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Technical note for the FS-MMSE-IC receiver

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1 Definitions

For any scalars or vectors $x$ and $y$ with convenient sizes, we define:

\[
\begin{cases}
x_n &= e_n^\top x \\
R_{xy} &= E_{x,y}[xy^\top] \\
R_x &= R_{xx}
\end{cases}
\]

where $e_n$ is the $(n+1)$th element of the canonic base. The operator $E_{x,y}[xy^\top]$ represents the statistical average with respect to $x$ and $y$.

For any matrix $A$, we define $A^\ast$ the conjugate matrix of $A$, $A^T$ its transpose matrix and $A^\dagger = (A^T)^\ast$.

2 Context & Problem formulation

We suppose having the following expressions:

\[
\begin{cases}
r &= \text{HU}s + w \\
I(f, p) &= f^\dagger r + p \\
J(f, p) &= E_{I(f, p)} |I(f, p) - s_n|^2
\end{cases}
\]

where:

- $f$: a $(2N \times 1)$ vector,
- $p$: a scalar,
- $H$: a $(2N \times 2Ns)$ deterministic convolution matrix,
- $U$: a $(2Ns \times Ns)$ up-sampling matrix,
- $D = U^\dagger$: a down-sampling matrix,
- $s$: a $(Ns \times 1)$ probabilistic vector such that $E[s] = s^d$ and $R_{s_n,s_m} = (v_n^d + |s_n^d|^2)\delta(n-m) \implies E[(s - s^d)(s - s^d)^\dagger] = V^d = \text{diag}(v^d),$
- $w$: a $(2N \times 1)$ probabilistic vector such that $E[w] = 0_N$ and $R_{w} = \sigma_w^2 I_N,$
Typically, the vector \( r \) represents a band-limited received signal after the convolution operation of the shaping filter and the channel \( H \), corrupted by a centered circular white noise \( w \) with average power \( \sigma_w^2 \). Moreover, this holds if and only if the shaping filter bandwidth is bounded by \( \frac{2}{T_s} \). Hence, \( r \) represents a Fractionally-Spaced (FS) system model where \( I(f, p) \) is the estimator of \( s \) with Mean Square Error (MSE) \( J(f, p) \). Furthermore, considering the prior information \((s^d, V^d)\) on \( s \) leads to an Interference-Cancellation (IC) structure.

**Problem formulation:** We search for \( \hat{s}^c_n = I(f_n, p_n) \), \( \hat{v}^c_n = J(f_n, p_n) \) such that:

\[
(f_n, p_n) = \operatorname{argmin}_{(f, p)} J(f, p)
\]

Consequently, \( \hat{v}^c_n \) represents the Minimum MSE of the considered system model (FS-MMSE-IC) and with estimator \( \hat{s}^c_n \) of \( s_n \).

### 3 Problem resolution

#### 3.1 Expression of the MSE minimizers

\[
J(f, p) = \mathbb{E}[I(f, p)]^2 - s_n^2 = f^\dagger R_f f + f^\dagger \mathbb{E}[r] p^* f + p \mathbb{E}[|r|^2] + 2 \mathbb{E}[s_n^*] R_s - \mathbb{E}[s_n] p^* + \sigma_w^2 \]

Minimizing \( J(f, p) \) leads us to nullify its the first partial derivatives.

\[
\frac{\partial J(f, p)}{\partial f} = 0 \iff 2R_f f + 2\mathbb{E}[r] p^* - 2R_{rs} = 0 \iff f = R_f^{-1}(R_{rs} \mathbb{E}[r] p^*)
\]

\[
\frac{\partial J(f, p)}{\partial p^*} = 0 \iff 2\mathbb{E}[r] f + 2p^* - 2\mathbb{E}[s_n^*] = 0 \iff p = \mathbb{E}[s_n] - f^\dagger \mathbb{E}[r]
\]

We check that such an extremum corresponds to a minimum:

\[
\begin{cases}
\frac{\partial^2 J(f, p)}{\partial f^2} = 2R_f \geq 0 \\
\frac{\partial^2 J(f, p)}{\partial p^*} = 2 \geq 0
\end{cases}
\]

#### 3.2 Computation of the MSE minimizers

Let us first compute:

\[
\begin{align*}
\mathbb{E}[s_n] &= s_n^d \\
\mathbb{E}[r] &= HUs^d \\
R_f &= HUR_s \mathbb{E}[r] + \sigma_w^2 I_N \\
R_{rs} &= HUR_{rs_n}
\end{align*}
\]
Then, the MSE minimizers can be computed:

\[
f_n^\dagger = \left( R_{s_n r} - p_n E[r^\dagger] \right) R_r^{-1} \\
= \left( R_{s_n r} - E[s_n E[r^\dagger]] + f_n^\dagger \| E[r] \|^2 \right) R_r^{-1} \\
= \left( R_{s_n r} - E[s_n E[r^\dagger]] \right) R_r^{-1} \left( I_N - \| E[r] \|^2 R_r^{-1} \right)^{-1} \\
= \left( R_{s_n r} - E[s_n E[r^\dagger]] \right) \left( R_r^{-1} - \| E[r] \|^2 \right)^{-1} \\
= \left( R_{s_n r} - E[s_n E[s^\dagger]] \right) \left( DH \left( HUV d^\dagger DH + \sigma_w^2 I_{2N} \right) \right)^{-1} \\
= v_n^d e_n^\dagger DH^\dagger \left( HUV d^\dagger DH + \sigma_w^2 I_{2N} \right)^{-1}
\]

Using twice the Woodbury identity on any \((K_1 \times K_2)\) matrix \(A\) and \((K_2 \times K_1)\) matrix \(B\):

\[
A(BA + I_{K_2})^{-1} = A(BI_{K_1} A + I_{K_2})^{-1} \\
= A(I_{K_2} - B(I_{K_1} + AB)^{-1} A) \\
= (I_{K_2} - AB(I_{K_1} + AB)^{-1} A) \\
= (ABI_{K_1} + I_{K_1})^{-1} A \\
= (AB + I_{K_1})^{-1} A
\]

We identify \(A = \sigma_w^{-2} DH^\dagger\) leading to:

\[
f_n^\dagger = v_n^d e_n^\dagger DH^\dagger \left( HUV d^\dagger DH + \sigma_w^2 I_{2N} \right)^{-1} \\
= v_n^d e_n^\dagger \left( DH^\dagger HUV d^\dagger + \sigma_w^2 I_N \right)^{-1} DH^\dagger \\
= v_n^d e_n^\dagger \Sigma^{-1} DH^\dagger
\]

where \(\Sigma = GV^d + \sigma_w^2 I_N\) is the MMSE equalization matrix and \(G = DH^\dagger HU\) is also called global filter and gathers the up-sampling, channel and matched filtering and down-sampling operations.

\[
p_n = E[s_n] - f_n^\dagger E[r] \\
= s_n^d - v_n^d e_n^\dagger \Sigma^{-1} Gs^d
\]

### 3.3 Computation of the FS-MMSE-IC and its estimator

The FS-MMSE-IC estimator is given by:

\[
\hat{s}_n^c = f_n^\dagger r + p_n \\
= s_n^d + v_n^d e_n^\dagger \Sigma^{-1} (DH^\dagger r - Gs^d) \\
= s_n^d + v_n^d e_n^\dagger \Sigma^{-1} (y - Gs^d)
\]

where \(y = DH^\dagger r\) represents the received signal after matched filtering and down-sampling.
The FS-MMSE-IC $\hat{c}_n^c$ of the estimator $\hat{s}_n^c$ can be computed by first deriving:

$$R_{\hat{c}^c_n} = |s_n^d|^2 + 2(s_n^d)^*v_n^d e_n^d \text{Re} (\Sigma^{-1}E_y([y - G s^d])) + (v_n^d)^2 e_n^d \Sigma^{-1}E_y((y - G s^d)(y - G s^d)^\dagger) \Sigma^{-1} e_n$$

$$= |s_n^d|^2 + (v_n^d)^2 e_n^d \Sigma^{-1}E_s, w |G(s - s^d) + DH^I w|^2 \Sigma^{-1} e_n$$

$$= |s_n^d|^2 + (v_n^d)^2 e_n^d \Sigma^{-1}DH^I E_s, w |HU(s - s^d) + w|^2 \Sigma^{-1} e_n$$

$$= |s_n^d|^2 + (v_n^d)^2 e_n^d \Sigma^{-1}DH^I (HU^d DH^I + \sigma_w^2 I_N) HU \Sigma^{-1} e_n$$

$$= |s_n^d|^2 + (v_n^d)^2 e_n^d \Sigma^{-1} (DH^I HU^d + \sigma_w^2 I_N) DH^I HU \Sigma^{-1} e_n$$

$$= |s_n^d|^2 + (v_n^d)^2 e_n^d \Sigma^{-1} Ge_n$$

$$R_{x_n, \tilde{s}_n} = E_y[(s_n^d + v_n^d e_n^d \Sigma^{-1}(y - G s^d)) s_n^*]$$

$$= |s_n^d|^2 + (v_n^d)^2 e_n^d \Sigma^{-1} Ge_n (s - s^d) s_n^*$$

$$= |s_n^d|^2 + (v_n^d)^2 e_n^d \Sigma^{-1} Ge_n$$

$$R_{s_n} = |s_n^d|^2 + v_n^d$$

This leads to the FS-MMSE-IC $\hat{c}_n^c$ expression:

$$\hat{c}_n^c = E_{s_n^c} \left[ |s_n^c - s_n|^2 \right]$$

$$= R_{\hat{c}^c_n} - R_{x_n, \tilde{s}_n} - R_{s_n, \tilde{s}_n} + R_{s_n}$$

$$= R_{s_n} - R_{s_n, \tilde{s}_n}^c$$

$$= |s_n^d|^2 + v_n^d - |s_n^d|^2 - (v_n^d)^2 e_n^d \Sigma^{-1} Ge_n$$

$$= v_n^d (1 - v_n^d c_n)$$

where $c_n = e_n^d \Sigma^{-1} Ge_n$.

### 3.4 Unbiased FS-MMSE-IC and its estimator

We can note that the estimator $\hat{s}_n^c$ of $s_n$ has a residual bias:

$$E[s_n^d | E[s_n] = \mu] = E_{s_n \sim (n), w} [s_n^d + v_n^d e_n^d \Sigma^{-1} (y - G s^d) | E[s_n] = \mu]$$

$$= s_n^d + v_n^d e_n^d \Sigma^{-1} Ge_n (s - s^d | E[s_n] = \mu)$$

$$= s_n^d + (\mu - s_n^d) v_n^d c_n$$

$$= v_n^d c_n \mu + s_n^d (1 - v_n^d c_n)$$

where the vector $s_n \sim (n)$ represents all the components of $s$ but $s_n$.

Removing the additive term $s_n^d (1 - v_n^d c_n)$ and dividing the multiplicative term $v_n^d c_n$ leads to:

$$s_n^c = (v_n^d c_n)^{-1} (s_n^c - s_n^d (1 - v_n^d c_n))$$

$$= s_n^d + c_n^{-1} e_n^d \Sigma^{-1} (y - G s^d)$$
We compute the new statistical matrices:

\[
R_{s_n} = |s_n|^2 + c_n e_n^\dagger \Sigma^{-1} \mathbb{E}_Y [(y - Gs^d) (y - Gs^d)^\dagger] \Sigma^{-1} e_n
\]

\[
= |s_n|^2 + c_n^{-1}
\]

\[
R_{s_n, s_n} = \mathbb{E}_Y [(s_n^d + c_n e_n^\dagger \Sigma^{-1} (y - Gs^d)) s_n^\dagger]
\]

\[
= |s_n|^2 + c_n e_n^\dagger \Sigma^{-1} G \mathbb{E}_s [(s - s^d) s_n^\dagger]
\]

\[
= |s_n|^2 + v_n^d
\]

\[
R_{s_n} = |s_n|^2 + v_n^d
\]

And finally we find the unbiased FS-MMSE-IC:

\[
v_n^c = \mathbb{E}_{s_n} |s_n^c - s_n|^2
\]

\[
= R_{s_n} - R_{s_n, s_n} - R_{s_n, s_n} + R_{s_n}
\]

\[
= |s_n|^2 + c_n^{-1} - |s_n^d|^2 - v_n^d - |s_n|^2 + v_n^d + |s_n^d|^2 + v_n^d
\]

\[
= c_n^{-1} - v_n^d
\]

4 Conclusion

The biased FS-MMSE-IC receiver is given by:

\[
\begin{align*}
\tilde{s}_n^c &= s_n^d + v_n^d e_n^\dagger \Sigma^{-1} (y - Gs^d) \\
\tilde{v}_n^c &= v_n^d (1 - v_n^d c_n)
\end{align*}
\]

where \(c_n = e_n^\dagger \Sigma^{-1} G e_n\) is a scalar, \(y = DH^\dagger r\) is the receive signal after matched filtering and down-sampling, \(G = DH^\dagger H U\) is the global effect including up-sampling, channel and matched filtering and down-sampling, \(\Sigma = GV^d + \sigma^2_w I_N\) is the MMSE equalization matrix.

Moreover, the unbiased FS-MMSE-IC receiver is given by:

\[
\begin{align*}
\tilde{s}_n^c &= s_n^d + c_n^{-1} e_n^\dagger \Sigma^{-1} (y - Gs^d) \\
\tilde{v}_n^c &= c_n^{-1} - v_n^d
\end{align*}
\]

Consequently, the FS-MMSE-IC receiver leads to a FS matched filter followed by a symbol time equalizer.