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Abstract

We discuss the asymptotic behavior of the Vlasov-Poisson system in the framework of the magnetic confinement, that is, under a strong external magnetic field. We concentrate on curved three dimensional magnetic fields. We derive second order approximations, when the magnetic field becomes large, for general initial particle densities.

Keywords: Vlasov-Poisson system, averaging, homogenization.

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1 Introduction

The asymptotic analysis of the transport of charged particles under strong magnetic fields is a very important topic in plasma physics [7, 8, 9, 10, 14, 15, 11, 12, 13, 1, 2, 3, 4, 5]. It is related to real life applications, such that the energy production through magnetic confinement. When the particle velocities are small with respect to the light speed, the evolution of the particle density $f = f(t,x,v)$ is described by the Vlasov-Poisson system

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \frac{q}{m}\{E[f^\varepsilon(t)](x) + v\wedge \mathbf{B}^\varepsilon(x)\}\cdot\nabla_v f^\varepsilon = 0,\ \ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3. \quad (1)$$

$$E[f^\varepsilon(t)] = -\nabla_x\Phi[f^\varepsilon(t)],\ \ \Phi[f^\varepsilon(t)](x) = \frac{q}{4\pi\epsilon_0}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f^\varepsilon(t,x',v')}{|x-x'|}\,\mathrm{d}v'\mathrm{d}x'$$

where $\varepsilon > 0$ is a small parameter, entering the strong external non vanishing magnetic field

$$\mathbf{B}^\varepsilon(x) = B^\varepsilon(x)e(x),\ \ B^\varepsilon(x) = \frac{B(x)}{\varepsilon},\ \ |e(x)| = 1,\ \ x\in\mathbb{R}^3.$$

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The potential $\Phi[f^\varepsilon]$ satisfies the Poisson equation

$$-\Delta_x \Phi[f^\varepsilon(t)] = \frac{q}{\epsilon_0} \int_{\mathbb{R}^3} f^\varepsilon(t,x,v) \, \mathrm{d}v, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

whose fundamental solution is $z \to \frac{1}{4\pi|z|}, z \in \mathbb{R}^3 \setminus \{0\}$. Here ϵ_0 represents the electric permittivity. For any particle density $f = f(x,v)$, the notation $E[f]$ stands for the Poisson electric field

$$E[f](x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x',v') \frac{x-x'}{|x-x'|^3} \, \mathrm{d}v'\mathrm{d}x' \tag{2}$$

and $\rho[f], j[f]$ are the charge and current densities respectively

$$\rho[f] = q \int_{\mathbb{R}^3} f(\cdot,v) \, \mathrm{d}v, \quad j[f] = q \int_{\mathbb{R}^3} f(\cdot,v) v \, \mathrm{d}v.$$

The above system is supplemented by the initial condition

$$f^\varepsilon(0,x,v) = f_{\mathrm{in}}(x,v), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

In [6] a regular reformulation (when $\varepsilon \searrow 0$) of the Vlasov-Poisson system has been derived, in the three dimensional setting, for well prepared initial particle densities. In this work we extend the previous analysis to general initial particle densities. Considering general initial conditions leads to fast oscillations in time. In order to describe the asymptotic behavior (when $\varepsilon \searrow 0$), we need to introduce a fast time variable $s = t/\varepsilon$. The analysis follows closely that in [6] and the arguments rely on averaging along the flow of a vector field. As a fast time variable has been introduced, we need to consider the extended phase space (s,x,v) for averaging functions and vector fields.

Our paper is organized as follows. The average operators on the extended phase space and main properties are discussed in Section 2. The regular reformulation of the Vlasov-Poisson problem with strong external magnetic field is derived in Section 3 and revisited in the last Section 4.

2 Average operators and main properties

As in [6], we introduce the relative velocity with respect to the electric cross field drift

$$\tilde{v} = v - \varepsilon \frac{E^\varepsilon(t,x) \wedge e(x)}{B(x)}.$$

Accordingly, at any time $t \in [0,T]$, we consider the new particle density

$$\tilde{f}^\varepsilon(t,x,\tilde{v}) = f^\varepsilon\left(t,x,\tilde{v} + \varepsilon \frac{E[f^\varepsilon(t)](x) \wedge e(x)}{B(x)}\right), \quad (x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

The particle densities $f^\varepsilon, \tilde{f}^\varepsilon$ have the same charge density

$$\rho[\tilde{f}^\varepsilon(t)] = q \int_{\mathbb{R}^3} \tilde{f}^\varepsilon(t,\cdot,\tilde{v}) \, \mathrm{d}\tilde{v} = q \int_{\mathbb{R}^3} f^\varepsilon(t,\cdot,v) \, \mathrm{d}v = \rho[f^\varepsilon(t)], \quad t \in [0,T]$$

implying that the Poisson electric fields corresponding to $f^\varepsilon, \tilde{f}^\varepsilon$ coincide

$$E[f^\varepsilon(t)] = E[\tilde{f}^\varepsilon(t)], \quad t \in [0,T].$$

Therefore we can use the same notation $E^\varepsilon(t)$ for denoting them. We assume that the magnetic field satisfies

$$B_0 := \inf_{x\in\mathbb{R}^3} |B(x)| > 0 \text{ or equivalently } \omega_0 := \inf_{x\in\mathbb{R}^3} |\omega_c(x)| > 0.$$

The new particle densities $(\tilde{f}^\varepsilon)_{\varepsilon>0}$ verify

$$\begin{aligned}\partial_t \tilde{f}^\varepsilon + \left(\tilde{v} + \varepsilon \frac{E^\varepsilon \wedge e}{B}\right) \cdot \nabla_x \tilde{f}^\varepsilon - \varepsilon \left[\frac{\partial_t E^\varepsilon \wedge e}{B} + \partial_x \left(\frac{E^\varepsilon \wedge e}{B}\right)\left(\tilde{v} + \varepsilon \frac{E^\varepsilon \wedge e}{B}\right)\right] \cdot \nabla_{\tilde{v}} \tilde{f}^\varepsilon \\ + \left[\frac{\omega_c}{\varepsilon}\, \tilde{v} \wedge e + \frac{q}{m}(E^\varepsilon \cdot e)\, e\right] \cdot \nabla_{\tilde{v}} \tilde{f}^\varepsilon = 0, \quad (t,x,\tilde{v}) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \qquad (3)\\ \tilde{f}^\varepsilon(0,x,\tilde{v}) = f_{\text{in}}\left(x, \tilde{v} + \varepsilon \frac{E[f_{\text{in}}](x) \wedge e(x)}{B(x)}\right), \quad (x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3.\end{aligned}$$

As in [6], thanks to the continuity equation

$$\partial_t \rho[f^\varepsilon] + \operatorname{div}_x j[f^\varepsilon] = 0$$

we obtain the following representation for the time derivative of the electric field E^ε, in terms of the particle density $\tilde{f}^\varepsilon$

$$\partial_t E[f^\varepsilon] = -\frac{1}{4\pi\epsilon_0} \operatorname{div}_x \int_{\mathbb{R}^3} \frac{x-x'}{|x-x'|^3} \otimes \left(j[\tilde{f}^\varepsilon(t)](x') + \varepsilon \rho[\tilde{f}^\varepsilon(t)](x') \frac{E^\varepsilon(t,x') \wedge e(x')}{B(x')}\right) \,\mathrm{d}x'.$$

We introduce the new Larmor center $\tilde{x} = x + \varepsilon \frac{\tilde{v}\wedge e(x)}{\omega_c(x)}$, which is a second order approximation of the Larmor center $x + \varepsilon \frac{v \wedge e(x)}{\omega_c(x)}$. We decompose the transport field in the Vlasov equation in such a way that $\tilde{x}$ remains invariant with respect to the fast dynamics. We will distinguish between the orthogonal and parallel directions, taking as reference direction the magnetic line passing through the new Larmor center $\tilde{x}$, that is $e(\tilde{x})$ (which is left invariant with respect to the fast dynamics)

$$\tilde{v} = [\tilde{v} - (\tilde{v} \cdot e(\tilde{x}))e(\tilde{x})] + (\tilde{v} \cdot e(\tilde{x}))e(\tilde{x}).$$

Finally the Vlasov equation (3) writes

$$\partial_t \tilde{f}^\varepsilon + c^\varepsilon[\tilde{f}^\varepsilon(t)] \cdot \nabla_{x,\tilde{v}}\, \tilde{f}^\varepsilon + \varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)] \cdot \nabla_{x,\tilde{v}}\, \tilde{f}^\varepsilon + \frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde{v}}\, \tilde{f}^\varepsilon = 0, \quad (t,x,\tilde{v}) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \qquad (4)$$

where the autonomous vector field $\frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde{v}}$ is given by

$$\frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde{v}} = [\tilde{v} - (\tilde{v} \cdot e(\tilde{x}))\, e(\tilde{x}) + \varepsilon A_x^\varepsilon(x,\tilde{v})] \cdot \nabla_x + \frac{\omega_c(x)}{\varepsilon}\, (\tilde{v} \wedge e(\tilde{x})) \cdot \nabla_{\tilde{v}}$$

and for any particle density $\tilde{f}$, $a^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$, $c^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$ stand for the vector fields

$$\begin{aligned}a^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}} &= \left(\frac{E[\tilde{f}] \wedge e}{B} - A_x^\varepsilon\right) \cdot \nabla_x + \left[-\partial_x \left(\frac{E[\tilde{f}] \wedge e}{B}\right)\left(\tilde{v} + \varepsilon \frac{E[\tilde{f}] \wedge e}{B}\right)\right. \\ &\quad \left. + \frac{1}{4\pi\epsilon_0 B} \operatorname{div}_x \int_{\mathbb{R}^3} \frac{x-x'}{|x-x'|^3} \otimes \left(j[\tilde{f}] + \varepsilon \rho[\tilde{f}] \frac{E[\tilde{f}] \wedge e}{B}\right)(x')\, \mathrm{d}x' \wedge e(x)\right] \cdot \nabla_{\tilde{v}}\end{aligned}$$

$$
\begin{aligned}
c^\varepsilon[\tilde f]\cdot\nabla_{x,\tilde v} &= (\tilde v\cdot e(\tilde x))\, e(\tilde x)\cdot\nabla_x + \left[\omega_c(x)\tilde v\wedge\frac{e(x)-e(\tilde x)}{\varepsilon} + \frac{q}{m}(E[\tilde f]\cdot e(x))\, e(x)\right]\cdot\nabla_{\tilde v}\\
&= \left[\frac{q}{m}(E[\tilde f]\cdot e(x))\, e(x) - \omega_c\tilde v\wedge\int_0^1 \partial_x e\left(x+\varepsilon s\frac{\tilde v\wedge e(x)}{\omega_c(x)}\right)\frac{\tilde v\wedge e(x)}{\omega_c(x)}\,\mathrm{d}s\right]\cdot\nabla_{\tilde v}\\
&+ (\tilde v\cdot e(\tilde x))\, e(\tilde x)\cdot\nabla_x.
\end{aligned}
$$

The vector field $A_x^\varepsilon(x,\tilde v)\cdot\nabla_x$ will be determined by imposing that the Larmor center $\tilde x$ is left invariant by the fast dynamics

$$
b^\varepsilon\cdot\nabla_{x,\tilde v}\left(x+\varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)}\right)=0
$$

that is

$$
\left[I_3+\varepsilon\partial_x\left(\frac{\tilde v\wedge e}{\omega_c}\right)\right]A_x^\varepsilon(x,\tilde v) = -\partial_x\left(\frac{\tilde v\wedge e}{\omega_c}\right)[\tilde v-(\tilde v\cdot e(\tilde x))\, e(\tilde x)] - \frac{e(\tilde x)-e(x)}{\varepsilon}\wedge(\tilde v\wedge e(\tilde x)).
$$

The method employed in [6] applies as well when the initial particle density is not well prepared. In this case we deal with two time scales: the slow time variable t and the fast time variable $s=t/\varepsilon$. We need to average in the extended phase space$(s,x,\tilde v)$. We say that a function $u=u(s,x,\tilde v)$ is $S=S(x,\tilde v)$ periodic with respect to s iff

$$
u(s+S(x,\tilde v),x,\tilde v)=u(s,x,\tilde v),\ \ (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3.
$$

Similarly, we say that a function $u=u(s,x,\tilde v)$ is $S^\varepsilon=S^\varepsilon(x,\tilde v)$ periodic with respect to s iff

$$
u(s+S^\varepsilon(x,\tilde v),x,\tilde v)=u(s,x,\tilde v),\ \ (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3.
$$

With the notations in [6] Propositions 3.1, 3.2, we observe that if u is S periodic with respect to s, therefore the function $(s,x,\tilde v)\to u(\Lambda^\varepsilon(s;x,\tilde v),T^\varepsilon(x,\tilde v))$ is S^ε periodic with respect to s. For establishing that, notice that

$$
\Lambda^\varepsilon(s+S^\varepsilon(x,\tilde v);x,\tilde v)=\Lambda^\varepsilon(s;x,\tilde v)+S(T^\varepsilon(x,\tilde v)).
$$

Indeed, we have, thanks to Proposition 3.2 [6]

$$
\begin{aligned}
\Lambda^\varepsilon(s+S^\varepsilon(x,\tilde v);x,\tilde v) &= \int_0^{s+S^\varepsilon(x,\tilde v)}\lambda^\varepsilon(\mathcal{X}^\varepsilon(\sigma;x,\tilde v),\tilde{\mathcal{V}}^\varepsilon(\sigma;x,\tilde v))\,\mathrm{d}\sigma\\
&= \int_0^s\lambda^\varepsilon(\mathcal{X}^\varepsilon(\sigma;x,\tilde v),\tilde{\mathcal{V}}^\varepsilon(\sigma;x,\tilde v))\,\mathrm{d}\sigma + \int_0^{S^\varepsilon(x,\tilde v)}\lambda^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\tau;(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde v)))\,\mathrm{d}\tau\\
&= \Lambda^\varepsilon(s;x,\tilde v) + \int_0^{S^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde v))}\lambda^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\tau;(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde v)))\,\mathrm{d}\tau\\
&= \Lambda^\varepsilon(s;x,\tilde v) + \Lambda^\varepsilon(S^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde v));(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde v))\\
&= \Lambda^\varepsilon(s;x,\tilde v) + S(T^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde v)))\\
&= \Lambda^\varepsilon(s;x,\tilde v) + S((\mathcal{X},\tilde{\mathcal{V}})(\Lambda^\varepsilon(s;x,\tilde v);T^\varepsilon(x,\tilde v)))\\
&= \Lambda^\varepsilon(s;x,\tilde v) + S(T^\varepsilon(x,\tilde v)).
\end{aligned}
$$

It is easily seen that

$$\begin{aligned}u(\Lambda^\varepsilon(s+S^\varepsilon(x,\tilde v);x,\tilde v),T^\varepsilon(x,\tilde v)) &= u(\Lambda^\varepsilon(s;x,\tilde v)+S(T^\varepsilon(x,\tilde v)),T^\varepsilon(x,\tilde v))\\ &= u(\Lambda^\varepsilon(s;x,\tilde v),T^\varepsilon(x,\tilde v))\end{aligned}$$

saying that the function $(s,x,\tilde v)\to u(\Lambda^\varepsilon(s;x,\tilde v),T^\varepsilon(x,\tilde v))$ is S^ε periodic with respect to s.

Observe that for any $(s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$, the characteristics of $\partial_s+b\cdot\nabla_{x,\tilde v}$, $\partial_s+b^\varepsilon\cdot\nabla_{x,\tilde v}$ issued from $(s,x,\tilde v)$ are

$$(s+\sigma,\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v)),\ \ (s+\sigma,\mathcal{X}^\varepsilon(\sigma;x,\tilde v),\tilde{\mathcal{V}}^\varepsilon(\sigma;x,\tilde v))$$

respectively. We define the average operators for continuous S periodic, S^ε periodic functions by

$$\langle u\rangle(s,x,\tilde v)=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}u(s+\sigma,\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma,\ \ (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$$

$$\langle u\rangle_\varepsilon(s,x,\tilde v)=\frac{1}{S^\varepsilon(x,\tilde v)}\int_0^{S^\varepsilon(x,\tilde v)}u(s+\sigma,\mathcal{X}^\varepsilon(\sigma;x,\tilde v),\tilde{\mathcal{V}}^\varepsilon(\sigma;x,\tilde v))\,\mathrm{d}\sigma,\ \ (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3.$$

Notice that the above operators extend the corresponding average operators defined in Proposition 3.1 [6] for continuous functions, not depending on s. As in Proposition 3.2 [6], we establish a relation between the average operators $\langle\cdot\rangle,\langle\cdot\rangle_\varepsilon$. We will work under the hypothesis $\nabla_x\omega_c=0$, implying that $S(x,\tilde v)=S^\varepsilon(x,\tilde v)=2\pi/\omega_c$, $\lambda^\varepsilon(x,\tilde v)=1$, $\Lambda^\varepsilon(s;x,\tilde v)=s$, $(s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$.

Proposition 2.1
Let $u\in C(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$ be a S periodic function with respect to s such that supp $u\subset\{(s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\ :\ |\tilde v|\le R\}$ *for some $R>0$. For any $\varepsilon>0$ satisfying $\varepsilon R\|\partial_x e\|_{L^\infty}/|\omega_c|<1$ we have*

$$\langle u(\cdot,T^\varepsilon)\rangle_\varepsilon=\langle u\rangle(\cdot,T^\varepsilon).$$

Proof.
It is enough to consider $(s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$ such that $|\tilde v|\le R$. In that case we have, cf. Proposition 3.2 [6]

$$\begin{aligned}\langle u(\cdot,T^\varepsilon)\rangle_\varepsilon(s,x,\tilde v)&=\frac{1}{S}\int_0^S u(s+\sigma,T^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\sigma;x,\tilde v)))\,\mathrm{d}\sigma\\ &=\frac{1}{S}\int_0^S u(s+\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;T^\varepsilon(x,\tilde v)))\,\mathrm{d}\sigma\\ &=\langle u\rangle(s,T^\varepsilon(x,\tilde v)).\end{aligned}$$

$\square$

We also need to adapt the result in Proposition 3.3 [6] for S periodic functions.

Proposition 2.2
Let $z\in C(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$ be a S periodic function of zero average

$$\langle z\rangle(s,x,\tilde v)=\frac{1}{S}\int_0^S z(s+\sigma,x,\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma=0,\ \ (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3.$$

1. *There is a unique continuous S periodic function u of zero average whose derivative along the flow of $\partial_s + b\cdot\nabla_{x,\tilde v}$ is z*

$$(\partial_s + b\cdot\nabla_{x,\tilde v})u = z,\ \ \langle u\rangle = 0.$$

   If z is bounded, so is u and

$$\|u\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))} \leq \frac{S}{2}\|z\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))}$$

   for any $R_x, R_{\tilde v} > 0$. If $\operatorname{supp} z \subset \mathbb{R}\times B(R_x)\times B(R_{\tilde v})$, *then* $\operatorname{supp} u \subset \mathbb{R}\times B(R_x)\times B(R_{\tilde v})$.

2. *If z is of class C^1, then so is u and we have for any $R_x, R_{\tilde v} > 0$*

$$\|\nabla_{\tilde v}u\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))} \leq S\sqrt{3}\|\nabla_{\tilde v}z\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))}$$

$$\|\nabla_x u\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))} \leq C\left(\|\nabla_x z\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))} + R_{\tilde v}\|\nabla_{\tilde v}z\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))}\right)$$

$$\|\partial_s u\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))} \leq \|z\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))} + 2\sqrt{3}R_{\tilde v}\|\nabla_{\tilde v}z\|_{C(\mathbb{R}\times B(R_x)\times B(R_{\tilde v}))}$$

   for some constant C depending on $\|\partial_x e\|_{L^\infty}$ and S.

Proof.
1. Take

$$u(s,x,\tilde v) = \frac{1}{S}\int_0^S (\sigma - S)z(s+\sigma, (\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde v))\,\mathrm{d}\sigma,\ \ (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3.$$

2. Use the vector fields $(c^i\cdot\nabla_{x,\tilde v})_{1\leq i\leq 6}$, see Proposition 3.1 [6], which are in involution with $\partial_s + b\cdot\nabla_{x,\tilde v}$, since $\nabla_x\omega_c = 0$. $\square$

3 The limit model and convergence result

We are ready to investigate the limit model in (4) as $\varepsilon \searrow 0$. In this case we intend to capture the fast oscillations due to the operator $\partial_t + \frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde v}$. We are looking for a development whose dominant term belongs to the kernel of $\partial_t + \frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde v}$. It is easily seen that for any function $u \in \ker(\partial_s + b\cdot\nabla_{x,\tilde v})$, we have $u(\cdot, T^\varepsilon) \in \ker(\partial_s + b^\varepsilon\cdot\nabla_{x,\tilde v})$, since

$$u(s+\sigma, T^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\sigma;x,\tilde v))) = u(s+\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;T^\varepsilon(x,\tilde v))) = u(s,T^\varepsilon(x,\tilde v)).$$

Similarly, for any S periodic function of zero average $\langle u\rangle = 0$, the S periodic function $u(\cdot,T^\varepsilon)$ has zero average

$$\langle u(\cdot,T^\varepsilon)\rangle_\varepsilon = \langle u\rangle(\cdot,T^\varepsilon) = 0.$$

The previous discussion suggests to consider the Ansatz

$$\tilde f^\varepsilon(t) = \tilde f_\varepsilon(t,t/\varepsilon)\circ T^\varepsilon + \varepsilon\tilde f^1_\varepsilon(t,t/\varepsilon)\circ T^\varepsilon + \varepsilon^2\tilde f^2_\varepsilon(t,t/\varepsilon)\circ T^\varepsilon + \dots \tag{5}$$

where $(\partial_s + b\cdot\nabla_{x,\tilde v})\tilde f_\varepsilon = 0$, $\left\langle \tilde f^1_\varepsilon\right\rangle = 0$. As in [6], at the leading order, the particle density $\tilde f^\varepsilon$ has no fluctuation (with respect to the extended average operator), and the averages at the orders $\mathcal{O}(\varepsilon^0), \mathcal{O}(\varepsilon)$ combine together in $\tilde{f}_\varepsilon(t, t/\varepsilon) \circ T^\varepsilon$. Notice also that the constraint $(\partial_s + b \cdot \nabla_{x,\tilde{v}})\tilde{f}_\varepsilon = 0$ is equivalent to $\tilde{f}_\varepsilon(t, s, x, \tilde{v}) = \tilde{F}_\varepsilon(t, (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}))$, for some function $\tilde{F}_\varepsilon(t)$. We are looking for a closure determining $\tilde{f}_\varepsilon, \tilde{f}^1_\varepsilon$. The error estimate will require to introduce the second order correction $\varepsilon^2 \tilde{f}^2_\varepsilon(t, t/\varepsilon) \circ T^\varepsilon$. Plugging the Ansatz (5) in (4) we obtain

$$
\begin{aligned}
&\frac{1}{\varepsilon}\partial_s \tilde{f}_\varepsilon(t,s) \circ T^\varepsilon + \partial_t \tilde{f}_\varepsilon(t,s) \circ T^\varepsilon + \partial_s \tilde{f}^1_\varepsilon(t,s) \circ T^\varepsilon + \varepsilon \partial_t \tilde{f}^1_\varepsilon(t,s) \circ T^\varepsilon + \varepsilon \partial_s \tilde{f}^2_\varepsilon(t,s) \circ T^\varepsilon + \ldots \\
&\qquad + c^\varepsilon[\tilde{f}_\varepsilon(t,s) \circ T^\varepsilon + \varepsilon \tilde{f}^1_\varepsilon(t,s) \circ T^\varepsilon + \ldots] \cdot \nabla[(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon + \ldots)(t,s) \circ T^\varepsilon] \\
&\qquad + \varepsilon a^\varepsilon[(\tilde{f}_\varepsilon + \ldots)(t,s) \circ T^\varepsilon] \cdot \nabla[(\tilde{f}_\varepsilon + \ldots)(t,s) \circ T^\varepsilon] \\
&\qquad + \frac{b^\varepsilon}{\varepsilon} \cdot \nabla[(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon + \varepsilon^2 \tilde{f}^2_\varepsilon + \ldots)(t,s) \circ T^\varepsilon] = 0.
\end{aligned}
$$

By construction we have $(\partial_s + b^\varepsilon \cdot \nabla)(\tilde{f}_\varepsilon \circ T^\varepsilon) = 0$, and therefore we deduce

$$
\begin{aligned}
\partial_t \tilde{f}_\varepsilon(t,s) \circ T^\varepsilon &+ (\partial_s + b^\varepsilon \cdot \nabla)((\tilde{f}^1_\varepsilon + \varepsilon \tilde{f}^2_\varepsilon)(t,s) \circ T^\varepsilon) + \varepsilon \partial_t \tilde{f}^1_\varepsilon(t,s) \circ T^\varepsilon \\
&+ c^\varepsilon[(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon)(t,s) \circ T^\varepsilon] \cdot \nabla[(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon)(t,s) \circ T^\varepsilon] \\
&+ \varepsilon a^\varepsilon[\tilde{f}_\varepsilon(t,s) \circ T^\varepsilon] \cdot \nabla[\tilde{f}_\varepsilon(t,s) \circ T^\varepsilon] = \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{6}
$$

We will take the (extended) average of (6) by discarding all second order contributions. Obviously we have

$$
\left\langle \partial_t \tilde{f}_\varepsilon(t,\cdot) \circ T^\varepsilon \right\rangle_\varepsilon = \partial_t \tilde{f}_\varepsilon(t,\cdot) \circ T^\varepsilon
$$

$$
\left\langle (\partial_s + b^\varepsilon \cdot \nabla_{x,\tilde{v}})(\tilde{f}^1_\varepsilon + \varepsilon \tilde{f}^2_\varepsilon)(t,\cdot) \circ T^\varepsilon) \right\rangle_\varepsilon = 0
$$

$$
\left\langle \partial_t \tilde{f}^1_\varepsilon(t,\cdot) \circ T^\varepsilon \right\rangle_\varepsilon = \left\langle \partial_t \tilde{f}^1_\varepsilon(t,\cdot) \right\rangle \circ T^\varepsilon = \partial_t \left\langle \tilde{f}^1_\varepsilon(t,\cdot) \right\rangle \circ T^\varepsilon = 0
$$

and

$$
\varepsilon a^\varepsilon[\tilde{f}_\varepsilon(t,s) \circ T^\varepsilon] \cdot \nabla_{x,\tilde{v}}[\tilde{f}_\varepsilon(t,s) \circ T^\varepsilon] = \varepsilon (a[\tilde{f}_\varepsilon(t,s)] \cdot \nabla_{x,\tilde{v}} \tilde{f}_\varepsilon(t,s)) \circ T^\varepsilon + \mathcal{O}(\varepsilon^2)
$$

which implies, cf. Proposition 2.1

$$
\begin{aligned}
\left\langle \varepsilon a^\varepsilon[\tilde{f}_\varepsilon(t,\cdot) \circ T^\varepsilon] \cdot \nabla_{x,\tilde{v}}[\tilde{f}_\varepsilon(t,\cdot) \circ T^\varepsilon] \right\rangle_\varepsilon &= \varepsilon \left\langle (a[\tilde{f}_\varepsilon(t,\cdot)] \cdot \nabla_{x,\tilde{v}} \tilde{f}_\varepsilon(t,\cdot)) \circ T^\varepsilon \right\rangle_\varepsilon + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon \left\langle a[\tilde{f}_\varepsilon(t,\cdot)] \cdot \nabla_{x,\tilde{v}} \tilde{f}_\varepsilon(t,\cdot) \right\rangle \circ T^\varepsilon + \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

We concentrate now on the term corresponding to the vector field $c^\varepsilon \cdot \nabla_{x,\tilde{v}}$

$$
\begin{aligned}
&c^\varepsilon[(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon)(t,s) \circ T^\varepsilon] \cdot [(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon)(t,s) \circ T^\varepsilon] = c^\varepsilon[\tilde{f}_\varepsilon(t,s) \circ T^\varepsilon] \cdot \nabla(\tilde{f}_\varepsilon(t,s) \circ T^\varepsilon) \\
&\quad + \varepsilon (c^\varepsilon[\tilde{f}_\varepsilon(t,s)] \cdot \nabla \tilde{f}^1_\varepsilon(t,s)) \circ T^\varepsilon + \varepsilon \frac{q}{m}[(E[\tilde{f}^1_\varepsilon(t,s)] \cdot e)(e \cdot \nabla_{\tilde{v}} \tilde{f}_\varepsilon(t,s)] \circ T^\varepsilon + \mathcal{O}(\varepsilon^2) \\
&\quad = (c_0[\tilde{f}_\varepsilon(t,s)] \cdot \nabla \tilde{f}_\varepsilon(t,s)) \circ T^\varepsilon + \varepsilon (c_1[\tilde{f}_\varepsilon(t,s)] \cdot \nabla \tilde{f}_\varepsilon(t,s)) \circ T^\varepsilon \\
&\quad + \varepsilon (c_0[\tilde{f}_\varepsilon(t,s)] \cdot \nabla \tilde{f}^1_\varepsilon(t,s)) \circ T^\varepsilon + \varepsilon \frac{q}{m}[(E[\tilde{f}^1_\varepsilon(t,s)] \cdot e)(e \cdot \nabla_{\tilde{v}} \tilde{f}_\varepsilon)] \circ T^\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

where

$$
c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \cdot e) e \cdot \nabla_x + \frac{q}{m}(E[\tilde{f}] \cdot e) e \cdot \nabla_{\tilde{v}} - [\tilde{v} \wedge \partial_x e(\tilde{v} \wedge e)] \cdot \nabla_{\tilde{v}}.
$$

We claim that the average along the flow of $\partial_s + b\cdot\nabla_{x,\tilde v}$ of $(E[\tilde f^1_\varepsilon(t,s)]\cdot e)e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t,s)$ vanishes. Indeed, as $e\cdot\nabla_{\tilde v}$ is in involution with respect to $\partial_s + b\cdot\nabla_{x,\tilde v}$, we have

$$(\partial_s + b\cdot\nabla_{x,\tilde v})(e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t)) = e\cdot\nabla_{\tilde v}((\partial_s + b\cdot\nabla_{x,\tilde v})\tilde f_\varepsilon(t)) = 0$$

implying that

$$\left\langle (E[\tilde f^1_\varepsilon(t,\cdot)]\cdot e)(e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t))\right\rangle = \left\langle E[\tilde f^1_\varepsilon(t)]\cdot e\right\rangle e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t) = 0$$

because the charge density of $\tilde f^1_\varepsilon(t)$ has zero average

$$\begin{aligned}\left\langle \rho[\tilde f^1_\varepsilon(t)]\right\rangle(s,x) &= \frac{q}{S}\int_0^S\int_{\mathbb{R}^3}\tilde f^1_\varepsilon(t,s+\sigma,x,\tilde v)\,\mathrm{d}\tilde v\mathrm{d}\sigma\\ &= \frac{q}{S}\int_0^S\int_{\mathbb{R}^3}\tilde f^1_\varepsilon(t,s+\sigma,x,\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\tilde v\mathrm{d}\sigma\\ &= q\int_{\mathbb{R}^3}\left\langle \tilde f^1_\varepsilon(t)\right\rangle(s,x,\tilde v)\,\mathrm{d}\tilde v = 0.\end{aligned}$$

Therefore, thanks to Proposition 2.1 we obtain

$$\left\langle [(E[\tilde f^1_\varepsilon(t)]\cdot e)(e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t))]\circ T^\varepsilon\right\rangle_\varepsilon = \left\langle (E[\tilde f^1_\varepsilon(t)]\cdot e)(e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t))\right\rangle\circ T^\varepsilon = 0$$

and thus

$$\begin{aligned}&\left\langle c^\varepsilon[(\tilde f_\varepsilon(t,\cdot)+\varepsilon\tilde f^1_\varepsilon(t,\cdot))\circ T^\varepsilon]\cdot\nabla[(\tilde f_\varepsilon(t,\cdot)+\varepsilon\tilde f^1_\varepsilon(t,\cdot))\circ T^\varepsilon]\right\rangle_\varepsilon = \left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle\circ T^\varepsilon\\ &+\varepsilon\left\langle c_1[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle\circ T^\varepsilon + \varepsilon\left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f^1_\varepsilon(t,\cdot)\right\rangle\circ T^\varepsilon + \mathcal{O}(\varepsilon^2).\end{aligned}$$

The previous computations lead to the following model for the particle density $\tilde f_\varepsilon$

$$\begin{aligned}\partial_t\tilde f_\varepsilon(t,s) + \left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle + \varepsilon\left\langle (a[\tilde f_\varepsilon(t,\cdot)] + c_1[\tilde f_\varepsilon(t,\cdot)])\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle\\ +\varepsilon\left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f^1_\varepsilon(t,\cdot)\right\rangle = 0,\ \ (t,s,x,\tilde v)\in[0,T]\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\end{aligned}\tag{7}$$

together with the constraint $(\partial_s + b\cdot\nabla_{x,\tilde v})\tilde f_\varepsilon = 0$. The equation for the fluctuation $\tilde f^1_\varepsilon$ comes by comparing (6) with respect to (7). Indeed, we have

$$\begin{aligned}(\partial_s + b^\varepsilon\cdot\nabla)[(\tilde f^1_\varepsilon + \varepsilon\tilde f^2_\varepsilon)(t,s)\circ T^\varepsilon] &= \partial_s(\tilde f^1_\varepsilon + \varepsilon\tilde f^2_\varepsilon)\circ T^\varepsilon + \underbrace{\partial T^\varepsilon b^\varepsilon}_{b\circ T^\varepsilon}\cdot[\nabla(\tilde f^1_\varepsilon + \varepsilon\tilde f^2_\varepsilon)]\circ T^\varepsilon\\ &= [(\partial_s + b\cdot\nabla)(\tilde f^1_\varepsilon + \varepsilon\tilde f^2_\varepsilon)]\circ T^\varepsilon\end{aligned}$$

and therefore (6) also writes

$$\begin{aligned}\partial_t\tilde f_\varepsilon(t,s) + (\partial_s + b\cdot\nabla)(\tilde f^1_\varepsilon + \varepsilon\tilde f^2_\varepsilon) + \varepsilon\partial_t\tilde f^1_\varepsilon + c_0[\tilde f_\varepsilon(t,s)]\cdot\nabla\tilde f_\varepsilon(t,s)\\ +\varepsilon(a[\tilde f_\varepsilon(t,s)] + c_1[\tilde f_\varepsilon(t,s)])\cdot\nabla\tilde f_\varepsilon(t,s) + \varepsilon c_0[\tilde f_\varepsilon(t,s)]\cdot\nabla\tilde f^1_\varepsilon(t,s)\\ +\varepsilon\frac{q}{m}(E[\tilde f^1_\varepsilon(t,s)]\cdot)(e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t,s)) = \mathcal{O}(\varepsilon^2).\end{aligned}\tag{8}$$

Taking the difference between (8), (7) yields

$$
\begin{aligned}
c_0[\tilde f_\varepsilon(t,s)]\cdot\nabla\tilde f_\varepsilon(t,s) &- \left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle + \varepsilon(a[\tilde f_\varepsilon(t,s)]+c_1[\tilde f_\varepsilon(t,s)])\cdot\nabla\tilde f_\varepsilon(t,s)\\
&- \varepsilon\left\langle (a[\tilde f_\varepsilon(t,\cdot)]+c_1[\tilde f_\varepsilon(t,\cdot)])\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle\\
&+ \varepsilon c_0[\tilde f_\varepsilon(t,s)]\cdot\nabla\tilde f^1_\varepsilon(t,s) - \varepsilon\left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f^1_\varepsilon(t,\cdot)\right\rangle\\
&+ \varepsilon\frac{q}{m}(E[\tilde f^1_\varepsilon(t,s)]\cdot e)(e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t,s)) + (\partial_s+b\cdot\nabla)(\tilde f^1_\varepsilon+\varepsilon\tilde f^2_\varepsilon) + \varepsilon\partial_t\tilde f^1_\varepsilon = \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

The above equality suggests to determine the fluctuation $\tilde f^1_\varepsilon$ by

$$
c_0[\tilde f_\varepsilon(t,s)]\cdot\nabla\tilde f_\varepsilon(t,s) - \left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle + (\partial_s+b\cdot\nabla)\tilde f^1_\varepsilon(t,s) = 0,\ \ \left\langle \tilde f^1_\varepsilon\right\rangle = 0 \quad (9)
$$

and to consider the corrector $\tilde f^2_\varepsilon$ such that

$$
\begin{aligned}
(a[\tilde f_\varepsilon(t,s)]+c_1[\tilde f_\varepsilon(t,s)])\cdot\nabla\tilde f_\varepsilon(t,s) &- \left\langle (a[\tilde f_\varepsilon(t,\cdot)]+c_1[\tilde f_\varepsilon(t,\cdot)])\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle\\
&+ c_0[\tilde f_\varepsilon(t,s)]\cdot\nabla\tilde f^1_\varepsilon(t,s) - \left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f^1_\varepsilon(t,\cdot)\right\rangle\\
&+ \frac{q}{m}(E[\tilde f^1_\varepsilon(t,s)]\cdot e)(e\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t,s)) + \partial_t\tilde f^1_\varepsilon + (\partial_s+b\cdot\nabla)\tilde f^2_\varepsilon = 0,\ \ \left\langle \tilde f^2_\varepsilon\right\rangle = 0.
\end{aligned}
$$

The well posedness of (7), (9) is stated in Section 4, see Theorem 4.1. As in Theorem 1.2 [6], we can establish the following error estimate. The proof details are left to the reader.

Theorem 3.1
Let $\mathbf{B} = Be \in C^4_b(\mathbb{R}^3)$ *be a smooth magnetic field such that* $\nabla_x B = 0, \operatorname{div}_x\mathbf{B} = 0$. *Consider a non negative, smooth, compactly supported initial particle density* $\tilde G \in C^3_c(\mathbb{R}^3\times\mathbb{R}^3)$ *and* $\tilde g(s,x,\tilde v) = \tilde G(x,\tilde{\mathcal{V}}(-s;x,\tilde v)), (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$. *We denote by* $(f^\varepsilon)_{0<\varepsilon\le 1}$ *the solutions of the Vlasov-Poisson equations with external magnetic field* (1), (2) *on* $[0,T], 0<T<T(f_{\mathrm{in}})$, *cf. Theorem 2.1 [6], corresponding to the initial condition*

$$
f^\varepsilon(0,x,v) = (\tilde g+\varepsilon\tilde g^1)\left(0, x+\varepsilon\frac{v\wedge(x)}{\omega_c}, v-\varepsilon\frac{E[\tilde G]\wedge e(x)}{B}\right),\ \ (x,v)\in\mathbb{R}^3\times\mathbb{R}^3
$$

where

$$
c_0[\tilde g]\cdot\nabla\tilde g - \langle c_0[\tilde g]\cdot\nabla\tilde g\rangle + (\partial_s+b\cdot\nabla)\tilde g^1 = 0,\ \ \left\langle\tilde g^1\right\rangle = 0.
$$

For $\varepsilon\in]0,\varepsilon_T]$ *small enough, we consider the solution* $\tilde f_\varepsilon = \tilde f_\varepsilon(t,s,x,\tilde v)$ *on* $[0,T]$ *of the problem*

$$
\begin{aligned}
\partial_t\tilde f_\varepsilon(t,s) + \left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle &+ \varepsilon\left\langle (a[\tilde f_\varepsilon(t,\cdot)]+c_1[\tilde f_\varepsilon(t,\cdot)])\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle\\
&+ \varepsilon\left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f^1_\varepsilon(t,\cdot)\right\rangle = 0,\ \ (t,s,x,\tilde v)\in[0,T]\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3
\end{aligned}
$$

$$
c_0[\tilde f_\varepsilon(t,s)]\cdot\nabla\tilde f_\varepsilon(t,s) - \left\langle c_0[\tilde f_\varepsilon(t,\cdot)]\cdot\nabla\tilde f_\varepsilon(t,\cdot)\right\rangle + (\partial_s+b\cdot\nabla)\tilde f^1_\varepsilon(t,s) = 0,\ \ \left\langle\tilde f^1_\varepsilon\right\rangle = 0
$$

corresponding to the initial condition

$$\tilde{f}_\varepsilon(0,s,x,\tilde{v}) = \tilde{G}(x,\tilde{\mathcal{V}}(-s;x,\tilde{v})) = \tilde{g}(s,x,\tilde{v}), \quad (s,x,\tilde{v}) \in \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3.$$

Therefore there is a constant C_T such that for any $0<\varepsilon\leq\varepsilon_T$

$$\sup_{t\in[0,T]}\left\{\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}\left|f^\varepsilon(t,x,v) - (\tilde{f}_\varepsilon+\varepsilon\tilde{f}^1_\varepsilon)\left(t,t/\varepsilon,x+\varepsilon\frac{v\wedge e}{\omega_c},v-\varepsilon\frac{E[\tilde{f}_\varepsilon(t,t/\varepsilon)]\wedge e}{B}\right)\right|^2 \mathrm{d}v\mathrm{d}x\right\}^{1/2}$$
$$\leq C_T\varepsilon^2.$$

4 Equivalent formulation of the limit model

We determine now the equivalent formulation for (7), (9) by computing the average of the vector fields entering this model. Most of the computations has been performed in [6], where the formulae for $\langle a\rangle\cdot\nabla$, $\langle c_0\rangle\cdot\nabla$, $\langle c_1\rangle\cdot\nabla$ are detailed, and we only need to complete them by treating the extra terms due to the presence of the fast time variable.

Proposition 4.1
Assume that $e\in C^2(\mathbb{R}^3), \nabla_x\omega_c = 0, \mathrm{div}_x e = 0$ and let us consider $\tilde{f}_\varepsilon\in C^2([0,T]\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$. Then $\tilde{f}_\varepsilon$ solves

$$\partial_t\tilde{f}_\varepsilon + \left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla\tilde{f}_\varepsilon\right\rangle + \varepsilon\left\langle (a[\tilde{f}_\varepsilon]+c_1[\tilde{f}_\varepsilon])\cdot\nabla\tilde{f}_\varepsilon\right\rangle + \varepsilon\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla\tilde{f}^1_\varepsilon\right\rangle = 0, (\partial_s+b\cdot\nabla)\tilde{f}_\varepsilon = 0 \tag{10}$$

$$c_0[\tilde{f}_\varepsilon]\cdot\nabla\tilde{f}_\varepsilon - \left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla\tilde{f}_\varepsilon\right\rangle + (\partial_s+b\cdot\nabla)\tilde{f}^1_\varepsilon = 0, \quad \left\langle\tilde{f}^1_\varepsilon\right\rangle = 0 \tag{11}$$

iff $\tilde{f}_\varepsilon$ satisfies $\tilde{f}_\varepsilon(t,s,x,\tilde{v}) = \tilde{F}_\varepsilon(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))$, where

$$\partial_t\tilde{F}_\varepsilon + \left\langle c_0[\tilde{F}_\varepsilon]\right\rangle\cdot\nabla\tilde{F}_\varepsilon + \varepsilon\left(\left\langle a[\tilde{F}_\varepsilon]\right\rangle + \left\langle c_1[\tilde{F}_\varepsilon]\right\rangle\right)\cdot\nabla\tilde{F}_\varepsilon + \varepsilon D[\tilde{F}_\varepsilon]\cdot\nabla\tilde{F}_\varepsilon = 0 \tag{12}$$

$$D[\tilde{F}]\cdot\nabla_{X,\tilde{V}} = \left[\frac{j[\tilde{F}]\wedge e(X)}{3\epsilon_0 B} + \frac{\int_{\mathbb{R}^3}N(e(X),X-X',e(X'))j[\tilde{F}](X')\,\mathrm{d}X'}{8\pi\epsilon_0 B}\wedge e(X)\right]\cdot\nabla_{\tilde{V}}$$
$$+\left[\frac{q}{m}e\otimes e\left(E\left[(e\cdot\mathrm{rot}_X e)\frac{(\tilde{V}\cdot e)}{\omega_c}\tilde{F}\right] - E\left[\frac{\tilde{V}\wedge e}{\omega_c}\cdot\nabla_X\tilde{F}\right]\right) + d_{\tilde{V}}(X,\tilde{V})\right]\cdot\nabla_{\tilde{V}}$$

$$d_{\tilde{V}}(X,\tilde{V}) = \left\langle(\overline{c}_0\cdot\nabla_{x,\tilde{v}}\,)\sum_{k=1}^{3}[(\mathcal{A}_k)_{-s}\cos(ks\omega_c)+(\mathcal{B}_k)_{-s}\sin(ks\omega_c)]\right\rangle(0,X,\tilde{V})$$

$$\tilde{f}^1_\varepsilon(t,s,x,\tilde{v}) = \sum_{k=1}^{3}[\mathcal{A}_k((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))\cos(ks\omega_c)+\mathcal{B}_k((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))\sin(ks\omega_c)]$$
$$\cdot(\nabla_{\tilde{V}}\tilde{F}_\varepsilon(t))((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})) \tag{13}$$

where $N(e,z,e') = (I_3 - e\otimes e)K(z)(I_3 - e'\otimes e') - M[e]K(z)M[e'], K(z) = \left(I_3 - 3z\otimes z/|z|^2\right)/|z|^3$

$$\mathcal{A}_k(X,\tilde{V}) = \frac{1}{k\pi}\int_0^S \mathcal{F}(s,X,\tilde{V})\sin(ks\omega_c)\,\mathrm{d}s,\quad \mathcal{B}_k(X,\tilde{V}) = -\frac{1}{k\pi}\int_0^S \mathcal{F}(s,X,\tilde{V})\cos(ks\omega_c)\,\mathrm{d}s$$

for $k\in\{1,2,3\}$ *and*

$$\mathcal{F}(s,X,\tilde{V}) = \partial_{x,\tilde{v}}\tilde{\mathcal{V}}(-s;(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))\overline{c}_0((\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))$$

$$\overline{c}_0(x,\tilde{v})\cdot\nabla_{x,\tilde{v}} = (\tilde{v}\cdot e)e\cdot\nabla_x - [\tilde{v}\wedge\partial_x e(\tilde{v}\wedge e)]\cdot\nabla_{\tilde{v}}.$$

Proof.
Clearly the constraint $(\partial_s + b\cdot\nabla)\tilde{f}_\varepsilon = 0$ writes $\tilde{f}_\varepsilon(t,s,x,\tilde{v}) = \tilde{F}_\varepsilon(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))$ for some function $\tilde{F}_\varepsilon = \tilde{F}_\varepsilon(t,X,\tilde{V})\in C^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$. We need to compute the averages $\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla\tilde{f}_\varepsilon\right\rangle$, $\left\langle (a[\tilde{f}_\varepsilon] + c_1[\tilde{f}_\varepsilon])\cdot\nabla\tilde{f}_\varepsilon\right\rangle$, $\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla\tilde{f}^1_\varepsilon\right\rangle$ along the flow of $\partial_s + b\cdot\nabla_{x,\tilde{v}}$ and to invert the operator $\partial_s + b\cdot\nabla_{x,\tilde{v}}$ on zero average functions, in order to solve (11). Recall that for any particle density $\tilde{f}$, the vector field $a[\tilde{f}]\cdot\nabla_{x,\tilde{v}}$ writes

$$\begin{aligned} a[\tilde{f}]\cdot\nabla_{x,\tilde{v}} &= \left(\frac{E[\tilde{f}]\wedge e}{B} - A_x(x,\tilde{v})\right)\cdot\nabla_x - \partial_x\left(\frac{E[\tilde{f}]\wedge e}{B}\right)\tilde{v}\cdot\nabla_{\tilde{v}} \\ &\quad + \frac{1}{4\pi\epsilon_0 B}\left(\mathrm{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes j[\tilde{f}](x')\,\mathrm{d}x'\wedge e(x)\right)\cdot\nabla_{\tilde{v}}. \end{aligned}$$

We have $\rho[\tilde{f}_\varepsilon(t,s)] = \rho[\tilde{F}_\varepsilon(t)]$ implying that $E[\tilde{f}_\varepsilon(t,s)] = E[\tilde{F}_\varepsilon(t)]$ and

$$j[\tilde{f}_\varepsilon(t,s)](x) = \mathcal{R}(-s\omega_c, e(x))j[\tilde{F}_\varepsilon(t)](x)$$

and therefore

$$a[\tilde{f}_\varepsilon(t,s)]\cdot\nabla_{x,\tilde{v}} = a[\tilde{F}_\varepsilon(t)]\cdot\nabla_{x,\tilde{v}} + \overline{a}_s[\tilde{F}_\varepsilon(t)]\cdot\nabla_{\tilde{v}}$$

where

$$\overline{a}_s[\tilde{F}_\varepsilon]\cdot\nabla_{\tilde{v}} = \frac{1}{4\pi\epsilon_0 B}\left(\mathrm{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes[\mathcal{R}(-s\omega_c,e(x')) - I_3]\,j[\tilde{F}_\varepsilon](x')\,\mathrm{d}x'\wedge e(x)\right)\cdot\nabla_{\tilde{v}}.$$

Thanks to the equality $\tilde{f}_\varepsilon(t,s,x,\tilde{v}) = \tilde{f}_\varepsilon(t,s+\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))$ we have

$$\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon(t,s,x,\tilde{v}) = {}^t\partial(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v})(\nabla\tilde{f}_\varepsilon)(t,s+\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))$$

implying that

$$(\nabla\tilde{f}_\varepsilon)(t,s+\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v})) = {}^t\partial(\mathcal{X},\tilde{\mathcal{V}})(-\sigma;(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))\nabla\tilde{f}_\varepsilon(t,s,x,\tilde{v})$$

and

$$\begin{aligned} &(a[\tilde{f}_\varepsilon]\cdot\nabla\tilde{f}_\varepsilon)(t,s+\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v})) \\ &\qquad = \partial(\mathcal{X},\tilde{\mathcal{V}})(-\sigma;(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))a[\tilde{F}_\varepsilon(t)]((\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))\cdot\nabla\tilde{f}_\varepsilon(t,s,x,\tilde{v}) \\ &\qquad + \mathcal{R}(\sigma\omega_c,e(x))\overline{a}_{s+\sigma}[\tilde{F}_\varepsilon(t)](x)\cdot\nabla_{\tilde{v}}\tilde{f}_\varepsilon(t,s,x,\tilde{v}). \end{aligned}$$

Averaging with respect to σ, we obtain cf. Proposition 5.1 [6]

$$\left\langle a[\tilde f_\varepsilon]\cdot\nabla \tilde f_\varepsilon\right\rangle(t,s,x,\tilde v) = \frac{1}{S}\int_0^S \partial(\mathcal{X},\tilde{\mathcal{V}})(-\sigma;(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde v))a[\tilde F_\varepsilon(t)]((\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde v))\ \mathrm{d}\sigma$$

$$\cdot\nabla_{x,\tilde v}\ \tilde f_\varepsilon(t,s,x,\tilde v) + \frac{1}{S}\int_0^S \mathcal{R}(\sigma\omega_c,e(x))\overline{a}_{s+\sigma}[\tilde F_\varepsilon(t)](x)\ \mathrm{d}\sigma\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t,s,x,\tilde v)$$

$$= \left\langle a[\tilde F_\varepsilon(t)]\right\rangle(x,\tilde v)\cdot\nabla_{x,\tilde v}\ \tilde f_\varepsilon(t,s,x,\tilde v) + \frac{1}{S}\int_0^S \mathcal{R}(\sigma\omega_c,e(x))\overline{a}_{s+\sigma}[\tilde F_\varepsilon(t)](x)\ \mathrm{d}\sigma\cdot\nabla_{\tilde v}\tilde f_\varepsilon(t,s,x,\tilde v)$$

$$= \partial(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde v)\left\langle a[\tilde F_\varepsilon(t)]\right\rangle(x,\tilde v)\cdot(\nabla_{X,\tilde V}\tilde F_\varepsilon)(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde v))$$

$$+\frac{1}{S}\int_0^S \mathcal{R}((s+\sigma)\omega_c,e(x))\overline{a}_{s+\sigma}[\tilde F_\varepsilon(t)](x)\ \mathrm{d}\sigma\cdot(\nabla_{\tilde V}\tilde F_\varepsilon)(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde v))$$

$$= \left\langle a[\tilde F_\varepsilon(t)]\right\rangle((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde v))\cdot(\nabla_{X,\tilde V}\tilde F_\varepsilon)(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde v))$$

$$+\frac{1}{S}\int_0^S \mathcal{R}(\sigma\omega_c,e(x))\overline{a}_{\sigma}[\tilde F_\varepsilon(t)](x)\ \mathrm{d}\sigma\cdot(\nabla_{\tilde V}\tilde F_\varepsilon)(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde v))$$

where $\langle a\rangle\cdot\nabla$ has been computed in Proposition 5.4 [6]. Notice that in the last equality we have used the involution of $\left\langle a[\tilde F_\varepsilon(t)]\right\rangle\cdot\nabla$ with respect to the vector field $b\cdot\nabla$, cf. Proposition 5.1 [6]. For the last average, observe that

$$\frac{1}{S}\int_0^S \mathcal{R}(\sigma\omega_c,e(x))\overline{a}_\sigma[\tilde F_\varepsilon(t)](x)\ \mathrm{d}\sigma$$

$$= \frac{1}{4\pi\epsilon_0 BS}\int_0^S \mathcal{R}(\sigma\omega_c,e(x))\left(\mathrm{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes\mathcal{R}(-\sigma\omega_c,e(x'))j[\tilde F_\varepsilon(t)](x')\ \mathrm{d}x'\wedge e(x)\right)\mathrm{d}\sigma$$

$$= \frac{1}{4\pi\epsilon_0 BS}\int_0^S [\cos(\sigma\omega_c)(I_3-e(x)\otimes e(x))+\sin(\sigma\omega_c)M[e(x)]+e(x)\otimes e(x)]$$

$$\mathrm{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes[\cos(\sigma\omega_c)(I_3-e(x')\otimes e(x'))-\sin(\sigma\omega_c)M[e(x')]+e(x')\otimes e(x')]j[\tilde F_\varepsilon(t)]\ \mathrm{d}x'$$

$$\wedge\, e(x)\ \mathrm{d}\sigma$$

$$= \frac{1}{4\pi\epsilon_0 B}\frac{I_3-e\otimes e}{2}\left(\mathrm{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes[I_3-e(x')\otimes e(x')]j[\tilde F_\varepsilon(t)](x')\ \mathrm{d}x'\wedge e(x)\right)$$

$$-\frac{1}{4\pi\epsilon_0 B}\frac{M[e]}{2}\left(\mathrm{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes M[e(x')]j[\tilde F_\varepsilon(t)](x')\ \mathrm{d}x'\wedge e(x)\right)$$

$$= -\frac{M[e]}{8\pi\epsilon_0 B}\lim_{\delta\searrow 0}\int_{|x-x'|>\delta}\frac{[I_3-e(x)\otimes e(x)](x-x')}{|x-x'|^3}\mathrm{div}\{[I_3-e(x')\otimes e(x')]j[\tilde F_\varepsilon(t)](x')\}\ \mathrm{d}x'$$

$$+\frac{M[e]}{8\pi\epsilon_0 B}\lim_{\delta\searrow 0}\int_{|x-x'|>\delta}\frac{M[e(x)](x-x')}{|x-x'|^3}\mathrm{div}\{M[e(x')]j[\tilde F_\varepsilon(t)](x')\}\ \mathrm{d}x'$$

$$= -\frac{M[e(x)]}{8\pi\epsilon_0 B}\lim_{\delta\searrow 0}\int_{|x-x'|>\delta}N(e(x),x-x',e(x'))j[\tilde F_\varepsilon(t)](x')\ \mathrm{d}x'$$

$$-\frac{M[e(x)]}{8\pi\epsilon_0 B}\lim_{\delta\searrow 0}\int_{|x-x'|=\delta}\{[I_3-e(x)\otimes e(x)]\frac{x-x'}{|x-x'|^3}\otimes[I_3-e(x')\otimes e(x')]\frac{x-x'}{|x-x'|}$$

$$+M[e(x)]\frac{x-x'}{|x-x'|^3}\otimes M[e(x')]\frac{x-x'}{|x-x'|}\}j[\tilde F_\varepsilon(t)](x')\ \mathrm{d}\sigma(x')$$

where $K(z) = (I_3 - 3\frac{z}{|z|} \otimes \frac{z}{|z|})/|z|^3, z \in \mathbb{R}^3 \setminus \{0\},\ N(e,z,e') = (I_3 - e\otimes e)K(z)(I_3 - e'\otimes e') - M[e]K(z)M[e'], e, z, e' \in \mathbb{R}^3 \setminus \{0\}, |e| = |e'| = 1$. Performing the change of variable $x' = x + \delta z$ in the last integral yields

$$\int_{|x-x'|=\delta} \{(I_3 - e(x)\otimes e(x))\frac{x-x'}{|x-x'|^3} \otimes (I_3 - e(x')\otimes e(x'))\frac{x-x'}{|x-x'|}$$
$$+ M[e(x)]\frac{x-x'}{|x-x'|^3} \otimes M[e(x')]\frac{x-x'}{|x-x'|}\} j[\tilde{F}_\varepsilon(t)](x')\mathrm{d}\sigma(x')$$
$$= \int_{|z|=1} \{(I_3 - e(x)\otimes e(x))z \otimes (I_3 - e(x+\delta z)\otimes e(x+\delta z))z$$
$$+ M[e(x)]z \otimes M[e(x+\delta z)]z\} j[\tilde{F}_\varepsilon(t)](x+\delta z)\mathrm{d}\sigma(z)$$
$$\underset{\delta\searrow 0}{\longrightarrow} \int_{|z|=1} \{(I_3 - e(x)\otimes e(x))z \otimes (I_3 - e(x)\otimes e(x))z + M[e(x)]z\otimes M[e(x)]z\}\mathrm{d}\sigma(z)\ j[\tilde{F}_\varepsilon(t)](x)$$
$$= \int_{|z|=1} |z\wedge e(x)|^2 (I_3 - e(x)\otimes e(x))\mathrm{d}\sigma\ j[\tilde{F}_\varepsilon(t)](x)$$
$$= \frac{8\pi}{3}(I_3 - e(x)\otimes e(x))\ j[\tilde{F}_\varepsilon(t)](x).$$

Therefore the average of $a[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon$ writes

$$\left\langle a[\tilde{f}_\varepsilon]\cdot\nabla \tilde{f}_\varepsilon\right\rangle(t,s,x,\tilde{v}) = \left(\left\langle a[\tilde{F}_\varepsilon(t)]\right\rangle \cdot \nabla_{X,\tilde{V}}\tilde{F}_\varepsilon(t)\right)((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})) \tag{14}$$
$$- \left(\frac{e(x)}{8\pi\epsilon_0 B}\wedge \lim_{\delta\searrow 0}\int_{|x-x'|>\delta} N(e(x), x-x', e(x'))j[\tilde{F}_\varepsilon(t)](x')\ \mathrm{d}x'\right)\cdot(\nabla_{\tilde{V}}\tilde{F}_\varepsilon(t))((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))$$
$$+ \frac{j[\tilde{F}_\varepsilon(t)](x)\wedge e(x)}{3\epsilon_0 B}\cdot(\nabla_{\tilde{V}}\tilde{F}_\varepsilon(t))((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})).$$

We inquire about the convergence, when $\delta\searrow 0$ of

$$\int_{|x-x'|>\delta} N(e(x), x-x', e(x'))j[\tilde{F}_\varepsilon](x')\ \mathrm{d}x' = \int_{\delta<|x-x'|<R} N(e(x), x-x', e(x'))(j[\tilde{F}_\varepsilon](x') - j[\tilde{F}_\varepsilon](x))\ \mathrm{d}x'$$
$$+ \int_{\delta<|x-x'|<R} N(e(x), x-x', e(x'))\ \mathrm{d}x'\ j[\tilde{F}_\varepsilon](x)$$

for R large enough. We are done if we establish the convergence of $\int \mathbf{1}_{\{\delta<|x-x'|<R\}}N(e(x), x-x', e(x'))\ \mathrm{d}x'$ when $\delta\searrow 0$. But we can write

$$\int_{\delta<|x-x'|<R} N(e(x), x-x', e(x'))\ \mathrm{d}x' = \int_{\delta<|x-x'|<R} (N(e(x), x-x', e(x')) - N(e(x), x-x', e(x)))\ \mathrm{d}x'$$
$$+ \int_{\delta<|x-x'|<R} N(e(x), x-x', e(x))\ \mathrm{d}x'$$

and therefore it is enough to prove the convergence, as $\delta\searrow 0$, of $\int \mathbf{1}_{\{\delta<|x-x'|<R\}}N(e(x), x-x', e(x))\ \mathrm{d}x'$. Actually, for any $r>0$ we have $\int_{|z|=r} N(e(x), z, e(x))\ \mathrm{d}\sigma(z) = 0$. This is a consequence of the fact that $K(z)$ has zero trace for any $z\in\mathbb{R}^3\setminus\{0\}$. Indeed, for any $\xi\in\mathbb{R}^3$ we have

$$K(z) : (I_3 - e\otimes e)\xi\otimes(I_3 - e\otimes e)\xi + K(z) : M[e]\xi\otimes M[e]\xi + |\xi\wedge e|^2 K(z) : e\otimes e$$
$$= |\xi\wedge e|^2 \mathrm{trace}K(z) = 0,\quad z\in\mathbb{R}^3\setminus\{0\}$$

implying that

$$[N(e,z,e) + (K(z)e \cdot e)(I_3 - e \otimes e)] : \xi \otimes \xi = 0, \ \ \xi \in \mathbb{R}^3.$$

As the matrix $N(e,z,e) + (K(z)e \cdot e)(I_3 - e \otimes e)$ is symmetric, we obtain $N(e,z,e) = -(K(z)e \cdot e)(I_3 - e \otimes e)$. By direct computation we obtain $\int_{|z|=r} (K(z)e \cdot e) \, \mathrm{d}\sigma(z) = 0, r > 0$. Similarly we have

$$\left\langle c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}} \ \tilde{f}_\varepsilon \right\rangle (t,s,x,\tilde{v}) = \left(\left\langle c_0[\tilde{F}_\varepsilon(t)] \right\rangle \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon(t) \right) ((\mathcal{X}, \tilde{\mathcal{V}})(-s;x,\tilde{v})) \tag{15}$$

where $\langle c_0 \rangle \cdot \nabla$ has been computed in Proposition 5.5 [6]. In order to treat the average of $c_1[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}} \ \tilde{f}_\varepsilon$ we need to compute $E[(\tilde{v} \wedge e) \cdot \nabla_x \tilde{f}_\varepsilon]$ and therefore the charge density

$$\begin{aligned} q \int_{\mathbb{R}^3} (\tilde{v} \wedge e) \cdot \nabla_x \tilde{f}_\varepsilon(t,s,x,\tilde{v}) \, \mathrm{d}\tilde{v} &= q \, \mathrm{div}_x \int_{\mathbb{R}^3} \tilde{f}_\varepsilon(t,s,x,\tilde{v})(\tilde{v} \wedge e) \, \mathrm{d}\tilde{v} \\ &\quad - q \int_{\mathbb{R}^3} \tilde{f}_\varepsilon(t,s,x,\tilde{v}) \mathrm{div}_x(\tilde{v} \wedge e) \, \mathrm{d}\tilde{v} \\ &= \mathrm{div}_x(j[\tilde{f}_\varepsilon(t,s)] \wedge e) + j[\tilde{f}_\varepsilon(t,s)] \cdot \mathrm{rot}_x e \\ &= e \cdot \mathrm{rot}_x(\mathcal{R}(-s\omega_c, e(x)) j[\tilde{F}_\varepsilon(t)]). \end{aligned}$$

We obtain the following formula for the vector field $c_1 \cdot \nabla$

$$c_1[\tilde{f}_\varepsilon(t,s)] \cdot \nabla_{x,\tilde{v}} \ = c_1[\tilde{F}_\varepsilon(t)] \cdot \nabla_{x,\tilde{v}} \ + \overline{c}_{1s}[\tilde{F}_\varepsilon(t)] \cdot \nabla_{\tilde{v}}$$

where

$$\overline{c}_{1s}[\tilde{F}_\varepsilon] \cdot \nabla_{\tilde{v}} = \frac{1}{B} \Big(E[e \cdot \mathrm{rot}_x((\mathcal{R}(-s\omega_c, e(x)) - I_3)\tilde{v}\tilde{F}_\varepsilon)] \cdot e \Big) \ e \cdot \nabla_{\tilde{v}}$$

and therefore

$$\begin{aligned} \left\langle c_1[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon \right\rangle (t,s,x,\tilde{v}) &= \left(\left\langle c_1[\tilde{F}_\varepsilon(t)] \right\rangle \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon(t) \right) ((\mathcal{X}, \tilde{\mathcal{V}})(-s;x,\tilde{v})) \\ &\quad + \frac{1}{S} \int_0^S \mathcal{R}(\sigma\omega_c, e(x)) \overline{c}_{1\sigma}[\tilde{F}_\varepsilon(t)](x) \, \mathrm{d}\sigma \cdot (\nabla_{\tilde{V}} \tilde{F}_\varepsilon(t))((\mathcal{X}, \tilde{\mathcal{V}})(-s;x,\tilde{v})) \end{aligned}$$

where $\langle c_1 \rangle \cdot \nabla$ has been computed in Proposition 5.6 [6]. As before, the last average writes

$$\begin{aligned} \frac{1}{S} \int_0^S \mathcal{R}(\sigma\omega_c, e(x)) \overline{c}_{1\sigma}[\tilde{F}_\varepsilon(t)](x) \, \mathrm{d}\sigma &= \frac{e \otimes e}{BS} \int_0^S E[e \cdot \mathrm{rot}_x((\mathcal{R}(-\sigma\omega_c, e) - I_3)\tilde{v}\tilde{F}_\varepsilon)] \, \mathrm{d}\sigma \\ &= \frac{e \otimes e}{B} E[e \cdot \mathrm{rot}_x((e \otimes e - I_3)\tilde{v}\tilde{F}_\varepsilon(t))] \\ &= \frac{e \otimes e}{B} E[(e \cdot \mathrm{rot}_x e)(\tilde{v} \cdot e)\tilde{F}_\varepsilon(t)] - \frac{e \otimes e}{B} E[(\tilde{v} \wedge e) \cdot \nabla_x \tilde{F}_\varepsilon(t)]. \end{aligned}$$

Finally the average of $c_1[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}} \ \tilde{f}_\varepsilon$ is given by

$$\begin{aligned} \left\langle c_1[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}} \ \tilde{f}_\varepsilon \right\rangle (t,s,x,\tilde{v}) &= \left(\left\langle c_1[\tilde{F}_\varepsilon(t)] \right\rangle \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon(t) \right) ((\mathcal{X}, \tilde{\mathcal{V}})(-s;x,\tilde{v})) \\ + \frac{q}{m} \left\{ E\left[(e \cdot \mathrm{rot}_x e) \frac{(\tilde{v} \cdot e)}{\omega_c} \tilde{F}_\varepsilon(t) \right] \cdot e - E\left[\frac{\tilde{v} \wedge e}{\omega_c} \cdot \nabla_X \tilde{F}_\varepsilon(t) \right] \cdot e \right\} &\ e \cdot (\nabla_{\tilde{V}} \tilde{F}_\varepsilon(t))((\mathcal{X}, \tilde{\mathcal{V}})(-s;x,\tilde{v})). \end{aligned} \tag{16}$$

We concentrate now on the average of $c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}} \ \tilde{f}^1_\varepsilon$. Before ending the proof of Proposition 4.1, we need to generalize the Proposition 5.2 [6] to the periodic case.

Proposition 4.2
Assume that $e \in C^2(\mathbb{R}^3), \nabla_x \omega_c = 0, \mathrm{div}_x e = 0$. *Let* $\chi \cdot \nabla_{s,x,\tilde{v}} = \chi_s \partial_s + \chi_x \cdot \nabla_x + \chi_{\tilde{v}} \cdot \nabla_{\tilde{v}}$ *be a* C^1 *vector field on* $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$, S *periodic with respect to* s. *There is a continuous vector field* $\xi \cdot \nabla_{s,x,\tilde{v}} = \xi_s \partial_s + \xi_x \cdot \nabla_x + \xi_{\tilde{v}} \cdot \nabla_{\tilde{v}}$ *in involution with respect to* $\partial_s + b \cdot \nabla_{x,\tilde{v}}$ *, such that for any* S *periodic function* $u \in C^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(\partial_s + b \cdot \nabla_{x,\tilde{v}}\ )$

$$\left\langle \chi \cdot \nabla_{s,x,\tilde{v}} u^1 \right\rangle = \xi \cdot \nabla_{s,x,\tilde{v}} u \tag{17}$$

where

$$\chi \cdot \nabla_{s,x,\tilde{v}} u - \langle \chi \cdot \nabla_{s,x,\tilde{v}} u \rangle + (\partial_s + b \cdot \nabla_{x,\tilde{v}}\ ) u^1 = 0, \ \ \left\langle u^1 \right\rangle = 0$$

and

$$\langle \chi \cdot \nabla_{s,x,\tilde{v}} \varphi \rangle = \xi_s \tag{18}$$

where

$$\chi_s - \langle \chi_s \rangle + (\partial_s + b \cdot \nabla_{x,\tilde{v}}\ ) \varphi = 0, \ \ \langle \varphi \rangle = 0.$$

Proof.
The vector field ξ is uniquely determined by imposing (17) with $(\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) \in \ker(\partial_s + b \cdot \nabla_{x,\tilde{v}}\ )$ and (18)

$$\left\langle \chi \cdot \nabla_{s,x,\tilde{v}} U^1 \right\rangle = -\xi_s b((\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v})) + \partial_{x,\tilde{v}} (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) \xi_{x,\tilde{v}}$$

where U^1 is the unique solution of

$$\chi \cdot \nabla_{s,x,\tilde{v}} (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) - \left\langle \chi \cdot \nabla_{s,x,\tilde{v}} (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) \right\rangle + (\partial_s + b \cdot \nabla_{x,\tilde{v}}\ ) U^1 = 0, \ \ \left\langle U^1 \right\rangle = 0 \tag{19}$$

and

$$\langle \chi \cdot \nabla_{s,x,\tilde{v}} \varphi \rangle = \xi_s, \ \ \chi_s - \langle \chi_s \rangle + (\partial_s + b \cdot \nabla_{x,\tilde{v}}\ ) \varphi = 0, \ \ \langle \varphi \rangle = 0.$$

It remains to check that (17) holds true for any $u(s, x, \tilde{v}) = U(z), z = (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}), U \in C^2$. We use the notation $f_s(x, \tilde{v}) = f((\mathcal{X}, \tilde{\mathcal{V}})(s; x, \tilde{v}))$, for any function f. As $\chi \cdot \nabla_{s,x,\tilde{v}} u = (\nabla_z U)_{-s} \cdot (\chi \cdot \nabla_{s,x,\tilde{v}})(\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v})$, it comes that the solution u^1 of

$$\chi \cdot \nabla_{s,x,\tilde{v}} u - \langle \chi \cdot \nabla_{s,x,\tilde{v}} u \rangle + (\partial_s + b \cdot \nabla_{x,\tilde{v}}\ ) u^1 = 0, \ \ \left\langle u^1 \right\rangle = 0$$

is given by $u^1 = (\nabla_z U)_{-s} \cdot U^1$, where U^1 is the unique solution of (19). Therefore we have

$$\begin{aligned} \left\langle \chi \cdot \nabla_{s,x,\tilde{v}} u^1 \right\rangle &= (\nabla_z U)_{-s} \cdot \left\langle \chi \cdot \nabla_{s,x,\tilde{v}} U^1 \right\rangle + \left\langle (\chi \cdot \nabla_{s,x,\tilde{v}}) (\nabla_z U)_{-s} \cdot U^1 \right\rangle \\ &= (\nabla_z U)_{-s} \cdot (\xi \cdot \nabla_{s,x,\tilde{v}}) (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) + \left\langle (\chi \cdot \nabla_{s,x,\tilde{v}}) (\nabla_z U)_{-s} \cdot U^1 \right\rangle \\ &= \xi \cdot \nabla_{s,x,\tilde{v}} u + \left\langle (\chi \cdot \nabla_{s,x,\tilde{v}}) (\nabla_z U)_{-s} \cdot U^1 \right\rangle \end{aligned}$$

and we are done provided that the last average vanishes. Indeed, as $\langle U^1 \rangle = 0$ we write, thanks to the symmetry of the Hessian matrix $(\partial_z^2 U)_{-s}$

$$\begin{aligned} &\left\langle (\chi \cdot \nabla_{s,x,\tilde{v}}) (\nabla_z U)_{-s} \cdot U^1 \right\rangle = \left\langle (\partial_z^2 U)_{-s} (\chi \cdot \nabla_{s,x,\tilde{v}}) (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) \cdot U^1 \right\rangle \\ &= \left\langle (\partial_z^2 U)_{-s} \left[\chi \cdot \nabla_{s,x,\tilde{v}} (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) - \left\langle \chi \cdot \nabla_{s,x,\tilde{v}} (\mathcal{X}, \tilde{\mathcal{V}})(-s; x, \tilde{v}) \right\rangle \right] \cdot U^1 \right\rangle \\ &= -\frac{1}{2} \left\langle (\partial_s + b \cdot \nabla_{x,\tilde{v}}\ ) \left[(\partial_z^2 U)_{-s} U^1 \cdot U^1 \right] \right\rangle = 0. \end{aligned}$$

$\square$

Remark 4.1
We check that if $\chi_s = 0$*, then* $\xi_s = 0$ *and if* $\partial_s \chi = 0$*, then* $\partial_s \xi = 0$*. Therefore if* $\chi(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}}$ *is a* C^1 *vector field, there is a continuous vector field* $\xi(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}}$ *in involution with respect to* $\partial_s + b \cdot \nabla_{x,\tilde{v}}$ *(and therefore with respect to* $b \cdot \nabla_{x,\tilde{v}}$ *) such that for any S periodic function* $u \in C^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(\partial_s + b \cdot \nabla_{x,\tilde{v}})$ *we have*

$$\left\langle \chi \cdot \nabla_{x,\tilde{v}}\, u^1 \right\rangle = \xi \cdot \nabla_{x,\tilde{v}}\, u$$

where

$$\chi \cdot \nabla_{x,\tilde{v}}\, u - \langle \chi \cdot \nabla_{x,\tilde{v}}\, u\rangle + (\partial_s + b \cdot \nabla_{x,\tilde{v}})u^1 = 0, \;\; \left\langle u^1 \right\rangle = 0.$$

In particular, for any function $u \in C^2(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(b \cdot \nabla_{x,\tilde{v}})$ *we have*

$$\left\langle \chi \cdot \nabla_{x,\tilde{v}}\, u^1 \right\rangle = \xi \cdot \nabla_{x,\tilde{v}}\, u$$

where $u^1 = u^1(x,\tilde{v})$ *is the unique solution of*

$$\chi \cdot \nabla_{x,\tilde{v}}\, u - \langle \chi \cdot \nabla_{x,\tilde{v}}\, u\rangle + b \cdot \nabla_{x,\tilde{v}}\, u^1 = 0, \;\; \left\langle u^1 \right\rangle = 0$$

see also Proposition 5.2 [6].

Thanks to the previous result, we compute the average of $c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\, \tilde{f}^1_\varepsilon$.

Proposition 4.3
Assume that $e \in C^2(\mathbb{R}^3), \nabla_x \omega_c = 0, \mathrm{div}_x e = 0$*. There is a vector field* $d(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}}$ *in involution with respect to* $b \cdot \nabla_{x,\tilde{v}}$ *such that for any S periodic* $\tilde{f} \in C^2_c(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(\partial_s + b \cdot \nabla_{x,\tilde{v}})$ *we have*

$$\left\langle c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\, \tilde{f}^1 \right\rangle = d(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}}\, \tilde{f}$$

where $\tilde{f}^1$ *solves*

$$c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\, \tilde{f} - \left\langle c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\, \tilde{f} \right\rangle + (\partial_s + b \cdot \nabla_{x,\tilde{v}})\tilde{f}^1 = 0, \;\; \left\langle \tilde{f}^1 \right\rangle = 0.$$

Proof.
Recall that $c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \cdot e)e \cdot \nabla_x + \frac{q}{m}(E[\tilde{f}] \cdot e)e \cdot \nabla_{\tilde{v}} - [\tilde{v} \wedge \partial_x e(\tilde{v} \wedge e)] \cdot \nabla_{\tilde{v}}$. As $\tilde{f} \in \ker(\partial_s + b \cdot \nabla_{x,\tilde{v}})$, we deduce that $\partial_s \rho[\tilde{f}] = 0$ and therefore the vector field $(E[\tilde{f}] \cdot e)e \cdot \nabla_{\tilde{v}}$ is in involution with respect to $b \cdot \nabla_{x,\tilde{v}}$ and $\partial_s + b \cdot \nabla_{x,\tilde{v}}$, implying that

$$\frac{q}{m}(E[\tilde{f}] \cdot e)e \cdot \nabla_{\tilde{v}}\tilde{f} - \left\langle \frac{q}{m}(E[\tilde{f}] \cdot e)e \cdot \nabla_{\tilde{v}}\tilde{f} \right\rangle = 0$$

and

$$\left\langle \frac{q}{m}(E[\tilde{f}] \cdot e)e \cdot \nabla_{\tilde{v}}\tilde{f}^1 \right\rangle = \frac{q}{m}(E[\tilde{f}] \cdot e)e \cdot \nabla_{\tilde{v}} \left\langle \tilde{f}^1 \right\rangle = 0.$$

Our conclusion follows by applying Proposition 4.2 with the vector field $\overline{c}_0 \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \cdot e)e \cdot \nabla_x - [\tilde{v} \wedge \partial_x e(\tilde{v} \wedge e)] \cdot \nabla_{\tilde{v}}$, whose average is cf. Proposition 5.5 [6], see also Remark 4.1

$$\langle \overline{c}_0 \rangle \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \cdot e)\left[e \cdot \nabla_x + (\partial_x e e \otimes e - e \otimes \partial_x e e)\tilde{v} \cdot \nabla_{\tilde{v}} + \frac{(\mathrm{rot}_x e \cdot e)}{2}(\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}} \right].$$

In order to express $\tilde{f}^1$ in terms of $\tilde{f}(s,x,\tilde{v}) = \tilde{F}((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))$, observe that

$$
\begin{aligned}
(\overline{c}_0 \cdot \nabla_{x,\tilde{v}}\ \tilde{f})(s+\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v})) &= \overline{c}_0((\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))\\
&\cdot\ {}^t\partial(\mathcal{X},\tilde{\mathcal{V}})(-\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))\ {}^t\partial(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})(\nabla\tilde{F})_{-s}\\
&= \partial(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})\partial(\mathcal{X},\tilde{\mathcal{V}})(-\sigma,(\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))\overline{c}_0((\mathcal{X},\tilde{\mathcal{V}})(\sigma;x,\tilde{v}))\cdot(\nabla\tilde{F})_{-s}
\end{aligned}
$$

$$
\left\langle \overline{c}_0 \cdot \nabla_{x,\tilde{v}}\ \tilde{f}\right\rangle(s,x,\tilde{v}) = \partial(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})\left\langle \overline{c}_0\right\rangle(x,\tilde{v})\cdot(\nabla\tilde{F})_{-s} = (\langle\overline{c}_0\rangle\cdot\nabla\tilde{F})_{-s}
$$

implying that

$$
\partial(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})\overline{c}_0(x,\tilde{v})\cdot\left(\nabla_{X,\tilde{V}}\tilde{F}\right)_{-s} - \left(\langle\overline{c}_0\rangle\cdot\nabla_{X,\tilde{V}}\tilde{F}\right)_{-s} + (\partial_s + b\cdot\nabla_{x,\tilde{v}}\ )\tilde{f}^1 = 0
$$

or equivalently

$$
\begin{aligned}
&[\partial(\mathcal{X},\tilde{\mathcal{V}})(-s;(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))\overline{c}_0((\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V})) - \langle\overline{c}_0\rangle(X,\tilde{V})]\cdot\nabla_{X,\tilde{V}}\tilde{F}\\
&\qquad\qquad\qquad\qquad + \frac{\mathrm{d}}{\mathrm{d}s}\tilde{f}^1(s,(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V})) = 0.
\end{aligned}
$$

Clearly we have

$$
\partial_{x,\tilde{v}}\mathcal{X}(-s;(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))\overline{c}_0((\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V})) - \langle\overline{c}_0\rangle_x(X,\tilde{V}) = 0
$$

and therefore the previous equality becomes

$$
\begin{aligned}
&[\partial_{x,\tilde{v}}\tilde{\mathcal{V}}(-s;(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))\overline{c}_0((\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V})) - \langle\overline{c}_0\rangle_{\tilde{v}}(X,\tilde{V})]\cdot\nabla_{\tilde{V}}\tilde{F}\\
&\qquad\qquad\qquad\qquad + \frac{\mathrm{d}}{\mathrm{d}s}\tilde{f}^1(s,(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V})) = 0.
\end{aligned}
$$

Let us introduce the function

$$
\begin{aligned}
\mathcal{F}(s,X,\tilde{V}) &:= \partial_{x,\tilde{v}}\tilde{\mathcal{V}}(-s;(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))\overline{c}_0((\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))\\
&= \partial_x\tilde{\mathcal{V}}(-s;(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))(\overline{c}_{0x})_s + \partial_{\tilde{v}}\tilde{\mathcal{V}}(-s;(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))(\overline{c}_{0\tilde{v}})_s\\
&= (\tilde{V}\cdot e)[(1-\cos(s\omega_c))(\partial_x ee\otimes e + e\otimes\partial_x ee)\tilde{\mathcal{V}}(s;X,\tilde{V}) + \sin(s\omega_c)\partial_x ee\wedge\tilde{\mathcal{V}}(s;X,\tilde{V})]\\
&- [\cos(s\omega_c)(I_3 - e\otimes e) + \sin(s\omega_c)M[e] + e\otimes e](\tilde{\mathcal{V}}(s;X,\tilde{V})\wedge\partial_x e(\tilde{\mathcal{V}}(s;X,\tilde{V})\wedge e))
\end{aligned}
$$

and therefore the S periodic function $s\to\tilde{f}^1(s,(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))$ has no Fourier modes at the frequencies $k\omega_c, k\geq 4$. Moreover, since

$$
\frac{1}{S}\int_0^S \tilde{f}^1(s,(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V}))\,\mathrm{d}s = \left\langle\tilde{f}^1\right\rangle(0,X,\tilde{V}) = 0
$$

we deduce that

$$
\tilde{f}^1(s,(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V})) = \sum_{k=1}^{3}[\mathcal{A}_k(X,\tilde{V})\cos(ks\omega_c) + \mathcal{B}_k(X,\tilde{V})\sin(ks\omega_c)]\cdot\nabla_{\tilde{V}}\tilde{F}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}s}\tilde{f}^1(s,(\mathcal{X},\tilde{\mathcal{V}})(s;X,\tilde{V})) = \sum_{k=1}^{3}[-k\omega_c\mathcal{A}_k(X,\tilde{V})\sin(ks\omega_c)+k\omega_c\mathcal{B}_k(X,\tilde{V})\cos(ks\omega_c)]\cdot\nabla_{\tilde{V}}\tilde{F}
$$

for some vector fields $(\mathcal{A}_k, \mathcal{B}_k)_{1\leq k\leq 3}$ to be determined. We have for $k \in \{1,2,3\}$

$$\mathcal{A}_k(X,\tilde{V}) = \frac{1}{k\pi}\int_0^S \mathcal{F}(s,X,\tilde{V})\sin(ks\omega_c)\ \mathrm{d}s,\ \mathcal{B}_k(X,\tilde{V}) = -\frac{1}{k\pi}\int_0^S \mathcal{F}(s,X,\tilde{V})\cos(ks\omega_c)\ \mathrm{d}s$$

and finally we obtain

$$\begin{aligned}\tilde{f}^1(s,x,\tilde{v}) = &\sum_{k=1}^{3}[\mathcal{A}_k((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))\cos(ks\omega_c) + \mathcal{B}_k((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))\sin(ks\omega_c)]\\ &\cdot (\nabla_{\tilde{V}}\tilde{F})((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})).\end{aligned}$$

In particular the solution of

$$\overline{c}_0\cdot\nabla_{x,\tilde{v}}\ (\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}) - \left\langle \overline{c}_0\cdot\nabla_{x,\tilde{v}}\ (\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})\right\rangle + (\partial_s + b\cdot\nabla_{x,\tilde{v}}\ )U^1 = 0,\ \ \langle U^1\rangle = 0$$

is given by

$$U^1 = (0, \sum_{k=1}^{3}[\mathcal{A}_k((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))\cos(ks\omega_c) + \mathcal{B}_k((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))\sin(ks\omega_c)])$$

and the vector field $d(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}$ writes, cf. Proposition 4.2, Remark 4.1

$$\begin{aligned}d(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}\ &= d_{\tilde{v}}\cdot\nabla_{\tilde{v}} = \left\langle(\overline{c}_0\cdot\nabla_{x,\tilde{v}}\ )U^1_{\tilde{v}}\right\rangle(0,x,\tilde{v})\cdot\nabla_{\tilde{v}}\\ &= \left\langle(\overline{c}_0\cdot\nabla_{x,\tilde{v}}\ )\sum_{k=1}^{3}[(\mathcal{A}_k)_{-s}\cos(ks\omega_c) + (\mathcal{B}_k)_{-s}\sin(ks\omega_c)]\right\rangle(0,x,\tilde{v})\cdot\nabla_{\tilde{v}}.\end{aligned}$$

Actually the vector field $d\cdot\nabla_{x,\tilde{v}}$ is parallel to the vector field $b\cdot\nabla_{x,\tilde{v}}$. If $\tilde{f}(s,x,\tilde{v}) = \frac{|\tilde{v}|^2}{2} \in \ker(\partial_s + b\cdot\nabla_{x,\tilde{v}}\ )$, we have

$$\overline{c}_0\cdot\nabla_{x,\tilde{v}}\ \tilde{f} - \left\langle\overline{c}_0\cdot\nabla_{x,\tilde{v}}\ \tilde{f}\right\rangle = 0$$

and $\tilde{f}^1 = 0$. As in the proof of Proposition 5.7 [6] we deduce that $d_{\tilde{v}}(x,\tilde{v})\cdot\tilde{v} = 0$. Similarly, considering $\tilde{f}(s,x,\tilde{v}) = (\tilde{v}\cdot e(x)) \in \ker(\partial_s + b\cdot\nabla_{x,\tilde{v}}\ )$, we obtain

$$\tilde{f}^1(s,x,\tilde{v}) = \frac{(\tilde{v}\cdot e)}{\omega_c}\partial_x ee\cdot(\tilde{v}\wedge e) - \partial_x e : \frac{(\tilde{v}\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)e) + (\tilde{v}-(\tilde{v}\cdot e)e)\otimes(\tilde{v}\wedge e)}{4\omega_c}$$

and $d_{\tilde{v}}(x,\tilde{v})\cdot e = 0$, cf. Proposition 5.7 [6]. □

We come back to the proof of Proposition 4.1. We know by Proposition 4.3 that $\tilde{f}_\varepsilon \in \ker(\partial_s + b\cdot\nabla_{x,\tilde{v}}\ )$ and

$$c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon - \left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle + (\partial_s + b\cdot\nabla_{x,\tilde{v}}\ )\tilde{f}^1_\varepsilon = 0,\ \ \left\langle\tilde{f}^1_\varepsilon\right\rangle = 0$$

imply (use also the involution of $d\cdot\nabla$ with respect to $b\cdot\nabla$)

$$\begin{aligned}\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon\right\rangle &= d(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\\ &= \partial(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})d(x,\tilde{v})\cdot(\nabla_{X,\tilde{V}}\tilde{F}_\varepsilon(t))((\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))\\ &= (d\cdot\nabla_{X,\tilde{V}}\tilde{F}_\varepsilon(t))_{-s}.\end{aligned} \tag{20}$$

The equation (12) follows by the equation (10), thanks to the equalities (14), (15), (16), (20), which ends the proof of Proposition 4.1. □

Remark 4.2
When the particle density $\tilde{F}_\varepsilon$ satisfies the constraint $b \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon = 0$, we have $\tilde{f}^\varepsilon(t,x,\tilde{v}) \approx \tilde{f}_\varepsilon(t, t/\varepsilon, x, \tilde{v}) = \tilde{F}_\varepsilon(t, (\mathcal{X}, \tilde{\mathcal{V}})(-t/\varepsilon; x, \tilde{v})) = \tilde{F}_\varepsilon(t,x,\tilde{v})$. Therefore the model (12) reduces to the model with well prepared initial conditions (64) [6]. Indeed, if the particle density verifies $b \cdot \nabla_{X,\tilde{V}} \tilde{F} = 0$, we claim that $D[\tilde{F}] = 0$. Clearly we have $j[\tilde{F}] = \rho[(\tilde{V} \cdot e)\tilde{F}]e$, implying that

$$j[\tilde{F}] \wedge e = 0, \quad N(e(X), X - X', e(X'))j[\tilde{F}(X')] = 0$$

and (see also Remark 5.3 [6])

$$\rho \left[(e \cdot \mathrm{rot}_x e) \frac{(\tilde{V} \cdot e)}{\omega_c} \tilde{F} \right] = \rho \left[\frac{\tilde{V} \wedge e}{\omega_c} \cdot \nabla_X \tilde{F} \right], \quad E \left[(e \cdot \mathrm{rot}_x e) \frac{(\tilde{V} \cdot e)}{\omega_c} \tilde{F} \right] = E \left[\frac{\tilde{V} \wedge e}{\omega_c} \cdot \nabla_X \tilde{F} \right].$$

As we know that $d \cdot \nabla_{x,\tilde{v}}$ is parallel to $b \cdot \nabla_{x,\tilde{v}}$, we also have $d \cdot \nabla_{X,\tilde{V}} \tilde{F} = 0$. By construction, the vector fields $\left\langle a[\tilde{F}] \right\rangle \cdot \nabla_{X,\tilde{V}}, \left\langle c_0[\tilde{F}] \right\rangle \cdot \nabla_{X,\tilde{V}}, \left\langle c_1[\tilde{F}] \right\rangle \cdot \nabla_{X,\tilde{V}}, D[\tilde{F}] \cdot \nabla_{X,\tilde{V}}$ are in involution with respect to $b \cdot \nabla_{X,\tilde{V}}$. Therefore, if $\tilde{F}_\varepsilon(0)$ satisfies the constraint $b \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon(0) = 0$, so is at any time $t \in [0,T]$, i.e., $b \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon(t) = 0, t \in [0,T]$.

The well posedness of the model (10), (11), or equivalently (12), (13), thanks to Proposition 4.1, follows by standard arguments when analyzing Vlasov-Poisson like equations. The details are left to the reader.

Theorem 4.1
Consider a non negative, smooth, compactly supported initial particle density $\tilde{F}_{\mathrm{in}} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$ and a smooth magnetic field $\mathbf{B}^\varepsilon = \frac{\mathbf{B}}{\varepsilon}, \mathbf{B} = Be \in C^2_b(\mathbb{R}^3), \nabla_x B = 0, B \neq 0, \mathrm{div}_x e = 0$. Let T be any positive time. Then it exists $\varepsilon_T > 0$ such that for any $0 < \varepsilon \leq \varepsilon_T$ there is a unique particle density $\tilde{F}_\varepsilon \in C^1_c([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ satisfying for any $(t, X, \tilde{V}) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$

$$\partial_t \tilde{F}_\varepsilon + \left\langle c_0[\tilde{F}_\varepsilon] \right\rangle \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon + \varepsilon \left(\left\langle a[\tilde{F}_\varepsilon] \right\rangle + \left\langle c_1[\tilde{F}_\varepsilon] \right\rangle + D[\tilde{F}_\varepsilon] \right) \cdot \nabla_{X,\tilde{V}} \tilde{F}_\varepsilon = 0$$

and

$$\tilde{F}_\varepsilon(0, X, \tilde{V}) = \tilde{F}_{\mathrm{in}}(X, \tilde{V}), \quad (X, \tilde{V}) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

If for some integer $k \geq 2$ we have $\tilde{F}_{\mathrm{in}} \in C^k_c(\mathbb{R}^3 \times \mathbb{R}^3), e \in C^{k+1}(\mathbb{R}^3)$, then $\tilde{F} \in C^k_c([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$.

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