Perspective-12-Quadric: An analytical solution to the camera pose estimation problem from conic - quadric correspondences

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Goal: To estimate the pose of a perspective camera given a set of n 3D quadrics in the world and their corresponding 2D projections (conics) in the image.

1 Problem statement.

- **2D:** Homogeneous quadratic form of a conic equation:

$$u^\top C_i u = 0,$$

where $u \in \mathbb{R}^3$ is the homogeneous vector of a generic 2D point belonging to the conic defined by the symmetric matrix $C_i \in \mathbb{R}^{3\times3}$. The conic has 5 degrees of freedom given by the 6 elements of the lower triangular part of the symmetric matrix C_i, except one for the scale. The dual conic is defined by $C_i^* = adj(C_i)$.

- **3D:** Homogeneous quadratic form of a quadric equation:

$$x^\top Q_i x = 0,$$

where $x \in \mathbb{R}^4$ is the homogeneous vector of a generic 3D point belonging to the quadric defined by the symmetric matrix $Q_i \in \mathbb{R}^{4\times4}$. The quadric has 9 degrees of freedom given by the 10 elements of the lower triangular part of the symmetric matrix Q_i, except one for the scale. The dual quadric is defined by $Q_i^* = adj(Q_i)$.

- **Projection equation:** Let's consider the projection matrix

$$P = K[R|t] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}$$

The projection equation [1] is:

$$\beta_i C_i^* = P Q_i^* P^\top \tag{1}$$

2 Solution.

In order to recover P in closed form from the set of conic-quadric pairs, we have to re-arrange Eq. (1) into a linear system [2, 3]. Let us define the operator $vec()$ that serialises all the elements of a generic matrix, and the operator $vech()$ that serialises the elements of the lower triangular part of a symmetric matrix. Then let us define $v_i^* = vech(Q_i^*)$ and $c_i^* = vech(C_i^*)$. Then let us arrange the product of the elements of P and $P^\top$ in a single matrix $G \in \mathbb{R}^{6\times 10}$ as follows:

$$G = D(P \otimes P)E$$

where $\otimes$ is the Kronecker product, and matrices $D \in \mathbb{R}^{6\times 9}$ and $E \in \mathbb{R}^{16\times 10}$, are two matrices such that $vech(X) = Dvec(X)$ and $vec(Y) = Evech(Y)$ respectively, where $X \in \mathbb{R}^{9\times 9}$ and $Y \in \mathbb{R}^{16\times 16}$ are two symmetric matrices.

$$D = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}, E = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$\text{G}=\begin{bmatrix} p_{11}^2 & 2p_{11}p_{12} & p_{12}^2 & 2p_{11}p_{13} & 2p_{12}p_{13} & p_{13}^2 & 2p_{11}p_{14} & 2p_{12}p_{14} & 2p_{13}p_{14} & p_{14}^2 \\ p_{11}p_{21} & p_{11}p_{22}+p_{12}p_{21} & p_{12}p_{22} & p_{11}p_{23}+p_{13}p_{21} & p_{12}p_{23}+p_{13}p_{22} & p_{13}p_{23} & p_{11}p_{24}+p_{14}p_{21} & p_{12}p_{24}+p_{14}p_{22} & p_{13}p_{24}+p_{14}p_{23} & p_{14}p_{24} \\ p_{21}^2 & 2p_{21}p_{22} & p_{22}^2 & 2p_{21}p_{23} & 2p_{22}p_{23} & p_{23}^2 & 2p_{21}p_{24} & 2p_{22}p_{24} & 2p_{23}p_{24} & p_{24}^2 \\ p_{11}p_{31} & p_{11}p_{32}+p_{12}p_{31} & p_{12}p_{32} & p_{11}p_{33}+p_{13}p_{31} & p_{12}p_{33}+p_{13}p_{32} & p_{13}p_{33} & p_{11}p_{34}+p_{14}p_{31} & p_{12}p_{34}+p_{14}p_{32} & p_{13}p_{34}+p_{14}p_{33} & p_{14}p_{34} \\ p_{21}p_{31} & p_{21}p_{32}+p_{22}p_{31} & p_{22}p_{32} & p_{21}p_{33}+p_{23}p_{31} & p_{22}p_{33}+p_{23}p_{32} & p_{23}p_{33} & p_{21}p_{34}+p_{24}p_{31} & p_{22}p_{34}+p_{24}p_{32} & p_{23}p_{34}+p_{24}p_{33} & p_{24}p_{34} \\ p_{31}^2 & 2p_{31}p_{32} & p_{32}^2 & 2p_{31}p_{33} & 2p_{32}p_{33} & p_{33}^2 & 2p_{31}p_{34} & 2p_{32}p_{34} & 2p_{33}p_{34} & p_{34}^2 \end{bmatrix}$$

Given G we can rewrite Eq. (1) as a linear equation:

$$\beta_i c_i^* = G v_i^* \tag{2}$$

Eq. (2) can thus be solved using DLT algorithm, *e.g.* can be written:

$$0 = Bg,$$

where $g = vec(G) = [g_{11} \cdots g_{610}]^\top \in \mathbb{R}^{60}$, and $B \in \mathbb{R}^{15n \times 60}$ is the concatenation of matrices $b_i \in \mathbb{R}^{15 \times 60}$:

$$b_i = \begin{bmatrix}
c_{2i}\mathbf{v_i}^\top & -c_{1i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & c_{3i}\mathbf{v_i}^\top & -c_{2i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & c_{4i}\mathbf{v_i}^\top & -c_{3i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & c_{5i}\mathbf{v_i}^\top & -c_{4i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & c_{6i}\mathbf{v_i}^\top & -c_{5i}\mathbf{v_i}^\top \\
-c_{3i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & c_{1i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & -c_{4i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & c_{2i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & -c_{5i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & c_{3i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & -c_{6i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & c_{4i}\mathbf{v_i}^\top \\
c_{4i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & -c_{1i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & c_{5i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & -c_{2i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & c_{6i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & -c_{4i}\mathbf{v_i}^\top \\
-c_{5i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & c_{1i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top \\
\mathbf{0_{10}}^\top & -c_{6i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & c_{2i}\mathbf{v_i}^\top \\
c_{6i}\mathbf{v_i}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & \mathbf{0_{10}}^\top & -c_{1i}\mathbf{v_i}^\top
\end{bmatrix},$$

with:

$$\begin{aligned}
\mathbf{c_i}^\top &= \begin{bmatrix} c_{1i} & c_{2i} & c_{3i} & c_{4i} & c_{5i} & c_{6i} \end{bmatrix}, \\
\mathbf{v_i}^\top &= \begin{bmatrix} v_{1i} & v_{2i} & v_{3i} & v_{4i} & v_{5i} & v_{6i} & v_{7i} & v_{8i} & v_{9i} & v_{10i} \end{bmatrix}, \\
\mathbf{0_{10}}^\top &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Each pair of conic-quadric provides 15 homogeneous linear equations in the unknown elements of G. However, only 5 are linearly independent. Thus 12 pairs of 3D-2D correspondences are required to solve the system. A total least squares solution can then be found by choosing g as a right singular vector corresponding to the smallest singular value of B. Finally, P can be inferred from G.

References

[1] R.I. Hartley and A. Zisserman, *Multiple View Geometry in Computer Vision*, Cambridge University Press, second edition, 2004.

[2] M. Crocco and C. Rubino and A. Del Bue, *Structure from Motion with Objects*, CVPR, 2016.

[3] C. Rubino and M. Crocco and A. Del Bue, *3D Object Localisation from Multi-View Image Detections*, TPAMI, 2018.