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On the Antimagic Labeling of (1,q)-polar and (1,q)-decomposable Graphs

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In this paper the graphs yielded by the Algebraic Graph Decomposition theory are used to study the Hartsfield-Ringel conjecture on the antimagicness of connected graphs. This way some results on the conjecture are obtained, namely the antimagicness of connected (1, 2)-polar and (1, 2)-decomposable graphs, as well as connected (1, q)-polar and (1, q)-decomposable graphs satisfying some specific conditions.

Keywords: antimagic graphs, Hartsfield-Ringel conjecture, Algebraic Graph Decomposition, (1, 2)-polar graphs, (1, 2)-decomposable graphs, (1, q)-polar graphs, (1, q)-decomposable graphs

In early 90s [Hartsfield and Ringel (1990)] introduced the notion of an antimagic numeration of graph edges. The graphs, which accept such a numeration, were named antimagic. A conjecture was also made, that all the connected graphs are antimagic.

In general this conjecture is still neither proven nor disproven, though a lot of articles on the subject do exist. This tells us, that the conjecture in question is interesting to the specialists in graph theory, but is difficult enough to be unresolved for almost 30 years.

The Algebraic Graph Decomposition theory (or AGD) was developed by R. Tyshkevich and her students to solve some algorithmic problems and count some characteristics of graphs. But recent results show, that this theory can also be a tool for studying conjectures, and there is a possibility, that the AGD can help prove or disprove the Hartsfield-Ringel conjecture as well.

1 Introduction

Let $G = (V, E)$ be an $(n, m)$-graph and $\varphi : E \rightarrow \{1, 2, \ldots, m\}$ be some injective function.

Let $f$ be a function on $V$, such that for all $v \in V f(v) = \sum e \varphi(e)$, where $e$ runs through all the edges, incident to $v$ (if $v$ is isolated, let $f(v) = 0$). If such an $f$ is also injective, then $\varphi$ is called an antimagic numeration and graph $G$, which has such a numeration, is called an antimagic graph.

Conjecture 1 Any connected graph, except $K_2$, is antimagic.

Note that $K_2$ is obviously not antimagic: it has only one edge and one numeration, which does not satisfy our condition.

All of the results on the antimagicness of graphs up to date are achieved by narrowing the problem to some smaller classes of graphs.
This way, Barrus (2010) has proven that connected split and 1-decomposable graphs are antimagic. This result is interesting due to it being the first try to use the graphs yielded by the AGD to study the Hartsfield-Ringel conjecture – and a successful one at that. Barrus used only the most simple form of decomposition, which nevertheless led to an elegant and substantial result. It brings us to the conclusion, that the AGD has a potential in studying the conjecture (as well as other conjectures, see Tyshkevich et al. (2013) for more information), which proves to be true, as shown below.

2 Preliminary information

If graph $G = (V, E)$ and $H \subseteq V$, then by $G(H)$ we’ll denote the induced subgraph of $G$ on the vertices from $H$.

By $X = Y \sqcup Z$ we’ll denote the disjoined partition of some set $X$ into its subsets $Y$ and $Z$, i.e. such a partition, where $Y \cap Z = \emptyset$.

A graph $G$ is called split, if there is such a partition $V = A \sqcup B$, that $G(A)$ is a clique and $G(B)$ is an independent set of vertices. A graph $G$ is called 1-decomposable, if there is such a partition $V = A \sqcup B \sqcup C$, that $G(A)$ is a clique, $G(B)$ is an independent set of vertices and $G(C)$ is an arbitrary graph, connected to $G(A)$ by all the possible edges (i.e. the edges of a complete bipartite graph with parts $A$ and $C$).

Let us consider two more general classes of graphs, namely, $(1, q)$-polar and $(1, q)$-decomposable graphs.

Let $q$ be a positive integer.

A graph $G$ is called $(1, q)$-polar, if there is such a partition $V = A \sqcup B$, that $G(A)$ is a clique and $G(B)$ is a disjoint union of cliques of order no greater than $q$.

A graph $G$ is called $(1, q)$-decomposable, if there is such a partition $V = A \sqcup B \sqcup C$, that $G(A)$ is a clique, $G(B)$ is a disjoint union of cliques of order no greater than $q$ and $G(C)$ is an arbitrary graph, connected to $G(A)$ by all the possible edges (i.e. the edges of a complete bipartite graph with parts $A$ and $C$).

It’s easy to see, that split graphs and 1-decomposable graphs are, in fact, $(1, q)$-polar graphs and $(1, q)$-decomposable graphs respectively with $q = 1$. Thus, $(1, q)$-polar and $(1, q)$-decomposable graphs are indeed generalizations of split and 1-decomposable ones.

We’ll call $G(A)$ the upper part and $G(B)$ the lower part of the graph $G$.

The algorithm, presented by Barrus (2010), can, with a few modifications, be applied to connected $(1, 2)$-polar and $(1, 2)$-decomposable graphs as well, as shown below.

The further attempts to generalize this algorithm for $q \geq 3$, however, have not been so successful. As it turned out, the algorithm requires some certain properties of the graph to work correctly. And these properties, while holding for $q = 1$ and $q = 2$, begin to fail for $q = 3$, and the larger is $q$ the worse it gets. We’ll discuss the specifics of this problem in the corresponding section.

And now let’s move on to the algorithm for $q = 2$.

3 The algorithm for $(1,2)$-polar and $(1,2)$-decomposable graphs

Let $q = 2$.

Note that we can assume that $G(C)$ does not contain cliques of order less or equal than 2, isolated in $G(C)$, otherwise they can be included into $G(B)$.
If \( C = \emptyset \) we can similarly assume that \( B \) does not contain vertices, adjacent to every vertex in \( A \), otherwise they can be included into \( A \).

In case \( C = \emptyset \) we can also assume \( |A| \geq 2 \), otherwise we can take any edge \( ab \), where \( a \in A \) and \( b \in B \) and include \( b \) into \( G(A) \). In case \( C \neq \emptyset \) any \( |A| \geq 1 \) is possible.

Having made these assumptions we can now guarantee that \( \deg b \leq \deg c \leq \deg a \) for any arbitrary \( a \in A, b \in B, c \in C \).

**Algorithm 1.**

**Input:** \( G = (V, E) \) – a connected \((1,2)\)-polar or \((1,2)\)-decomposable \((n,m)\)-graph, \( V = A \sqcup B \sqcup C \).

**Output:** an antimagic numeration of \( G \) in form of an injective \( \varphi : E \rightarrow \{1,2,\ldots,m\} \).

**Begin**

**Step 1.** Let \( A = \{a_1, \ldots, a_{|A|}\}, B = \{b_1, \ldots, b_{|B|}\}, C = \{c_1, \ldots, c_{|C|}\} \), where the vertices are indexed in non-decreasing order of degrees.

Let \( B^1 \subseteq B \) be the subset of vertices isolated in \( G(B) \). Let \( B^2 \subseteq B \) be the subset of vertices belonging to all the \( K_2 \) in \( G(B) \). Note that \( B = B^1 \sqcup B^2 \).

The vertices from \( B^i \) will be denoted as \( b^i_j, i \in \{1,2\}, j \in \{1,\ldots,|B^i|\} \).

**Step 2.** Find all the vertices from \( B^2 \) of degree 1. Sort their incident edges \( b^2_1b^2_2 \) lexicographically; assign to them the first numbers in order. Function \( \varphi \) for such edges is constructed.

Find all the vertices from \( B^1 \) of degree 1. Sort their incident edges \( b^1_1a_j \) lexicographically; assign to them the next numbers in order. Function \( \varphi \) for such edges is constructed.

Note: all the edges incident to endpoints are labeled.

**Step 3.** Sort all the remaining edges \( b^2_1b^2_2 \) lexicographically; assign to them the next numbers in order. Function \( \varphi \) for such edges is constructed.

Define \( g^2_B : B^2 \rightarrow \mathbb{N} \) to be the sum of the numbers on the already labeled edges, incident to each \( b^2 \in B^2 \).

**Step 4.** Re-order \( B \) by vertex degrees in the non-decreasing order with three additional conditions:

4.1 If the degrees of the vertices from \( B^1 \) and \( B^2 \) are equal, first we index the vertices from \( B^2 \).

4.2 If the degrees of the vertices from \( B^2 \) are equal, we index the vertices by the non-increasing order of \( g^2_B \).

4.3 If the degrees of the vertices from \( B^1 \) are equal, we index the vertices by the non-increasing order of the sum of the indices, belonging to the vertices from \( A \), adjacent to those vertices from \( B^1 \).

So, after the re-ordering, we get \( B = \{b_1, \ldots, b_{|B|}\} \), where the vertices are indexed in non-decreasing order of degrees with additional conditions 4.1-4.3.

**Step 5.** Sort all the edges \( b_ia_j \) lexicographically; assign to them the next numbers in order (note that the endpoint edges are already labeled, so they are skipped). Function \( \varphi \) for such edges is constructed.
Step 6. Define $g_A : A \rightarrow \mathbb{N} \cup \{0\}$ to be the sum of the labels, assigned to the already labeled edges, which are incident to each $a \in A$, but don’t belong to $G(A)$ (not all of them are labeled at the moment, it’s even possible that none are). Re-order $A$ by the non-decreasing order of $g_A$: 
\[ A = \{a_1, \ldots, a_{|A|}\}, \text{where } g_A(a_i) \leq g_A(a_j) \text{ if } i < j. \]

Step 7. Sort all the edges $c_i c_j$ lexicographically; assign to them the next numbers in order. Function $\varphi$ for such edges is constructed.

Define $g_C : C \rightarrow \mathbb{N}$ to be the sum of the labels, assigned to the edges from $G(C)$, incident to each $c \in C$. Note that the vertices of $C$ are indexed by the non-decreasing order of $g_C$.

Step 8. Sort all the edges $c_i a_j$ lexicographically; assign to them the next numbers in order. Function $\varphi$ for such edges is constructed. Note that the vertices of $A$ are still indexed by the non-decreasing order of $g_A$.

Step 9. If $|A| = 1$ (which is possible only if $C \neq \emptyset$), then return $\varphi$ and end the algorithm (all the edges are already labeled).

Otherwise, sort all the edges $a_i a_j$ lexicographically; assign to them the remaining numbers in order. Function $\varphi$ for such edges is constructed.

Step 10. Return $\varphi$.

End

4 The proof of antimagicness

Statement 1 Let $G = (V, E)$ be a connected $(1, 2)$-polar or $(1, 2)$-decomposable $(n, m)$-graph and let $\varphi : E \rightarrow \{1, 2, \ldots, m\}$ be its numeration, constructed by Algorithm 1.

Then $f(v) = \sum_e \varphi(e)$ is injective on $V$ and $\varphi$ is antimagic.

Proof:

- For all $b_i, b_j \in B, i < j$ we get:
  
  If $b_i, b_j \in B^1$ or $b_i \in B^2, b_j \in B^1$, then $f(b_i) < f(b_j)$, because $\deg b_i \leq \deg b_j$ and all the edges incident to $b_j$ get their numbers later than all the edges incident to $b_i$.
  
  If $b_i \in B^1, b_j \in B^2$, then $f(b_i) < f(b_j)$ because $\deg b_i < \deg b_j$ (otherwise $b_i$ would be indexed later than $b_j$ and $i > j$) and there are at least $\deg b_i$ edges, incident to $b_j$, getting their numbers later than all the $\deg b_i$ edges, incident to $b_i$.
  
  So, $f$ is injective on $B$.

- For all $a \in A, b \in B$ we have $\deg b \leq \deg a$ as stated earlier. Vertex $a$ is adjacent to $|A| - 1$ vertices from $A$ and possibly some vertices from $B$, and vertex $b$ can be adjacent to $|A| - 1$ or less vertices from $A$ and possibly some vertices from $B$. Meanwhile, the numbers on $ba_i$ are strictly less than ones on $aa_i$, and possible numbers on $bb_i$ are strictly less than those on $ab_j$, which in their turn
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are lesser than numbers on \( ab_k \). Therefore, for each edge incident to \( b \), there is an edge incident to \( a \) with a greater number (except, possibly, the edge \( ab \), contributing equally to \( f(a) \) and \( f(b) \) both). So, \( f(b) < f(a) \).

- For all \( b \in B, c \in C \) we have \( \text{deg} b \leq \text{deg} c \) as stated earlier. The edges incident to \( b \) get their numbers earlier, than those incident to \( c \), therefore \( f(b) < f(c) \).

- For all \( c_i, c_j \in C \) with \( i < j \) we have \( g_C(c_i) \leq g_C(c_j) \), and because for any \( a \in A \) the edge \( c_i a \) get its number earlier, than \( c_j a \), we get \( f(c_i) < f(c_j) \), i.e. \( f \) is injective on \( C \).

- For any \( a \in A, c \in C \) we have \( \text{deg} c \leq \text{deg} a \), and the edges incident to \( c \) get their numbers earlier, than those incident to \( a \), so \( f(c) < f(a) \).

- If \( |A| = 1 \) then all the sums are already checked to be different and the numeration is antimagic. Otherwise, it remains only to check the injectivity of \( f \) on \( A \) (when \( |A| \geq 2 \)).

First we consider \( |A| \geq 3 \).

For all \( a_i, a_j \in A \) with \( i < j \) we have \( g_A(a_i) \leq g_A(a_j) \), while for any \( a_k \in A \), where \( k \neq i, k \neq j \) the edge \( a_i a_k \) gets its number earlier than \( a_j a_k \). So \( f(a_i) < f(a_j) \) and \( f \) is injective on \( A \) (if \( |A| \geq 3 \)).

Let’s get back to \( |A| = 2 \).

We’ll have to consider two cases here, namely \( C = \emptyset \) and \( C \neq \emptyset \) (i.e., (1,2)-polar graphs and (1,2)-decomposable graphs separately).

That’s because if \( C = \emptyset \), then after Step 6 of the algorithm we go straight to Step 9, and due to \( |A| = 2 \) we have only one edge in the upper part, which cannot affect the relation between \( f(a_1) \) and \( f(a_2) \), and those can generally turn out to be equal after Step 6. It leads to the necessity of considering eight small subcases, which can be rather easily solved as is shown in the next section.

Meanwhile for \( C \neq \emptyset \) this problem does not arise at all: after Step 6 we have \( g_A(a_1) \leq g_A(a_2) \), and after Step 8 for any \( c \in C \) the edge \( ca_1 \) gets its number earlier than \( ca_2 \). Therefore, \( f(a_1) < f(a_2) \).

So, if \( |A| = 2 \), \( f \) is also injective on \( A \) (probably, after some adjustments to \( \varphi \), see the next section).

Hence, our numeration is antimagic. The following theorem is thus proven. \( \square \)

**Theorem 1** Connected (1,2)-polar and (1,2)-decomposable graphs are antimagic.

5 Special case: (1,2)-polar graphs with 2 vertices in \( A \)

To finish the proof of Theorem 1 we need to consider the case \( C = \emptyset \) and \( |A| = 2 \).

More specifically, we’ll prove the next statement:

**Statement 2** Let \( G = (V, E) \) be a connected (1,2)-polar \((n, m)\)-graph with \( |A| = 2 \) and let \( \varphi : E \to \{1, 2, \ldots, m\} \) be its numeration, constructed by Algorithm 1.

Then we can exchange some numbers on the edges of \( G \) so that \( f(v) = \sum e \varphi(e) \) is injective on \( V \) and \( \varphi \) is antimagic.
Proof: Note, that due to the proof of Theorem 1 so far, $a_1$ and $a_2$ are the only vertices, where $f$ can have equal values. So, if after the algorithm has finished working we have $f(a_1) \neq f(a_2)$, then $f$ is injective on $V$ and our numeration is antimagic.

If $f(a_1) = f(a_2)$, we’ll have to consider all the possible subcases.

Subcase 1 The neighborhoods of $a_1$ and $a_2$ have at least one endpoint (see Fig. 1).

![Fig. 1: Subcase 1](image)

Without loss of generalization, $i < j$ and $i_1 < j_1$.

Exchange the indices between $b_i$ and $b_{i_1}$, and also the numbers $i_1$ and $j_1$ at the corresponding edges. We get: all the corresponding inequalities from the proof of Theorem 1 hold (note that $\deg b \leq 2$ for any $b \in B$), and $f(a_1) > f(a_2)$ due to $i_1 < j_1$. The numeration is antimagic now.

Subcase 2 The neighborhoods of $a_1$ and $a_2$ have at least one chain $P_3$ each.

Subcase 3 The neighborhoods of $a_1$ and $a_2$ have at least one triangle $C_3$ each.

Subcase 4 Vertices $a_1$ and $a_2$ belong to a cycle $C_4$.

These subcases are very similar to the first one and are omitted for briefness.

Note that if $f(a_1) = f(a_2)$, then it is not possible to have endpoints in the neighborhood of $a_1$ but not in the neighborhood of $a_2$, otherwise all the vertices adjacent to $a_2$ would have degrees at least 2 and their incident edges would be labeled later, than those of $a_1$. This would mean $f(a_1) < f(a_2)$.

Due to this note, we are left with 4 more subcases to consider.

Subcase 5 The 2-neighborhood of $a_1$ doesn’t have any endpoints, but has a $P_3$, and the neighborhood of $a_2$ has an endpoint. Let $k \geq 1$ be the largest index of a vertex belonging to any $P_3$ from the 2-neighborhood of $a_1$, then the smallest index of an endpoint adjacent to $a_2$ is equal to $k + 1$ (see Fig. 2).

Exchange the indices between $b_k$ and $b_{k+1}$, and also the numbers $k$ and $k + 1$ at the corresponding edges. We get: for all the endpoints from $B$ the bounds for $f$ hold; for vertices of degree 2 from $B$ only $f(b_1)$ has changed (increased by 1). But $f(b_1)$ was already the largest among all the $f(b_i)$ for vertices $b_i$ belonging to all the chains $P_3$, so it stays the largest. In the meantime, if $a_1$ or $a_2$ had any triangles in their neighborhoods, then minimal $f(b_j)$ for vertices
b_j belonging to all the triangles would be greater than f(b_l) at least by 2 (because the endpoint edges are numbered by the algorithm earlier than all the others, and b_{k+1} \in B^1 does exist). So, f stays injective on B. Moreover, after we exchange indices and numbers, f(b_l) = k_l + k + 1 and f(a_2) = a + k + x, where x \geq 0. We have k_l + 1 \leq a, therefore f(b_l) \leq f(a_2), and the equality is possible if and only if x = 0 and k_l + 1 = a. But then 2 = \deg a_2 = \deg a_1, and we get that our whole graph is just a P_4, which is obviously antimagic (and also f(a_1) \neq f(a_2) in the first place, so this case isn’t even possible). So f(b_l) < f(a_2). And f(a_2) < f(a_1) due to k < k + 1. The numeration is antimagic now.

Subcase 6 The 2-neighborhood of a_1 doesn’t have any endpoints or chains P_3, but has a triangle C_3, the 2-neighborhood of a_2 has an endpoint and a P_3, but no triangles. Let k \geq 1 be the largest index of a vertex belonging to any P_3 from the 2-neighborhood of a_2, k + l (where l \geq 1) be the largest index of an endpoint adjacent to a_2, t \geq 1 be the number of triangles, containing a_1 (note, that 2l \leq k + l) (see Fig. 3).

Exchange the numbers 2k + l + t and k + l + t + 1. We get: for all the endpoints from
Subcase 8 The 2-neighborhood of \(a\) the bounds for \(f\) hold; for vertices of degree 2 from \(B\) only \(f(b_{2k+l})\) and \(f(b_{2k+l+1})\) have changed (increased and decreased by 1 respectively). After we exchange the numbers: \(f(b_{2k+l}) = 2k+l+t+1+k = 3k+l+t+1\); \(f(b_{2k+l+1}) = 2k+l+t+k+l+1 = 3k+2l+t+1\). And due to \(l \geq 1\) we have \(f(b_{2k+l}) < f(b_{2k+l+1})\), here the bounds for \(f\) also hold. So, all the bounds for \(f\) hold, \(f(a_1) < f(a_2)\) due to \(2k+l+t < 2k+l+t+1\). The numeration is antimagic now.

Subcase 7 The 2-neighborhood of \(a_1\) doesn’t have any endpoints or chains \(P_3\), but has a triangle \(C_3\), the 2-neighborhood of \(a_2\) has an endpoint, but no chains \(P_3\) or triangles. Let \(k \geq 1\) be the largest index of an endpoint adjacent to \(a_2\), \(t \geq 1\) be the number of triangles, containing \(a_1\) (note, that \(2t \leq k\)) (see Fig. 4).

![Fig. 4: Subcase 7](image)

"Shift" the numbers: \(k + 3t\) replaces \(k\), all numbers on the triangles decrease by 1 (\(k\) instead of \(k + 1\), \(k + 1\) instead of \(k + 2\) and so forth). We get: for all the endpoints from \(B\) the bounds for \(f\) hold, only the largest of them has increased by \(3t\); for vertices of degree 2 from \(B\) the bounds for \(f\) hold (they have all decreased by the same amount). After we exchange the numbers: \(f(b_k) = k + 3t\), \(f(b_{k+1}) = 2k + t\). Knowing \(2t \leq k\), we get \(f(b_k) \leq f(b_{k+1})\), and the equality is possible if and only if \(2t = k\). Suppose it is. Evaluating \(f(a_1)\) and \(f(a_2)\), writing the equality \(f(a_1) = f(a_2)\) and simplifying it we get \(t - 9t^2 = 0\) for an integer \(t\); that means \(t = 0\), but \(t \geq 1\), contradiction. Therefore, \(f(b_k) < f(b_{k+1})\) and all the bounds for \(f\) hold. And \(f(a_1) < f(a_2)\) due to the first sum decreasing by \(\frac{3t}{2}+1\) and the second one increasing by \(3t\). The numeration is antimagic now.

Subcase 8 The 2-neighborhood of \(a_1\) doesn’t have any endpoints or chains \(P_3\), but has a triangle \(C_3\), the 2-neighborhood of \(a_2\) has a \(P_3\), but no endpoints or triangles. Let \(k \geq 1\) be the largest index of a vertex belonging to any \(P_3\) from the 2-neighborhood of \(a_2\), \(t \geq 1\) be the number of triangles, containing \(a_1\) (note again, that \(2t \leq k\)) (see Fig. 5).

Exchange the numbers \(k + t\) and \(k + t + 1\). We get: for all the endpoints from \(B\) the bounds for \(f\) hold; for vertices of degree 2 from \(B\) only \(f(b_{2k+2t})\) and \(f(b_{k+1})\) have changed (increased and decreased by 1 respectively). After we exchange the numbers: \(f(b_k) = k\), \(f(b_{k+1}) =
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Fig. 5: Subcase 8

$k + t + 1$, the decreased label doesn’t cause any troubles. The increased $f(b_{2k+2t})$ doesn’t affect the situation either, because it was already the largest among all the $f(b_i)$, so it stays this way. So, $f$ is injective on $B$. Moreover, after we exchange the numbers: $f(b_{2k+2t}) = k + t + 1 + 2k + 3t = 3k + 4t + 1 \leq 5k + 1$ and $f(a_2) = k + t + k(t + k) + \frac{2(k+1)}{2}$. For all integer $k, t$ (where $k \geq 2t, t \geq 1$) the inequality $f(b_{2k+2t}) < f(a_2)$ holds. And $f(a_2) < f(a_1)$ due to $k + t < k + t + 1$. Once again, the numeration is antimagic.

All possible cases are considered.

So, all the possible problems and the ways to solve them are shown for $C = \emptyset$ and $|A| = 2$. Thus Statement 2 is proven. \hfill \Box

6 On (1,q)-polar and (1,q)-decomposable graphs

So far we know that connected split and 1-decomposable graphs are antimagic, and we have proven connected (1,2)-polar and (1,2)-decomposable graphs to be antimagic too. The next step would naturally be to further generalize our algorithm for $q \geq 3$, but here is where the fundamental flaws of this approach start to show.

First of all, we have the exact same problem with the case $C = \emptyset$ and $|A| = 2$. The algorithm does not guarantee that $f(a_1) \neq f(a_2)$ after we enumerate all the other edges, and the last number assigned to $a_1a_2$ does not change anything either.

For $q = 1$ this is solved by considering only one simple subcase, identical to our Subcase 1 from the previous section (M. Barrus has actually skipped this possibility in his article, but it’s quite obvious and easy to fix). For $q = 2$ we already had to solve eight subcases, which is quite an increase, considering $q$ had simply went from 1 to 2.

For $q = 3$ the number of different subcases to consider is over twenty, and for $q \geq 4$ it even becomes problematic to find and count all those subcases, not even speaking of actually solving them. And even if the increase in the number of such subcases was not so rapid, it still wouldn’t be possible to truly generalize this approach for an arbitrary $q$. 
That said, we could simply exclude the case $C = \emptyset$ and $|A| = 2$, and build our algorithm for the rest of $(1, q)$-polar and $(1, q)$-decomposable graphs, but that’s not all of our troubles yet.

The second problem is that the algorithm and its possible generalizations work only if $\deg b \leq \deg a$ and $\deg b \leq \deg c$ for any $a \in A$, $b \in B$, $c \in C$, which is generally wrong if $q \geq 3$.

Let’s find the exact conditions of this happening:

Let $G = (V, E)$ be a connected $(1, q)$-polar or $(1, q)$-decomposable graph. Let $V = A \cup B \cup C$, where $G(A)$ is upper part, $G(B)$ is the lower part, $G(C)$ is an arbitrary graph. If $G$ is a $(1, q)$-polar graph we just assume that $C = \emptyset$ and ignore all the matters, concerning it.

Let $A = \{a_1, \ldots, a_{|A|}\}$, $B = \{b_1, \ldots, b_{|B|}\}$, $C = \{c_1, \ldots, c_{|C|}\}$, where the vertices are indexed in non-decreasing order of degrees.

Let $r_i$ be the number of vertices from $A$ not adjacent to $b_i$, $i \in \{1, \ldots, |B|\}$. Let $r = \min\{r_1, \ldots, r_{|B|}\}$, (note that $r \in \{1, \ldots, |A| − 1\}$).

Next we’ll prove some statements.

**Lemma 1** Let $G$ be a connected $(1, q)$-polar or $(1, q)$-decomposable graph and $a, b$ and $c$ be arbitrary vertices from $A$, $B$ and $C$ respectively. Then:

1) $\deg b \leq \deg a$ if and only if the inequality $q − r \leq |C| + \deg_{G(B∪\{a\})} a$ holds;

2) $\deg b \leq \deg c$ if and only if the inequality $q − r \leq \deg_{G(C)} c + 1$ holds.

**Proof:** $\deg b$ equals the sum of degrees of vertex $b$ inside $G(B)$ and outside of it, the first one being less or equal to $q − 1$ and the second one being less or equal to $|A| − r$ by the definition of $r$.

$\deg a$ equals the sum of degrees of vertex $a$ inside $G(A)$ and outside of it, the first one being equal to $|A| − 1$ and the second one being equal to $|C| + \deg_{G(B∪\{a\})} a$.

$\deg c$ equals the sum of degrees of vertex $c$ inside $G(C)$ and outside of it, the second one being equal to $|A|$.

So:

\[
\deg b \leq \deg a \iff (q − 1) + (|A| − r) \leq (|A| − 1) + |C| + \deg_{G(B∪\{a\})} a \iff q − r \leq |C| + \deg_{G(B∪\{a\})} a,
\]

\[
\deg b \leq \deg c \iff (q − 1) + (|A| − r) \leq \deg_{G(C)} c + |A| \iff q − r \leq \deg_{G(C)} c + 1.
\]

**Corollary 1** Both $\deg b \leq \deg a$ and $\deg b \leq \deg c$ hold if and only if one of the following conditions is met:

1) If $C = \emptyset$ then $\deg_{G(B∪\{a\})} a \geq q − r$;

2) If $C \neq \emptyset$ then $\deg_{G(C)} c + 1 \geq q − r$.

**Proof:** If $C = \emptyset$ then only the first statement of Lemma 1 is valid and $|C| = 0$. So $\deg b \leq \deg a \iff q − r \leq 0 + \deg_{G(B∪\{a\})} a = \deg_{G(B∪\{a\})} a$.

If $C \neq \emptyset$ then both statements of Lemma 1 are valid, the second one being actually stronger than the first: $\deg_{G(C)} c + 1 \leq (|C| − 1) + 1 = |C| \leq |C| + \deg_{G(B∪\{a\})} a$. So we can leave only the second one: $q − r \leq \deg_{G(C)} c + 1$.

Joining both of these mutually exclusive cases, we get our statement.

Thus, the $(1, q)$-polar and $(1, q)$-decomposable graphs, not satisfying the conditions of Corollary 1, have to be excluded from our consideration too.
Still, even that’s not all of it.

Our third and possibly main problem is the increasing complexity of $G(B)$ and its connections to $G(A)$ as well. More specifically, let’s consider the following subproblem:

Build such an edge numbering of $G(B)$ and all the edges between $B$ and $A$ by the first available numbers in order, that all the sums of labels for each $b \in B$ are different (and sums for $a \in A$ don’t matter).

Even such a variant of our problem, where we don’t care about $A$ or $C$, is difficult to solve, due to $G(B)$ and the edges between $B$ and $A$ being both arbitrary and independent of each other. All the general algorithms of numerating such a construction can and will have counterexamples based on its randomness, and this task seems no easier than just building an antimagic numbering for an arbitrary graph.

For $q \leq 2$ it is possible to do such a thing because of the small number of variants existing, but for $q \geq 3$ the problem becomes too complex to generalize.

7 Conclusions and open questions

In this article we have proven, that connected $(1,2)$-polar and $(1,2)$-decomposable graphs are antimagic and showed the difficulties, that arise when trying to expand this method to $(1,q)$-polar and $(1,q)$-decomposable graphs with $q \geq 3$.

Naturally, the next question remains open:

**Question 1** Are connected $(1,q)$-polar and $(1,q)$-decomposable graphs for $q \geq 3$ antimagic?

Supposedly they are, but the proof is yet to be found.

Moreover, $(1,q)$-polar and $(1,q)$-decomposable graphs themselves are a part of a larger class of graphs, namely $(p,q)$-polar and $(p,q)$-decomposable graphs (see Tyshkevich et al. (2013)), defined as follows:

Let $p, q$ be positive integers.

A graph $G$ is called $(p,q)$-**polar**, if there is such a partition $V = A \sqcup B$, that $G(A)$ is a graph, complementary to a disjoint union of cliques of order no greater than $p$, and $G(B)$ is a disjoint union of cliques of order no greater than $q$.

A graph $G$ is called $(p,q)$-**decomposable**, if there is such a partition $V = A \sqcup B \sqcup C$, that $G(A)$ is a graph, complementary to a disjoint union of cliques of order no greater than $p$, $G(B)$ is a disjoint union of cliques of order no greater than $q$ and $G(C)$ is an arbitrary graph, connected to $G(A)$ by all the possible edges (i.e. the edges of a complete bipartite graph with parts $A$ and $C$).

Assuming $p = 1$ and $q = 1$ we get split and 1-decomposable graphs, $p = 1$ and $q = 2$ lead to $(1,2)$-polar and $(1,2)$-decomposable graphs and so on.

Moving from split and 1-decomposable graphs as a starting point, we have considered making the lower part more complex, but it might be interesting to study the graphs with complex upper parts as well:

**Question 2** Are connected $(p,1)$-polar and $(p,1)$-decomposable graphs for $p \geq 2$ antimagic?

And having solved this one, next generalisation follows:

**Question 3** Are connected $(p,q)$-polar and $(p,q)$-decomposable graphs antimagic?

Even finding some of such pairs $(p,q)$ seems to be a difficult and interesting task.

Finally, the AGD, which all these decompositions are based on, allows us to decompose the graphs in many different ways, so if some of them have proven to work, there may be other ones waiting to be found:
Question 4  Are there other ways to decompose graphs based on the AGD, useful for studying the Hartsfield-Ringel conjecture?

This is the main direction of our further studies.

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