldsymbol{H}\cdot\boldsymbol{n})_{//} = 0. \tag{4}$$

where $\boldsymbol{n}$ stands for the exterior normal to the domain and $\mathrm{div}_{//}$ for the surface divergence operator on the boundary†. We refer to[20–23] for the interpretation of the first equation in terms of double-forces, the dependence of surface force on the curvature of the surface (through $\mathrm{div}_{//}\,\boldsymbol{n}$) as well as the possible presence of edge forces in case of a non smooth boundary (not presented here). In these articles the case when non vanishing forces or double-forces are applied on the boundary are detailed and the dual conditions (in terms of imposed displacement and normal derivative of the displacement) are also considered.

To understand the effect of the different terms and of the boundary conditions on the equilibrium, let us consider, for the time being, a one-dimensional case: a bar made of a strain-gradient material is fixed at its center while a non vanishing displacement u_0 is imposed at its left hand extremity (see Figure 1). Thus the left hand half is expanding while the right half of the bar is left free. In ordinary (first gradient) elasticity the solution is obvious: the left half

*Throughout this article, we use tensorial notation. Tensor product of vectors and contraction products of tensors are defined, using Einstein summation convention, by

$$(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} := \boldsymbol{a}_i\boldsymbol{b}_j, \qquad (\boldsymbol{A}\cdot\boldsymbol{B})_{i_1\ldots i_n} := \boldsymbol{A}_{i_1\ldots i_p\,j}\boldsymbol{B}_{j\,i_{p+1}\ldots i_n}$$
$$(\boldsymbol{A}:\boldsymbol{B})_{i_1\ldots i_n} := \boldsymbol{A}_{i_1\ldots i_p\,j\,k}\boldsymbol{B}_{k\,j\,i_{p+1}\ldots i_n}, \qquad (\boldsymbol{A}\,\vdots\,\boldsymbol{B})_{i_1\ldots i_n} := \boldsymbol{A}_{i_1\ldots,i_p\,j\,k\,l}\boldsymbol{B}_{l\,k\,j\,i_{p+1}\ldots i_n}.$$

†For any second-order tensor M, notation $M_{//}$ stands for the projection $M_{//} := M - (M\cdot n)\otimes n$

of the bar is subjected to a uniform dilatation while the right half remains at rest (see the case $\ell = 0$ in Figure 2a). For a strain-gradient material the situation is more complex: the dilation tends to propagate in the free right part of the bar and when a coupling term is present this propagation has its own behavior. Let us perform completely the minimization of the total energy for this one-dimensional example: the domain $\Omega = (-L, L)$ is an interval and the total energy reads

$$\int_{-L}^{L} Q(u', u'')\, dx = \int_{-L}^{L} \left(\frac{1}{2}a(u'(x))^2 + bu'(x)u''(x) + \frac{1}{2}c(u''(x))^2\right) dx$$

under the constraints $u(0) = 0,\ u(-L) = u_0$. Introducing the intrinsic length $\ell := \frac{c}{\sqrt{ac-b^2}}$ and adimensional parameter $k := \frac{\ell b}{c}$, the energy is proportional to

$$\frac{1}{2}\int_{-L}^{L} \left((u'(x))^2 + (\ell u''(x) + ku'(x))^2\right) dx.$$

We can see that two terms are in competition: the classical term, proportional to $(u')^2$, tends to minimize the strain while the other one tends, when $k = 0$ to minimize the variation of the strain, that is to favor a constant strain and when $k \neq 0$ to favor an exponentially varying strain. The equilibrium is the solution on $(-L, 0)$ and $(0, L)$ of the linear ordinary differential equation

$$\ell^2 u'''' - (1 + k^2)u'' = 0$$

with the boundary and continuity conditions $u(-L) = u_0,\ \ell u''(-L) + ku'(-L) = 0,\ \ell^2 u'''(L) - (1 + k^2)u'(L) = 0,\ \ell u''(L) + ku'(L) = 0,\ u(0^-) = u(0^+) = 0,\ u'(0^+) = u'(0^-)$ and $\ell u''(0^-) + ku'(0^-) = \ell u''(0^+) + ku'(0^+)$. The influence of the parameters ℓ and k over the equilibrium are shown in Figure 2. The effect of length ℓ is clear. The expansion imposed to the left half decreases exponentially in the free right half. This length is characteristic of that decreasing. The effect of the adimensional coupling parameter k is more intriguing.

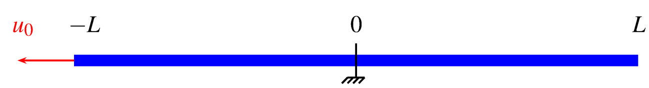
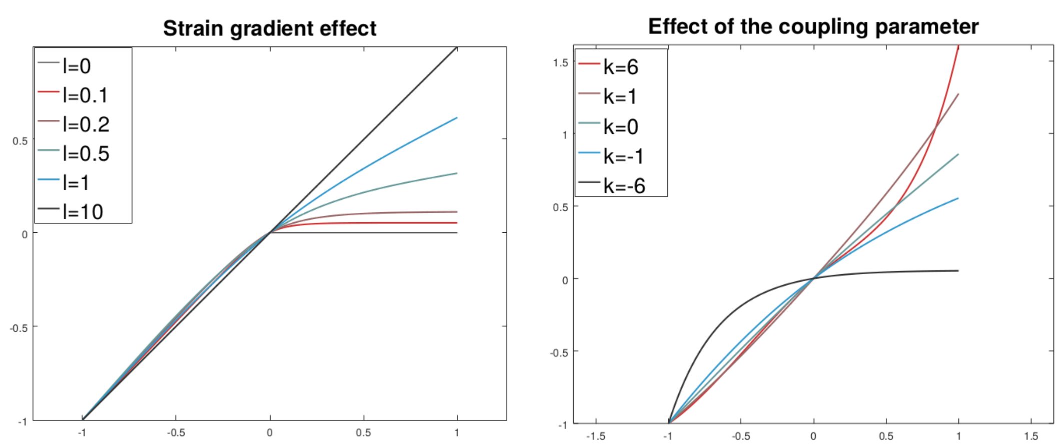


Figure 1. Imposing an extension on the left half of a bar.



(a) Effect of intrinsic length ℓ in absence of coupling ($k = 0$)

(b) Effect of coupling parameter k ($\ell = 2$)

Figure 2. Displacement field of a strain-gradient bar when extension is imposed on its left half only. In this drawing we have set $L = 1$ and $u_0 = -1$.

Pantographic structures

Even if the results of[3] give a straightforward procedure for analyzing the strain-gradient effective behavior of periodic frame lattices, it is not obvious to exhibit structures which actually show such a behavior. Indeed one has to conceive structures which are able to propagate the strain and, up to now, very few structures are known to do so. Pantographic structures are among them[4,5,19,27,32–34,36–38] : they lead through homogenization process to interesting strain-gradient models. A way for building them is to consider a periodic array of cells made of six nodes, linked by elastic beams as represented in Figure 3. The "pantographic" beam thus obtained can be used to build 2D or 3D materials by arranging many parallel pantographic beams and linking them in a suitable way (see Figures 4-5). Let us be more specific: the considered periodic lattice is made of nodes $\boldsymbol{y}_{(i,j,k),s} := \varepsilon(\boldsymbol{y}_s + i\boldsymbol{t}_1 + j\boldsymbol{t}_2 + k\boldsymbol{t}_3)$ where $\boldsymbol{y}_s$ stands for the position of node s ($s \in \{1,\dots,6\}$) in the prototype rescaled cell, $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, $\boldsymbol{t}_3$ are three independent periodicity vectors and i, j and k vary from 1 to ε^{-1} (which is assumed to be an integer). Fixing

$$\boldsymbol{y}_1 = \Big(\frac{1}{4},0,0\Big),\ \ \boldsymbol{y}_2 = \Big(0,-\frac{1}{4},0\Big),\ \ \boldsymbol{y}_3 = \Big(0,\frac{1}{4},0\Big),\ \ \boldsymbol{y}_4 = \Big(\frac{1}{2},\frac{1}{2},0\Big),\ \ \boldsymbol{y}_5 = \Big(\frac{1}{2},-\frac{1}{2},0\Big),\ \ \boldsymbol{y}_6 = \Big(\frac{3}{4},0,0\Big), \tag{5}$$

assuming that, inside each cell, only pairs of nodes $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,5)$, $(3,4)$, $(4,6)$, $(5,6)$ are interacting, assuming that nodes 4 and 5 of each cell are respectively interacting with nodes 3 and 2 of the next cell following $\varepsilon\boldsymbol{t}_1$ (with $\boldsymbol{t}_1 = (1,0,0)$) and assuming moreover that node 6 of each cell is interacting with both nodes 3 and 2 of the next cell following εt_1, we get a pantographic beam as shown in Figure 6. Assuming moreover that node 1 of each cell is interacting with node 1 of the next cells following $\varepsilon\boldsymbol{t}_2$ and $\varepsilon\boldsymbol{t}_3$ (with $\boldsymbol{t}_2 = (0,\sqrt{3},-1)$ and $\boldsymbol{t}_3 = (0,\sqrt{3},1)$) we get a 3D structure as shown in Figures 4-5.

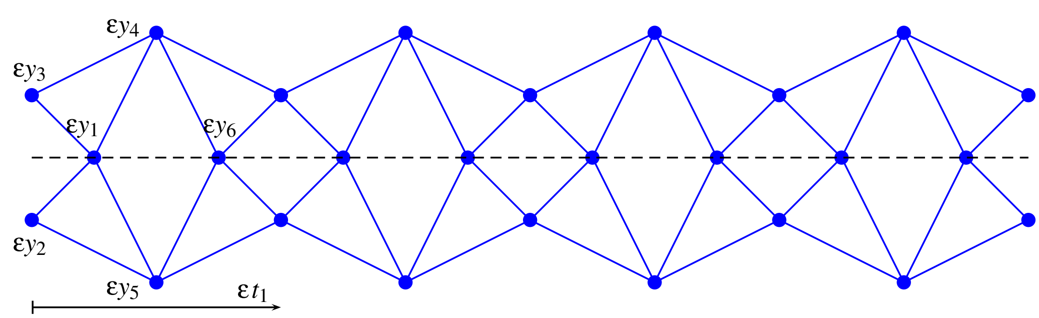


Figure 3. Standard pantographic beam

The procedure described in[3] gives the effective behavior of this structure which can be considered as a micro-structured material. The homogenized elastic energy is a quadratic functional of the displacement field $\boldsymbol{u}$. It reads

$$\mathcal{E}(\boldsymbol{u}) = \frac{1}{2}\int_\Omega \Big(\varpi\,(\boldsymbol{e}_{13}(\boldsymbol{u}))^2 + \sigma\,(\boldsymbol{e}_{12}(\boldsymbol{u}))^2 + \gamma\,(\boldsymbol{e}_{11}(\boldsymbol{u}))^2 + \kappa\Big(\big(\frac{\partial^2\boldsymbol{u}_1}{\partial x_1^2}\big)^2 + \big(\frac{\partial^2\boldsymbol{u}_2}{\partial x_1^2}\big)^2\Big)\Big)\,d\boldsymbol{x}$$

under the constraints $\boldsymbol{e}_{22}(\boldsymbol{u}) = \boldsymbol{e}_{33}(\boldsymbol{u}) = \boldsymbol{e}_{23}(\boldsymbol{u}) = 0$. The values of material parameters ϖ, σ, γ, κ are explicit in terms of the rigidities of the bars the structure is made of. Note also that the second partial derivatives could be written in terms of the strain-gradient. As noticed in[3] this energy does not couple strain and strain-gradient. For such a coupling, a term like

$$\left(\frac{\partial^2\boldsymbol{u}_1}{\partial x_1^2} - \frac{\partial\boldsymbol{u}_1}{\partial x_1}\right)^2$$

would be necessary. As, at equilibrium, the continuum tends to minimizes its energy, the displacement field would tend to fulfill the equation

$$\frac{\partial^2\boldsymbol{u}_1}{\partial x_1^2} - \frac{\partial\boldsymbol{u}_1}{\partial x_1} = 0$$

that is would tend to increase exponentially in the direction $\boldsymbol{t}_1$. How could that be possible ? The answer is natural: modifying the pantograph we just described, making it asymmetrical, must lead to a structure which not only propagates the dilatation but also modifies it from cell to cell.

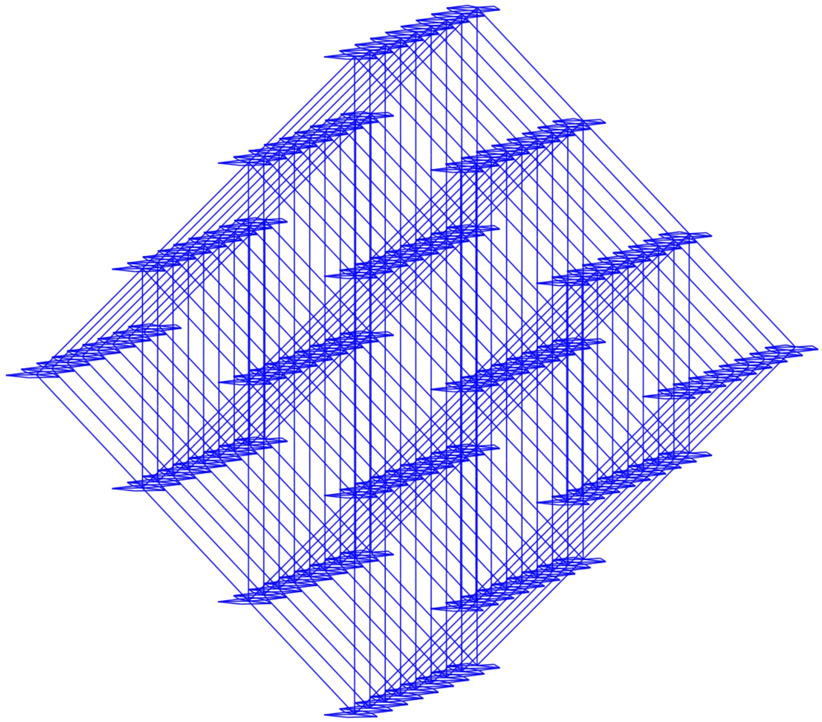
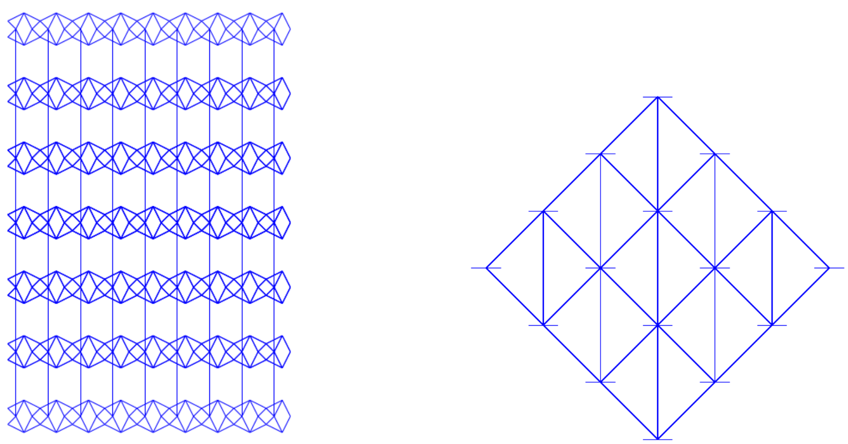


Figure 4. Perspective view of the 3D structure



(a) Projection onto the plane (e_1, e_2)

(b) Projection onto the plane (e_2, e_3)

Figure 5. Projections of the 3D structure

However the attempt to simply modify the coordinates of the nodes given in (5) at order $O(1)$ in the rescaled cell is bound to fail: indeed one would remain in the framework studied in[3] where it is proved that the homogenized energy density is the sum of a quadratic form of the strain-gradient due to the extensional rigidities of the bars and of a quadratic form of the classical strain due to the torsion or bending rigidities of the bars. Hence no coupling can arise. The remark on which we base our study is that, even though it seems a natural idea, there is no reason for assuming that the positions $\boldsymbol{y}_s$ of the nodes inside the rescaled cell do not depend on ε. We thus propose to modify the positions $\boldsymbol{y}_4$ and $\boldsymbol{y}_5$ of nodes 4 and 5 as

$$\boldsymbol{y}_4^\varepsilon := \Big(\frac{1}{2} + \alpha\,\varepsilon, \frac{1}{2}, 0\Big) \quad \text{and} \quad \boldsymbol{y}_5^\varepsilon := \Big(\frac{1}{2} + \alpha\,\varepsilon, -\frac{1}{2}, 0\Big),$$

respectively, where α is a parameter which allows us to tune the asymmetry of our structure.

The new pantographic structure is now very slightly asymmetrical as shown in Figure 6 but there is no need to draw the new 3D structure which remains essentially similar to the one represented in Figures 4-5.

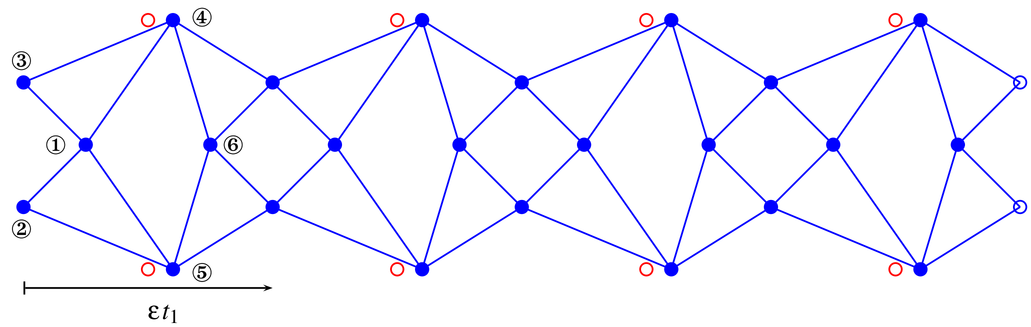


Figure 6. Asymmetrical pantographic structure. Nodes 4 and 5 have been slightly moved on the right from their previous positions (here in red) of Figure 3.

Next section is devoted to the study of the effective properties of periodic lattices when the position of the nodes in the rescaled cell is allowed still to depend on ε. Later on, the homogenization result will be applied to the asymmetrical pantographic structure we just described and we will indeed verify that the effective model is a strain-gradient model with coupling.

Statement of the homogenization problem

Considered geometry

We restrict ourselves to 3D structures leading to 3D continuum models. The study of structures leading to 2D or 1D models, that is to membranes, plates and beams can be treated in a very similar way. This has been done in[3] for non-coupling cases.

So we consider in $\mathbb{R}^3$ a periodic structure with three independent vectors of periodicity $\boldsymbol{t_1}$, $\boldsymbol{t_2}$ and $\boldsymbol{t_3}$. The domain Ω is the parallelepiped $\Omega := \{x_1\boldsymbol{t_1} + x_2\boldsymbol{t_2} + x_3\boldsymbol{t_3} :\ (x_1, x_2, x_3) \in [0,1]^3\}$. We consider a discrete lattice made of a finite number of points (called "nodes") repeated periodically following the vectors $\boldsymbol{t_i}$. We use $I := (i_1, i_2, i_3) \in \mathbb{Z}^3$ to label the different cells of the lattice and $s \in \{1, \dots, K\}$ to label the different nodes in each cell. The geometry of the lattice depends on a small parameter‡ ε and the position of node s of cell I is assumed to be of the form

$$\boldsymbol{y}^{\varepsilon}_{I,s} := \varepsilon(\boldsymbol{y}_s + \varepsilon\boldsymbol{z}_s + i_1\boldsymbol{t_1} + i_2\boldsymbol{t_2} + i_3\boldsymbol{t_3}), \tag{6}$$

where $(\boldsymbol{y}_s)$ and $(\boldsymbol{z}_s)$ are two fixed families of K vectors of $\mathbb{R}^3$. The considered lattice is made of those nodes which lie inside Ω and is therefore divided in ε^{-3} cells similar to $\varepsilon\Omega$.

Note that the role of the parameter ε is twofold: first it measures the ratio of the sizes of the periodic cell and of the considered domain (in that sense, the fact that ε is very small is the standard hypothesis on which homogenization is based); secondly one can see in (6) that it measures the speed of convergence of the nodes positions $\boldsymbol{y}_s + \varepsilon\boldsymbol{z}_s$ in the rescaled prototype cell towards their limit positions $\boldsymbol{y}_s$. Here lies the main difference with previous works[2,3] where these rescaled positions were assumed not to depend on ε.

We introduce $\boldsymbol{y}^{\varepsilon}_I := \varepsilon(i_1\boldsymbol{t_1} + i_2\boldsymbol{t_2} + i_3\boldsymbol{t_3})$ as a reference point in the cell I and

$$\mathcal{I}^{\varepsilon} := \left\{(i_1, i_2, i_3) \in \left(0, \varepsilon^{-1}\right)^3\right\}$$

the set of indices of the cells of the structure. In the sequel, we will use the notation $\sum_I \phi_{I,s}$ to denote the mean value of any field $\phi_{I,s}$ defined at the nodes of the lattice:

$$\sum_I \phi_{I,s} := \varepsilon^3 \sum_{I \in \mathcal{I}^{\varepsilon}} \phi_{I,s}.$$

‡For sake of simplicity, ε^{-1} is assumed to be an integer.

In order to describe the interactions between the nodes of a generic cell I and the interactions between the nodes of this cell with the nodes of its 26 closest neighbor cells $I+p$ (actually, due to periodicity, only half of them have to be considered), we introduce the set

$$\mathcal{P} := \{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(1,-1,0),(0,0,1),(0,1,1),(1,0,1),\\ (0,1,-1),(-1,0,1),(1,1,1),(1,1,-1),(-1,1,1),(1,-1,1)\}$$

that we identify with $\{1,\dots,14\}$. For any $p := (p_1,p_2,p_3) \in \mathcal{P}$ we denote $\boldsymbol{p} := p_1\boldsymbol{t}_1 + p_2\boldsymbol{t}_2 + p_3\boldsymbol{t}_3$ the corresponding vector so that $\boldsymbol{y}^\varepsilon_{I+p,s} = \boldsymbol{y}^\varepsilon_{I,s} + \varepsilon\boldsymbol{p}$. For any pair of distinct nodes $(\boldsymbol{y}^\varepsilon_{I,s}, \boldsymbol{y}^\varepsilon_{I+p,s'})$, we introduce the associated rescaled length and direction:

$$\ell^\varepsilon_{p,s,s'} := \varepsilon^{-1}\|\boldsymbol{y}^\varepsilon_{I+p,s'} - \boldsymbol{y}^\varepsilon_{I,s}\|, \qquad \boldsymbol{\tau}^\varepsilon_{p,s,s'} := \frac{\boldsymbol{y}^\varepsilon_{I+p,s'} - \boldsymbol{y}^\varepsilon_{I,s}}{\varepsilon\ell^\varepsilon_{p,s,s'}}.$$

Using (6) we get the expression

$$\boldsymbol{\tau}^\varepsilon_{p,s,s'} = \frac{\boldsymbol{y}_{s'} - \boldsymbol{y}_s + \boldsymbol{p} + \varepsilon(\boldsymbol{z}_{s'} - \boldsymbol{z}_s)}{\|\boldsymbol{y}_{s'} - \boldsymbol{y}_s + \boldsymbol{p} + \varepsilon(\boldsymbol{z}_{s'} - \boldsymbol{z}_s)\|}.$$

A Taylor series expansion in ε gives

$$\boldsymbol{\tau}^\varepsilon_{p,s,s'} = \boldsymbol{\tau}_{p,s,s'} + \varepsilon\boldsymbol{\rho}_{p,s,s'} + \varepsilon^2\boldsymbol{\zeta}^\varepsilon_{p,s,s'}, \tag{7}$$

with $\|\boldsymbol{\zeta}^\varepsilon_{p,s,s'}\| \le C$ (C being a constant independent of ε),

$$\boldsymbol{\tau}_{p,s,s'} := (\boldsymbol{y}_{s'} - \boldsymbol{y}_s + \boldsymbol{p})/\ell_{p,s,s'}, \qquad \boldsymbol{\rho}_{p,s,s'} := \frac{1}{\ell_{p,s,s'}}(\mathrm{Id} - \boldsymbol{\tau}_{p,s,s'} \otimes \boldsymbol{\tau}_{p,s,s'}) \cdot (\boldsymbol{z}_{s'} - \boldsymbol{z}_s), \tag{8}$$

and $\ell_{p,s,s'} := \|\boldsymbol{y}_{s'} - \boldsymbol{y}_s + \boldsymbol{p}\|$, where Id stands for the identity tensor.

Mechanical interactions

The nodes of the structure are linked by elastic bars made of a homogeneous isotropic elastic material. These bars are slender cylinders with circular cross-section whose radius is of the form $r^\varepsilon := \beta\varepsilon^2$, so that flexural and torsional energies are of order ε^2 compared to the extensional energy. We will not do here the asymptotic study of slender cylinders for which extension, bending and torsional rigidities can be found in many mechanics textbooks, for instance in [26]. We will neither do the integration along the bars (see [2]) which reduces to a quadratic energy involving rigid motions associated to the two interacting nodes. Instead we will start directly from this quadratic interaction energy. Any node of the structure is endowed with a rigid motion, that is a couple of vectors $(\boldsymbol{U}_{I,s}, \boldsymbol{\theta}_{I,s})$ (remember that in small deformations, rotations are represented by vectors). Hence the kinematics of the structure is represented by the two families of vectors (i.e. discrete fields) $\boldsymbol{U} := (\boldsymbol{U}_{I,s})$ and $\boldsymbol{\theta} := (\boldsymbol{\theta}_{I,s})$.

For any interacting pair of distinct nodes $\big((I,s),(I+p,s')\big)$, we introduce the *extension* and the *global rotation* of the bar by setting respectively

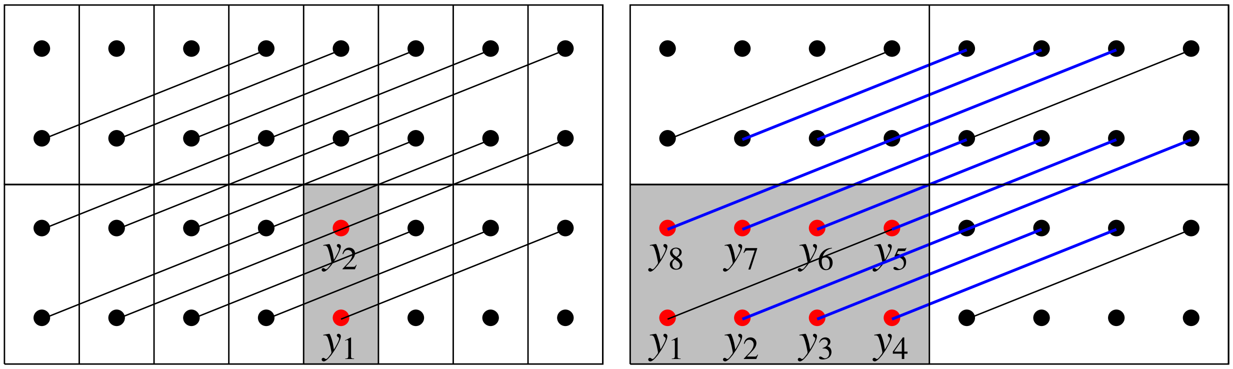
$$(\boldsymbol{\rho_u})_{I,p,s,s'} := \frac{\boldsymbol{U}_{I+p,s'} - \boldsymbol{U}_{I,s}}{\varepsilon} \cdot \boldsymbol{\tau}^\varepsilon_{p,s,s'} \quad \text{and} \quad (\boldsymbol{\alpha_u})_{I,p,s,s'} := \frac{\boldsymbol{U}_{I+p,s'} - \boldsymbol{U}_{I,s}}{\varepsilon\ell_{p,s,s'}} \times \boldsymbol{\tau}^\varepsilon_{p,s,s'}.$$

Since the bar linking the nodes (I,s) and $(I+p,s')$ is elastic, an energy proportional to $(\boldsymbol{\rho_u})^2_{I,p,s,s'}$ is needed for extending it while the energy needed to bend and twist it is a quadratic form of $(\boldsymbol{\alpha_u})_{I,p,s,s'} - \boldsymbol{\theta}_{I,s}$ and $(\boldsymbol{\alpha_u})_{I,p,s,s'} - \boldsymbol{\theta}_{I+p,s'}$.

Let us denote $\mathcal{A} \subset \mathcal{P} \times \{1,\dots,K\}^2$ the set of indices (p,s,s') such that nodes (I,s) and $(I+p,s')$ are linked by a bar. Without loss of generality, we assume that a cell is in interaction only with its closest neighbors. Indeed, the periodic cell can always been chosen large enough for this assumption to be satisfied as illustrated in Figure 7.

We are not interested in disconnected structures. We make, like in [2,3], the "connectedness assumption" that the bars connect all the nodes of the lattice. More precisely[§] we assume that, for any $p \in \mathcal{P}$ and any $(s,s') \in \{1,\dots,K\}^2$, there exist a finite path in the structure which connects the node (I,s) to the node $(I+p,s')$ that is a finite sequence

§For nodes (I,s) and $(I+p,s')$ lying sufficiently inside Ω, the joining path which is assumed to be independent of I, lies inside Ω and connectedness is thus assured. Some nodes lying near the boundary may remain disconnected from the structure. In that case, such nodes must be deleted otherwise compactness could not be assured.



(a) Long-range interactions: 16 cells (b) Interactions between neighboring cells (blue lines): 4 cells

Figure 7. Two points of view for describing the same lattice: In figure (a), we have a "small cell" with long-range interactions while in figure (b) we have chosen the cell large enough for our "next cell hypothesis" to be satisfied.

$(s_1, \ldots, s_{r+1})$ in $\{1, \ldots, K\}$, $(p_1, \ldots, p_r)$ in $\mathcal{P}$, $(\epsilon_1, \ldots, \epsilon_r)$ in $\{-1, 1\}$ such that $s_1 = s$, $s_{r+1} = s'$, $\sum_{i=1}^{r} \epsilon_i \boldsymbol{p}_i = \boldsymbol{p}$,

$$\epsilon_i > 0 \implies (p_i, s_i, s_{i+1}) \in \mathcal{A} \qquad \text{and} \qquad \epsilon_i < 0 \implies (p_i, s_{i+1}, s_i) \in \mathcal{A}.$$

The global extensional energy reads

$$E_\varepsilon(\boldsymbol{U}) := \varepsilon^{-2} \sum_I \sum_{(p,s,s')\in\mathcal{A}} \frac{a_{p,s,s'}}{2} (\rho_{\boldsymbol{u}})^2_{I,p,s,s'}. \tag{9}$$

Here

$$a_{p,s,s'} = \frac{\mathcal{Y}\pi\beta\varepsilon^3}{\ell_{p,s,s'}} \tag{10}$$

where $\mathcal{Y}$ stands for the Young modulus of the material the bars are made of. We assume that the Young modulus is of order ε^{-3}. This simply means that we have chosen the energy unit in such a way that the positive coefficients $a_{p,s,s'}$ are of order one.

The global bending/torsional energy reads

$$\begin{aligned} F_\varepsilon(\boldsymbol{U}, \boldsymbol{\theta}) := \sum_I \sum_{(p,s,s')\in\mathcal{A}} \Big[& (\boldsymbol{\theta}_{I,s} - (\boldsymbol{\alpha}_{\boldsymbol{u}})_{I,p,s,s'}) \cdot \frac{\boldsymbol{B}_{p,s,s'}}{2} \cdot (\boldsymbol{\theta}_{I,s} - (\boldsymbol{\alpha}_{\boldsymbol{u}})_{I,p,s,s'}) \\ & + (\boldsymbol{\theta}_{I,s} - (\boldsymbol{\alpha}_{\boldsymbol{u}})_{I,p,s,s'}) \cdot \boldsymbol{C}_{p,s,s'} \cdot (\boldsymbol{\theta}_{I+p,s'} - (\boldsymbol{\alpha}_{\boldsymbol{u}})_{I,p,s,s'}) \\ & + (\boldsymbol{\theta}_{I+p,s'} - (\boldsymbol{\alpha}_{\boldsymbol{u}})_{I,p,s,s'}) \cdot \frac{\boldsymbol{D}_{p,s,s'}}{2} \cdot (\boldsymbol{\theta}_{I+p,s'} - (\boldsymbol{\alpha}_{\boldsymbol{u}})_{I,p,s,s'}) \Big]. \end{aligned} \tag{11}$$

Here

$$\boldsymbol{B}_{p,s,s'} = \boldsymbol{D}_{p,s,s'} = \beta^2 a_{p,s,s'} \left(\mathrm{Id} - \frac{3+4\nu}{4+4\nu} \boldsymbol{\tau}_{p,s,s'} \otimes \boldsymbol{\tau}_{p,s,s'} \right), \ \boldsymbol{C}_{p,s,s'} = \beta^2 \frac{a_{p,s,s'}}{2} \left(\mathrm{Id} - \frac{3+2\nu}{2+2\nu} \boldsymbol{\tau}_{p,s,s'} \otimes \boldsymbol{\tau}_{p,s,s'} \right),$$

where ν is the Poisson ratio of the material the bars are made of. For any $(p, s, s') \in \mathcal{A}$, the block matrix $\begin{pmatrix} \boldsymbol{B}_{p,s,s'} & \boldsymbol{C}_{p,s,s'} \\ \boldsymbol{C}^T_{p,s,s'} & \boldsymbol{D}_{p,s,s'} \end{pmatrix}$ is positive.

To summarize, the total elastic energy of the considered structure reads

$$\mathcal{E}_\varepsilon(\boldsymbol{U}, \boldsymbol{\theta}) := E_\varepsilon(\boldsymbol{U}) + F_\varepsilon(\boldsymbol{U}, \boldsymbol{\theta}). \tag{12}$$

We associate to any sequence $\boldsymbol{U}^\varepsilon$ the families of vectors $\boldsymbol{m}_I^\varepsilon$, $\boldsymbol{v}_{I,s}^\varepsilon$ and $\boldsymbol{\chi}_{I,p}^\varepsilon$ defined by

$$\boldsymbol{m}_I^\varepsilon := \frac{1}{K} \sum_{s=1}^{K} \boldsymbol{U}_{I,s}^\varepsilon, \quad \boldsymbol{v}_{I,s}^\varepsilon := \varepsilon^{-1}(\boldsymbol{U}_{I,s}^\varepsilon - \boldsymbol{m}_I^\varepsilon), \quad \boldsymbol{\chi}_{I,p}^\varepsilon := \varepsilon^{-1}(\boldsymbol{m}_{I+p}^\varepsilon - \boldsymbol{m}_I^\varepsilon) \tag{13}$$

and the family of real numbers $\omega^\varepsilon_{I,p,s,s'}$ defined by

$$\omega^\varepsilon_{I,p,s,s'} := \begin{cases} \varepsilon^{-2}(\boldsymbol{U}^\varepsilon_{I+p,s'} - \boldsymbol{U}^\varepsilon_{I,s}) \cdot \boldsymbol{\tau}^\varepsilon_{p,s,s'} & \text{if } (p,s,s') \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Using this notation, we can rewrite the two addends $E_\varepsilon(\boldsymbol{U}^\varepsilon)$ and $F_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ of the total elastic energy $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ under the following forms:

$$E_\varepsilon(\boldsymbol{U}^\varepsilon) \equiv \bar{E}_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{\chi}^\varepsilon) = \varepsilon^{-2} \sum_I \sum_{(p,s,s')\in\mathcal{A}} \frac{a_{p,s,s'}}{2} \left((\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} + \boldsymbol{\chi}^\varepsilon_{I,p}) \cdot \boldsymbol{\tau}^\varepsilon_{p,s,s'}\right)^2, \tag{15}$$

$$\begin{aligned} F_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \equiv \bar{F}_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{\chi}^\varepsilon, \boldsymbol{\theta}^\varepsilon) = \sum_I \sum_{(p,s,s')\in\mathcal{A}} \Bigg[& \Big(\boldsymbol{\theta}^\varepsilon_{I,s} - (\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{I,p,s,s'}\Big) \cdot \frac{\boldsymbol{B}_{p,s,s'}}{2} \cdot \Big(\boldsymbol{\theta}^\varepsilon_{I,s} - (\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{I,p,s,s'}\Big) \\ & + \Big(\boldsymbol{\theta}^\varepsilon_{I,s} - (\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{I,p,s,s'}\Big) \cdot \boldsymbol{C}_{p,s,s'} \cdot \Big(\boldsymbol{\theta}^\varepsilon_{I+p,s'} - (\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{I,p,s,s'}\Big) \\ & + \Big(\boldsymbol{\theta}^\varepsilon_{I+p,s'} - (\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{I,p,s,s'}\Big) \cdot \frac{\boldsymbol{D}_{p,s,s'}}{2} \Big(\boldsymbol{\theta}^\varepsilon_{I+p,s'} - (\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{I,p,s,s'}\Big)\Bigg], \end{aligned} \tag{16}$$

where $(\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{I,p,s,s'} := \dfrac{\boldsymbol{\tau}^\varepsilon_{p,s,s'}}{\ell_{p,s,s'}} \times (\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} + \boldsymbol{\chi}^\varepsilon_{I,p})$.

Asymptotic expansion

In order to guess the limit of $\mathcal{E}_\varepsilon$ as ε tends to zero, let us first compute the limit of $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ for particular sequences of displacement fields of the form

$$\boldsymbol{U}^\varepsilon_{I,s} := \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I) + \varepsilon \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I) + \varepsilon^2 \boldsymbol{w}_s(\boldsymbol{y}^\varepsilon_I) \quad \text{and} \quad \boldsymbol{\theta}^\varepsilon_{I,s} := \boldsymbol{\theta}_s(\boldsymbol{y}^\varepsilon_I) \tag{17}$$

where $\boldsymbol{u}$, $\boldsymbol{v}_s$, $\boldsymbol{w}_s$, $\boldsymbol{\theta}_s$ are smooth functions satisfying $\sum_{s=1}^K \boldsymbol{v}_s = \sum_{s=1}^K \boldsymbol{w}_s = 0$. Remind that $\boldsymbol{y}^\varepsilon_I$ is the reference point in cell I. We still associate to $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ the quantities $\boldsymbol{m}^\varepsilon_I$, $\boldsymbol{v}^\varepsilon_{I,s}$, $\boldsymbol{\chi}^\varepsilon_{I,p}$ and $\omega^\varepsilon_{I,p,s,s'}$ as in (13) and (14). We have $\boldsymbol{m}^\varepsilon_I = \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I)$, $\boldsymbol{v}^\varepsilon_{I,s} = \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I) + \varepsilon \boldsymbol{w}_s(\boldsymbol{y}^\varepsilon_I)$,

$$\boldsymbol{\chi}^\varepsilon_{I,p} = \varepsilon^{-1}(\boldsymbol{u}(\boldsymbol{y}^\varepsilon_{I+p}) - \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I)) = \nabla \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I) \cdot \boldsymbol{p} + \frac{\varepsilon}{2} \nabla\nabla \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I) : (\boldsymbol{p} \otimes \boldsymbol{p}) + O(\varepsilon^2)$$

and

$$\begin{aligned} \boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} &= \boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_{I+p}) - \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I) + \varepsilon(\boldsymbol{w}_{s'}(\boldsymbol{y}^\varepsilon_{I+p}) - \boldsymbol{w}_s(\boldsymbol{y}^\varepsilon_I)) \\ &= \big(\boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_{I+p}) - \boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I)\big) + \big(\boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I) - \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I)\big) + \varepsilon\big(\boldsymbol{w}_{s'}(\boldsymbol{y}^\varepsilon_{I+p}) - \boldsymbol{w}_s(\boldsymbol{y}^\varepsilon_I)\big) \\ &= \varepsilon \nabla \boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I) \cdot \boldsymbol{p} + \boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I) - \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I) + \varepsilon\big(\boldsymbol{w}_{s'}(\boldsymbol{y}^\varepsilon_{I+p}) - \boldsymbol{w}_s(\boldsymbol{y}^\varepsilon_I)\big) + O(\varepsilon^2). \end{aligned}$$

Hence $(\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} + \boldsymbol{\chi}^\varepsilon_{I,p}) \cdot \boldsymbol{\tau}^\varepsilon_{p,s,s'} = (\nabla \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I) \cdot \boldsymbol{p} + \boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I) - \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I)) \cdot \boldsymbol{\tau}_{p,s,s'} + O(\varepsilon)$ and the extensional energy $E_\varepsilon(\boldsymbol{U}^\varepsilon)$ cannot remain finite unless the functions $\boldsymbol{u}$ and $\boldsymbol{v}_s$ satisfy the constraint

$$(\nabla \boldsymbol{u} \cdot \boldsymbol{p} + \boldsymbol{v}_{s'} - \boldsymbol{v}_s) \cdot \boldsymbol{\tau}_{p,s,s'} = 0. \tag{18}$$

So now we restrict ourselves to functions satisfying (18). In that case we get

$$\begin{aligned} (\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} + \boldsymbol{\chi}^\varepsilon_{I,p}) \cdot \boldsymbol{\tau}^\varepsilon_{p,s,s'} = & \Big(\frac{\varepsilon}{2} \nabla\nabla \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I) : (\boldsymbol{p} \otimes \boldsymbol{p}) + \varepsilon \nabla \boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I) \cdot \boldsymbol{p} + \varepsilon(\boldsymbol{w}_{s'}(\boldsymbol{y}^\varepsilon_I) - \boldsymbol{w}_s(\boldsymbol{y}^\varepsilon_I))\Big) \cdot \boldsymbol{\tau}_{p,s,s'} \\ & + \varepsilon\, (\nabla \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I) \cdot \boldsymbol{p} + \boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I) - \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I)) \cdot \boldsymbol{\rho}_{p,s,s'} + O(\varepsilon^2), \end{aligned}$$

with $\boldsymbol{\rho}_{p,s,s'}$ defined in (8). By Riemann sum,

$$\begin{aligned} \lim_{\varepsilon\to 0} \bar{E}_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{\chi}^\varepsilon) &= \lim_{\varepsilon\to 0} \varepsilon^{-2} \sum_I \sum_{(p,s,s')\in\mathcal{A}} \frac{a_{p,s,s'}}{2} \left((\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} + \boldsymbol{\chi}^\varepsilon_{I,p}) \cdot \boldsymbol{\tau}^\varepsilon_{p,s,s'}\right)^2 \\ &= \int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \frac{a_{p,s,s'}}{2} \Bigg(\Big(\frac{1}{2} \nabla\nabla \boldsymbol{u}(\boldsymbol{x}) : (\boldsymbol{p} \otimes \boldsymbol{p}) + \nabla \boldsymbol{v}_{s'}(\boldsymbol{x}) \cdot \boldsymbol{p} + \boldsymbol{w}_{s'}(\boldsymbol{x}) - \boldsymbol{w}_s(\boldsymbol{x})\Big) \cdot \boldsymbol{\tau}_{p,s,s'} \\ &\qquad + (\nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{p} + \boldsymbol{v}_{s'}(\boldsymbol{x}) - \boldsymbol{v}_s(\boldsymbol{x})) \cdot \boldsymbol{\rho}_{p,s,s'} \Bigg)^2 d\boldsymbol{x}. \end{aligned}$$

In order to write this result in a more synthetic way, let us fix some notation. For any functions $\boldsymbol{u}$, $\boldsymbol{v}_s$ and $\boldsymbol{w}_s$ in $\mathrm{L}^2(\mathbb{R}^3)$, we set in the sense of distributions, for any[¶] $(p,s) \in \mathcal{P} \times \{1,\ldots,K\}$,

$$(\boldsymbol{\eta}_{\boldsymbol{u}})_{p,s} := \nabla \boldsymbol{u} \cdot \boldsymbol{p}, \tag{19}$$

$$(\boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}})_{p,s} := \nabla \boldsymbol{v}_s \cdot \boldsymbol{p} + \frac{1}{2} \nabla\nabla \boldsymbol{u} : (\boldsymbol{p} \otimes \boldsymbol{p}) \tag{20}$$

and we define

$$\bar{E}(\boldsymbol{w},\boldsymbol{\xi},\boldsymbol{v},\boldsymbol{\eta}) := \int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \frac{a_{p,s,s'}}{2} \left((\boldsymbol{w}_{s'} - \boldsymbol{w}_s + \boldsymbol{\xi}_{p,s'}) \cdot \boldsymbol{\tau}_{p,s,s'} + (\boldsymbol{v}_{s'} - \boldsymbol{v}_s + \boldsymbol{\eta}_{p,s'}) \cdot \boldsymbol{\rho}_{p,s,s'}\right)^2 d\boldsymbol{x}. \tag{21}$$

Using this notation, the previous limit reads $\lim_{\varepsilon\to 0} \bar{E}_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{\chi}^\varepsilon) = \bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}, \boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}})$ and constraint (18) can be written $\bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0$.

Computing the bending-torsion energy is simpler: we have

$$(\boldsymbol{\alpha}^\varepsilon_{\boldsymbol{v},\boldsymbol{\chi}})_{p,s,s'} = \frac{\boldsymbol{\tau}^\varepsilon_{p,s,s'}}{\ell_{p,s,s'}} \times (\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} + \boldsymbol{\chi}^\varepsilon_{I,p}) = \frac{\boldsymbol{\tau}_{p,s,s'}}{\ell_{p,s,s'}} \times (\boldsymbol{v}_{s'}(\boldsymbol{y}^\varepsilon_I) - \boldsymbol{v}_s(\boldsymbol{y}^\varepsilon_I) + \nabla \boldsymbol{u}(\boldsymbol{y}^\varepsilon_I) \cdot \boldsymbol{p}) + O(\varepsilon).$$

Therefore, setting $(\boldsymbol{\alpha}_{\boldsymbol{v},\boldsymbol{\eta}_u})_{p,s,s'} := \dfrac{\boldsymbol{\tau}_{p,s,s'}}{\ell_{p,s,s'}} \times (\boldsymbol{v}_{s'} - \boldsymbol{v}_s + (\boldsymbol{\eta}_{\boldsymbol{u}})_{p,s})$ and

$$\begin{aligned}\bar{F}(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\theta}) = \int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \Bigg(&(\boldsymbol{\theta}_s - (\boldsymbol{\alpha}_{\boldsymbol{v},\boldsymbol{\eta}})_{p,s,s'}) \cdot \frac{\boldsymbol{B}_{p,s,s'}}{2} \cdot (\boldsymbol{\theta}_s - (\boldsymbol{\alpha}_{\boldsymbol{v},\boldsymbol{\eta}})_{p,s,s'}) \\ &+ (\boldsymbol{\theta}_s - (\boldsymbol{\alpha}_{\boldsymbol{v},\boldsymbol{\eta}})_{p,s,s'}) \cdot \boldsymbol{C}_{p,s,s'} \cdot (\boldsymbol{\theta}_{s'} - (\boldsymbol{\alpha}_{\boldsymbol{v},\boldsymbol{\eta}})_{p,s,s'}) \\ &+ (\boldsymbol{\theta}_{s'} - (\boldsymbol{\alpha}_{\boldsymbol{v},\boldsymbol{\eta}})_{p,s,s'}) \cdot \frac{\boldsymbol{D}_{p,s,s'}}{2} \cdot (\boldsymbol{\theta}_{s'} - (\boldsymbol{\alpha}_{\boldsymbol{v},\boldsymbol{\eta}})_{p,s,s'}) \Bigg) d\boldsymbol{x}\end{aligned} \tag{22}$$

we get $\lim_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) = \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta})$.

To conclude, the limit of the elastic energy $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ is $\bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}, \boldsymbol{v}, \boldsymbol{\eta}_u) + \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta})$ under the constraint $\bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0$. At this point the reader may consider that the effective behavior of the structure has been ascertained. That is not quite true as sequences of equilibrium solutions have no reason to fulfill the assumption (17). It remains to prove that the result remains true in the general case.

Homogenization framework

To state a homogenization result one has to fix the relationship between sequences of equilibrium displacement fields and their macroscopic limits. In the standard case this relationship is natural: all displacement fields belong to the same functional space (for instance $\mathrm{L}^2(\Omega)$) and their limit is taken is the sense of a suitable topology on this space. We cannot do that here, as, for different values of ε, the displacement field is defined on different sets of nodes. Moreover at fixed ε the displacement field is a finite family of vectors while we expect the effective displacement field to be a continuous function on Ω.

A usual way for transferring the problem on a fixed functional space is to introduce an extension over Ω of the fields which are defined only at the nodes of the structure. As such an extension is somehow arbitrary, we prefer to see the displacement field at fixed ε as a discrete measure and to define its continuous limit as the diffuse limit measure.

Definition 1. *Let $Z^\varepsilon = (Z^\varepsilon_I)_{I\in\mathcal{I}^\varepsilon}$ be a sequence of families of real numbers. We say that Z^ε converges to a measurable function z, and we write $Z^\varepsilon \rightharpoonup z$, when the following weak-* convergence of measures holds:*

$$\sum_I Z^\varepsilon_I \delta_{\boldsymbol{y}^\varepsilon_I} \overset{*}{\rightharpoonup} z(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \tag{23}$$

where $\delta_{\boldsymbol{y}^\varepsilon_I}$ stands for the Dirac measure at the point $\boldsymbol{y}^\varepsilon_I$. This definition is extended to vector fields by applying it component-wise.

[¶]Note that $(\boldsymbol{\eta}_{\boldsymbol{u}})_{p,s}$ actually does not depend on s. It is introduced in this way because the expression of the limit energy is simpler when $\boldsymbol{\eta}_{\boldsymbol{u}}$ and $\boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}$ share the same tensorial nature.

Remark 1. *The convergence (23) means that, for any continuous function φ on Ω, the following limit relation holds true:*

$$\lim_{\varepsilon\to 0}\sum_I Z_I^\varepsilon \varphi(\boldsymbol{y}_I^\varepsilon) = \int_\Omega z(\boldsymbol{x})\varphi(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}. \tag{24}$$

When applied to sequences (Z^ε) that satisfy $\sum_I \|Z_I^\varepsilon\|^2 \le C$ where C is a constant independent on ε, this notion of convergence ensures the existence of some function $z \in L^2(\Omega)$ such that, up to a sub-sequence, (Z^ε) converges to z (see Lemma 10.1 in[16]). In view of the relation (24), we note that we can replace in (23) the Dirac measure $\delta_{\boldsymbol{y}_I^\varepsilon}$ by $\delta_{\boldsymbol{y}_{I,s}^\varepsilon}$ or even $\delta_{\boldsymbol{y}_{I+p,s}^\varepsilon}$. Indeed, for any continuous function φ, we have $\varphi(\boldsymbol{y}_{I+p,s'}^\varepsilon) - \varphi(\boldsymbol{y}_{I,s}^\varepsilon) = o(1)$.

Remark 2. *When it holds for any $s \in \{1,\dots,K\}$, the convergence of measures $Z_s^\varepsilon \rightharpoonup z_s$ is closely related to the notion of two-scale convergence developed by Nguetseng[29] and Allaire[6]. The discrete variable s plays the role of the fast variable. In that case, for any convex lower semi-continuous function Φ we have*

$$\liminf_{\varepsilon\to 0}\sum_I \frac{1}{K}\sum_{s=1}^K \Phi(Z_{I,s}^\varepsilon) \ge \int_\Omega \Phi\left(\frac{1}{K}\sum_{s=1}^K z_s(\boldsymbol{x})\right)\ \mathrm{d}\boldsymbol{x}. \tag{25}$$

see Lemma 3.1 in[11].

Remark 3. *If Z^ε stands for a family $(Z_s^\varepsilon)_{s=1}^K$ and if, for any $s \in \{1,\dots,K\}$, $Z_s^\varepsilon \rightharpoonup z$ we simply write $Z^\varepsilon \rightharpoonup z$ as no confusion can arise. Equation (25) in the previous remark then reduces to $\liminf_{\varepsilon\to 0}\sum_I \frac{1}{K}\sum_{s=1}^K \Phi(Z_{I,s}^\varepsilon) \ge \int_\Omega \Phi\left(z(\boldsymbol{x})\right)\ \mathrm{d}\boldsymbol{x}$.*

The goal of homogenization is the determination of an elastic problem whose solution is the limit of the equilibrium states when ε tends to zero. A unique homogenized energy is expected to determine the effective elastic problem under the action of diverse external forces. The notion of Γ-convergence is one of the suitable tool for that aim.

Definition 2. *The sequence of energies $\mathcal{E}_\varepsilon$ is said to Γ-converge to $\mathcal{E}$ if*
i) for any family $\boldsymbol{U}^\varepsilon$ of displacements at the nodes of the structure,

$$\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u} \Longrightarrow \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(U^\varepsilon) \ge \mathcal{E}(\boldsymbol{u})$$

ii) for any $\boldsymbol{u}$, there exists a family $\boldsymbol{U}^\varepsilon$ such that

$$\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u} \quad \textit{and} \quad \limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon) \le \mathcal{E}(\boldsymbol{u}).$$

We refer to[14,18] for the definition and properties of Γ-convergence. The two most important properties for our purpose are the following: (i) provided that sequences $(\boldsymbol{U}^\varepsilon)$ with bounded energy $(\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon) < M)$ are relatively compact for the chosen convergence, then minimizers of $\mathcal{E}_\varepsilon$ converge to a minimizer of $\mathcal{E}$; (ii) provided that a given external force potential is a continuous perturbation of the elastic energy, one can add this potential to the elastic energy and recover the same potential in the limit energy. Let us assume that external forces $\boldsymbol{F}_I^\varepsilon$ are applied at the nodes $\boldsymbol{y}_{I,1}^\varepsilon$ of the structure and are of the type $\boldsymbol{F}_I^\varepsilon = \varepsilon^3 \boldsymbol{F}(\boldsymbol{y}_{I,1}^\varepsilon)$ where $\boldsymbol{F}$ is a continuous vector function. The associated potential satisfies

$$\lim_{\varepsilon\to 0}\sum_I \boldsymbol{F}_I^\varepsilon\cdot\boldsymbol{U}_{I,1}^\varepsilon = \lim_{\varepsilon\to 0}\sum_I \boldsymbol{F}(\boldsymbol{y}_{I,1}^\varepsilon)\cdot\boldsymbol{U}_{I,1}^\varepsilon = \lim_{\varepsilon\to 0}\int_\Omega \boldsymbol{F}(\boldsymbol{x})\ \mathrm{d}(\sum_I U_{I,1}^\varepsilon \delta_{\boldsymbol{y}_I^\varepsilon}(\boldsymbol{x})) = \int_\Omega \boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,d\boldsymbol{x}.$$

as soon as $\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u}$ and therefore is a continuous perturbation. A consequence of the two aforementioned properties is that the equilibrium displacement family $\boldsymbol{U}^\varepsilon$ under the action of $\boldsymbol{F}_{I,1}^\varepsilon$ converges to a minimum of $\mathcal{E}(\boldsymbol{u}) - \int_\Omega \boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,d\boldsymbol{x}$. Of course, the same also holds for other nodes $\boldsymbol{y}_{I,s}^\varepsilon$ with $s \ne 1$.

So Γ-convergence allows to study the homogenization of our structure without taking into account external forces. In that sense it follows closely the intuitive notion of effective behavior.

Note however that the action of more oscillating external forces as well as the action of forces concentrated along the boundary are not encompassed by the present results that must therefore be adapted to these cases. It is also,the case if the displacement is imposed on some nodes (on the boundary or not) of the structure. Therefore the particular problem in the section entitled "strain-gradient models in elasticity" and represented in Figure 1 does not rigorously enter the scope of our result: we have chosen it for illustrative purposes and the reader may be confident that sufficiently concentrated but diffuse forces at the vicinity of the points where the displacement is imposed would lead to solutions close to the ones presented in Figure 2.

Homogenization result

In this section we seek the Γ-limit $\mathcal{E}$ of the sequence of functionals $\mathcal{E}_\varepsilon$ defined in (12). We consider free boundary conditions. In this case, the equilibrium of the structure can be reached only when the applied external forces are balanced and the solution of equilibrium problems is defined up to a global rigid motion. In order to ensure uniqueness of the equilibrium solution, we forbid a global rigid motion by assuming

$$\sum_I \boldsymbol{U}^\varepsilon_{I,1} = 0 \quad \text{and} \quad \sum_I \boldsymbol{\theta}^\varepsilon_{I,1} = 0. \tag{26}$$

This condition, which in the sequel is referred to as the "zero mean rigid motion" assumption‖, is taken into account in the energy by setting $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) = +\infty$ whenever it is not satisfied.

We extend the functionals $\bar{E}$ and $\bar{F}$ defined in (21) and (22) onto whole $\mathrm{L}^2(\Omega)$ by setting $\bar{E} = +\infty$ or $\bar{F} = +\infty$ whenever the integrands are not square integrable. The homogenized energy corresponding to the microscopic energy $\mathcal{E}_\varepsilon$ is provided by the following theorem.

Theorem 1. *The sequence of energies $\mathcal{E}_\varepsilon$ defined in* (12) Γ*-converges to the energy $\mathcal{E}$ defined by*

$$\mathcal{E}(\boldsymbol{u}) := \inf_{\boldsymbol{w},\boldsymbol{v},\boldsymbol{\theta}} \left\{ \bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}, \boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}) + \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta}) : \bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0, \ \int_\Omega \boldsymbol{\theta}_1(\boldsymbol{x})\, d\boldsymbol{x} = 0 \right\} \tag{27}$$

if $\int_\Omega \boldsymbol{u}(\boldsymbol{x})\, d\boldsymbol{x} = 0$, and $\mathcal{E}(\boldsymbol{u}) := +\infty$ otherwise, where $\boldsymbol{\eta}_{\boldsymbol{u}}$, $\boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}$ are the quantities defined in (19) *and* (20)*. Here the infimum is taken on all functions $\boldsymbol{w}$, $\boldsymbol{v}$ and $\boldsymbol{\theta}$ in $\mathrm{L}^2(\Omega)$. More precisely we have*

(i) (Compactness) For all sequences $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ with bounded energy (i.e. $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \le M$), there exist $\boldsymbol{u}$ and $\boldsymbol{\theta}$ such that, up to a sub-sequence, $\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u}$ and $\boldsymbol{\theta}^\varepsilon \rightharpoonup \boldsymbol{\theta}$.

(ii) (Lower bound) For all sequences $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ such that $\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u}$ and $\boldsymbol{\theta}^\varepsilon \rightharpoonup \boldsymbol{\theta}$, we have

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \ge \mathcal{E}(\boldsymbol{u}).$$

(iii) (Upper bound) For any $\boldsymbol{u}$ satisfying $\mathcal{E}(\boldsymbol{u}) < +\infty$, there exists a sequence $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ such that

$$\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u} \quad \text{and} \quad \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \le \mathcal{E}(\boldsymbol{u}).$$

The next subsections are devoted to the proof of this theorem.

Compactness of sequences with bounded energy

In this subsection, we prove point (i) of Theorem 1, that is the relative compactness of sequences $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ with bounded energy. We also prove the relative compactness of the associated sequences $\boldsymbol{m}^\varepsilon$, $\boldsymbol{v}^\varepsilon$, $\boldsymbol{\chi}^\varepsilon$, $\omega^\varepsilon_{p,s,s'}$ defined in (13) and (14). This compactness result is necessary to ensure that any sequence of minima of $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ converges towards a minimum of $\mathcal{E}(\boldsymbol{u})$.

The next lemma is a discrete version of the Korn inequality. Its proof is essentially based on the classical Korn inequality applied to an affine interpolation of the nodes displacements.

Lemma 1. *There exists a positive constant C depending only on Ω such that, for any sequence of families $(\boldsymbol{U}^\varepsilon_I)_{I\in\mathcal{I}^\varepsilon}$, $(\boldsymbol{\theta}^\varepsilon_I)_{I\in\mathcal{I}^\varepsilon}$ satisfying, for any $p \in \mathcal{P}$,*

$$\sum_I \boldsymbol{U}^\varepsilon_I = 0, \quad \sum_I \boldsymbol{\theta}^\varepsilon_I = 0, \quad \sum_I \left\| \frac{\boldsymbol{U}^\varepsilon_{I+p} - \boldsymbol{U}^\varepsilon_I}{\varepsilon} - \boldsymbol{\theta}^\varepsilon_I \times \boldsymbol{p} \right\|^2 < 1, \tag{28}$$

one has

$$\sum_I \|\boldsymbol{\theta}^\varepsilon_I\|^2 < C, \quad \sum_I \left\| \frac{\boldsymbol{U}^\varepsilon_{I+p} - \boldsymbol{U}^\varepsilon_I}{\varepsilon} \right\| < C \quad \text{and} \quad \sum_I \|\boldsymbol{U}^\varepsilon_I\|^2 < C. \tag{29}$$

‖The choice of the node 1 in this assumption is arbitrary. A mean value of the displacements of all node could have been chosen instead. Rotation could have alternatively been forbidden by imposing a condition on the displacements.

Proof. Here C is a positive constant (depending only on Ω) whose value may vary from line to line.

We first divide Ω into ε^{-3} small parallelepipeds $\Omega_I^\varepsilon := \boldsymbol{y}_I^\varepsilon + \varepsilon\Omega$. Then we divide each Ω_I^ε into six tetrahedra by considering the six different triplets $(\boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_k)$ of distinct directions in $\{\boldsymbol{t_1}, \boldsymbol{t_2}, \boldsymbol{t_3}\}$ and the associated tetrahedra with vertices $(\boldsymbol{y}_I^\varepsilon, \boldsymbol{y}_{I+p_i}^\varepsilon, \boldsymbol{y}_{I+p_i+p_j}^\varepsilon, \boldsymbol{y}_{I+p_i+p_j+p_k}^\varepsilon)$. On each tetrahedron we define the affine interpolation $\mathcal{U}^\varepsilon$ of $\boldsymbol{U}^\varepsilon$ by setting for any $0 \le z \le y \le x \le 1$,

$$\mathcal{U}^\varepsilon(\boldsymbol{y}_I^\varepsilon + \varepsilon(x\boldsymbol{p}_i + y\boldsymbol{p}_j + z\boldsymbol{p}_k)) := (1-x)\boldsymbol{U}_I^\varepsilon + (x-y)\boldsymbol{U}_{I+p_i}^\varepsilon + (y-z)\boldsymbol{U}_{I+p_i+p_j}^\varepsilon + z\boldsymbol{U}_{I+p_i+p_j+p_k}^\varepsilon.$$

Checking that this piecewise affine function belongs to $H^1(\Omega)$ is straightforward. Moreover, there exists a constant C such that

$$\|\mathcal{U}^\varepsilon\|_{L^2(\Omega)}^2 = \varepsilon^3 |\det(\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3)| \sum_{I \in I_\varepsilon} \int_0^1 \int_0^1 \int_0^1 |\mathcal{U}^\varepsilon(\boldsymbol{y}_I^\varepsilon + \varepsilon(x\boldsymbol{p}_i + y\boldsymbol{p}_j + z\boldsymbol{p}_k))|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \ge C \sum_I \|\boldsymbol{U}_I^\varepsilon\|^2. \tag{30}$$

In the tetrahedron $(\boldsymbol{y}_I^\varepsilon, \boldsymbol{y}_{I+p_i}^\varepsilon, \boldsymbol{y}_{I+p_i+p_j}^\varepsilon, \boldsymbol{y}_{I+p_i+p_j+p_k}^\varepsilon)$ we have

$$\varepsilon\nabla\mathcal{U}^\varepsilon = (\boldsymbol{U}_{I+p_i}^\varepsilon - \boldsymbol{U}_I^\varepsilon) \otimes \boldsymbol{p}_i^* + (\boldsymbol{U}_{I+p_i+p_j}^\varepsilon - \boldsymbol{U}_{I+p_i}^\varepsilon) \otimes \boldsymbol{p}_j^* + (\boldsymbol{U}_{I+p_i+p_j+p_k}^\varepsilon - \boldsymbol{U}_{I+p_i+p_j}^\varepsilon) \otimes \boldsymbol{p}_k^*, \tag{31}$$

where $(\boldsymbol{p}_i^*, \boldsymbol{p}_j^*, \boldsymbol{p}_k^*)$ is the dual basis corresponding to $(\boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_k)$. Therefore, denoting by $e(\mathcal{U}^\varepsilon)$ the symmetric part of $\nabla\mathcal{U}^\varepsilon$, the following equalities hold:

$$\begin{aligned}
&\varepsilon e(\mathcal{U}^\varepsilon) : (\boldsymbol{p}_i \otimes \boldsymbol{p}_i) = (\boldsymbol{U}_{I+p_i}^\varepsilon - \boldsymbol{U}_I^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_i) \cdot \boldsymbol{p}_i,\\
&\varepsilon e(\mathcal{U}^\varepsilon) : (\boldsymbol{p}_j \otimes \boldsymbol{p}_j) = (\boldsymbol{U}_{I+p_i+p_j}^\varepsilon - \boldsymbol{U}_{I+p_i}^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_j) \cdot \boldsymbol{p}_j,\\
&\varepsilon e(\mathcal{U}^\varepsilon) : (\boldsymbol{p}_k \otimes \boldsymbol{p}_k) = (\boldsymbol{U}_{I+p_i+p_j+p_k}^\varepsilon - \boldsymbol{U}_{I+p_i+p_j}^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_k) \cdot \boldsymbol{p}_k,\\
&\varepsilon e(\mathcal{U}^\varepsilon) : (\boldsymbol{p}_i \otimes \boldsymbol{p}_j + \boldsymbol{p}_j \otimes \boldsymbol{p}_i) = (\boldsymbol{U}_{I+p_i}^\varepsilon - \boldsymbol{U}_I^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_i) \cdot \boldsymbol{p}_j + (\boldsymbol{U}_{I+p_i+p_j}^\varepsilon - \boldsymbol{U}_{I+p_i}^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_j) \cdot \boldsymbol{p}_i,\\
&\varepsilon e(\mathcal{U}^\varepsilon) : (\boldsymbol{p}_i \otimes \boldsymbol{p}_k + \boldsymbol{p}_k \otimes \boldsymbol{p}_i) = (\boldsymbol{U}_{I+p_i}^\varepsilon - \boldsymbol{U}_I^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_i) \cdot \boldsymbol{p}_k\\
&\qquad\qquad + (\boldsymbol{U}_{I+p_i+p_j+p_k}^\varepsilon - \boldsymbol{U}_{I+p_i+p_j}^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_k) \cdot \boldsymbol{p}_i,\\
&\varepsilon e(\mathcal{U}^\varepsilon) : (\boldsymbol{p}_j \otimes \boldsymbol{p}_k + \boldsymbol{p}_k \otimes \boldsymbol{p}_j) = (\boldsymbol{U}_{I+p_i+p_j}^\varepsilon - \boldsymbol{U}_{I+p_i}^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_j) \cdot \boldsymbol{p}_k\\
&\qquad\qquad + (\boldsymbol{U}_{I+p_i+p_j+p_k}^\varepsilon - \boldsymbol{U}_{I+p_i+p_j}^\varepsilon - \varepsilon\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_k) \cdot \boldsymbol{p}_j,
\end{aligned}$$

where we have used the identity $(\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}) \cdot \boldsymbol{q} + (\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{q}) \cdot \boldsymbol{p} = 0$ with $\boldsymbol{p}, \boldsymbol{q} \in \{\boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_k\}$. As $\boldsymbol{p}_i \otimes \boldsymbol{p}_i, \boldsymbol{p}_j \otimes \boldsymbol{p}_j, \boldsymbol{p}_k \otimes \boldsymbol{p}_k$, $(\boldsymbol{p}_i \otimes \boldsymbol{p}_j + \boldsymbol{p}_j \otimes \boldsymbol{p}_i)$, $(\boldsymbol{p}_i \otimes \boldsymbol{p}_k + \boldsymbol{p}_k \otimes \boldsymbol{p}_i)$, $(\boldsymbol{p}_j \otimes \boldsymbol{p}_k + \boldsymbol{p}_k \otimes \boldsymbol{p}_j)$ make a basis for symmetric matrices, then by using (28) and summing over all parallelepipeds Ω_I^ε, we obtain that $\|e(\mathcal{U}^\varepsilon)\|_{L^2(\Omega)}^2 \le C$. We can use classical Korn inequality on Ω (see Theorem 2.5 in[30]): there exist a constant C and a global rigid motion $\boldsymbol{R}^\varepsilon(\boldsymbol{y}) := \boldsymbol{a}^\varepsilon + B^\varepsilon \cdot \boldsymbol{y}$ (where $\boldsymbol{a}^\varepsilon$ is a constant vector and B^ε is a skew-symmetric matrix) such that $\|\mathcal{U}^\varepsilon - \boldsymbol{R}^\varepsilon\|_{H^1(\Omega)}^2 \le C$.

In particular we have

$$\|\nabla\mathcal{U}^\varepsilon - B^\varepsilon\|_{L^2(\Omega)}^2 \le C. \tag{32}$$

From (31) and (32), we get for any direction $\boldsymbol{p}_i$, the inequality

$$\sum_I \left\| \frac{\boldsymbol{U}_{I+p_i}^\varepsilon - \boldsymbol{U}_I^\varepsilon}{\varepsilon} - B^\varepsilon \cdot \boldsymbol{p}_i \right\|^2 \le C \tag{33}$$

which, together with assumption (28), gives using Jensen and triangle inequalities,

$$\left\| \sum_I \left(B^\varepsilon \cdot \boldsymbol{p}_i - \boldsymbol{\theta}_I^\varepsilon \cdot \boldsymbol{p}_i \right) \right\|^2 \le \sum_I \|B^\varepsilon \cdot \boldsymbol{p}_i - \boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_i\|^2 \le C. \tag{34}$$

We obtain $\left\| B^\varepsilon \cdot \boldsymbol{p}_i - \sum_I \boldsymbol{\theta}_I^\varepsilon \cdot \boldsymbol{p}_i \right\|^2 \le C$ which, since $\sum_I \boldsymbol{\theta}_I^\varepsilon = 0$, reduces to $\|B^\varepsilon \cdot \boldsymbol{p}_i\|^2 \le C$. Going back to (34) we obtain $\sum_I \|\boldsymbol{\theta}_I^\varepsilon \times \boldsymbol{p}_i\|^2 \le C$. These two inequalities, being true for any direction $\boldsymbol{p}_i$, we deduce that

$$\|B^\varepsilon\|^2 \le C \quad \text{and} \quad \sum_I \|\boldsymbol{\theta}_I^\varepsilon\|^2 \le C. \tag{35}$$

Using again the triangle inequality and inequality (33), we obtain the second desired bound:

$$\sum_I \left\| \frac{\boldsymbol{U}_{I+p_i}^\varepsilon - \boldsymbol{U}_I^\varepsilon}{\varepsilon} \right\|^2 \le C. \tag{36}$$

On the other hand, it is clear that $\mathcal{U}^\varepsilon - \boldsymbol{R}^\varepsilon$ is the affine interpolation of $\boldsymbol{U}_I^\varepsilon - \boldsymbol{R}^\varepsilon \cdot \boldsymbol{y}_I^\varepsilon$. We can then apply inequality (30) to these functions. This gives

$$C \sum_I \|\boldsymbol{U}_I^\varepsilon - \boldsymbol{R}^\varepsilon \cdot \boldsymbol{y}_I^\varepsilon\|^2 \leq \|\mathcal{U}^\varepsilon - \boldsymbol{R}^\varepsilon\|^2_{L^2(\Omega)}$$

and thus $\sum_I \|\boldsymbol{U}_I^\varepsilon - \boldsymbol{R}^\varepsilon \cdot \boldsymbol{y}_I^\varepsilon\|^2 \leq C$. Using again Jensen inequality and the assumption $\sum_I \boldsymbol{U}_I^\varepsilon = 0$, we get $\|\boldsymbol{R}^\varepsilon \cdot (\sum_I \boldsymbol{y}_I^\varepsilon)\|^2 \leq C$ and thus $\|\boldsymbol{R}^\varepsilon\|^2 \leq C$. This leads to the last desired bound:

$$\sum_I \|\boldsymbol{U}_I^\varepsilon\|^2 \leq C. \tag{37}$$

This concludes the proof of the lemma.

Lemma 2. Compactness. *Let $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ be a sequence of displacements satisfying $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \leq C$ and with zero mean rigid motion as in (26). Then the sequences $\Big(\sum_I \|\boldsymbol{U}_{I,s}^\varepsilon\|^2\Big)$, $\Big(\sum_I \|\boldsymbol{\theta}_{I,s}^\varepsilon\|^2\Big)$, $\Big(\sum_I \|\boldsymbol{m}_I^\varepsilon\|^2\Big)$, $\Big(\sum_I \|\boldsymbol{v}_{I,s}^\varepsilon\|^2\Big)$, $\Big(\sum_I \|\boldsymbol{\chi}_{I,p}^\varepsilon\|^2\Big)$ and $\Big(\sum_I (\omega_{I,p,s,s'}^\varepsilon)^2\Big)$ are bounded.*

Proof. Here C is a constant independent of ε whose value may vary from line to line.

The inequality $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \leq C$ implies that $E_\varepsilon(\boldsymbol{U}^\varepsilon) \leq C$ and $F_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \leq C$. The first bound $E_\varepsilon(\boldsymbol{U}^\varepsilon) \leq C$ leads directly to

$$\sum_I (\omega_{I,p,s,s'}^\varepsilon)^2 = \varepsilon^{-2} \sum_I \left(\frac{\boldsymbol{U}_{I+p,s'}^\varepsilon - \boldsymbol{U}_{I,s}^\varepsilon}{\varepsilon} \cdot \boldsymbol{\tau}_{p,s,s'}^\varepsilon \right)^2 \leq C. \tag{38}$$

We first consider any (p, s, s') in $\mathcal{A}$. The second bound $F_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \leq C$ implies

$$\sum_I \left\| \boldsymbol{\theta}_{I,s}^\varepsilon - \boldsymbol{\tau}_{p,s,s'}^\varepsilon \times \frac{\boldsymbol{U}_{I+p,s'}^\varepsilon - \boldsymbol{U}_{I,s}^\varepsilon}{\varepsilon \ell_{p,s,s'}} \right\|^2 \leq C \tag{39}$$

and

$$\sum_I \left\| \boldsymbol{\theta}_{I+p,s'}^\varepsilon - \boldsymbol{\tau}_{p,s,s'}^\varepsilon \times \frac{\boldsymbol{U}_{I+p,s'}^\varepsilon - \boldsymbol{U}_{I,s}^\varepsilon}{\varepsilon \ell_{p,s,s'}} \right\|^2 \leq C. \tag{40}$$

Using the triangle inequality, then (39) and (40) yield

$$\sum_I \|\boldsymbol{\theta}_{I+p,s'}^\varepsilon - \boldsymbol{\theta}_{I,s}^\varepsilon\|^2 \leq C.$$

Now let us consider any (p, s, s') in $\mathcal{P} \times \{1, \ldots, K\}^2$. Owing to the connectedness assumption we consider the path $(s_i, p_i, \epsilon_i)_{i=1,\ldots,r}$ connecting nodes (I, s) and $(I + p, s')$. We set $\tilde{p}_j := \sum_{i=1}^{j-1} \epsilon_i p_i$. Using the triangle inequality, we get

$$\sum_I \|\boldsymbol{\theta}_{I+\tilde{p}_j,s_j}^\varepsilon - \boldsymbol{\theta}_{I,s}^\varepsilon\|^2 \leq C. \tag{41}$$

Hence, from (39) or (40), we can deduce

$$\sum_I \left\| \epsilon_j \frac{\boldsymbol{U}_{I+\tilde{p}_{j+1},s_{j+1}}^\varepsilon - \boldsymbol{U}_{I+\tilde{p}_j,s_j}^\varepsilon}{\varepsilon \ell_{p_j,s_j,s_{j+1}}} \times \boldsymbol{\tau}_{p_j,s_j,s_{j+1}}^\varepsilon - \boldsymbol{\theta}_{I,s}^\varepsilon \right\|^2 \leq C. \tag{42}$$

Let us set $\widetilde{\boldsymbol{U}}_{I,s,J,s'}^\varepsilon := \boldsymbol{U}_{J,s'}^\varepsilon - \boldsymbol{\theta}_{I,s}^\varepsilon \times (\boldsymbol{y}_{J,s'}^\varepsilon - \boldsymbol{y}_{I,s}^\varepsilon)$ so that

$$\widetilde{\boldsymbol{U}}_{I,s,J+p,s''}^\varepsilon - \widetilde{\boldsymbol{U}}_{I,s,J,s'}^\varepsilon = \boldsymbol{U}_{J+p,s''}^\varepsilon - \boldsymbol{U}_{I,s'}^\varepsilon - \varepsilon \ell_{p,s',s''} \, \boldsymbol{\theta}_{I,s}^\varepsilon \times \boldsymbol{\tau}_{p,s',s''}^\varepsilon.$$

Then inequality (42) yields

$$\sum_I \left\| \frac{\widetilde{\boldsymbol{U}}_{I,s,I+\tilde{p}_{j+1},s_{j+1}}^\varepsilon - \widetilde{\boldsymbol{U}}_{I,s,I+\tilde{p}_j,s_j}^\varepsilon}{\varepsilon \ell_{p_j,s_j,s_{j+1}}} \times \boldsymbol{\tau}_{p_j,s_j,s_{j+1}}^\varepsilon \right\|^2 \leq C.$$

Since (38) also implies that

$$\varepsilon^{-2}\sum_I \left\| \left(\widetilde{\boldsymbol{U}}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_{j+1}} - \widetilde{\boldsymbol{U}}^\varepsilon_{I,s,I+\tilde{p}_j,s_j} \right) \cdot \boldsymbol{\tau}^\varepsilon_{p_j,s_j,s_{j+1}} \right\|^2 \le C,$$

we get $\varepsilon^{-2}\sum_I \left\| \widetilde{\boldsymbol{U}}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_{j+1}} - \widetilde{\boldsymbol{U}}^\varepsilon_{I,s,I+\tilde{p}_{j+1},s_j} \right\|^2 < C$ which leads, still using the triangle inequality, to

$$\varepsilon^{-2}\sum_I \left\| \widetilde{\boldsymbol{U}}^\varepsilon_{I,s,I+p,s'} - \widetilde{\boldsymbol{U}}^\varepsilon_{I,s,I,s} \right\|^2 \le C,$$

or equivalently to

$$\varepsilon^{-2}\sum_I \left\| \boldsymbol{U}^\varepsilon_{I+p,s'} - \boldsymbol{U}^\varepsilon_{I,s} - \boldsymbol{\theta}^\varepsilon_{I,s} \times (\boldsymbol{y}^\varepsilon_{I+p,s'} - \boldsymbol{y}^\varepsilon_{I,s}) \right\|^2 \le C. \tag{43}$$

We focus temporarily on the particular case $s = s' = 1$ which reads

$$\sum_I \left\| \frac{\boldsymbol{U}^\varepsilon_{I+p,1} - \boldsymbol{U}^\varepsilon_{I,1}}{\varepsilon} - \boldsymbol{\theta}^\varepsilon_{I,1} \times \boldsymbol{p} \right\|^2 < C. \tag{44}$$

Applying Lemma 1 we obtain $\sum_I \|\boldsymbol{\theta}^\varepsilon_{I,1}\|^2 \le C$. Owing to (41) this bound extends to any $s \in \{1,\ldots,K\}$:

$$\sum_I \|\boldsymbol{\theta}^\varepsilon_{I,s}\|^2 \le C. \tag{45}$$

Using inequality (43), we obtain for any $(p,s,s') \in \mathcal{P} \times \{1,\ldots,K\}^2$,

$$\varepsilon^{-2}\sum_I \|\boldsymbol{U}^\varepsilon_{I+p,s'} - \boldsymbol{U}^\varepsilon_{I,s}\|^2 \le C. \tag{46}$$

As Lemma 1 also gives $\sum_I \|\boldsymbol{U}^\varepsilon_{I,1}\|^2 \le C$, we get for any $s \in \{1,\ldots,K\}$, by choosing $p = 0$ and $s' = 1$,

$$\sum_I \|\boldsymbol{U}^\varepsilon_{I,s}\|^2 \le C. \tag{47}$$

Taking the mean value with respect to s in (47) and (46) (with $p = 0$) we get respectively

$$\sum_I \|\boldsymbol{m}^\varepsilon_I\|^2 \le C, \quad \sum_I \|\boldsymbol{v}^\varepsilon_{I,s}\|^2 \le C, \tag{48}$$

and taking the mean value with respect to s and s' in (46), we get

$$\sum_I \|\boldsymbol{\chi}^\varepsilon_{I,p}\|^2 \le C. \tag{49}$$

The proof is concluded by collecting (38), (45), (47), (48) and (49).

Properties of the double-scale limits

The estimates established in Lemma 2 imply that for any sequence $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ with bounded energy, there exist functions $\boldsymbol{\theta}$, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{\chi}_p$ and $\omega_{p,s,s'}$ in $L^2(\Omega)$ such that, for any $s \in \{1,\ldots,K\}$ and up to sub-sequences,

$$\boldsymbol{\theta}^\varepsilon_s \rightharpoonup \boldsymbol{\theta}_s, \quad \boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{u}, \quad \boldsymbol{v}^\varepsilon_s \rightharpoonup \boldsymbol{v}_s, \quad \boldsymbol{\chi}^\varepsilon_p \rightharpoonup \boldsymbol{\chi}_p \ \text{ and } \ \omega^\varepsilon_{p,s,s'} \rightharpoonup \omega_{p,s,s'}. \tag{50}$$

In the following lemma we establish some properties of these limits.

Lemma 3. *Let $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ be a sequence such that $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \le C$. We have*

$$\boldsymbol{U}^\varepsilon_s \rightharpoonup \boldsymbol{u}, \quad \int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = 0, \quad \int_\Omega \boldsymbol{\theta}_1\, d\boldsymbol{x} = 0, \quad \sum_{s=1}^K \boldsymbol{v}_s = 0 \ \text{ and } \ \boldsymbol{\chi}_p = \nabla \boldsymbol{u} \cdot \boldsymbol{p}. \tag{51}$$

Moreover there exist some fields $\boldsymbol{w}_s$ and $\boldsymbol{\lambda}$ in $L^2(\mathbb{R}^3)$ such that, for any $(p,s,s') \in \mathcal{A}$,

$$\omega_{p,s,s'} = \Big(\boldsymbol{w}_{s'} - \boldsymbol{w}_s + \nabla(\boldsymbol{v}_{s'} + \boldsymbol{\lambda}) \cdot \boldsymbol{p} + \frac{1}{2}\nabla\nabla\boldsymbol{u} : (\boldsymbol{p} \otimes \boldsymbol{p}) \Big) \cdot \boldsymbol{\tau}_{p,s,s'} + (\boldsymbol{v}_{s'} - \boldsymbol{v}_s + \nabla \boldsymbol{u} \cdot \boldsymbol{p}) \cdot \boldsymbol{\rho}_{p,s,s'}. \tag{52}$$

Proof. The convergence of $\boldsymbol{v}_s^\varepsilon \rightharpoonup \boldsymbol{v}_s$ implies that $(\boldsymbol{U}_s^\varepsilon - \boldsymbol{m}^\varepsilon) \rightharpoonup 0$ and so, for any s, $\boldsymbol{U}_s^\varepsilon \rightharpoonup \boldsymbol{u}$. The assumptions $\sum_I \boldsymbol{U}_{I,1}^\varepsilon = 0$ and $\sum_I \boldsymbol{\theta}_{I,1}^\varepsilon = 0$ also hold in the limit and give $\int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = 0$ and $\int_\Omega \boldsymbol{\theta}_1\, d\boldsymbol{x} = 0$. The fact that $\sum_{s=1}^K \boldsymbol{v}_{I,s}^\varepsilon = 0$ clearly implies that $\sum_{s=1}^K \boldsymbol{v}_s(\boldsymbol{x}) = 0$. For any smooth test field $\boldsymbol{\varphi}$ with compact support in Ω, we have

$$\begin{aligned}
\int_\Omega \boldsymbol{\chi}_p(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} &= \lim_{\varepsilon\to 0} \sum_I \varepsilon^{-1}(\boldsymbol{m}_{I+p}^\varepsilon - \boldsymbol{m}_I^\varepsilon) \cdot \boldsymbol{\varphi}(\boldsymbol{y}_I^\varepsilon) \\
&= \lim_{\varepsilon\to 0} \sum_I \boldsymbol{m}_I^\varepsilon \cdot \varepsilon^{-1}(\boldsymbol{\varphi}(\boldsymbol{y}_{I-p}^\varepsilon) - \boldsymbol{\varphi}(\boldsymbol{y}_I^\varepsilon)) \\
&= \lim_{\varepsilon\to 0} \sum_I \boldsymbol{m}_I^\varepsilon \cdot (-\nabla\boldsymbol{\varphi}(\boldsymbol{y}_I^\varepsilon) \cdot \boldsymbol{p} + O(\varepsilon)) \\
&= -\int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot (\nabla\boldsymbol{\varphi}(\boldsymbol{x}) \cdot \boldsymbol{p})\ \mathrm{d}\boldsymbol{x} = \int_\Omega (\nabla\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{p}) \cdot \boldsymbol{\varphi}(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

We deduce that $\boldsymbol{\chi}_p = \nabla\boldsymbol{u} \cdot \boldsymbol{p}$.

In order to characterize the limit $\omega_{p,s,s'}$, we introduce the set $\mathcal{D}_A$ of families of distributions in $H^{-1}(\mathbb{R}^3)$ defined by

$$\mathcal{D}_A := \left\{\psi_{p,s,s'} = (\boldsymbol{w}_{s'} - \boldsymbol{w}_s + \nabla\boldsymbol{\lambda} \cdot \boldsymbol{p}) \cdot \boldsymbol{\tau}_{p,s,s'} \text{ such that } (p,s,s') \in \mathcal{A}, \boldsymbol{w}_s \in L^2(\mathbb{R}^3), \boldsymbol{\lambda} \in L^2(\mathbb{R}^3)\right\}.$$

Its orthogonal $\mathcal{D}_A{}^\perp$ is the set of families $(\phi_{p,s,s'})_{(p,s,s')\in\mathcal{A}}$ of functions in $H^1(\mathbb{R}^3)$ such that, for all $\psi_{p,s,s'} \in \mathcal{D}_A$, $\sum_{(p,s,s')\in\mathcal{A}} \langle \psi_{p,s,s'}, \phi_{p,s,s'}\rangle = 0$. Let us remark that, for any $\phi \in \mathcal{D}_A{}^\perp$ we have

$$\sum_{(p,s,s')\in\mathcal{A}} (\nabla\phi_{p,s,s'} \cdot \boldsymbol{p})\, \boldsymbol{\tau}_{p,s,s'} = 0, \tag{53}$$

and for any $\boldsymbol{w} = (\boldsymbol{w}_s)_{s=1,\dots,K} \in (L^2(\mathbb{R}^3;\mathbb{R}^3))^K$,

$$\sum_{(p,s,s')\in\mathcal{A}} ((\boldsymbol{w}_{s'} - \boldsymbol{w}_s) \cdot \boldsymbol{\tau}_{p,s,s'})\, \phi_{p,s,s'} = 0. \tag{54}$$

We extend ϕ by setting $\phi_{p,s,s'} = 0$ whenever $(p,s,s') \notin \mathcal{A}$. Then we can rewrite the equation (54) as

$$\sum_{(p,s,s')} (\boldsymbol{\tau}_{p,s,s'}\phi_{p,s,s'} - \boldsymbol{\tau}_{p,s',s}\phi_{p,s',s}) = 0. \tag{55}$$

On the one hand, we have

$$\int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \omega_{p,s,s'}(\boldsymbol{x})\, \phi_{p,s,s'}(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} = \lim_{\varepsilon\to 0} \sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2}(\boldsymbol{U}_{I+p,s'}^\varepsilon - \boldsymbol{U}_{I,s}^\varepsilon) \cdot (\phi_{p,s,s'}(\boldsymbol{y}_I^\varepsilon)\, \boldsymbol{\tau}_{p,s,s'}^\varepsilon).$$

On the other hand, using functions $\phi_{p,s,s'}$ satisfying (53) and (55) and using the decomposition (7) of $\boldsymbol{\tau}_{p,s,s'}^\varepsilon$, we obtain

$$\begin{aligned}
&\sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2}(\boldsymbol{U}_{I+p,s'}^\varepsilon - \boldsymbol{U}_{I,s}^\varepsilon) \cdot (\phi_{p,s,s'}(\boldsymbol{y}_I^\varepsilon)\, \boldsymbol{\tau}_{p,s,s'}^\varepsilon) \\
&= \sum_I \sum_{(p,s,s')\in\mathcal{A}} (\varepsilon^{-1}(\boldsymbol{v}_{I+p,s'}^\varepsilon - \boldsymbol{v}_{I,s}^\varepsilon) + \varepsilon^{-2}(\boldsymbol{m}_{I+p}^\varepsilon - \boldsymbol{m}_I^\varepsilon)) \cdot (\phi_{p,s,s'}(\boldsymbol{y}_I^\varepsilon)\, \boldsymbol{\tau}_{p,s,s'}^\varepsilon) \\
&= \sum_I \sum_{(p,s,s')\in\mathcal{A}} (\varepsilon^{-1}(\boldsymbol{v}_{I+p,s'}^\varepsilon - \boldsymbol{v}_{I,s}^\varepsilon) + \varepsilon^{-2}(\boldsymbol{m}_{I+p}^\varepsilon - \boldsymbol{m}_I^\varepsilon)) \cdot (\phi_{p,s,s'}(\boldsymbol{y}_I^\varepsilon)\, \boldsymbol{\tau}_{p,s,s'}) \\
&+ \sum_I \sum_{(p,s,s')\in\mathcal{A}} (\boldsymbol{v}_{I+p,s'}^\varepsilon - \boldsymbol{v}_{I,s}^\varepsilon + \varepsilon^{-1}(\boldsymbol{m}_{I+p}^\varepsilon - \boldsymbol{m}_I^\varepsilon)) \cdot (\phi_{p,s,s'}(\boldsymbol{y}_I^\varepsilon)\, \boldsymbol{\rho}_{p,s,s'}) \\
&+ \sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon(\boldsymbol{v}_{I+p,s'}^\varepsilon - \boldsymbol{v}_{I,s}^\varepsilon + \varepsilon^{-1}(\boldsymbol{m}_{I+p}^\varepsilon - \boldsymbol{m}_I^\varepsilon)) \cdot (\phi_{p,s,s'}(\boldsymbol{y}_I^\varepsilon)\, \boldsymbol{\zeta}_{p,s,s'}^\varepsilon).
\end{aligned}$$

It is clear that the third addend in the last sum tends to zero, since $\boldsymbol{\zeta}_{p,s,s'}^\varepsilon$ is bounded and $\boldsymbol{v}_{I+p,s'}^\varepsilon - \boldsymbol{v}_{I,s}^\varepsilon + \varepsilon^{-1}(\boldsymbol{m}_{I+p}^\varepsilon - \boldsymbol{m}_I^\varepsilon)$ tends to $\boldsymbol{v}_{s'}(x) - \boldsymbol{v}_s(x) + \nabla\boldsymbol{u}(x) \cdot \boldsymbol{p}$ as $\varepsilon \to 0$.

Let us estimate separately the remaining two addends. To that aim we restrict ourselves to smooth functions $\phi_{p,s,s'}$ with compact support in Ω. Owing to Remark 1, we get

$$\lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} (\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s} + \varepsilon^{-1}(\boldsymbol{m}^\varepsilon_{I+p} - \boldsymbol{m}^\varepsilon_I)) \cdot (\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\,\boldsymbol{\rho}_{p,s,s'})$$
$$= \sum_{(p,s,s')\in\mathcal{A}} \Big\langle \boldsymbol{v}_{s'}(\boldsymbol{x}) - \boldsymbol{v}_s(\boldsymbol{x}) + \nabla \boldsymbol{u}(\boldsymbol{x})\cdot \boldsymbol{p}, (\phi_{p,s,s'}(\boldsymbol{x})\,\boldsymbol{\rho}_{p,s,s'})\Big\rangle.$$

Using (54), the first addend becomes

$$\sum_{(p,s,s')\in\mathcal{A}} (\varepsilon^{-1}(\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s}) + \varepsilon^{-2}(\boldsymbol{m}^\varepsilon_{I+p} - \boldsymbol{m}^\varepsilon_I)) \cdot (\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\,\boldsymbol{\tau}_{p,s,s'})$$
$$= \sum_{(p,s,s')\in\mathcal{A}} (\varepsilon^{-1}(\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s'}) + \varepsilon^{-1}(\boldsymbol{v}^\varepsilon_{I,s'} - \boldsymbol{v}^\varepsilon_{I,s}) + \varepsilon^{-2}(\boldsymbol{m}^\varepsilon_{I+p} - \boldsymbol{m}^\varepsilon_I)) \cdot (\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\,\boldsymbol{\tau}_{p,s,s'})$$
$$= \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-1}(\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s'}) \cdot (\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\,\boldsymbol{\tau}_{p,s,s'}) + \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2}(\boldsymbol{m}^\varepsilon_{I+p} - \boldsymbol{m}^\varepsilon_I) \cdot (\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\,\boldsymbol{\tau}_{p,s,s'}).$$

The first term yields

$$\lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-1}(\boldsymbol{v}^\varepsilon_{I+p,s'} - \boldsymbol{v}^\varepsilon_{I,s'}) \cdot (\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\,\boldsymbol{\tau}_{p,s,s'})$$
$$= \lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} \boldsymbol{v}^\varepsilon_{I,s'} \cdot (\varepsilon^{-1}(\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_{I-p}) - \phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I))\,\boldsymbol{\tau}_{p,s,s'})$$
$$= \lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} \boldsymbol{v}^\varepsilon_{I,s'} \cdot \Big[(-\nabla\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\cdot \boldsymbol{p})\,\boldsymbol{\tau}_{p,s,s'}\Big] + O(\varepsilon)$$
$$= \sum_{(p,s,s')\in\mathcal{A}} \Big\langle \boldsymbol{v}_{s'}(\boldsymbol{x}), ((-\nabla\phi_{p,s,s'}(\boldsymbol{x})\cdot \boldsymbol{p})\,\boldsymbol{\tau}_{p,s,s'})\Big\rangle$$
$$= \sum_{(p,s,s')\in\mathcal{A}} \Big\langle \nabla \boldsymbol{v}_{s'}(\boldsymbol{x})\cdot \boldsymbol{p}, (\phi_{p,s,s'}(\boldsymbol{x})\,\boldsymbol{\tau}_{p,s,s'})\Big\rangle.$$

For the second term, using (53) we get

$$\lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2}(\boldsymbol{m}^\varepsilon_{I+p} - \boldsymbol{m}^\varepsilon_I) \cdot (\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\,\boldsymbol{\tau}_{p,s,s'})$$
$$= \lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} \varepsilon^{-2}\boldsymbol{m}^\varepsilon_I \cdot \Big[(\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_{I-p}) - \phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I))\,\boldsymbol{\tau}_{p,s,s'}\Big]$$
$$= \lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} \boldsymbol{m}^\varepsilon_I \cdot \Big[\Big(-\varepsilon^{-1}\nabla\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I)\cdot \boldsymbol{p} + \frac{1}{2}\nabla\nabla\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I) : (\boldsymbol{p}\otimes\boldsymbol{p})\Big)\,\boldsymbol{\tau}_{p,s,s'}\Big] + O(\varepsilon)$$
$$= \lim_{\varepsilon\to 0}\sum_I \sum_{(p,s,s')\in\mathcal{A}} \boldsymbol{m}^\varepsilon_I \cdot \Big[\Big(\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(\boldsymbol{y}^\varepsilon_I) : (\boldsymbol{p}\otimes\boldsymbol{p})\Big)\boldsymbol{\tau}_{p,s,s'}\Big] + O(\varepsilon)$$
$$= \sum_{(p,s,s')\in\mathcal{A}} \Big\langle \boldsymbol{u}(\boldsymbol{x}), \Big(\Big(\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(\boldsymbol{x})\cdot \boldsymbol{p}\Big)\boldsymbol{\tau}_{p,s,s'}\Big)\Big\rangle$$
$$= \sum_{(p,s,s')\in\mathcal{A}} \Big\langle \frac{1}{2}\nabla\nabla \boldsymbol{u}(x) : (\boldsymbol{p}\otimes\boldsymbol{p}), (\phi_{p,s,s'}(\boldsymbol{x})\boldsymbol{\tau}_{p,s,s'})\Big\rangle.$$

The previous results prove that the distribution

$$\omega_{p,s,s'} - \Big(\nabla \boldsymbol{v}_{s'}\cdot\boldsymbol{p} + \frac{1}{2}\nabla\nabla\boldsymbol{u} : (\boldsymbol{p}\otimes\boldsymbol{p})\Big)\cdot\boldsymbol{\tau}_{p,s,s'} - (\boldsymbol{v}_{s'} - \boldsymbol{v}_s + \nabla\boldsymbol{u}\cdot\boldsymbol{p})\cdot\boldsymbol{\rho}_{p,s,s'}$$

is orthogonal to all smooth functions in ${\mathcal{D}_A}^\perp$ with compact support in Ω. As such functions are dense in ${\mathcal{D}_A}^\perp$, there exist some fields $\boldsymbol{w}_s$ and $\boldsymbol{\lambda}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that, for any $(p,s,s')\in\mathcal{A}$,

$$\omega_{p,s,s'} = \Big(\boldsymbol{w}_{s'} - \boldsymbol{w}_s + \nabla(\boldsymbol{v}_{s'} + \boldsymbol{\lambda})\cdot\boldsymbol{p} + \frac{1}{2}\nabla\nabla\boldsymbol{u} : (\boldsymbol{p}\otimes\boldsymbol{p})\Big)\cdot\boldsymbol{\tau}_{p,s,s'} + (\boldsymbol{v}_{s'} - \boldsymbol{v}_s + \nabla\boldsymbol{u}\cdot\boldsymbol{p})\cdot\boldsymbol{\rho}_{p,s,s'}.$$

This concludes the proof of the lemma.

Proof of the homogenization result

(i) Lower bound: Let $(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ be a sequence such that $\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u}$ and $\boldsymbol{\theta}^\varepsilon \rightharpoonup \boldsymbol{\theta}$. If $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) = +\infty$, there is nothing to prove. We then assume that $\mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \leq C$. From Lemma 3 we know that the constraints $\int_\Omega \boldsymbol{\theta}_1(\boldsymbol{x})\, d\boldsymbol{x} = 0$ and $\int_\Omega \boldsymbol{u}(\boldsymbol{x})\, d\boldsymbol{x} = 0$ are satisfied. From the same lemma we know that $\boldsymbol{v}_s^\varepsilon$ converges to some $\boldsymbol{v}_s$ and that $\boldsymbol{\chi}_p^\varepsilon \rightharpoonup \boldsymbol{\eta}_{\boldsymbol{u}}$. As $E_\varepsilon(\boldsymbol{U}^\varepsilon)$ is bounded, then $\varepsilon^2 E_\varepsilon(\boldsymbol{U}^\varepsilon)$ tends to zero. By virtue of Remark 2, we have

$$0 = \liminf_{\varepsilon\to 0}(\varepsilon^2 E_\varepsilon(\boldsymbol{U}^\varepsilon)) = \liminf_{\varepsilon\to 0}(\varepsilon^2 \bar{E}_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{\chi}^\varepsilon)) \geq \bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0)$$

which gives the constraint $\bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0$.

Definition (14) allows us to rewrite $E_\varepsilon(\boldsymbol{U}^\varepsilon)$ as $\sum_I \sum_{(p,s,s')\in\mathcal{A}} \frac{a_{p,s,s'}}{2}(\omega^\varepsilon_{I,p,s,s'})^2$. Therefore, using (50), (51), (52), Remarks 1 and 2, we get

$$\begin{aligned}\liminf_{\varepsilon\to 0}(\bar{E}_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{\chi}^\varepsilon) + \bar{F}_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{\chi}^\varepsilon, \boldsymbol{\theta}^\varepsilon)) &\geq \bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}+\boldsymbol{\lambda}}, \boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}) + \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta}) \\ &\geq \inf_{\boldsymbol{w},\boldsymbol{v},\boldsymbol{\theta}}\{\bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}+\boldsymbol{\lambda}}, \boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}) + \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta})\}. \end{aligned} \tag{56}$$

We obtain the desired inequality by noticing that $\bar{F}(\boldsymbol{v}-\boldsymbol{\lambda}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta}) = \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta})$ and $\bar{E}(\boldsymbol{v}+\boldsymbol{\lambda}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = \bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0$.

(ii) Upper bound: By a density argument, it is enough to consider a function $u \in C^\infty(\Omega)$ such that $\mathcal{E}(\boldsymbol{u}) < +\infty$. In virtue of coercivity and lower semi-continuity of the functionals $\bar{E}$ and $\bar{F}$, we can introduce the fields $(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\theta})$ belonging to $C^\infty(\Omega)$ such that $\mathcal{E}(\boldsymbol{u}) = \bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}, \boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}) + \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta})$, $\bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0$, $\int_\Omega \boldsymbol{u}(\boldsymbol{x})\, d\boldsymbol{x} = 0$ and $\int_\Omega \boldsymbol{\theta}_1(\boldsymbol{x})\, d\boldsymbol{x} = 0$. As the constraint $\bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0$ is equivalent to equation (18), we can apply the results of the asymptotic expansion which state that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) = \bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{u,v}, \boldsymbol{v}, \boldsymbol{\eta}_u) + \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta}) = \mathcal{E}(\boldsymbol{u})$$

where $\boldsymbol{U}^\varepsilon$ and $\boldsymbol{\theta}^\varepsilon$ are defined by

$$\boldsymbol{U}^\varepsilon_{I,s} := \boldsymbol{u}(\boldsymbol{y}_I^\varepsilon) + \varepsilon \boldsymbol{v}_s(\boldsymbol{y}_I^\varepsilon) + \varepsilon^2 \boldsymbol{w}_s(\boldsymbol{y}_I^\varepsilon) \quad \text{and} \quad \boldsymbol{\theta}^\varepsilon_{I,s} := \boldsymbol{\theta}_s(\boldsymbol{y}_I^\varepsilon). \tag{57}$$

However we must modify such families in order to fulfill the constraints $\sum_I \boldsymbol{U}^\varepsilon_{I,1} = 0$ and $\sum_I \boldsymbol{\theta}^\varepsilon_{I,1} = 0$. This is easy to do: as the functions $\boldsymbol{u}$, $\boldsymbol{v}_1$, $\boldsymbol{w}_1$, $\boldsymbol{\theta}_1$ are smooth and satisfy $\int_\Omega \boldsymbol{u}(\boldsymbol{x})\, d\boldsymbol{x} = 0$ and $\int_\Omega \boldsymbol{\theta}_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0$, the quantities

$$\boldsymbol{M}^\varepsilon := \sum_I \left(\boldsymbol{u}(\boldsymbol{y}_I^\varepsilon) + \varepsilon \boldsymbol{v}_1(\boldsymbol{y}_I^\varepsilon) + \varepsilon^2 \boldsymbol{w}_1(\boldsymbol{y}_I^\varepsilon)\right) \quad \text{and} \quad \boldsymbol{t}^\varepsilon := \sum_I \boldsymbol{\theta}_1(\boldsymbol{y}_I^\varepsilon)$$

are $O(\varepsilon)$. We now redefine $\boldsymbol{U}^\varepsilon$ and $\boldsymbol{\theta}^\varepsilon$ by setting

$$\boldsymbol{U}^\varepsilon_{I,s} := \boldsymbol{u}(\boldsymbol{y}_I^\varepsilon) + \varepsilon \boldsymbol{v}_s(\boldsymbol{y}_I^\varepsilon) + \varepsilon^2 \boldsymbol{w}_s(\boldsymbol{y}_I^\varepsilon) - \boldsymbol{M}^\varepsilon \quad \text{and} \quad \boldsymbol{\theta}^\varepsilon_{I,s} := \boldsymbol{\theta}_s(\boldsymbol{y}_I^\varepsilon) - \boldsymbol{t}^\varepsilon. \tag{58}$$

It is clear that $\sum_I \boldsymbol{U}^\varepsilon_{I,1} = \sum_I \boldsymbol{\theta}^\varepsilon_{I,1} = 0$ and that the limits $\boldsymbol{U}^\varepsilon \rightharpoonup \boldsymbol{u}$ and $\boldsymbol{\theta}_s^\varepsilon \rightharpoonup \boldsymbol{\theta}_s$ still hold. As the energy $\mathcal{E}^\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ is invariant by the addition of a rigid motion we still have $\lim_{\varepsilon\to 0} \mathcal{E}^\varepsilon(\boldsymbol{U}^\varepsilon, \boldsymbol{\theta}^\varepsilon) = \mathcal{E}(\boldsymbol{u})$. □

Explicit computation of the homogenized energy

Cell problems

In the homogenized energy we obtained, i.e.,

$$\mathcal{E}(\boldsymbol{u}) := \inf_{\boldsymbol{w},\boldsymbol{v},\boldsymbol{\theta}}\{\bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}, \boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}) + \bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta}) : \bar{E}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, 0, 0) = 0\},$$

one has to compute the minimum with respect to three extra kinematic variables $\boldsymbol{w}$, $\boldsymbol{v}$, $\boldsymbol{\theta}$. Let us explain how that can be done.

We first remark that no partial derivative of $\boldsymbol{w}$ appears in the expression of $\bar{E}(\boldsymbol{w}, \boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}, \boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}})$. Thus the minimization with respect to $\boldsymbol{w}$ is a local cell problem in which $\boldsymbol{\xi}_{\boldsymbol{u},\boldsymbol{v}}$, $\boldsymbol{v}$, $\boldsymbol{\eta}_{\boldsymbol{u}}$ are three vectorial parameters. The linear problem associated with the finite dimensional unknown $\boldsymbol{w}$ can easily be solved. Similarly, no partial derivative of $\boldsymbol{\theta}$ appears in the expression of $\bar{F}(\boldsymbol{v}, \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\theta})$. The minimization with respect to $\boldsymbol{\theta}$ is a local cell problem in which $\boldsymbol{\eta}_{\boldsymbol{u}}$

is a vectorial parameter. Again the linear problem associated with the finite dimensional unknown $\boldsymbol{\theta}$ can easily be solved.

Minimizing with respect to $\boldsymbol{v}$ is more subtle. The constraint $\bar{E}(\boldsymbol{v},\boldsymbol{\eta_u},0,0)=0$ imposes $\boldsymbol{v}$ to minimize $\bar{E}(\boldsymbol{v},\boldsymbol{\eta_u},0,0)$ and this problem can be solved like the two first ones. The unknown $\boldsymbol{v}$ depends linearly on the "parameter" $\boldsymbol{\eta_u}$ (or equivalently on $\nabla\boldsymbol{u}$) up to an element of the kernel of the functional $\bar{E}(\boldsymbol{v},\boldsymbol{\eta_u},0,0)$. Indeed, this kernel is never trivial as it contains at least the uniform fields (i.e. the families $(\boldsymbol{v}_s)$ which do not depend on s) but it may be much larger. Hence we get a linear operator L such that $v=L\cdot\nabla\boldsymbol{u}+\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ is any field in $(\mathrm{L}^2(\Omega;\mathbb{R}^3))^K$ with values in the kernel of $\bar{E}(\boldsymbol{v},\boldsymbol{\eta_u},0,0)$. A first consequence is that the constraint $\bar{E}(\boldsymbol{v},\boldsymbol{\eta_u},0,0)=0$ leads to the following constraint on $\nabla\boldsymbol{u}$:

$$\bar{E}(L\cdot\nabla\boldsymbol{u},\boldsymbol{\eta_u},0,0)=0$$

Then we can replace $\boldsymbol{w}$, $\boldsymbol{\theta}$ and $\boldsymbol{v}$ by the solutions of these cell problems. The definition (19) of $\boldsymbol{\eta_u}$ and the dependence of $\boldsymbol{\xi_{u,v}}$ on $\nabla\boldsymbol{v}$ through (20) imply that the homogenized energy can be written as a quadratic functional of $\nabla\boldsymbol{u}$, $\nabla\nabla\boldsymbol{u}$, $\boldsymbol{\lambda}$ and $\nabla\boldsymbol{\lambda}$ or equivalently of $\boldsymbol{e}(\boldsymbol{u})$, $\nabla\boldsymbol{e}(\boldsymbol{u})$, $\boldsymbol{\lambda}$ and $\nabla\boldsymbol{\lambda}$:

$$\mathcal{E}(\boldsymbol{u})=\inf_{\boldsymbol{\lambda}\in L^2(\Omega)}\int_\Omega Q(\boldsymbol{e}(\boldsymbol{u}),\nabla\boldsymbol{e}(\boldsymbol{u}),\boldsymbol{\lambda},\nabla\boldsymbol{\lambda}), \tag{59}$$

where Q is a non negative quadratic form.

An extra kinematic variable $\boldsymbol{\lambda}$ still remains in the expression (59) but the minimization with respect to this variable cannot be performed locally as its gradient is involved. Indeed, many examples provided in[3] show that the effective behavior of frame lattices may correspond to materials which must be described in terms of extra kinematic variables (e.g. Cosserat, Timoshenko or Reissner models). Then, in general the limit model obtained is both a generalized continuum model and a second gradient model[3]. The variable λ can be interpreted as a *microadjustement*.

A general algorithm for making explicit the homogenized energy $\mathcal{E}(\boldsymbol{u})$ has been given in[3] for the particular case where $\boldsymbol{z}_s=0$. The procedure is quite similar in the more general case that we consider here. Only the canonical form of the functional energy $\bar{E}$ has to be modified as the expression of the limit extensional energy $\bar{E}$ obtained here is different from the one obtained in[3].

Back to the asymmetrical pantographic structure

The asymmetrical structure described in the third section enters our general description of lattices. The family $(\boldsymbol{y_i})$ is the one defined in (5). Family $(\boldsymbol{z_i})$ is

$$\boldsymbol{z_1}=\boldsymbol{z_2}=\boldsymbol{z_3}=\boldsymbol{z_6}=0,\quad \boldsymbol{z_4}=\boldsymbol{z_5}=(\alpha,0,0)\,. \tag{60}$$

Considering the mechanical interactions which were described and drawn in Figures 4-5-6 implies that coefficients

$$a_{1,1,2},\ a_{1,1,3},\ a_{1,1,4},\ a_{1,1,5},\ a_{1,2,5},\ a_{1,3,4},\ a_{1,4,6},\ a_{1,5,6},\ a_{2,4,3},\ a_{2,5,2},\ a_{2,6,2},\ a_{2,6,3},\ a_{3,1,1},\ a_{6,1,1},\ a_{9,1,1}$$

are the only non vanishing interaction coefficients.

To fix ideas, we consider the case where all bars are made of the same material with Poisson coefficient $\nu=0.3$ and have the same circular section with radius ε^2. Hence the non vanishing coefficients $a_{p,s,r}$ are proportional to $\ell_{p,s,r}^{-1}$. A simple choice of energy unit allows us to fix $a_{p,s,r}=\ell_{p,s,r}^{-1}$. We have $\beta=1$, thus bending and torsional rigidities are fixed. Applying the procedure described in the previous section is straightforward but needs the help of a computer at least for the determination of bending and torsion parts. Minimization with respect to $\boldsymbol{\lambda}$ is obtained in this particular case with $\boldsymbol{\lambda}=0$. So no extra kinematic variable is needed and the effective energy (59) reduces to the form (1). More specifically we get the homogenized elastic energy

$$\mathcal{E}(\boldsymbol{u})=\frac{1}{2}\int_\Omega\Big(\varpi\,(\boldsymbol{e}_{13}(\boldsymbol{u}))^2+\sigma\,(\boldsymbol{e}_{12}(\boldsymbol{u}))^2+\gamma\,(\boldsymbol{e}_{11}(\boldsymbol{u}))^2+\kappa\big(\big(\big(\frac{\partial^2\boldsymbol{u}_1}{\partial x_1^2}-8\alpha(\boldsymbol{e}_{11}(\boldsymbol{u}))\big)^2+\big(\frac{\partial^2\boldsymbol{u}_2}{\partial x_1^2}\big)^2\big)\Big)\,d\boldsymbol{x}$$

while the constraint $\inf_{\boldsymbol{v}}\bar{E}(\boldsymbol{v},\boldsymbol{\eta_u},0,0)=0$ reads $\boldsymbol{e}_{22}(\boldsymbol{u})=\boldsymbol{e}_{33}(\boldsymbol{u})=\boldsymbol{e}_{23}(\boldsymbol{u})=0$. The effective material coefficients are $\varpi\approx4.3$, $\sigma\approx7.1$, $\gamma\approx307.5$, $\kappa=\frac{12}{47}$.

Conclusions

In this article we have provided a new rigorous homogenization result for periodic structures made of welded elastic bars. We have shown that allowing the positions of the nodes of the structure inside the rescaled periodic cell to depend on the small parameter ε, may lead to a strain-gradient model in which strain-gradient and classical strain terms are coupled. We have illustrated the astonishing effects of such a coupling on the equilibrium solutions. Note that, for this coupling to arise, the nodes positions usually assumed to be $\boldsymbol{y}_{(i,j,k),s} := \varepsilon(\boldsymbol{y}_s + i\boldsymbol{t}_1 + j\boldsymbol{t}_2 + k\boldsymbol{t}_3)$ are modified at the order ε^2 within each cell.

To summarize the results of the present study and those of[1], let consider a generic modification $\varepsilon^\alpha \boldsymbol{z}_s$ of the positions of the nodes within the periodic lattice.

- If $\alpha = 1$, then $\boldsymbol{y}_s$ is simply replaced by $\boldsymbol{y}_s + \boldsymbol{z}_s$. The new geometry can still lead to a homogenized uncoupled strain-gradient model but, in the generic case, the structure will become rigid (see[1]).
- If $1 < \alpha < 2$ then, in the generic case, the structure will again become rigid.
- When $\alpha = 2$, as in the present study, then an effective energy with a coupling between classical strain and strain-gradient terms can be obtained.
- A modification with $\alpha > 2$ will have no effect.

These results may explain how difficult it is to build in practice and characterize experimentally an efficient micro-architectured structure leading to a strain-gradient material. Indeed they show how sensitive is the effective model to the design of the lattice and to manufacturing defects.

Since second gradient effects are very difficult to measure experimentally, in order to get an experimental evidence of them it would be interesting to: (i) seek optimal topologies for frame structures, (ii) extend our study to non-linear elasticity to take into account geometrical non-linearities that may arise in micro-structures. Some works in these directions can be found in the literature[7,8] but, up to our knowledge, in the context of nonlinear elasticity, second gradient materials have only been obtained heuristically[19].

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