Symbolic models for incrementally stable singularly perturbed hybrid affine systems

Zohra Kader, Antoine Girard

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Abstract— **In this paper, we consider the problem of symbolic models design for the class of incrementally stable singularly perturbed hybrid affine systems. Contrarily to the existing results in the literature where only switching are taken into account, here we consider a more general class of hybrid systems including switches, impulsions and dynamics evolving in different timescales. Firstly, a discussion about incremental stability of the considered class of systems is given. Secondly, a new method for designing symbolic models for incrementally stable singularly perturbed hybrid affine systems is proposed. Inspired from singularly perturbed techniques based on decoupling the slow dynamics from the fast ones, the obtained symbolic abstraction is designed by discretizing only a part of the state space representing the slow dynamics. An ε-approximate bisimulation relation between the original singularly perturbed hybrid affine system and the symbolic model obtained by discretizing the slow dynamics is provided. Indeed, since the discrete abstraction is designed for a system of lower dimension, the number of its transitions is drastically reduced. Finally, an example is proposed in order to illustrate the efficiency of the proposed results.**

I. INTRODUCTION

Hybrid systems have been largely studied in the literature during the last decades [4], [5], [8]. Due to their heterogeneous nature, they are used for modelling physical systems that present discrete events during their continuous dynamics. Two types of events can be encountered in real processes: switches i.e., dynamics changes without state jumps and impulses i.e., jumps in the system's state.

Another phenomena that can occur in physical systems is the presence of processes evolving in different timescales [6], [7], [9]. Recently, a large interest has been given to the class of singularly perturbed hybrid systems. Different examples can motivate this interest, namely the design of fast controllers for hybrid systems [13] and the existence of physical systems in engineering presenting discrete events and different timescales [9], [12]. Moreover, in the presence of timescales separation, stability analysis and control design become more complex and singular perturbation theory must be utilized [6], [7]. Numerous results have been already proposed namely on the stability analysis and stabilization of this class of systems - see for instance [9], [12], [13]. However, technology advances demand that more complex control goals such as language and logic specifications, safety properties, obstacle avoidance be considered. This leads to several studies using symbolic models also called discrete abstractions, for controller design, see for instance [2], [14]. The main advantage when using symbolic models is that if the obtained symbolic model is finite then the problem of controller design can be efficiently solved using the mature methods for supervisory control design for discrete-event systems.

Symbolic models are very popular for hybrid systems design [2], [11], [14]. In particular, based on the Lyapunov theory, several approaches for designing symbolic models for incrementally stable hybrid systems have been proposed. For instance, we can cite the work proposed in [3] where a symbolic model has been designed using both state and time discretization. However, to the best of our knowledge, the existing results about symbolic models design consider only the class of switched systems. Moreover, those results do not take into account the case where the system's dynamics evolve in different timescales.

Here, we are interested in the design of symbolic models for a more general class of hybrid systems presenting switches, impulses, and dynamics that evolve in different timescales. First, global incremental asymptotic stability of hybrid affine systems is defined. A discussion about how the conditions for global asymptotic stability of singularly perturbed hybrid linear systems provided in [12] can be used in order to show incremental global uniform asymptotic stability of hybrid affine systems is provided. Then, a new method for designing symbolic models for incrementally stable singularly perturbed hybrid affine systems is proposed. Inspired from singularly perturbed techniques based on decoupling the slow dynamics from the fast ones, the obtained symbolic abstraction is designed by discretizing only a part of the state space representing the slow dynamics. We have equally shown that the original singularly perturbed hybrid affine system is related by an ε-approximate bisimulation relation to the symbolic model designed by discretizing the slow dynamics. Besides the fact that this design methodology takes into account the singular perturbation nature of the system, the main advantage of the proposed method is that the obtained symbolic model is of reduced size. Indeed, since the discrete abstraction is designed for a system of lower dimension, the number of its transitions is drastically reduced.

The paper is structured as follows: Section 2 provides a description of the considered singularly perturbed hybrid affine system. The notion of incremental stability of singularly

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perturbed hybrid affine systems is defined and definitions necessary for our study are given in Section 3. In Section 4, we propose a method for constructing symbolic models for incrementally stable singularly perturbed hybrid affine systems. In Section 5, a numerical example that illustrates the proposed results is provided. The paper ends with a brief conclusion.

Notations.: In this paper we use the notations $\mathbb{R}$, $\mathbb{R}_0^+$ and $\mathbb{R}^+$ to refer to the set of real, non-negative real, and positive real numbers, respectively. $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}^+$ refer to the sets of integers, of non-negative integers and of positive integers, respectively. card($\mathscr{S}$) refers to the cardinal of a set $\mathscr{S}$. $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$ and $x_{(i)}$ refers to its i-th row. I_n denotes the identity matrix of dimension n. 0 refers to a matrix of appropriate dimension whose elements are null. The notation M^T refers to the transpose of a matrix or vector M. M^{-1} indicates the inverse of a square matrix M. $M \succ 0$ (respectively $M \prec 0$) denotes a positive definite (respectively, negative definite) matrix. $M \succeq 0$ (respectively $M \preceq 0$) refers to a positive semidefinite (respectively, negative semidefinite) matrix. For a symmetric matrix $M \succeq 0$, $M^{\frac{1}{2}}$ is the unique symmetric matrix $S \succeq 0$ such that $S^2 = M$. For a positive definite matrix M, $\lambda_{\min}(M)$ (respectively, $\lambda_{\max}(M)$) stands for the minimum (respectively, maximum) eigenvalue of M. The matrix M is said to be Hurwitz if all its eigenvalues have negative real parts. M is said to be Schur if its eigenvalues lie strictly inside the unit disk. The notation $x(t_k^-) = \lim_{d \to 0, d>0} x(t-d)$ is equally used in the paper.

A continuous function γ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\gamma(0) = 0$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathscr{KL}$ if : for any fixed r, the map $\beta(.,s)$ belongs to the class $\mathscr{K}$, and for each fixed s the map $\beta(r,.)$ is strictly decreasing and $\beta(r,.)$ goes to zero as s tends to infinity. Given a function $l : (0, \bar{\theta}) \to \mathbb{R}$, we say that $l(\theta) = \mathscr{O}(\alpha(\theta))$ if and only if there exists $\theta_0 \in (0, \bar{\theta})$ and $c > 0$, such that for all $\theta \in (0, \theta_0), |l(\theta)| \leq c\alpha(\theta)$.

II. System description and problem statement

A. Singularly perturbed hybrid affine systems

Here, we are particularly interested in the construction of symbolic models for the class of singularly perturbed hybrid affine systems. The dynamics of the hybrid affine system Σ are defined as follows:

$$\begin{cases} \begin{bmatrix} \dot{\mathrm{x}}(t) \\ \theta\dot{\mathrm{z}}(t) \end{bmatrix} = A^{\mathbf{p}(t_k)} \begin{bmatrix} \mathrm{x}(t) \\ \mathrm{z}(t) \end{bmatrix} + \alpha^{\mathbf{p}(t_k)} = f^{\mathbf{p}(t_k)}(\mathrm{x}(t),\mathrm{z}(t)), \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N} \\ \begin{bmatrix} \mathrm{x}(t_k) \\ \mathrm{z}(t_k) \end{bmatrix} = C^{\mathbf{p}(t_k^-) \longrightarrow \mathbf{p}(t_k)} \begin{bmatrix} \mathrm{x}(t_k^-) \\ \mathrm{z}(t_k^-) \end{bmatrix} + d^{\mathbf{p}(t_k^-) \longrightarrow \mathbf{p}(t_k)} \\ \qquad = g^{\mathbf{p}(t_k^-) \longrightarrow \mathbf{p}(t_k)}(\mathrm{x}(t_k^-), \mathrm{z}(t_k^-)), \forall k \geq 1, \end{cases} \tag{1}$$

where $\mathrm{x}(t) \in \mathbb{R}^{n_x}$, $\mathrm{z}(t) \in \mathbb{R}^{n_z}$ are the slow and fast variables respectively. $n_x \in \mathbb{N}$ and $n_z \in \mathbb{N}$ are the dimensions of the slow and fast varying variables such that $n_x + n_z = n$ where $n \in \mathbb{N}$ is the system's order. $\theta > 0$ is the small parameter characterizing the time scale separation between the slow and the fast dynamics. $\mathbf{p} \in \mathscr{P}$ denotes the switching signal with $\mathscr{P}$ is a subset of $\mathscr{S}(\mathbb{R}_0^+, P)$ which refers to the set of piece-wise constant and right continuous functions $\mathbf{p}$ from $\mathbb{R}_0^+$ to the finite set of modes $P = \{1, 2, \ldots, m\}$, with a finite number of discontinuities on every bounded interval of $\mathbb{R}_0^+$. This guarantees the absence of Zeno behaviours. In the rest of our paper we will denote $p^- = \mathbf{p}(t_k^-)$ and $p = \mathbf{p}(t_k)$. For all $p, p^- \in P$, $A^p \in \mathbb{R}^{n \times n}$, $\alpha^p \in \mathbb{R}^{n \times 1}$, $C^{p^- \longrightarrow p} \in \mathbb{R}^{n \times n}$, and $d^{p^- \longrightarrow p} \in \mathbb{R}^{n \times 1}$ are matrices defining the continuous and impulsive dynamics.

B. Change of variables

Let for all $p^-, p \in P$

$$A^p = \begin{bmatrix} A_{11}^p & A_{12}^p \\ A_{21}^p & A_{22}^p \end{bmatrix}, \alpha^p = \begin{bmatrix} \alpha_1^p \\ \alpha_2^p \end{bmatrix},$$
$$C^{p^- \longrightarrow p} = \begin{bmatrix} C_{11}^{p^- \longrightarrow p} & C_{12}^{p^- \longrightarrow p} \\ C_{21}^{p^- \longrightarrow p} & C_{22}^{p^- \longrightarrow p} \end{bmatrix}, d^{p^- \longrightarrow p} = \begin{bmatrix} d_1^{p^- \longrightarrow p} \\ d_2^{p^- \longrightarrow p} \end{bmatrix}$$

where $A_{11}^p, C_{11}^{p^- \longrightarrow p} \in \mathbb{R}^{n_x \times n_x}$, $A_{22}^p, C_{22}^{p^- \longrightarrow p} \in \mathbb{R}^{n_z \times n_z}$, $\alpha_1^p, d_1^{p^- \longrightarrow p} \in \mathbb{R}^{n_x \times 1}$, $\alpha_2^p, d_2^{p^- \longrightarrow p} \in \mathbb{R}^{n_z \times 1}$, and A_{12}^p, A_{21}^p, $C_{12}^{p^- \longrightarrow p}$, $C_{21}^{p^- \longrightarrow p}$ are matrices of appropriate dimensions.

When we set $\theta = 0$ in (1) the dimension of the state equation reduces from $n_z + n_x$ to n_x because, for all $p \in P$ the differential equations of the fast dynamics z degenerate to

$$0 = A_{22}^p \mathrm{z} + A_{21}^p \mathrm{x} + \alpha_2^p. \tag{2}$$

Let us make the standard assumption in the singular perturbation theory framework [6], [7], in the following

Assumption 1: A_{22}^p are non-singular for all $p \in P$.

Under this assumption for all $p \in P$, the solutions of (2) are given by

$$\bar{\mathrm{z}} = h^p(\mathrm{x}) = -(A_{22}^p)^{-1} A_{21}^p \mathrm{x} - (A_{22}^p)^{-1} \alpha_2^p$$

which corresponds to the quasi-steady state of the fast dynamic of the respective mode p. It is more convenient and common in the literature of singular perturbed systems to perform the change of coordinates that renders the quasi-steady state of the fast dynamic null. Here, in order to take into account the hybrid nature of system (1), for all $p \in P$ we perform the following time-dependent change of coordinates

$$\mathrm{y}(t) = \mathrm{z}(t) - h^p(\mathrm{x}(t)), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{3}$$

to shift the quasi-steady state of z to the origin.

Using the change of coordinates (3), the continuous dynamics in (1) become

$$\begin{bmatrix} \dot{\mathrm{x}}(t) \\ \theta\dot{\mathrm{y}}(t) \end{bmatrix} = \begin{bmatrix} A_0^p & A_1^p \\ \theta A_2^p & A_4^p + \theta A_3^p \end{bmatrix} \begin{bmatrix} \mathrm{x}(t) \\ \mathrm{y}(t) \end{bmatrix} + \begin{bmatrix} B_1^p \\ \theta B_2^p \end{bmatrix}, \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N} \tag{4}$$

where for all $p \in P$:

$$A_0^p = A_{11}^p - A_{12}^p (A_{22}^p)^{-1} A_{21}^p,\ A_1^p = A_{12}^p$$
$$A_2^p = (A_{22}^p)^{-1} A_{21}^p A_0^p,\ A_3^p = (A_{22}^p)^{-1} A_{21}^p A_{12}^p,\ A_4^p = A_{22}^p,$$
$$B_1^p = \alpha_1^p - A_{12}^p (A_{22}^p)^{-1} \alpha_2^p,\ B_2^p = (A_{22}^p)^{-1} A_{21}^p B_1^p.$$

Likewise, the impulsive dynamics in (1) turn into

$$\begin{bmatrix} \mathrm{x}(t_k) \\ \mathrm{y}(t_k) \end{bmatrix} = \begin{bmatrix} R_{11}^{p^- \longrightarrow p} & R_{12}^{p^- \longrightarrow p} \\ R_{21}^{p^- \longrightarrow p} & R_{22}^{p^- \longrightarrow p} \end{bmatrix} \begin{bmatrix} \mathrm{x}(t_k^-) \\ \mathrm{y}(t_k^-) \end{bmatrix} + \begin{bmatrix} \mathscr{B}_1^{p^- \longrightarrow p} \\ \mathscr{B}_2^{p^- \longrightarrow p} \end{bmatrix}, \forall k \geq 1, \tag{5}$$

where for all $p^-, p \in P$

$$\begin{aligned} R_{11}^{p^-\longrightarrow p} &= C_{11}^{p^-\longrightarrow p} - C_{12}^{p^-\longrightarrow p}(A_{22}^{p^-})^{-1}A_{21}^{p^-};\\ R_{12}^{p^-\longrightarrow p} &= C_{12}^{p^-\longrightarrow p},\\ R_{21}^{p^-\longrightarrow p} &= C_{21}^{p^-\longrightarrow p} - C_{22}^{p^-\longrightarrow p}(A_{22}^{p^-})^{-1}A_{21}^{p^-}\\ &\quad + (A_{22}^{p})^{-1}A_{21}^{p}R_{11}^{p^-\longrightarrow p},\\ R_{22}^{p^-\longrightarrow p} &= C_{22}^{p^-\longrightarrow p} + (A_{22}^{p})^{-1}A_{21}^{p}C_{12}^{p^-\longrightarrow p},\\ \mathscr{B}_1^{p^-\longrightarrow p} &= d_1^{p^-\longrightarrow p} - C_{12}^{p^-\longrightarrow p}(A_{22}^{p^-})^{-1}\alpha_2^{p^-},\\ \mathscr{B}_2^{p^-\longrightarrow p} &= d_2^{p^-\longrightarrow p} + (A_{22}^{p})^{-1}A_{21}^{p}\mathscr{B}_1^{p^-\longrightarrow p}\\ &\quad + (A_{22}^{p})^{-1}\alpha_2^{p} - C_{22}^{p^-\longrightarrow p}(A_{22}^{p^-})^{-1}\alpha_2^{p^-}. \end{aligned}$$

In the rest of our paper we will denote by $\begin{bmatrix}\mathtt{x}(t,(x,y),\mathbf{p})\\ \mathtt{y}(t,(x,y),\mathbf{p})\end{bmatrix}$ the point reached at time t by the trajectory of system (4), (5) starting at $\mathtt{x}(0)=x$, $\mathtt{y}(0)=y$ under the switching signal $\mathbf{p}$. $\begin{bmatrix}\phi_{\mathtt{x}}^{p^-\longrightarrow p}(\tau,(x,y))\\ \phi_{\mathtt{y}}^{p^-\longrightarrow p}(\tau,(x,y))\end{bmatrix}$ will refer to the value of the solution of (4), (5) at time t_{k+1}^-, with $t_{k+1}=t_k+\tau$ starting from $\mathtt{x}(t_k^-)=x$ and $\mathtt{y}(t_k^-)=y$ under the switching signal $\mathbf{p}$ where $\mathbf{p}(t_k^-)=p^-$ and $\mathbf{p}(t_k)=\mathbf{p}(t_{k+1}^-)=p$.

III. Preliminaries

A. Incremental stability

It has been shown recently for different classes of systems such as switched nonlinear and networked systems that the construction of symbolic models can rely directly on the incremental stability notion [3], [10]. This notion has been presented for nonlinear systems in [1]. An extension of this result to the case of switched nonlinear systems has been provided in [3]. Hereafter, we adapt the definition of incremental stability to the class of singularly perturbed hybrid affine systems which is under study in this paper.

Definition 1: A singularly perturbed hybrid affine system Σ is said to be incrementally globally uniformly asymptotically stable (δ-GUAS) if there exists $\mathscr{K}\mathscr{L}$ function β such that for all $t\in\mathbb{R}_0^+$, for all $x,x'\in\mathbb{R}^{n_x}$, $y,y'\in\mathbb{R}^{n_z}$ and for all switching signal $\mathbf{p}\in\mathscr{P}$, the following condition holds:

$$\left\|\begin{bmatrix}\mathtt{x}(t,(x,y),\mathbf{p})-\mathtt{x}(t,(x',y'),\mathbf{p})\\ \mathtt{y}(t,(x,y),\mathbf{p})-\mathtt{y}(t,(x',y'),\mathbf{p})\end{bmatrix}\right\|\le\beta\left(\left\|\begin{bmatrix}x-x'\\ y-y'\end{bmatrix}\right\|,t\right) \quad (6)$$

Roughly speaking, incremental stability means that all the trajectories induced by the same switching signal converge to the same reference trajectory independently of their initial states.

Showing that system (4), (5) is incrementally globally uniformly asymptotically stable (δ-GUAS) leads to prove that the hybrid linear system

$$\begin{aligned}\begin{bmatrix}\dot{\mathsf{e}}_{\mathtt{x}}(t)\\ \theta\dot{\mathsf{e}}_{\mathtt{y}}(t)\end{bmatrix} &= \begin{bmatrix}A_0^p & A_1^p\\ \theta A_2^p & A_4^p+\theta A_3^p\end{bmatrix}\begin{bmatrix}\mathsf{e}_{\mathtt{x}}(t)\\ \mathsf{e}_{\mathtt{y}}(t)\end{bmatrix},\ \forall t\in[t_k,t_{k+1}),\ k\in\mathbb{N}\\ \begin{bmatrix}\mathsf{e}_{\mathtt{x}}(t_k)\\ \mathsf{e}_{\mathtt{y}}(t_k)\end{bmatrix} &= \begin{bmatrix}R_{11}^{p^-\longrightarrow p} & R_{12}^{p^-\longrightarrow p}\\ R_{21}^{p^-\longrightarrow p} & R_{22}^{p^-\longrightarrow p}\end{bmatrix}\begin{bmatrix}\mathsf{e}_{\mathtt{x}}(t_k^-)\\ \mathsf{e}_{\mathtt{y}}(t_k^-)\end{bmatrix},\ \forall k\ge 1\end{aligned} \quad (7)$$

where $e_{\mathtt{x}}(t)=\mathtt{x}(t)-\mathtt{x}'(t)$ and $e_{\mathtt{y}}(t)=\mathtt{y}(t)-\mathtt{y}'(t)$, is globally asymptotically stable. Recently, sufficient conditions for global asymptotic stability of singularly perturbed hybrid linear systems of the form (7) have been proposed in [12].

In order to construct symbolic models for the singularly perturbed hybrid affine system (1) (or equivalently system (4), (5)), we consider the following assumption:

Assumption 2: A_0^p and A_4^p are Hurwitz for all $p\in P$. This means that there exist symmetric positive definite matrices $Q_s^p\succeq I_{n_x}$ and $Q_f^p\succeq I_{n_z}$ and positive scalars λ_s^p and λ_f^p such that

$$\begin{aligned}(A_0^p)^TQ_s^p+Q_s^pA_0^p &\preceq -2\lambda_s^pQ_s^p,\\ (A_4^p)^TQ_f^p+Q_f^pA_4^p &\preceq -2\lambda_f^pQ_f^p.\end{aligned} \quad (8)$$

Under this assumption, it has been shown in [12] that if $\theta\in(0,\theta_1)$ where $\theta_1=\frac{\lambda_f}{\frac{(b_1+b_2)^2}{4\lambda_s}+b_3}$ with $\lambda_f=\min_{p\in P}\lambda_f^p$, $\lambda_s=\min_{p\in P}\lambda_s^p$, $b_1^p=\|(Q_s^p)^{\frac{1}{2}}A_1^p(Q_f^p)^{\frac{-1}{2}}\|$, $b_2^p=\|(Q_f^p)^{\frac{1}{2}}A_2^p(Q_s^p)^{\frac{-1}{2}}\|$, $b_3^p=\|(Q_f^p)^{\frac{1}{2}}A_3^p(Q_f^p)^{\frac{-1}{2}}\|$, and $b_j=\max_{p\in P}b_j^p$, $j\in\{1,2,3\}$, then

$$V_p(e_{\mathtt{x}},e_{\mathtt{y}})=e_{\mathtt{x}}^TQ_s^pe_{\mathtt{x}}+e_{\mathtt{y}}^TQ_f^pe_{\mathtt{y}},\forall p\in P \quad (9)$$

is a Lyapunov function in mode p. Thus, all the modes of (7) are Lyapunov stable. The asymptotic stability of system (7) and therefore the incremental stability of system (4), (5) has been shown while using the following functions

$$\begin{aligned}W_s^p(e_{\mathtt{x}}(t)) &= \sqrt{e_{\mathtt{x}}(t)^TQ_s^pe_{\mathtt{x}}(t)},\ \forall t\in[t_k,t_{k+1}),\ k\in\mathbb{N},\\ W_f^p(e_{\mathtt{y}}(t)) &= \sqrt{e_{\mathtt{y}}(t)^TQ_f^pe_{\mathtt{y}}(t)},\ \forall t\in[t_k,t_{k+1}),\ k\in\mathbb{N}.\end{aligned} \quad (10)$$

Considering $\theta_2\in(0,\theta_1)\cap(0,\frac{\lambda_f}{\lambda_s})$, $c_1=\frac{\sqrt{b_1^2+b_3^2}}{\lambda_f}$, $c_2=\frac{b_1}{\lambda_f-\theta_2\lambda_s}$, and $c_3=\frac{b_1c_1}{\lambda_s}$, the evolution of these functions between two events (switch or impulse) and when an event occurs are characterized in the following lemmas respectively.

Lemma 1 ([12]): Under Assumption 2, let $\theta\in(0,\theta_2]$, and $t_{k+1}-t_k=\tau>0$. Then for all $k\in\mathbb{N}$

$$\begin{aligned}W_s^p(e_{\mathtt{x}}(t_{k+1}^-)) &\le W_s^p(e_{\mathtt{x}}(t_k))(e^{-\lambda_s\tau}+\theta c_3)\\ &\quad + W_f^p(e_{\mathtt{y}}(t_k))\theta(c_2+c_3)\end{aligned}$$

$$W_f^p(e_{\mathtt{y}}(t_{k+1}^-))\le W_s^p(e_{\mathtt{x}}(t_k))\theta c_1+W_f^p(e_{\mathtt{y}}(t_k))(e^{-\frac{\lambda_f}{\theta}\tau}+\theta c_1).$$

Lemma 2 ([12]): For all $k\ge 0$,

$$\begin{aligned}W_s^p(e_{\mathtt{x}}(t_k)) &\le \psi_{11}W_s^{p^-}(e_{\mathtt{x}}(t_k^-))+\psi_{12}W_f^{p^-}(e_{\mathtt{x}}(t_k^-))\\ W_f^p(e_{\mathtt{y}}(t_k)) &\le \psi_{21}W_s^{p^-}(e_{\mathtt{x}}(t_k^-))+\psi_{22}W_f^{p^-}(e_{\mathtt{y}}(t_k^{-1})),\end{aligned} \quad (11)$$

where

$$\begin{aligned}\psi_{11} &= \max_{p^-,p\in P}\|(Q_s^p)^{\frac{1}{2}}R_{11}^{p^-\longrightarrow p}(Q_s^{p^-})^{-\frac{1}{2}}\|,\\ \psi_{12} &= \max_{p^-,p\in P}\|(Q_s^p)^{\frac{1}{2}}R_{12}^{p^-\longrightarrow p}(Q_f^{p^-})^{-\frac{1}{2}}\|,\\ \psi_{21} &= \max_{p^-,p\in P}\|(Q_f^p)^{\frac{1}{2}}R_{21}^{p^-\longrightarrow p}(Q_s^{p^-})^{-\frac{1}{2}}\|,\\ \psi_{22} &= \max_{p^-,p\in P}\|(Q_f^p)^{\frac{1}{2}}R_{22}^{p^-\longrightarrow p}(Q_f^{p^-})^{-\frac{1}{2}}\|.\end{aligned}$$

A sufficient condition for the global asymptotic stability of system (4), (5) has been given in [12]. It consists in the existence of a minimal dwell time τ^* for which the positive matrix $M_{\tau^*}\Psi$ is Schur with $M_{\tau^*}=$

$\begin{bmatrix} e^{-\lambda_s\tau^*}+\theta c_3 & \theta(c_2+c_3) \\ \theta c_1 & e^{-\frac{\lambda_f}{\theta}\tau^*}+\theta c_1 \end{bmatrix}$ and $\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}$. Here, we consider the case where ψ_{11} satisfies the following assumption:

Assumption 3:

$$\psi_{11} \leq 1. \tag{12}$$

Under this assumption sufficient conditions for deriving the values of τ^* have been provided in [12] and are recalled hereafter:

- if $\psi_{11} = 1$ and $\psi_{12} \neq 0$ then

$$\begin{aligned} \tau^* > & -\frac{\theta}{\lambda_f} ln(\theta) \\ & +\frac{\theta}{\lambda_f} ln\left(\frac{\psi 12 + a\theta\psi_{22}}{a-\psi_{11}(c_1+c_3)-\psi_{12}c_1-a\theta(\psi_{21}(c_2+c_3)+\psi_{22}c_1}\right) \\ & = \mathscr{O}(\theta ln(\theta)) \end{aligned} \tag{13}$$

  with $a > 0$ chosen such that $a > \psi_{11}(c_2+c_3)+\psi_{12}c_1$;
- if $\psi_{11} = 1$ and $\psi_{12} = 0$ then

$$\tau^* > l_3(\theta) = \mathscr{O}(\theta) \tag{14}$$

  where $l_3(\theta) = \max(l_1(\theta), l_2(\theta))$ with

$$\begin{aligned} l_1(\theta) = & \frac{1}{\lambda_s}\left(ln(1+\frac{a\theta\psi_{21}}{\psi_{11}})\right. \\ & \left. - ln(1-\theta(\psi_{11}c_3+\psi_{12}c_1)-a\theta^2(\psi_{21}c_3+\psi_{22}c_1))\right), \end{aligned}$$

$$\begin{aligned} & l_2(\theta) = \\ & -\frac{\theta}{\lambda_f} ln\left(\frac{a-\psi_{11}(c_2+c_3)-\psi_{12}c_1-a\theta(\psi_{21}(c_2+c_3)+\psi_{22}c_1)}{a\psi_{22}}\right), \end{aligned}$$

  and $a > 0$ chosen such that $a > \psi_{11}(c_2+c_3)+\psi_{12}c_1$;
- finally, if $\psi_{11} < 1$, then

$$\tau^* > l_4(\theta), \tag{15}$$

  where

$$\begin{aligned} & l_4(\theta) = \\ & \frac{\theta}{\lambda_f} ln\left(\frac{\psi_{12}+a\psi_{22}}{a-\theta(\psi_{11}(c_2+c_3)+\psi_{12}c_1)-a\theta(\psi_{21}(c_2+c_3)+\psi_{22}c_1)}\right), \end{aligned}$$

  with $a > 0$ chosen such that $\psi_{11}+a\psi_{21} < 1$.

In order to provide our result about symbolic models construction for singularly perturbed hybrid affine systems, we will consider the following properties of functions W_s^p in the rest of our paper :

$$\forall p \in P, \sqrt{\lambda_{\min}^s}\|e_x\| \leq W_s^p(e_x) \leq \sqrt{\lambda_{\max}^s}\|e_x\|, \tag{16}$$

and for all $p \in P$, for all $\mathrm{x}, \mathrm{x}', \bar{\mathrm{x}} \in \mathbb{R}^{n_x}$

$$|W_s^p(\mathrm{x}-\mathrm{x}') - W_s^p(\mathrm{x}-\bar{\mathrm{x}})| \leq \sqrt{\lambda_{\max}^s}\|\mathrm{x}'-\bar{\mathrm{x}}\| \tag{17}$$

where $\lambda_{\min}^s = \min_{p\in P}\lambda_{\min}(Q_s^p)$ and $\lambda_{\max}^s = \max_{p\in P}\lambda_{\max}(Q_s^p)$.

Likewise for W_f^p we have

$$\forall p \in P, \sqrt{\lambda_{\min}^f}\|e_y\| \leq W_f^p(e_{y(t)}) \leq \sqrt{\lambda_{\max}^f}\|e_y\|, \tag{18}$$

and for all $p \in P$, for all $\mathrm{y}, \mathrm{y}', \bar{\mathrm{y}} \in \mathbb{R}^{n_z}$

$$|W_f^p(\mathrm{y}-\mathrm{y}') - W_f^p(\mathrm{y}-\bar{\mathrm{y}})| \leq \sqrt{\lambda_{\max}^f}\|\mathrm{y}'-\bar{\mathrm{y}}\| \tag{19}$$

where $\lambda_{\min}^f = \min_{p\in P}\lambda_{\min}(Q_f^p)$ and $\lambda_{\max}^f = \max_{p\in P}\lambda_{\max}(Q_f^p)$.

B. Transition systems

We are interested in the computation of discrete abstractions for singularly perturbed hybrid affine systems. In what follows, we present the concept of transition systems that allows us to describe both hybrid systems and symbolic models in a common framework:

Definition 2: A transition system is a tuple $T = (Q, U, O, \Delta, I)$ where:

- Q is a set of states ;
- U is a set of inputs ;
- O is a set of outputs ;
- $\Delta \subseteq Q \times U \times Q \times O$ is a transition relation;
- $I \subseteq Q$ is a set of initial states .

T is said to be *metric* if the set of outputs O is equipped with a metric d such that $d(o_1, o_2) = \|o_1 - o_2\|$, *symbolic* if Q and U are finite or countable sets.
$(x', o) \in \Delta(x, u)$ will refer to the transition $(x, u, x', o) \in \Delta$. This means that by applying the input u the trajectory of the transition system will evolve from the state x to the state x' while providing the output o. Given a state $x \in Q$, an input $u \in U$ is said to belong to the set of *enabled* inputs, denoted by $Enab(x)$, if $\Delta(x, u) \neq \emptyset$. A state $x \in Q$ is said to be blocking if $Enab(x) = \emptyset$, it is said non-blocking otherwise. T is said to be *deterministic* if for all $x \in Q$ and for all $u \in Enab(x)$, $\mathrm{card}(\Delta(x, u)) = 1$.

Definition 3: Let $T_1 = (Q_1, U, O, \Delta_1, I_1)$, $T_2 = (Q_2, U, O, \Delta_2, I_2)$ be two metric transition systems with the same input set U and the same output set O equipped with the metric d. Let $\varepsilon \geq 0$ be a given precision. A relation $\mathscr{R} \subseteq Q_1 \times Q_2$ is said to be an ε-approximate bisimulation relation between T_1 and T_2 if for all $(x_1, x_2) \in \mathscr{R}$, $Enab(x_1) = Enab(x_2)$ and for all $u \in Enab(x_1)$:

$\forall (x_1', o_1) \in \Delta_1(x_1, u), \exists (x_2', o_2) \in \Delta_2(x_2, u)$ such that $d(o_1, o_2) \leq \varepsilon$ and $(x_1', x_2') \in \mathscr{R}$;

$\forall (x_2', o_2) \in \Delta_2(x_2, u), \exists (x_1', o_1) \in \Delta_1(x_1, u)$ such that $d(o_1, o_2) \leq \varepsilon$ and $(x_1', x_2') \in \mathscr{R}$.

IV. Approximate bisimilar model design

Given a singularly perturbed hybrid affine system Σ with $\mathscr{P} = \mathscr{S}(\mathbb{R}_0^+, P)$, and a time sampling parameter $\tau_s \in \mathbb{R}^+$ we define the associated transition system $T_{\tau_s}(\Sigma) = (Q_1, U, O, \Delta, I_1)$ where:

- $Q_1 = \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times P$ is the set of states;
- $U = P$ is the set of inputs;
- $O = \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ is the set of outputs;
- $\Delta \subseteq Q_1 \times U \times Q_1 \times O$ is the transition relation defined as follows: $\forall u \in U$, $\forall (x, y, p^-) \in Q_1$, $((x', y', p), o_1) \in \Delta(x, y, p^-, u)$ if and only if
  $u = p$, $(x', y') = (\phi_{\mathrm{x}}^{p^- \longrightarrow p}(\tau_s, (x, y)), \phi_{\mathrm{y}}^{p^- \longrightarrow p}(\tau_s, (x, y)))$, and $o_1 = (x, y)$;
- $I_1 = \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times P$ is the set of initial states.

The reduced state space $\mathbb{R}^{n_x}$ is then approximated by the lattice:

$$[\mathbb{R}^{n_x}]_\eta = \left\{ q_x \in \mathbb{R}^{n_x} | q_{x(i)} = k_i \frac{2\eta}{\sqrt{n_x}},\ k_i \in \mathbb{Z},\ i = 1, \dots, n_x \right\},$$

where $\eta \in \mathbb{R}^+$ is the reduced state space sampling parameter.

The quantizer $\mathscr{Q}_\eta : \mathbb{R}^{n_x} \to [\mathbb{R}^{n_x}]_\eta$ is defined by $\mathscr{Q}_\eta(x) = q_x$ if and only if

$$\forall i = 1,\dots,n_x,\ q_{x(i)} - \frac{\eta}{\sqrt{n_x}} \le x_{(i)} < q_{x(i)} + \frac{\eta}{\sqrt{n_x}}. \quad (20)$$

It can be easily verified that for all $x \in \mathbb{R}^{n_x}$, $\|\mathscr{Q}_\eta(x) - x\| \le \eta$.

We define the symbolic model $T^s_{\tau_s,\eta}(\Sigma) = (Q_\eta, U, O_\eta, \Delta_\eta, I_\eta)$ as follows:

- the set of states is $Q_\eta = ([\mathbb{R}^{n_x}]_\eta \cap \mathscr{C}) \times P$, where $\mathscr{C}$ is a compact set in $\mathbb{R}^{n_x}$;
- the set of labels (inputs) is $U = P$;
- the set of outputs is $O_\eta = \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$;
- the transition relation $\Delta_\eta \subseteq Q_\eta \times U \times Q_\eta \times O_\eta$ is given as follows: $\forall u \in U$, $\forall (q_x, p^-) \in Q_\eta$, $(q_x^{'}, p, o_2) \in \Delta_\eta(q_x, p^-, u)$ if and only if
$$u = p,\ q_x^{'} = \mathscr{Q}_\eta(\phi_x^{p^- \longrightarrow p}(\tau_s, (q_x, y_{eq}^{p^-}))) \text{ and } o_2 = \begin{bmatrix} q_x & y_{eq}^{p^-} \end{bmatrix}^T;$$
where $y_{eq}^{p^-}$ is such that
$$\begin{bmatrix} x_{eq}^{p^-} \\ y_{eq}^{p^-} \end{bmatrix} = -\begin{bmatrix} A_0^{p^-} & A_1^{p^-} \\ \theta A_2^{p^-} & A_4^{p^-} + \theta A_3^{p^-} \end{bmatrix}^{-1} \begin{bmatrix} B_1^{p^-} \\ \theta B_2^{p^-} \end{bmatrix},$$
i.e., $(x_{eq}^{p^-}, y_{eq}^{p^-})$ is the equilibrium point of the system in mode p^-.
- the set of initial states is $I_\eta = ([\mathbb{R}^{n_x}]_\eta \cap \mathscr{C}) \times P$.

Theorem 1: Consider system (4), (5) under Assumptions 2 and 3. Let $\tau_s \ge \tau^* > 0$ such that the positive matrix $M_{\tau_s}\Psi$ is Schur. Consider $\eta > 0$ and let $\varepsilon_s > 0$ and $\varepsilon_f > 0$ such that

$$\begin{bmatrix} \sqrt{\lambda^s_{\min}}\varepsilon_s \\ \sqrt{\lambda^f_{\min}}\varepsilon_f \end{bmatrix} \ge (I - M_{\tau_s}\Psi)^{-1} \begin{bmatrix} \sqrt{\lambda^f_{\max}}\eta \\ \bar{\gamma}_f(\theta) \end{bmatrix} \quad (21)$$

where

$$\bar{\gamma}_f(\theta) = \frac{\sqrt{\lambda^f_{max}}}{\sqrt{\lambda^f_{\min}}} \left(\theta c_1 \overline{W}_s + (e^{-\frac{\lambda_f \tau_s}{\theta}} + \theta c_1)\overline{W}_f \right)$$

with

$$\overline{W}_s = \max_{p^-, p \in P, q_x \in \mathscr{C}} \sqrt{\lambda^s_{\max}} \|x^p_{eq} - R_{11}^{p^- \to p} q_x - R_{12}^{p^- \to p} y_{eq}^{p^-} - \mathscr{B}_1^{p^- \to p}\|, \quad (22)$$

and

$$\overline{W}_f = \max_{p^-, p \in P, q_x \in \mathscr{C}} \sqrt{\lambda^f_{max}} \|y^p_{eq} - R_{21}^{p^- \to p} q_x - R_{22}^{p^- \to p} y_{eq}^{p^-} - \mathscr{B}_2^{p^- \to p}\|, \quad (23)$$

where (x^p_{eq}, y^p_{eq}) is the equilibrium point in mode p, then

$$\mathscr{R} = \left\{ ((x,y,p^-),(q_x,p^-)) \in Q_1 \times Q_\eta | \begin{bmatrix} W_s^{p^-}(x - q_x) \\ W_f^{p^-}(y - y_{eq}^{p^-}) \end{bmatrix} \le \begin{bmatrix} \sqrt{\lambda^s_{\min}}\varepsilon_s \\ \sqrt{\lambda^f_{\min}}\varepsilon_f \end{bmatrix} \right\} \quad (24)$$

is an ε-approximate bisimulation relation between $T_{\tau_s}(\Sigma)$ and $T^s_{\tau_s,\eta}(\Sigma)$ with $\varepsilon = \sqrt{\varepsilon_s^2 + \varepsilon_f^2}$.

Proof: See the Appendix. ■

Remark 1: The result in Theorem 1 is constructive. We may remark that ε_f depends on $\bar{\gamma}_f(\theta)$ which can be computed numerically. ε_f is a function of θ and η which are small parameters, therefore the obtained precision ε_f for the fast dynamic is sufficiently small.

Remark 2: In the case of switched affine systems $\Psi = \begin{bmatrix} I_{n_x} & 0 \\ \psi_{21} & I_{n_z} \end{bmatrix}$ and when $\theta \to 0$ we have $M_{\tau_s} = \begin{bmatrix} e^{-\lambda_s \tau_s} & 0 \\ 0 & 0 \end{bmatrix}$. Thus, inequality (21) leads to

$$\begin{bmatrix} \sqrt{\lambda^s_{\min}}\varepsilon_s \\ \sqrt{\lambda^f_{\min}}\varepsilon_f \end{bmatrix} \ge \begin{bmatrix} [1 - e^{-\lambda_s \tau_s}]^{-1} \sqrt{\lambda^s_{\max}}\eta \\ 0 \end{bmatrix}. \quad (25)$$

From this inequality we can remark that the obtained precision for the slow dynamics is $\sqrt{\lambda^s_{\min}}\varepsilon_s \ge [1 - e^{-\lambda_s \tau_s}]^{-1} \sqrt{\lambda^s_{\max}}\eta$. This expression recalls the one obtained in [3] for δ-GUAS switched systems when using the classical method for symbolic models design.

In this paper we provide new design method of symbolic models based on the singular perturbation theory for globally incrementally stable hybrid affine systems. The main advantage of this approach is that since the fast dynamic y vanishes very quickly to zero then instead of discretizing the system of dimension n the symbolic model is designed while discretizing only the slow dynamics which reduces drastically its number of transitions.

V. ILLUSTRATIVE EXAMPLE:ROOM TEMPERATURE REGULATION

Consider the thermal dynamics of a room-heater system modelled as a singularly perturbed switched affine system with two modes as follows:

$$\frac{d}{dt}\begin{bmatrix} T_1 \\ \theta T_2 \end{bmatrix} = \begin{bmatrix} -\kappa_{12} - \kappa_{10} & \kappa_{12} \\ \kappa_{21} & -\kappa_{21} - \kappa_{2f}u \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} \kappa_{10} T_0 \\ \kappa_{2f} T_f u \end{bmatrix} \quad (26)$$

where T_1 is the thermal dynamic of the room, T_2 is the thermal dynamic of the heater. T_1 and T_2 are the two continuous-time states variables. u is the discrete variable which takes values $u = 1$ and $u = 0$ that correspond to the positions "switch ON" and " switch OFF" of the heater, respectively. T_0 is the ambient temperature and T_f is the maximal temperature of the heater. κ_{12}, κ_{10}, κ_{21}, κ_{2f} are the heat transfer coefficients. θ is the time-scale separation parameter between the fast dynamic T_2 and the slow dynamic T_1.

Considering $\mathrm{x} = T_1$ and $\mathrm{z} = T_2$, system (26) can be written in the form (1) as follows

$$\begin{bmatrix} \dot{\mathrm{x}} \\ \theta \dot{\mathrm{z}} \end{bmatrix} = A^p \begin{bmatrix} \mathrm{x} \\ \mathrm{z} \end{bmatrix} + \alpha^p, p \in \{1,2\} \quad (27)$$

where $A^1 = \begin{bmatrix} -\kappa_{12} - \kappa_{10} & \kappa_{12} \\ \kappa_{21} & -\kappa_{21} - \kappa_{2f} \end{bmatrix}$, $A^2 = \begin{bmatrix} -\kappa_{12} - \kappa_{10} & \kappa_{12} \\ \kappa_{21} & -\kappa_{21} \end{bmatrix}$, $\alpha^1 = \begin{bmatrix} \kappa_{10} T_0 \\ \kappa_{2f} T_f \end{bmatrix}$, $\alpha^2 = \begin{bmatrix} \kappa_{10} T_0 \\ 0 \end{bmatrix}$. Here, we consider the following numerical values of the parameters : $\kappa_{12} = 4$, $\kappa_{10} = 1$, $\kappa_{21} = 1$, $\kappa_{2f} = 0.5$ and $\theta = 10^{-3}$. Matrices A^1_{22} and A^2_{22} are not singular. Then, Assumption 1 holds and the time-dependent change of coordinates given in (3) is derived as follows

$$\begin{aligned} \mathrm{y}(t) &= \mathrm{z}(t) - 0.6667\mathrm{x}(t) - 16.6667,\ p = 1; \\ \mathrm{y}(t) &= \mathrm{z}(t) - \mathrm{x}(t),\ p = 2. \end{aligned} \quad (28)$$

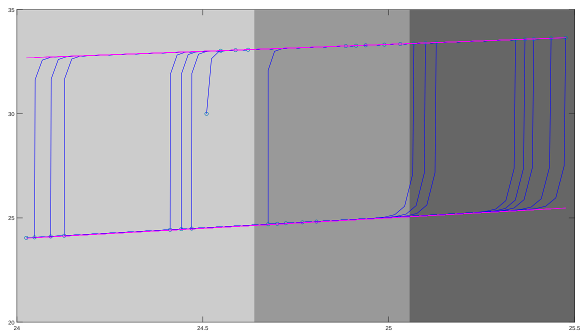

Fig. 1. Symbolic abstraction for room-heater system (26) : dark gray: mode 1, light gray: mode 2, medium gray: both modes are acceptable. Slow manifolds (lines in magenta). Trajectories of the closed loop system originating at $[x,z]^T = [24.51, 30]^T$ (blue line).

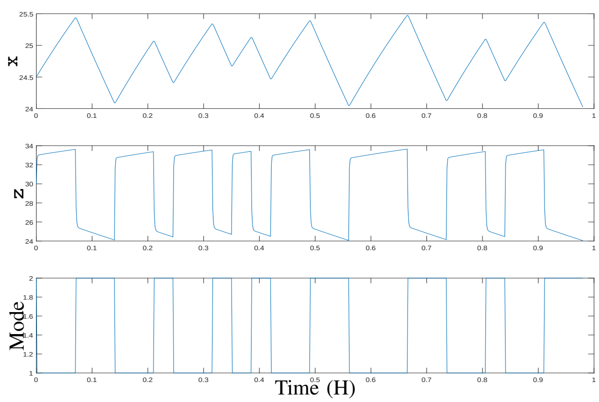

Fig. 2. Trajectories of system (26) starting at $[x,z]^T = [24.51, 30]^T$. Active modes of system (26)

Assumption 2 holds. LMIs (8) are feasible for $\lambda_f = \lambda_s = 1$ and $Q_s^p = Q_f^p = 1$, $\forall p \in \{1,2\}$. $\psi_{11} = 1$ and $\psi_{12} = 0$ (i.e., $\Psi = I_2$) thus from (14) we obtain $\tau^* = 0.0322h$. For simulations we consider $\tau_s = 0.035h > \tau^*$ such that the positive matrix $M_{\tau_s}\Psi$ is Schur. Therefore, under these conditions system (26) is δ-GUAS. Thus, we are now able to design our symbolic model for system (26).

We restrict the dynamics of the system to the compact set $\mathscr{C} = \begin{bmatrix} 24 & 25.5 \end{bmatrix}$. The state space sampling parameter is taken as $\eta = 0.0002$. The regulation goal is to maintain the room temperature around 25 degrees i.e., $T_1 \in \mathscr{C}$. The obtained precision is $\varepsilon = 1.7276$ i.e., $\varepsilon_f = 0.1790$ and $\varepsilon_s = 1.7184$. We can observe from Figure 1 that the obtained symbolic model does not have blocking states. Therefore, all the transitions of the obtained abstraction are safe. We can remark that the trajectory of the system remains inside the safe set $\mathscr{C} = \begin{bmatrix} 24.5 & 25.5 \end{bmatrix}$.

VI. Conclusion

This paper has provided a new method for symbolic models design for the class of incrementally stable singularly perturbed hybrid affine systems. The proposed method is inspired from singularly perturbed techniques based on decoupling the slow dynamics from the fast ones. Thus, the obtained symbolic abstraction is designed by discretizing only a part of the state space representing the slow dynamics. An ε-approximate bisimulation relation between the original singularly perturbed hybrid affine system and the symbolic model obtained by discritizing the slow dynamics has been provided. It has been shown that since the discrete abstraction is designed for a system of lower dimension, the number of its transitions is drastically reduced. Finally, simulations have been performed for a room temperature regulation system in order to assess the efficiency of the proposed approach.

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Appendix
Proof of Theorem 1

Proof: The proof of Theorem 1 follows the steps of Definition 3. First, let $((x,y,p^-),(q_x,p^-)) \in \mathscr{R}$ and let $((x^{'},y^{'},p),o_1) \in \Delta$. There exists $(q_x^{'},p) \in Q_\eta$ such that $((q_x^{'},p),o_2) \in \Delta_\eta(q_x,p^-,p)$. Then, let verify that $d(o_1,o_2) \leq \varepsilon$.

We have $((x,y,p^-),(q_x,p^-)) \in \mathscr{R}$, and from (16), we obtain

$$\sqrt{\lambda_{\min}^s}\|x - q_x\| \leq W_s^{p^-}(x - q_x) \leq \sqrt{\lambda_{\min}^s}\varepsilon_s. \tag{29}$$

This leads to

$$\|x-q_x\| \le \frac{1}{\sqrt{\lambda_{\min}^s}} W_s^{p^-}(x-q_x) \le \varepsilon_s. \quad (30)$$

Likewise, for the fast dynamics, from (18) we obtain

$$\sqrt{\lambda_{\min}^f}\|y-y_{eq}^{p^-}\| \le W_f^{p^-}(y-y_{eq}^{p^-}) \le \sqrt{\lambda_{\min}^f}\varepsilon_f \quad (31)$$

where $\lambda_{\min}^f = \min_{p\in P} \lambda_{\min}(Q_f^p)$.

Thus,

$$\|y-y_{eq}^{p^-}\| \le \frac{1}{\sqrt{\lambda_{\min}^f}} W_f^{p^-}(y-y_{eq}^{p^-}) \le \varepsilon_f. \quad (32)$$

Now we should verify that $d(o_1,o_2) \le \varepsilon$.

$$d(o_1,o_2) = \left\| \begin{bmatrix} x-q_x \\ y-y_{eq}^{p^-} \end{bmatrix} \right\| \le \sqrt{\|x-q_x\|^2 + \|y-y_{eq}^{p^-}\|^2}. \quad (33)$$

From (30) and (32), we obtain

$$d(o_1,o_2) \le \sqrt{\|x-q_x\|^2 + \|y-y_{eq}^{p^-}\|^2} \le \sqrt{\varepsilon_s^2+\varepsilon_f^2} = \varepsilon. \quad (34)$$

Therefore, the first condition in Definition 3 holds.

Now let $((x,y,p^-),(q_x,p^-)) \in \mathscr{R}$. To show that $((x',y',p),(q_x',p)) \in \mathscr{R}$ it is sufficient to prove that

$$\begin{bmatrix} W_s^p(x'-q_x') \\ W_f^p(y'-y_{eq}^p) \end{bmatrix} \le \begin{bmatrix} \sqrt{\lambda_{\min}^s}\varepsilon_s \\ \sqrt{\lambda_{\min}^f}\varepsilon_f \end{bmatrix}. \quad (35)$$

From (17), (19) we have

$$\begin{bmatrix} W_s^p(x'-q_x') \\ W_f^p(y'-y_{eq}^p) \end{bmatrix} \le \begin{bmatrix} W_s^{p^-}(\phi_x^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-})) - \phi_x^{p^-\longrightarrow p}(\tau_s,(x,y))) \\ W_f^{p^-}(\phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-})) - \phi_y^{p^-\longrightarrow p}(\tau_s,(x,y))) \end{bmatrix} + \begin{bmatrix} \sqrt{\lambda_{\max}^s}\|q_x' - \phi_x^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \\ \sqrt{\lambda_{\max}^f}\|y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \end{bmatrix}. \quad (36)$$

From Lemma 1 and Lemma 2, for all $\tau_s \ge \tau^*$ we have

$$\begin{bmatrix} W_s^p(x'-q_x') \\ W_f^p(y'-y_{eq}^p) \end{bmatrix} \le M_{\tau_s}\Psi \begin{bmatrix} \sqrt{\lambda_{\min}^s}\varepsilon_s \\ \sqrt{\lambda_{\min}^f}\varepsilon_f \end{bmatrix} + \begin{bmatrix} \sqrt{\lambda_{\max}^s}\eta \\ \sqrt{\lambda_{\max}^f}\|y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \end{bmatrix}. \quad (37)$$

From (18), we obtain

$$\|y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \le \frac{W_f^p(y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-})))}{\sqrt{\lambda_{\min}^f}}. \quad (38)$$

From Lemma 1, we have

$$\begin{aligned} & W_f^p(y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))) \\ \le & W_s^p(x_{eq}^p - R_{11}^{p^-\to p}q_x - R_{12}^{p^-\to p}y_{eq}^{p^-} - \mathscr{B}_1^{p^-\to p})\theta c_1 \\ & + W_f^p(y_{eq}^p - R_{21}^{p^-\to p}q_x - R_{22}^{p^-\to p}y_{eq}^{p^-} - \mathscr{B}_2^{p^-\to p})(e^{-\frac{\lambda_f\tau_s}{\theta}} + \theta c_1), \end{aligned} \quad (39)$$

where (x_{eq}^p, y_{eq}^p) is the equilibrium point in mode p.

From this last inequality, (38) leads to

$$\begin{aligned} & \|y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \\ & \le \frac{1}{\sqrt{\lambda_{\min}^f}}\Big(W_s^p(x_{eq}^p - R_{11}^{p^-\to p}q_x - R_{12}^{p^-\to p}y_{eq}^{p^-} - \mathscr{B}_1^{p^-\to p})\theta c_1 \\ & + W_f^p(y_{eq}^p - R_{21}^{p^-\to p}q_x - R_{22}^{p^-\to p}y_{eq}^{p^-} - \mathscr{B}_2^{p^-\to p})(e^{-\frac{\lambda_f\tau_s}{\theta}} + \theta c_1)\Big). \end{aligned} \quad (40)$$

From (16) and (18), (40) becomes

$$\begin{aligned} & \|y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \\ & \le \frac{1}{\sqrt{\lambda_{\min}^f}}\Big(\sqrt{\lambda_{\max}^s}\|x_{eq}^p - R_{11}^{p^-\to p}q_x - R_{12}^{p^-\to p}y_{eq}^{p^-} - \mathscr{B}_1^{p^-\to p}\|\theta c_1 \\ & + \sqrt{\lambda_{max}^f}\|y_{eq}^p - R_{21}^{p^-\to p}q_x - R_{22}^{p^-\to p}y_{eq}^{p^-} - \mathscr{B}_2^{p^-\to p}\|(e^{-\frac{\lambda_f\tau_s}{\theta}} + \theta c_1)\Big). \end{aligned} \quad (41)$$

By definition of $\overline{W}_s$ and $\overline{W}_f$, (41) leads to

$$\|y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \le \frac{\theta c_1}{\sqrt{\lambda_{\min}^f}}\overline{W}_s + \frac{(e^{-\frac{\lambda_f\tau_s}{\theta}} + \theta c_1)}{\sqrt{\lambda_{\min}^f}}\overline{W}_f. \quad (42)$$

Thus

$$\begin{aligned} & \sqrt{\lambda_{\max}^f}\|y_{eq}^p - \phi_y^{p^-\longrightarrow p}(\tau_s,(q_x,y_{eq}^{p^-}))\| \\ & \le \frac{\sqrt{\lambda_{max}^f}}{\sqrt{\lambda_{\min}^f}}\Big(\theta c_1\overline{W}_s + (e^{-\frac{\lambda_f\tau_s}{\theta}} + \theta c_1)\overline{W}_f\Big) = \bar{\gamma}_f(\theta) \end{aligned} \quad (43)$$

Thanks to the last inequality, (37) becomes

$$\begin{bmatrix} W_s^p(x'-q_x') \\ W_f^p(y'-y_{eq}^p) \end{bmatrix} \le M_{\tau_s}\Psi \begin{bmatrix} \sqrt{\lambda_{\min}^s}\varepsilon_s \\ \sqrt{\lambda_{\min}^f}\varepsilon_f \end{bmatrix} + \begin{bmatrix} \sqrt{\lambda_{\max}^s}\eta \\ \bar{\gamma}_f(\theta) \end{bmatrix}. \quad (44)$$

Therefore, in order to show that $((x',y',p),(q_x',p)) \in \mathscr{R}$ it is sufficient to prove that

$$\begin{bmatrix} W_s^p(x'-q_x') \\ W_f^p(y'-y_{eq}^p) \end{bmatrix} \le M_{\tau_s}\Psi \begin{bmatrix} \sqrt{\lambda_{\min}^s}\varepsilon_s \\ \sqrt{\lambda_{\min}^f}\varepsilon_f \end{bmatrix} + \begin{bmatrix} \sqrt{\lambda_{\max}^s}\eta \\ \bar{\gamma}_f(\theta) \end{bmatrix} \le \begin{bmatrix} \sqrt{\lambda_{\min}^s}\varepsilon_s \\ \sqrt{\lambda_{\min}^f}\varepsilon_f \end{bmatrix}. \quad (45)$$

The second term of inequality (45) is equivalent to

$$\begin{bmatrix} \sqrt{\lambda_{\max}^s}\eta \\ \bar{\gamma}_f(\theta) \end{bmatrix} \le (I - M_{\tau_s}\Psi)\begin{bmatrix} \sqrt{\lambda_{\min}^s}\varepsilon_s \\ \sqrt{\lambda_{\min}^f}\varepsilon_f \end{bmatrix} \quad (46)$$

Since $M_{\tau_s}\Psi$ is a positive and Schur matrix and from the Neumann series properties, the matrix $I - M_{\tau_s}\Psi$ is invertible and its inverse $(I - M_{\tau_s}\Psi)^{-1}$ is positive. Therefore, (46) leads to

$$\begin{bmatrix} \sqrt{\lambda_{\min}^s}\varepsilon_s \\ \sqrt{\lambda_{\min}^f}\varepsilon_f \end{bmatrix} \ge (I - M_{\tau_s}\Psi)^{-1}\begin{bmatrix} \sqrt{\lambda_{\max}^s}\eta \\ \bar{\gamma}_f(\theta) \end{bmatrix}. \quad (47)$$

Thus, from the last inequality and from (45), (35) is verified. Therefore, $((x',y',p),(q_x',p)) \in \mathscr{R}$. Then, $\mathscr{R}$ is an ε-approximate bisimulation relation between $T_{\tau_s}(\Sigma)$ and $T_{\tau_s,\eta}^s(\Sigma)$. ■