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Ramsey for complete graphs with a dropped edge or a triangle

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Abstract

Let $K_{[k,t]}$ be the complete graph on $k$ vertices from which a set of edges, induced by a clique of order $t$, has been dropped (note that $K_{[k,1]}$ is just $K_k$). In this paper we study $R(K_{[k_1,t_1]}, \ldots, K_{[k_r,t_r]})$ (the smallest integer $n$ such that for any $r$-edge coloring of $K_n$ there always occurs a monochromatic $K_{[k_i,t_i]}$ for some $i$).

We first present a general upper bound (containing the well-known Graham-Rödl upper bound for complete graphs in the particular case when $t_i = 1$ for all $i$). We then focus our attention when $r = 2$ and dropped cliques of order 2 and 3 (edges and triangles). We give the exact value for $R(K_{[n,2]}, K_{[4,3]})$ and $R(K_{[n,3]}, K_{[4,3]})$ for all $n \geq 2$.

Keywords: Ramsey number, recursive formula.
1 Introduction

Let $K_n$ be a complete graph and let $r \geq 2$ be an integer. A $r$-edge coloring of a graph is a surjection from $E(G)$ to $\{0, \ldots, r - 1\}$ (and thus each color class is not empty). Let $k \geq t \geq 1$ be positive integers. We denote by $K_{[k,t]}$ the complete graph on $k$ vertices from which a set of edges, induced by a clique of order $t$, has been dropped, see Figure 1.

![Fig. 1. (a) $K_{[5,3]}$ and (b) $K_{[4,2]}$](image)

Let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers with $k_i \geq t_i$ for all $i \in \{1, \ldots, r\}$. Let $R([k_1, t_1], \ldots, [k_r, t_r])$ be the smallest integer $n$ such that for any $r$-edge coloring of $K_n$ there always occurs a monochromatic $K_{[k_i,t_i]}$ for some $i$. In the case when $k_i = t_i$ for some $i$, we set

$$R([k_1, t_1], \ldots, [k_{i-1}, t_{i-1}], [t_i, t_i], [k_{i+1}, t_{i+1}], \ldots, [k_r, t_r]) \leq t_i.$$  

We note that equality is reached at $\min \{t_i | t_i = k_i\}$. Since the set of all the edges of $K_{[t_i,t_i]}$ (which is empty) can always be colored with color $i$. We also notice that the case $R([k_1, 1], \ldots, [k_r, 1])$ is exactly the classical Ramsey number $r(k_1, \ldots, k_r)$ (the smallest integer $n$ such that for any $r$-edge coloring of $K_n$ there always occurs a monochromatic $K_{k_i}$ for some $i$). We refer the reader to the excellent survey [6] on Ramsey numbers for small values. In this paper, we investigate $R([k_1, t_1], \ldots, [k_r, t_r])$.

2 General upper bound

In this section we present a recursive formula (Lemma 2.1) that yields to an explicit general upper bound (Theorem 2.2). The latter contains the well-known explicit general upper bound for $R([k_1, 1], \ldots, [k_r, 1])$ due to Graham and Rödl [3] (see Equation (4)).

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The following recursive inequality is classical in Ramsey theory

\[(1) \quad r(k_1, k_2, \ldots, k_r) \leq r(k_1 - 1, k_2, \ldots, k_r) + r(k_2 - 1, \ldots, k_r) + \cdots + r(k_1, k_2, \ldots, k_r - 1) - (r - 2)\]

In the same spirit, we have the following.

**Lemma 2.1** Let \( r \geq 2 \) and let \( k_1, \ldots, k_r \) and \( t_1, \ldots, t_r \) be positive integers with \( k_i \geq t_i + 1 \geq 2 \) for all \( i \). Then,

\[
R([k_1, t_1], \ldots, [k_r, t_r]) \leq R([k_1 - 1, t_1], [k_2, t_2], \ldots, [k_r, t_r])
+ R([k_1, t_1], [k_2 - 1, t_2], \ldots, [k_r, t_r])
+ \cdots
+ R([k_1, t_1], [k_2, t_2], \ldots, [k_r - 1, t_r]) - (r - 2).
\]

A similar recursive inequality has been treated in [7] in a much more general setting in which a family of graphs are intrinsically constructed via two operations disjoin unions and joins (see also [4] for the case \( r = 2 \)). However, it is not clear how the latter could be used to obtain Lemma 2.1 that allows us to give the following general upper bound for \( R([k_1, t_1], \ldots, [k_r, t_r]) \) (which was not considered in [7]).

**Theorem 2.2** Let \( r \geq 2 \) be a positive integer and let \( k_1, \ldots, k_r \) and \( t_1, \ldots, t_r \) be positive integers such that \( k_i \geq t_i \) for all \( i \in \{1, \ldots, r\} \). Then,

\[
R([k_1, t_1], \ldots, [k_r, t_r]) \leq \max_{1 \leq i \leq r} \left\{ t_i \right\} \binom{k_1 + \cdots + k_r - (t_1 + \cdots + t_r)}{k_1 - t_1, k_2 - t_2, \ldots, k_r - t_r}
\]

where \( \binom{n_1 + n_2 + \cdots + n_r}{n_1, n_2, \ldots, n_r} \) is the multinomial coefficient defined by \( \binom{n_1 + n_2 + \cdots + n_r}{n_1, n_2, \ldots, n_r} = \frac{(n_1 + n_2 + \cdots + n_r)!}{n_1! n_2! \cdots n_r!} \), for all nonnegative integers \( n_1, \ldots, n_r \).

Theorem 2.2 is a natural generalization of the well-known explicit upper bound for classical Ramsey numbers. Indeed, an immediate consequence of Theorem 2.2 (by taking \( t_i = 1 \) for all \( i \)) is the following classical upper bound due to Graham and Rödl [3, (2.48)]

\[(2) \quad R([k_1, 1], \ldots, [k_r, 1]) \leq \binom{k_1 + \cdots + k_r - r}{k_1 - 1, \ldots, k_r - 1}.
\]

Let \( k \geq t \geq 2 \) and \( r \geq 2 \) be integers and let \( R_r([k, t]) = R([k, t], \ldots, [k, t]) \).

An immediate consequence of Theorem 2.2 (by taking \( k = k_1 = \cdots = k_n \) and
$t = t_1 = \cdots = t_n$) is the following inequality

$$R_r([k,t]) \leq t \binom{r(k-t)}{k-t, \ldots, k-t}$$

Moreover, if $t = 1$ then

$$R_r([k,1]) \leq \frac{(rk-r)!}{((k-1)!)^r}.$$ 

3 Exact values

By the so-called Chvátal’s result [2], we know that the exact value of the Ramsey number of $K_{[4,3]}$ (a star) versus cliques is given by $R([n,1],[4,3]) = 3n-2$ for all $n \geq 1$. We then naturally focus our attention to the Ramsey number of $K_{[4,3]}$ versus cliques with either a dropped edge or a dropped triangle, see [1] where $R([m,1],[n,2])$ has been computed for numerous cases. We provide the new following exact values of Ramsey numbers.

**Theorem 3.1** Let $n \geq 2$ be an integer. Then,

- $R([n,2],[4,3]) = 2$ for $n = 2$,
- $R([n,2],[4,3]) = 5$ for $n = 3$,
- $R([n,2],[4,3]) = 3n-5$ for $n \geq 4$.

**Theorem 3.2** Let $n \geq 2$ be an integer. Then,

- $R([n,3],[4,3]) = 3$ for $n = 3$,
- $R([n,3],[4,3]) = 6$ for $n = 4$,
- $R([n,3],[4,3]) = 8$ for $n = 5$,
- $R([n,3],[4,3]) = 11$ for $n = 6$,
- $R([n,3],[4,3]) = 3n-8$ for $n \geq 7$.

3.1 An estimation for $R([n,2],[5,3])$

By considering $K_{[5,3]}$ as the book graph $B_3$, it was proved in [5,8] that

$$R([n,1],[5,3]) \leq \frac{3n^2}{\log(n/e)},$$

for all positive integers $n$.

The following result is a first estimation for the value $R([n,2],[5,3])$.

**Theorem 3.3** Let $n \geq 2$ be an integer. Then,
• $R([n, 2], [5, 3]) = 2$ for $n = 2$,
• $R([n, 2], [5, 3]) = 7$ for $n = 3$,
• $R([n, 2], [5, 3]) \leq 3\binom{n+1}{2} - 5n + 4$ for $n \geq 4$.

References


