THE SOLUTIONS OF THE 3 RD AND 4 TH CLAY MILLENNIUM PROBLEMS

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PROLOGUE

In this treatise we present the solutions of the 3rd Clay Millennium problem in the Computational Complexity and the 4th Clay Millennium problem in classical fluid dynamics.

The solution of the 3rd Clay Millennium problem has already been published in *International Journal of Pure and Applied Mathematics Volume 120 No. 3 2018, pp 497-510 ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version) url: http://www.ijpam.eu doi: 10.12732/ijpam.v120i3.1* (see also part A of the current treatise)

The solution of the 4thClay Millennium problem is in two papers the first of which has already been published , and the 2nd which is the final solution is still under referees review. (see part B of the current treatise)

Of course it is not a single man's mind only that decides but the collective intelligence that eventually validates and makes solutions accepted.

It seems that at the beginning of each century has become a tradition to state a list of significant and usually difficult problems in the mathematics, that it is considered that their solution will advance significantly the mathematical sciences. At the begging of the 20th century (1900) it was D. Hilbert who formulated and listed 23 problems that most of them have been solved till today (see e.g. https://en.wikipedia.org/wiki/Hilbert%27s_problems) . Those problems from the 23 that have been solved proved to be indeed critical for the overall evolution of mathematics and their applications. Continuing this tradition, the Clay Mathematical Instituted formulated in 2000, 7 critical problems and this time there is a monetary award for their solution (see e.g. http://www.claymath.org/millennium-problems) . From them, the 6th problem (Poincare Hypothesis) it has been accepted that it has been solved by Grigoriy Perelman in 2003. It is not presented here a common or joint method of solution of the 3rd and 4th Clay millenniums problems. It is only because I am an interdisciplinary researcher that I have worked , on both of them. And of course I had both the advantages and disadvantages of an interdisciplinary researcher. The disadvantage was that I had to sharpen by specialized knowledge in two different areas of Computer science and Mathematical physics , that specialist would not need not do it, while the advantage , that turned out to be more important, were that "I was not blinded by the trees so as to see the forest"; In other words I used new heuristic methods from other disciplines to discover the correct direction of solutions and afterwards I worked out a standard classical proof for each one of them. This is well known in the history of mathematics. E.g. Archimedes found at first the correct formulae of volumes of the sphere, cylinder etc with water, sand and balanced moments of forces experiments before he worked out logically complete proofs of them in the context of

Euclidean geometry. Similarly, Newton discovered the laws of gravitation for earth, sun, moon etc with his, at that time unpublished calculus of fluxes or infinitesimals, and then worked strict proofs within Euclidean geometry in his famous Principia Mathematica.

Similarly, I used myself a heuristic methodology that I call **"Digital Mathematics without the infinite"** that two initial papers on it can be found here

1) Digital continuous Euclidean Geometry without the infinite
   http://cris.teiep.gr/jspui/handle/123456789/1590
   or
   https://drive.google.com/open?id=1i4DnLBmf_2QDPGe-ZZO1iavuwyMogpdp

2) Digital differential and integral calculus
   http://cris.teiep.gr/jspui/handle/123456789/1679

or

https://drive.google.com/open?id=1lcgQ3YZJpfRZj7Wn0UC5P9ajVtEbPX-o

The above tools of digital mathematics are I believe a major millennium upgrade of the classical mathematics which is still in infancy, with unpredictable good applications in Artificial Intelligence too. Nevertheless, strictly speaking not results or theorems of the above two papers were used in the proofs of the solutions of the 3rd and 4th Clay Millennium problems that are stated in an independent and self-contained manner within classical mathematics.

Roughly speaking, my heuristic methodology was to re-formulate the two problems in the context of digital mathematics that do not allow for the infinite which complicated the ontology a lot, , discover the true direction of validity of their solution, diagnose the nature of their difficulty and then attach them for a valid proof in the context of classical mathematics with the infinite. Both problems had at least two different directions of solution. For the 3rd Clay Millennium problem it is

1) that the non-deterministic polynomial complexity symbolized by NP is equal to a polynomial complexity symbolized by P (in which case the usual encryption of passwords and messages might be unsafe) or

2) to a higher e.g. EXPTIME (in which case the usual encryption of passwords and messages is as expected safe). The heuristic analysis gave that it should hold NP=EXPTIME, which was eventually proved. And for the 4th Clay Millennium problem two different directions of solution would be that

1) There exist a Blow-up of velocities in finite time

2) No blow-up exist in finite time and the solutions of the Navier-Stokes equations are regular.

The heuristic analysis within digital mathematics gave that because of finite initial energy and energy conservation there cannot be a Blow-up which was eventually proved within the context of classical fluid dynamics that allows for infinite limits etc. More on the logic and strategy of proof for each problem in the next two parts of this treatise.

PART A.

THE SOLUTION OF THE 3RD CLAY MILLENNIUM PROBLEM

Prologue

The standard formulation of the 3rd Clay Millennium problem can be found in (**Cook, Stephen** ***April 2000*** *The P versus NP Problem (PDF), Clay Mathematics Institute site*. http://www.claymath.org/millennium-problems/p-vs-np-problem **http://www.claymath.org/sites/default/files/pvsnp.pdf*)***
My initial heuristic analysis of the problem gave in a few months that it was more likely (as most mathematicians were expecting) that the complexity class of non-determinist polynomial complexity NP was non-reducible to polynomial complexity P. And it seemed to me that the difficulty in proving it, was hidden in that we already use the countable infinite in the classical computational complexity theory, which is more or less formulated with (Zermelo-Frankel) set theory. Strictly speaking there is nothing infinite in the functions of a computer, but the mathematical theory about computational complexity is traditional to use the countable infinite as valid ontology. So it seemed to me that the missing key abstraction was that we had to start with an abstract EXPTIME computational complete problem, and ***transform*** it with classical and elementary set theoretic methods to an NP-computational complete problem, proving therefore that NP=EXPTIME. And the existence of an abstract EXPTIME computational complete problem is guaranteed by the classical Time hierarchy theorem of Computational Complexity. The proof of the P versus NP problem in the direction P ≠ NP, is supposed also to mean that the standard practice of encryption in the internet, is safe.

So here was my strategy.

1) The P versus NP is a difficult problem, that has troubled the scientific community for some decades
2) It may have simple proofs of a few paragraphs, hopefully not longer than the proof of the Time Hierarchy theorem, which seems to be a deeper result.
3) But it can also have very lengthily and complex proofs, that may take dozens of pages.

What the final proof in the next published is or is not:

1) It does not introduce new theoretical concepts in computational complexity theory so as to solve the P versus NP.
2) It does not use relativization and oracles
3) It does not use diagonalization arguments, although the main proof, utilizes results from the time hierarchy theorem
4) It is not based on improvements of previous bounds of complexity on circuits
5) It is proved with the method of counter-example. Thus it is transparent short and "simple". It takes any Exptime-complete DTM decision problem, and from it, it derives in the context of deterministic Turing machines a decision problem language which it is apparent that it belongs in the NP class decision problems while it does not belong the class P of decision problems.
6) It seems a "simple" proof because it chooses the right context to make the arguments and constructions and the key-abstraction mentioned above. So it helps that the scientific community will accept that this 3rd Clay Millennium problem has already been solved.

The next paper is a preliminary version (submitted August 2017) of the finally published paper in *International Journal of Pure and Applied Mathematics Volume 120 No. 3 2018, pp 497-510 ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version) url: http://www.ijpam.eu doi: 10.12732/ijpam.v120i3.1* (see also part A of the current treatise)

- Before this final version there was one more version submitted during April 2017, where it was only proved that P is not equal to NP but it was not proved that NP=EXPTIME, and was published in the proceedings of the Conference: 1st INTERNATIONAL CONFERENCE ON QUANTITATIVE, SOCIAL, BIOMEDICAL AND ECONOMIC ISSUES 2017 ,June 29-30, http://icqsbei2017.weebly.com At: STANLEY HOTEL,KARAISKAKI SQUARE,METAXOURGIO,ATHENS,GREECE Volume: V 1,2

https://books.google.gr/books?id=BSUsDwAAQBAJ&pg=PP1&lpg=PP1&dq=Proceedings+1st+conference+frangos&source=bl&ots=D4_tetxuxK&sig=ACfU3U3uOSgx1HQvKXQJkk3mbetw3FH_1w&hl=el&sa=X&ved=2ahUKEwjVqc66qcPgAhUEKFAKHQ75DIcQ6AEwAXoECAUQAQ#v=onepage&q=Proceedings%201st%20conference%20frangos&f=false

THE SOLUTION OF THE 3RD CLAY MILLENNIUM PROBLEM.

A SHORT PROOF THAT P ≠ NP=EXPTIME IN THE CONTEXT OF ZERMELO-FRANKEL SET THEORY.

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ABSTRACT

In this paper I provide a very short but decisive proof that P ≠ NP, and NP=EXPTIME in the context of the Zermelo-Frankel set theory and deterministic Turing machines. We discuss also the subtle implications of considering the P versus NP problem, in different axiomatic theories. The results of the current paper definitely solve the 3rd Clay Millennium problem P versus NP, in a simple and transparent away that the general scientific community, but also the experts of the area, can follow, understand and therefore become able to accept.

Key words: *3rd Clay Millennium problem*, *EXPTIME-complete problems, NP-complexity, P-complexity*

Mathematical Subject Classification: 68Q15

1. Introduction

In the history of mathematics, it is known that difficult problems that have troubled a lot the mathematicians, turned out to have different proofs one simple and one very complex. Such an example is if the general 5th order polynomial equation can be solved with addition, subtraction, multiplication , division and extraction of radicals starting from the coefficients. This was a problem that troubled the mathematicians for centuries! No doubt a very difficult problem. Yet it was the famous mathematician Niels Henrik Abel who gave a very simple proof that a general the order polynomial equations cannot be solved so, in about 5-6 pages! On the other hand the proof of the same, by the E. Galois theory, is a whole book of dozens of pages! This is not strange, as it depends on the right twist of semantics and symbols in mathematics. The proof in the way of E. Galois is much longer as it shows also how to solve any such equation when it is solvable! The fact that N. H. Abel solved it in only 5-6 pages should not make us think that it was an easy problem!

We may make the metaphor that the proof that a general polynomial equation of the 5^{th} order cannot be solved with radicals, is like a mountain path. It is not long , it is say only one kilometer, but at a particular point in order to cross it , small bridge is required , that no-one has ever constructed for...centuries! The unconstructed bridge is the metaphor of a particular type of abstract thinking, that non-one was able to compose so far, therefore no-one has been able to walk this mountain path, or in other words non-one has been able to prove the particular problem. It was N.H.Abel who though of the abstraction of the structure of the group of permutations of 5 symbols (the missing bridge) who was able to prove it! The proof of E. Galois may correspond to our metaphor, to another long mountain path, thus passes from the start and end points of the N.H. Abel's path, but also climbs up all the mountain and it is of course longer say 10 kilometers.

It is the same with the solution of the P versus NP problem in this paper. We will utilize in our proofs, the ***key abstraction*** of the existence of an EXPTIME complete language, (it is known that it exists) without specifying which one, which will simplify much the arguments. Then we synthesize other languages and arguments over it, that will solve the problem.

A second issue that is important to mention, is a statement, that is usually attributed to the famous mathematician Yuri Manin, that "A correct proof in mathematics is considered a proof only if it has passed the social barrier of being accepted and understood by the scientific community and published in accepted Journals"

Passing the obstruction of the social barrier, sometimes is more difficult than solving the mathematical problem itself!

An example in the history of mathematics is the Hyperbolic or Bolyai–Lobachevskian geometry. If it was not that the famous and well established J.C.F. Gauss , assured the mathematic community that he himself had also discovered this geometry, the scientific community, would not accepted it so easily. Gauss also mentioned that he refrain from publishing it for obvious reasons. It seems that he was afraid that he might be ridiculed, and so although he himself seemed that he had solved the very famous and century old problem of the independence of the 5^{th} axiom of parallel lines in the Euclidean geometry, he did not dare to try to pass also the social barrier.

These two observations seem to apply also in the famous 3^{rd} Clay millennium problem of P versus NP.

We must notice here that the P versus NP problem , is in fact a set of different problems within different axiomatic systems. And in the context of what axiomatic system is the Complexity Theory of Turing machines? Since the complexity theory of Turing machines requires entities like infinite sets of words etc then it is in the context of some axiomatic set theory, together with the axiom of infinite. So we notice that the next are different problems:

1) The P versus NP problem in the Zermelo-Frankel axiomatic system of sets without the axiom of choice and this axiomatic system formulated in the 2rd order formal languages.
2) The P versus NP problem in the Zermelo-Frankel axiomatic system of sets with the axiom of choice and this axiomatic system formulated in the 2rd order formal languages.
3) Etc

We might try to think of the P versus NP problem within the context of the axiomatic system of Peano Arithmetic with or without the axiom of induction and within second order formal languages. But to do so, we must carefully define, what additional axioms or definitions give the existence of infinite subsets of natural numbers that are used in the Complexity Theory.

My main hidden guiding idea in searching for such a simple proof, was that what the "arbitrary human-like free-will" of a non-deterministic Turing machine as human-machine interactive software (e.g. in password setting), can do in polynomial time cannot be done by a purely mechanical deterministic Turing machine in polynomial time. (See also beginning of paragraph 4) After the Key-abstraction mentioned above I had to find the right simple argumentsto make a valid proof of this idea. The proof of the P versus NP problem in the direction P $\neq$ NP, is supposed also to mean that the standard practice of encryption in the internet , is safe.

We notice also that the P versus NP :

4) It is a difficult problem, that has troubled the scientific community for some decades
5) It may have simple proofs of a few paragraphs, hopefully not longer than the proof of the Time Hierarchy theorem, which seems to be a deeper result.
6) But it can also have very lengthily and complex proofs, than may take dozens of pages.
7) There many researchers (tens and tens of them) that have claimed to have solved it, either as P=NP, or as P ≠ NP, and even as suggestions that neither are provable, but only a handful of them seem to have been able to pass the preliminary social barrier and publish their solution in conferences or Journals with referees. The rest of them have published online only preprints (see e.g. the [16] P versus NP page). It seems to me though that it is not probable that all of them have correct solutions. Especially in the direction P=NP, there is a common confusion and mistake, that has been pointed out by Yannakakis M. 1998 in [17]. Furthermore this confusing situation has contributed so that although there are publications in respectable Journals, the experts and the scientific community does not seem of being able to decide if the P versus NP problem has been solved or not. This is reasonable, as there are proofs of close to 100 pages, and no average reader would feel comfortable to go through them, and decide for himself if there a flaw or error somewhere. Still it is better to have published results than non-published, and then let the large number of readers to try to find errors or flaws in the solutions if there are any.

So here comes the need of a more challenging problem: Not only to solve the P versus NP problem, but also solve it in such an simple, elegant and short way, so that the researchers will know a decisive proof that can understand and control that P ≠ NP or not , so short that anyone familiar with the area, would discover any flaw or error if it existed.

This is I believe the value of the present paper that provides such a proof in the context of the Zermelo-Frankel set theory (we do not use the axiom of choice), formulated within 2nd order formal languages.

What this proof is or is not:

7) It does not introduce new theoretical concepts in computational complexity theory so as to solve the P versus NP.
8) It does not use relativization and oracles
9) It does not use diagonalization arguments, although the main proof, utilizes results from the time hierarchy theorem
10) It is not based on improvements of previous bounds of complexity on circuits
11) It is proved with the method of counter-example. Thus it is transparent short and "simple". It takes any Exptime-complete DTM decision problem, and from it, it derives in the context of deterministic Turing machines a decision problem language which it is apparent that it belongs in the NP class decision problems while it does not belong the class P of decision problems.

12) It seems a "simple" proof because it chooses the right context to make the arguments and constructions and the key-abstraction mentioned above. So it helps that the scientific community will accept that this 3rd Clay Millennium problem has already been solved.

In relation to the use of oracles, in arguments of complexity theory, we must notice, that their use sometimes may be equivalent to the introduction of new axioms that guarantee their existence in complexity theory within the context of ZFC set theory, which sometimes may lead to contradictions and non-consistent axiomatic system, that can prove anything. It is known that often are claimed by authors oracles that decide non-decidable sets. In this paper we do not use in the arguments oracles.

In the paragraph 4, we give an advanced, full proof that P ≠ NP, in the standard context of deterministic Turing machines, solving thus the 3rd Clay Millennium problem.

2. Preliminary concepts, and the formulation of the 3rd Clay millennium problem, P versus NP.

In this paragraph , for the sake of the reader, we will just mention the basics to understand the formulation of the 3rd Clay Millennium problem. The official formulation is found in **[3]** (**Cook, Stephen** ***(April 2000)***, *The P versus NP Problem (PDF), Clay Mathematics Institute site)*. Together with an appendix where there is concise definition of whar are the Determinstic Turing machines, that is considered that they formulate, in Computational Complexity theory , the notion and ontology of the software computer programs.

In the same paper are also defined the computational complexity ***classes P, NP***.

The elements of the classes P, NP etc strictly speaking are not only sets of words denoted by L, that is not only languages, but also for each such set of words or language L at least one DTM , M that decides it, in the specified complexity so they are pairs (L,M). Two such pairs (L_1, M_1) (L_2, M_2) are called ***equidecidable*** if $L_1 = L_2$ although it may happen that $M_1 \neq M_2$. E.g. if the complexity of M_1 is polynomial-time while that of M_2 exponential-time choosing the first pair instead of the second means that we have turned an high complexity problem to a low complexity feasible problem.

The definition of other computational complexity classes like **EXPTIME** etc can be found in standard books like [6],[10],[11]. In the official formulation [3] there is also the definition of the concept of ***a decision problem language in polynomial time reducible to another decision problem language***.

Based on this definition it is defined that an EXPTIME-complete decision language of EXPTIME is EXPTIME-complete, when all other decision problems languages of EXPTIME have a polynomial time reduction to it. Here is the exact definition

Definition 2.1 *Suppose that Li is a language over all words Σ_i , i = 1, 2. Then $L_1 \leq p$ L_2 (L_1 is p-reducible to L_2) iff there is a polynomial-time computable function $f : \Sigma_1 \to \Sigma_2$ such that $x \in L_1$ if and only if $f(x) \in L_2$, for all $x \in \Sigma_1$.*

In the same books [6],[10],[11] can be found the concepts and definitions of ***NP-complete and EXPTIME-compete decision problems***. See also [7], [11] where its proved that specific decision problems are EXPTIME-complete.

For simplicity we will consider here only binary alphabets {0,1} and binary set of words Σ.

3. Well known results that will be used.

We will not use too many results of the computational complexity theory for our proof that P ≠ NP.
A very deep theorem in the Computational Complexity is the ***Time Hierarchy Theorem*** (see e.g. [6],[10],[11],[9],[13]. This theorem gives the existence of decision problems that cannot be decided by any other deterministic Turing machine in less complexity than a specified.
Based on this theorem , it is proved that:

Proposition 3.1 *There is at least one EXPTIME-complete decision problem, that cannot be decided in polynomial time, thus P ≠EXPTIME.*

The next two propositions indicate what is necessary to prove in order to give the solution of the P versus NP problem.
Proposition 3.2 *If the class NP contains a language L which cannot be decided with a polynomial time algorithm, then P ≠ NP.*
Proposition 3.3 *If the class NP contains a language L which is EXPTIME complete, then NP=EXPTIME.*

4. The solution: P ≠ NP=EXPTIME in the context of deterministic Turing machines.

We will prove in this paragraph that P ≠ NP in the context of second order formal language of the Zermelo-Frankel set theory.

Since we are obliged to take strictly the official formulation of the problem , rather than text books about it, we make the next clarifications.

We will use the next conditions for a Language to be in the class NP, as stated in the standard formulation of the P versus NP problem (see **[3] Cook, Stephen** ***(April 2000)****, The P versus NP Problem (PDF), Clay Mathematics Institute.).*
We denote by *Σ*all the words of an alphabet Σ.*

Definition 4.1 *A language L of binary words is in the class NP if and only if the next conditions hold*

1) *There is a deterministic Turing machine M that decides L. In other words for any word x in L, when x is given as input to M, then M accepts it and if x does not belong to L then M rejects it.*
   In symbols: Ǝ a deterministic Turing machine M, such that Ɐx$\in\Sigma^*$ *, x is either accepted or rejected by M and if M accepts x → x*$\in$*L, and if M reject x → x* $\notin$*L*
2) *There is a polynomial-time checkable relation R(x,y), and a natural number k of N, so that for every word x , x belongs to L if and only if there is a word y , with* $|y| <= |x|^k$*, and R(x,y) holds.*

In symbols:∃ relation R which is polynomial-time checkable ,and ∃ kϵN, such that ∀xϵΣ,*
xϵL↔ (∃ yϵ Σ, $|y|<=|x|^k$ and R(x,y) holds).*

Remark 4.1. In the official statement of the P versus NP problem (see **[3] Cook, Stephen** ***(April 2000)****, The P versus NP Problem (PDF), Clay Mathematics Institute)* the condition 1) is not mentioned. But anyone that has studied complexity theory, knows that it is required. The condition 2) alone cannot guarantee that there is a deterministic Turing machine that decides the language., as the polynomial checkable relation works only if we provide it with certificate y, and not with only x as input. Indeed we shall see below at the end of the proposition in **Remark 4.4,** that there is even an undecidable language L , for which nevertheless there is a polynomial checkable relation R, so that condition R is satisfied. The languages of NP cannot be semidecidable (or undecidable). The NP class is also defined as NP =∪kϵN NTIME(n^k), but this definition is also in the context of non-deterministic Turing Machines. The situation with P, is more clear, because the mere requirement that a language of P is of polynomial time complexity as it standard to define it , involves already that there exist a deterministic Turing machines that for every input word, it halts within polynomial time steps and either accepts or rejects it, therefore it decides it. And not that is simply the language of a deterministic Turing machine , and therefore maybe only semi-decidable.

Remark 4.2. Notice that in the condition 2) the k depends on the relation R and is not changing as the certificate y changes. In other words k does not depend on y and we ***did not*** state the next:

There is a polynomial-time checkable relation R(x,y), so that for every word x , x belongs to L if and only if there is a word y , and k in N ,with $|y|<=|x|^k$, and R(x,y) holds. In symbols: ∃ relation R which is polynomial-time checkable , such that ∀xϵΣ, xϵL↔ (∃yϵΣ* and ∃kϵN such that $|y|<=|x|^k$ and R(x,y) holds).*

In the official statement of the P versus NP problem (see **[3] Cook, Stephen** ***(April 2000)****, The P versus NP Problem (PDF), Clay Mathematics Institute)* this is not made clear, in the natural language that the definition is stated. But that k does not depend on the certificate, but on the polynomial checkable relation becomes clear, when we look at the proof in any good textbook about complexity theory, of how a non-deterministic Turing machine which runs in polynomial time, can define a deterministic Turing machine with a polynomial time checkable relation, which is considered that replaces it.

Remark 4.3 : My main intuition to find a proof that *P ≠ NP=EXPTIME .***The password setting.**

Le us make the next thought-experiment: Imagine a human Mr H who has available infinite time , and has infinite mental capabilities. No the world asks Mr H to set passwords on all lenths of words! So Mr H sets a password p(l) for words of length l=1,2,3,...n,...etc. Next let us imagine the problem of finding the password of length say x=153. Mr H has an arbitrary free will and he is honest not to give his passwords , in addition Mr H has provided us with a device D(l) for each length l, that unlocks if we give to it the password p(l), so we will know if w is the password or not. So the only way to discover if particular word w of length |w|= l=153 is the password p(153) or not, it is to search all the words of length l in an exhaustive way and try them on the device D(l). This is of course a an EXPTIME complexity problem, that *cannot be reduced to a polynomial time problem. Therefore finding the language LP of passwords p(l) of Mr H, cannot be a problem of polynomial time complexity. If in addition , we assume that the blind exhaustive search of all words of length l, is an EXPTIME-complete complexity problem on the initial data l, then finding the language of passwords of Mr H is also an EXPTIME-complete problem. Nevertheless for each word w of length l, the Device(l) is the polynomial time on the length l, checkable relation (certificate for each word w) , that can decide if w is a password or not, therefore the problem of finding the language L, is in the NP class complexity. But the above then after Propositions 3.2 and 3,3, indicate that $P \neq NP=EXPTIME$.*

Now this intuitive idea, is obviously not a formal proof at all, as we are taking about "human Mr H" , "arbitrary human free will" etc. Besides we are taking about the complexity of problems here rather than the complexity of languages. How can we turn this intuition to strict and formal proof, without using oracles, or non-formal arguments? The solution is the key-abstraction that I mentioned in the Introduction, that is to start with the existence of an EXPTIME-complete complexity language, that we know it exists, without specifying which one. Then define other languages over it and make simple arguments that solve to P versus NP problem.

The strategy to do so is quite simple: We will start with an exptime-complete decision problem and its language L_{exp} and we will derive from it an NP class decision problem than cannot be solved in the polynomial time (it does not belong to the class P).

<u>The next proposition sets the existence of an EXPTIME-complete complexity language of the EXPTIME complexity class (Proposition 3.1) in a convenient form, that can be used for further compositions of other languages over it.</u>

Proposition 4.1. *There is at least one infinite binary sequence, that can be computed and decided in exptime-complete complexity.*

Proof.

Let an exptime-complete decision problem A , that its existence is guaranteed by **Proposition 3.1** ,and its language $L_{exp} \in$ EXPTIME. We will need for the sake of symbolic convenience this language and decision problem , in the form of a binary sequence. If Σ^* is the set of all words of the binary alphabet Σ of the language L_{exp} , then we give a linear order to the binary alphabet Σ={0,1} 0<1, and then the inherited linear lexicographic order to the set of words Σ^*. Since Σ^* is linearly and well ordered with a first element and after excluding all words with a left sequence of consecutive zeros (which is obviously a polynomial time decision on the length of the words) reducing to the set denoted by Σ^{**} ,we fix the identity map as an arithmetization with an 1-1 and on to correspondence F: Σ^{**} -> **N** to the set of natural numbers, so that the language L_{exp} can be considered after this fixed arithmetization identity mapping correspondence F, as a subset of the natural numbers. So let Char(L_{exp}) : **N**->{0,1} be the characteristic function of the set L_{exp} in the Natural numbers encoded thus in a binary base. Then Char(L_{exp}) consists of d_i, for $i \in$ **N**, and d_i is binary digit, that is equal to 0 or

1. A first finite 7-digits segment of it, would seem for example like (0010110...). Since L_{exp} is an exptime-complete decision problem L_{exp} ϵ EXPTIME, its characteristic function is computable with an exptime-complexity too on the length of the binary words , and conversely any Turing machine computation of this characteristic function and also infinite binary sequence Char(L_{exp}) : **N**->{0,1}: consisting from d_i/ for all i ϵ **N**, and d_i is binary digit, that is equal to 0 or 1, is also a Turing machine decision computation of the language L_{exp}. Therefore there is no polynomial time complexity computation of this infinite binary sequence, as this would make EXPTIME=P and we know that P ≠EXPTIME. For the sake of intuitive understanding of the following arguments we call this binary sequence "*An exptime-compete binary DNA sequence*" and we denote it by DNA_{exp}. This simplification from the original exptime-complete decision problem and Language L_{exp} of Σ^* to the DNA_{exp} of N can be considered also as a polynomial time reduction of decision problem and languages L_{exp} *≤p* DNA_{exp} *(*L_{exp} *is p-reducible to* DNA_{exp})(see Definition 2.1). QED.

Proposition 4.2 (3rd Clay Millennium problem) *There is at least one decision problem language of the class NP which is not also in the class P. Therefore P ≠ NP.*

1st Proof.

In the next we show that there is a language L_{np} belonging to the class NP that cannot also belong to the class P without making, the previous binary sequence in the proof of the Proposition 4.1 called "*exptime-complete binary DNA sequence*" and denoted by DNA_{exp}, computable in polynomial time complexity! To ensure that a language L_{np} belongs to the class NP it must hold that there is a polynomial-time checkable relation R(x,y) and a natural number k, so that for every word x, it holds that x belongs to the language L_{np} if and only if there is another word y, called "certificate" with length $|y|<=|x|^k$,so that R(x,y) holds. Here by |x| we denote the length of the word x, which is a natural number.

Now comes the intuition behind calling the binary sequence DNA_{exp} of the previous proof, a DNA sequence: The trick here is to define this language denoted by L_{np} with the information encoded in the binary sequence DNA_{exp} so that, although a human with deterministic Turing machines and exptime-time complexity can compute DNA_{exp} and therefore decide L_{np} , no deterministic Turing machine within polynomial-time complexity can compute and decide the L_{np}. In addition for every word x, if a human will give to such deterministic machines the necessary information in the form of a "certificate" y, then a deterministic Turing machine can decide if x belongs or not to L_{np} within polynomial-time complexity.

We define such a language L_{np} with the previous requirements simply as follows:

For any word x ϵ Σ^* ,x ϵL_{np} if and only if , the word is an initial word of the infinite binary sequence DNA_{exp} and the $d_{|x|}$ =1 , where $d_{|x|}$ is the |x|-order binary digit of the infinite binary sequence DNA_{exp} . And of course x does not belong to L_{np} if and only if this does not happen. If we re-phrase the condition "$d_{|x|}$ =1", as "DNA_{exp} acceptable" , then the definition can be rephrased as that a word x belongs to the language L_{np} if its length and itself is DNA_{exp} acceptable.

Then we define as “certificate” y of the word x, the finite sequence $y=(d_1, d_2,...,d_{|x|})$, and as polynomial time checkable relation R(x,y), and that R(x,y) holds , the fact that given x , and y, the last digit of y is 1 and the rest of the digits agree too. Notice that here a human gives a lot of information to a Turing machine that will check if x belongs or not to L_{np} , in the form of the |x|-length initial segment y of the infinite binary sequence DNA_{exp} that we know that no Turing machine can compute within polynomial-time complexity.

That this relation R(x,y) is checkable in polynomial time relative to the length |x| of x, is obvious as the Turing machine with input x and y, will have only go through |x|-many steps to check the last digit of y.

Now no deterministic Turing machine M can decide the language L_{np} , in other words decide given as input only the word x (without its “certificate” y), if x ϵL_{np} or not. And this is so, because if it exist such a deterministic Turing machine M, then it could also decide (or compute) the digit $d_{|x|}$ of DNA_{exp} which we know that is not computable in polynomial-time complexity. Thus L_{np} does not belong to P, and therefore P ≠ NP

QED.

2[nd] Proof

We may define, in a simpler way, the language $L_{passwords}$ (the index passwords , is so as to follow the intuition of password setting as in the **Remark 4.3**) as the set of all binary words, that are the successive n-initial segments of the infinite binary sequence $DNA_{exp.}$ Then this language is obviously (after **Proposition 4.1**) an EXPTIME-complete language. Nevertheless the language $L_{passwords}$ also belongs to the class NP, because for each word w, of length |w|=n, a “certificate” y of it is the word w itself y=w, and the polynomial time checkable relation R(w,y) , y=w , is checkable in polynomial time, relative to the length |w|=n. Notice that we have here one only word w for each word-length n. But then from the **Proposition 3.3** NP=EXPTIME, and thus P ≠ NP. QED.

Corollary 4.1. *It holds that NP=EXPTIME*

Proof: Direct from the **Proposition 3.3,** and that the language $L_{passwords}$ in the **2[nd] proof** of the **Proposition 4.2** is also EXPTIME-complete language , besides belonging in the class NP.

QED

Remark 4.4 Notice that instead of taking, the characteristic function DNA_{exp} of an exptime-complete language, we could have taken the characteristic function DNA_{und} of an undecidable language and we know that, there is at least one, and repeat the definition of the Language L_{np} , deriving thus an undecidable language , which still it has a polynomial time checkable relation, that nevertheless works only if a human feeds it with a certificate y and there is not a Turing machine that can decide it by taking as input the word x alone. This confirms that in the definition of NP in, Definition 4.1, the condition 1) is required. Alternatively we may prove the same thing in a different way. By using the axiom of choice of the ZFC set theory we may define for example an arbitrary infinite sequence L_p of passwords p_n , each one of length exactly n, from the infinite set of the sets of words Σ^n of length n. It is known that the axiom of choice of the ZFC set theory, gives no information at all about what are the elements of such a set, besides that each p_n belongs to Σ^n . We cannot expect that any such infinite choice L_p of n-length passwords p_n can be decided by a deterministic Turing machine. If it was so, as such Turing machines are countable, we order all such languages $L_{p,i}$,i϶N in a sequence and with the diagonal method we define a new and different such language L_0 of passwords , differing to at least one password from all those $L_{p,i}$, thus this L_0 is undecidable. Still again there is a polynomial-time checkable relation $R(x,p_{|x|})$, which simply is checking if $x= p_{|x|}$,so that for every word x, there is a "certificate", here the password $p_{|x|}$, and x belongs to the language L_0 of passwords iff $R(x,p_{|x|})$ holds.

5. Conclusions

Sometimes great problems have relatively short and elegant solutions provided we find the **key-abstractions** and convenient context , symbols and semantics to solve them. But even relatively simple paths of reasoning, may be difficult to travel, if there is not, at a certain point of them, the necessary "bridge" , that is the necessary key-abstraction or the right conceptual "coins" of symbols and semantics to exchange and convert. Here the key-abstraction was to start from the class EXPTIME and an EXPTIME-complete language of it , witgout specifying which one instead starting from the class NP. If the P versus NP problem is researched without a main strategy, that would require a short proof, it might become a very complex problem to solve. ***My main hidden guiding idea in searching for such a simple proof, was that what the "arbitrary human-like free-will" of a non-deterministic Turing machine as human-machine interactive software (e.g. in password setting), can do in polynomial time cannot be done by a purely mechanical deterministic Turing machine in polynomial time.*** Since in my opinion the Hierarchy Theorem is a deeper result than the P versus NP problem, in principle there should exist a not much more complicated proof of the P versus NP problem, compared to the proof of the Hierarchy Theorem. The proof of the P versus NP problem in the direction P ≠ NP, is supposed also to mean that the standard practice of encryption in the internet , is safe.

References

[1] Conway J.H. *On numbers and games* , Academic press 1976

[2] Cook, Stephen A. (1972). *"A hierarchy for nondeterministic time complexity"*. Proceedings of the fourth annual ACM symposium on Theory of computing. STOC '72. Denver, Colorado, United States: ACM. pp. 187–192

[3] Cook, Stephen ***(April 2000)***, *The P versus NP Problem (PDF), Clay Mathematics Institute site*.

[4] Diduch Rodrigo Gilberto (2012), *P vs NP,* International Journal of Computer Science and Network Security (IJCSNS) Volume 2, pp 165-167.

[5] Gram Seenil 2001 *"Redundancy, Obscurity, Self-Containment & Independence"* by The 3rd International Conference on Information and Communications Security (ICICS 2001) took place in Xian, China, November 13-16, 2001. Proceedings of ICICS as Volume 2229 of Springer Lecture Notes in Computer Science. Pages 495-501.

[6] Harry R. Lewis and Christos H. Papadimitriou *Elements of the Theory of Computation*, Prentice-Hall, Englewood Cliffs, New Jersey, 1981, ISBN 0-13-273417-6.

[7] Hartmanis, J.; Stearns, R. E. ***(1 May 1965)***. *"On the computational complexity of algorithms".* Transactions of the American Mathematical Society*. American Mathematical Society.* ***117****: 285–306. .* ISSN 0002-9947. JSTOR 1994208. MR 0170805.

[8] Kyritisis C. *On the solution of the 3rd Clay Millennium problem.A short and elegant proof that P ≠ NP in the context of deterministic Turing machines and Zermelo-Frankel set theory.* Proceedings of the first ICQSBEI 2017 conference, Athens, Greece, pp 170-181

[9] Luca Trevisan, *Notes on Hierarchy Theorems*, U.C. Berkeley.

[10] John C. Martin *(1997). Introduction to Languages and the Theory of Computation (2nd ed.).* McGraw-Hill. ISBN 0-07-040845-9.

[11] Papadimitriou Christos (1994)*. Computational Complexity.* Addison-Wesley. ISBN 0-201-53082-1.

[12]Rustem Chingizovich Valeyev 2013 *The Lower Border of Complexity of Algorithm of the Elementary NP-Complete Task (The Most Condensed Version)* World Applied Sciences Journal 24 (8): 1072-1083, 2013 ISSN 1818-4952 © IDOSI Publications, 2013.

[13]Stanislav, Žák (October 1983). *"A Turing machine time hierarchy".* Theoretical Computer Science. Elsevier Science B.V. **26** (3): 327–333.

[14]A. A. Tsay, W. S. Lovejoy, David R. Karger, *Random Sampling in Cut, Flow, and Network Design Problems*, Mathematics of Operations Research, 24(2):383–413, 1999.

[15]Ivanov Viktor V. 2014*, A short proof that NP is not P*. International Journal of Pure and Applied Mathematics IJPAM pp 81-88 .

[16] Woeginger GJ (2016) *The P versus NP page* ,https://www.win.tue.nl/~gwoegi/P-versus-NP.htm

[17]Yannakakis M. 1998 *"Expressing combinatorial optimization problems by linear programs"* Proceedings of STOC 1988, pp. 223-228.

PART B THE SOLUTON OF THE 4^{TH} CLAY MILLENNIUM PROBLEM

Prologue.

The standard formulation of the 4^{th} Clay Millennium problem can be found in the site of the Clay Mathematical Institute here http://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation

http://www.claymath.org/sites/default/files/navierstokes.pdf

Roughly speaking it asks if in classical 3 dimensional incompressible fluids , (governed by the Navier-Stokes equations) with finite initial energy and smooth initial conditions (with pressures and velocities falling to zero faster than all polynomial powers as we go to infinite distances away or in short smooth Schwartz initial conditions) the flow will continuous forever smooth or would there be a finite time, where velocities and pressures will blow-up to infinite and smoothness will break? The standard formulation is both with periodic initial conditions or not periodic.

Most of the mathematicians were expecting that, since it has been proved that there is no blow-up in 2-dimensions , this should hold in 3 dimensions too. But as more than half a century has passed with not being able to prove it many researchers sarted believing that because of the vortex stretching which is possible only in 3-dimasions and not in 2-dimensions a blow-up might exist.

Because it was easier to do at the beginning, I spent about half a year discovering more than a dozen of explicitly formulated cases of axial symmetric flows that lead to blow-up in finite time. Nevertheless, for all of them, it was necessary that they start with infinite initial energy.

So I went back to the more probable case that no Blow-up can occur in finite time.

My heuristic analysis which took 1-2 years, with statistical mechanics and classical fluid dynamics in digital differential and integral calculus suggested to me that there should not exist in finite time a blow-up. The naïve and simple argument was that a blow up would give that at least one particle of the fluid (and in statistical mechanics or classical fluid dynamics in digital differential and integral calculus, finite many finite particles do exist) would exhibit infinite kinetic energy. Nevertheless, what is easy to prove in heuristic context is not at all easy to prove in the classical context of fluid dynamics where there are not finite many particles of finite and lower bounded size, but infinite many points with zero size.

In this strategy my interdisciplinary approach was an advantage. I did not considered as consistent for sciences that e.g. statistical mechanics would give that there is no-blow up in finite time, while classical fluid dynamics would prove that there is a blow-up in finite time.

The next table makes the comparisons in statistical mechanics and classical fluid dynamics

Table 0

COMPARISON AND MUTUAL SIGNIFICANCE OF DIFFERENT TYPES OF MATHEMATICAL MODELS FOR THE 4^{TH} CLAY PROBLEM (NO EXTERNAL FORCE)	**CONTINUOUS FLUID MECHANICS MODEL**	**STATISTICAL MECHANICS MODEL**
SMOOTH SCHWARTZ INITIAL CONDITIONS	YES	POSSIBLE TO IMPOSE
FINITE INITIAL ENERGY	YES	YES
CONSERVATION OF THE PARTICES	***YES(NON-OBVIOUS FORMULATION)***	***YES (OBVIOUS FORMULATION)***
LOCAL SMOOTH EVOLUTION IN A INITIAL FINITE TIME INTERVAL	YES	POSSIBLE TO DERIVE
EMERGENCE OF A BLOW-UP IN FINITE TIME	IMPOSSIBLE TO OCCUR	IMPOSSIBLE TO OCCUR

So as it was easy to prove in statistical mechanics that there is no blow-up in finite time, I thought , so as to increase our confidence for the correct side of the solution of the problem , to add hypotheses to the standard formulation of the 4^{th} Clay Millennium problem that correspond to the conservation of particles during the flow, and which would lead to an accessible solution of this problem (that there is no Blow-up in finite) dew to finite initial energy and energy and particle conservation. This of course was not the solution of the 4^{th} Clay Millennium problem, and the solution finally is presented in the 3^{rd} and last paper in this part B of this treatise.

The next 1^{st} paper here is an initial version (uploaded in) of the published paper in the Journal of Scientific Research and Studies Vol. 4(11), pp. 304-317, November, 2017 ISSN 2375-8791 Copyright © 2017

So once my confidence was in strength that the correct solution is that there is no Blow-up in finite time I started attacking the problem for a proof in the classical fluid dynamics only with the hypotheses of the standard formulation of the 4th Clay Millennium problem.

The first thing to do was to get rid of the infinite space in the initial conditions of the fluid, and substitute them with smooth compact support initial conditions. In many books of fluid dynamics where most of the results are stated for smooth Schwartz initial conditions and infinite space , the authors often make arguments that as they say "for simplicity we assume compact support initial conditions" It is therefore a common expectation in fluid dynamics although I found no proof for this anywhere. Happily, a rather recent work by Terence Tao gave to me the idea of how this could be proved by arguments that are used in wavelet theory and in particular here in the theorem 12.2 of Tao's paper (TAO, T. 2013 Localisation and compactness properties of the Navier-Stokes global regularity problem. *Analysis & PDE 6 (2013), 25-107*)

The next 2nd paper is a preprint version (submitted Wednesday, July 26, 2017 - 10:56:46 AM) of the published paper in the. *1st INTERNATIONAL CONFERENCE ON QUANTITATIVE, SOCIAL, BIOMEDICAL AND ECONOMIC ISSUES 2017 – ICQSBEI 2017. http://icqsbei2017.weebly.com/*, Jun 2017, Athens, Stanley Hotel, Greece. 1 (1), pp.146, 2017, PROCEEDINGS OF THE 26 Ιουν 2017 - 1st INTERNATIONAL CONFERENCE ON QUANTITATIVE, SOCIAL, BIOMEDICAL AND ECONOMIC ISSUES 2017 – ICQSBEI 2017.

Having reduced the 4th Clay Millennium problem to an equivalent with the same hypotheses of finite initial energy but on compact support initial conditions too, made all sorts of arguments easier or possible to do.

It was not obvious how the finite initial energy and the energy conservation could be used to prove the non-existence of a Blow-up in finite time. To surround carefully the problem I proved more than 8 different necessary and sufficient conditions of non-existence of a Blow-up in finite time. Finally, it was that the pressures must remain bounded in finite time intervals which proved that there cannot be a Blow-up in finite time. Anthe pressures must remain bounded because of the conservation of energy , the initial finite energy and that pressures as it known define a conservative field of forces in the fluid all the times.

The next 3rd paper which was completed and uploaded in the internet during 25 February 2018 is I believe the final solution of the 4th Clay Millennium problem, and it has been published in the *World Journal of Research and Review* https://www.wjrr.org/ during August 2021.

1st paper

On the solution of the 4th clay millennium problem. Proof of the regularity of the solutions of the Euler and Navier-Stokes equations, based on the conservation of particles as a local structure of the fluid, formulated in the context of continuous fluid mechanics.

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ABSTRACT

As more and more researchers tend to believe that with the hypotheses of the official formulation of the 4th Clay Millennium problem a blowup may occur, a new goal is set: to find the simplest and most physically natural enhancement of the hypotheses in the official formulation so that the regularity can be proved in the case of 3 dimensions too. The position of this paper is that the standard assumptions of the official formulation of the 4th Clay millennium problem, although they reflect, the finiteness and the conservation of momentum and energy and the smoothness of the incompressible physical flows, they do not reflect the conservation of particles as local structure. By formulating the later conservation and adding it to the hypotheses, we prove the regularity (global in time existence and smoothness) both for the Euler and the Navier-Stokes equations.

Key words: *Incompressible flows, regularity, Navier-Stokes equations, 4th Clay millennium problem*

Mathematical Subject Classification: 76A02

1. Introduction

The famous problem of the 4th Clay mathematical Institute as formulated in FEFFERMAN C. L. 2006 , is considered a significant challenge to the science of mathematical physics of fluids, not only because it has withstand the efforts of the scientific community for decades to prove it (or types of converses to it) but also because it is supposed to hide a significant missing perception about the nature of our mathematical formulations of the physical flows through the Euler and the Navier-Stokes equations.

When the 4th Clay Millennium problem was officially formulated the majority was hoping that the regularity was holding also in 3 dimensions as it had been proved to hold also in 2 dimensions. But as time passed more and more mathematicians started believing that a Blowup can occur with the hypotheses of the official formulation. Therefore a new goal is set to find the simplest and most physically natural enhancement of the hypotheses in the official formulation so that the regularity can be proved in the case of 3 dimensions too. This is done by the current paper.

After 3 years of research, in the 4th Clay Millennium problem, the author came to believe that, what most of the mathematicians would want, (and seemingly including the official formulators of the problem too), in other words a proof of the regularity in 3 dimensions as well, cannot be given merely by the assumptions of the official formulation of the problem. In other words a Blow-up may occur even with compact support smooth initial data with finite energy. But solving the 4th Clay Millennium problem, by designing such a case of Blow-up is I think not interesting from the physical point of view, as it is quite away from physical applications and a mathematical pathological curiosity. On the other hand discovering what physical aspect of the flows is not captured by the mathematical hypotheses, is I believe a more significant contribution to the science of mathematical physics in this area. Although the mathematical assumptions of the official formulation reflect, the finiteness and the conservation of momentum and energy and the smoothness of the incompressible physical flows, they do not reflect the conservation of particles as local structure. By adding this physical aspect formulated simply in the context of continuous fluid mechanics, the expected result of regularity can be proved.

In statistical mechanical models of incompressible flow, we have the realistic advantage of finite many particles, e.g. like balls B(x,r) with finite diameter r. These particles as they flow in time, **remain particles of the same nature and size** and the velocities and inside them remain approximately constant.

Because space and time dimensions in classical fluid dynamics goes in orders of smallness , smaller and at least as small as the real physical molecules , atoms and particles of the fluids, this might suggest imposing too, such conditions resembling uniform continuity conditions. In the case of continuous fluid dynamics models such natural conditions, emerging from the particle nature of material fluids, together with the energy conservation, the incompressibility and the momentum conservation, as laws conserved in time, may derive the regularity of the local smooth solutions of the Euler and Navier-Stokes equations. **For every atom or material particle of a material fluid, we may assume around it a ball of fixed radius, called *particle range* depending on the size of the atom or particle, that covers the particle and a little bit of the electromagnetic, gravitational or quantum vacuum field around it, that their velocities and space-time accelerations are affected by the motion of the molecule or particle.** E.g. for the case water, we are speaking here for molecules of H_2O, that are estimated to have a diameter of 2.75 angstroms or 2r= 2.75*10^(-10) meters, we may define as water molecule **particle range** the balls $B(r_0)$ of radius $r_{0=}$3*10^(-10) meters around the water molecule. As the fluid flows, especially in our case here of incompressible fluids, the shape and size of the molecules do not change much, neither there are significant differences of the velocities and space-time accelerations of parts of the molecule. Bounds δ_u δ_ω of such differences remain constant as the fluid flows. **We may call this effect as the *__principle of conservation of particles__* as a local structure.** This principle must be posed in equal setting as the energy conservation and incompressibility together with the Navier-Stokes or Euler equations. Of course if the fluid is say of solar plasma matter, such a description would not apply. Nevertheless then incompressibility is hardly a property of it. But if we are talking about incompressible fluids that the molecule is conserved as well as the atoms and do not change atomic number (as e.g. in fusion or fission) then this principle is physically valid. The principle of conservation of particles as a local structure, blocks the self-similarity effects of concentrating the energy and turbulence in very small areas and creating thus a Blow-up. It is the missing invariant in the discussion of many researchers about superctitical, critical and subcritical invariants in scale transformations of the solutions.

The exact definition of the conservation of particles as local structure Is in DEFINITION 5.1 and it is as follows:

(**__Conservation of particles as local structure in a fluid__**)

*Let a smooth solution of the Euler or Navier-Stokes equations for incompressible fluids, that exists in the time interval [0,T). We may assume initial data on all of R^3 or only on a connected compact support V_0. For simplicity let us concentrate only on the latter simpler case. Let us denote by F the displacement transformation of the flow. Let us also denote by g the partial derivatives of 1^{st} order in space and time , that is $\left|\partial_x^a \partial_t^{\,b} u\ (x)\right|$, $|\alpha|=1$, $|b|<=1$,and call then space-time accelerations . We say that there is **__conservation of the particles__** **__in the interval [0,T) in__** a derivatives homogenous setting, as a local structure of the solution if and only if:*

*There is a small radius r, and small constants δ_x , δ_u , δ_ω , >0 so that for all t in [0,T) there is a finite cover C_t (in the case of initial data on R^3 ' it is infinite cover, but finite on any compact subset) of V_t , from balls B(r) of radius r, called **__ranges of the particles__** , such that:*

1) *For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||F(x_1)-F(x_2)||<=r+ \delta_x$ for all t>=s in [0,T).*
2) *For an x_1 and x_2 in a ball B(r) of V_s,s in [0,T), $||u(F(x_1))-u(F(x_2))||<= \delta_u$ for all t >=s in [0,T).*
3) *For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||g(F(x_1))-g(F(x_2))||<= \delta_\omega$ for all t >=s in [0,T).*
   *If we state the same conditions 1) 2) 3) for all times t in [0,+∞) , then we say that we have the **__strong version__** of the conservation of particles as local structure.*

We prove in paragraph 5 in PROPOSITION 5.2 that indeed adding the above conservation of particles as local structure in the hypotheses of the official formulation of the 4^{th} Clay Millennium problem, we solve it, in the sense of proving the regularity (global in time smoothness) of the locally in time smooth solutions that are known to exist.

A short outline of the logical structure of the paper is the next.

1) The paragraph 3, contains the official formulation of the 4^{th} Cay millennium problem as in FEFFERMAN C. L. 2006. The official formulation is any one of 4 different conjectures, that two of them, assert the existence of blow-up in the periodic and non-periodic case, and two of them the non-existence of blow-up , that is the global in time regularity in the periodic and non-periodic case. We concentrate on to prove the regularity in the non-periodic case or conjecture (A) with is described by equations 1-6 after adding the conservation of particles as a local structure. The paragraph 3 contains definitions, and more modern symbolism introduced by T, Tao in TAO T. 2013. The current paper follows the formal and mathematical austerity standards that the official formulation has set, together with the suggested by the official formulation relevant results in the literature like in the book MAJDA A.J-BERTOZZI A. L. 2002 .
   But we try also not to lose the intuition of the physical interpretation, as we are in the area of mathematical physics rather than pure mathematics.

   The goal is that reader after reading a dozen of mathematical propositions and their proofs , he must be able at the end to have simple physical intuition , why the conjecture (A) of the 4^{th} Clay millennium together with the conservation of particles in the hypotheses problem holds.

2) The paragraph 4 contains some known theorems and results, that are to be used in this paper, so that the reader is not searching them in the literature and can have a direct, at a glance, image of what holds and what is proved. The most important are a list of necessary and sufficient conditions of regularity (PROPOSITIONS 4.5-4.10) The same paragraph contains also some well known and very relevant results that are not used directly but are there for a better understanding of the physics.
3) The paragraph 5 contains the main idea that the conservation of particles during the flow can be approximately formulated in the context of continuous fluid mechanics and that is the key missing concept of conservation that acts as subcritical invariant in other words blocks the self-similar concentration of energy and turbulence that would create a Blowup. With this new invariant we prove the regularity in the case of 3 dimensions: PROPOSITIONS 5.2 .
4) The paragraph 6 contains the idea of defining **a measure of turbulence** in the context of deterministic mechanics based on the **total variation** of the component functions or norms (DEFINITION 6.1) It is also made the significant observation that the smoothness of the solutions of the Euler and Navier-Stokes equations is not a general type of smoothness but one that would deserve the name **homogeneous smoothness** (Remark 6.2) .

According to CONSTANTIN P. 2007 "..The blowup problem for the Euler equations is a major open problem of PDE, theory of far greater physical importance that the blow-up problem of the Navier-Stokes equation, which is of course known to non specialists because of the Clay Millennium problem…"

Almost all of our proved propositions and in particular the regularity in paragraphs 4 , 5 and 6 (in particular PROPOSITION 4.11 and PROPOSITION 5.2) are stated not only for the Navier-Stokes but also for the Euler equations.

2. The ontology of the continuous fluid mechanics models versus the ontology of statistical mechanics models. The main physical idea of the proof of the regularity in 3 spatial dimensions.

All researchers discriminate between the physical reality with its natural physical ontology (e.g. atoms, fluids etc) from the mathematical ontology (e.g. sets, numbers, vector fields etc). If we do not do that much confusion will arise. The main difference of the physical reality ontology, from the mathematical reality ontology, is what the mathematician D. Hilbert had remarked in his writings about the infinite. He remarked that nowhere in the physical reality there is anything infinite, while the mathematical infinite, as formulated in a special axiom of the infinite in G. Cantor's theory of sets, is simply a convenient phenomenological abstraction, at a time that the atomic theory of matter was not well established yet in the mathematical community. In the physical reality ontology, as best captured by statistical mechanics models, the problem of the global 3-dimensional regularity seems easier to solve. For example it is known (See PROPOSITION 4.9 and PROPOSITION 4.12 maximum Cauchy development, and it is referred also in the official formulation of the Clay millennium problem in C. L. FEFFERMAN 2006) that if the global 3D regularity does not hold then the velocities become unbounded or tend in absolute value to infinite as time gets close to the finite Blow-up time. Now we know that a fluid consists from a finite number of atoms and molecules, which also have finite mass and with a lower bound in their size. **If such a phenomenon (Blowup) would occur, it would mean that for at least one particle the kinetic energy, is increasing in an unbounded way.** But from the assumptions (see paragraph 3) the initial energy is finite, so this could never happen. We conclude that the fluid is 3D globally in time regular. Unfortunately such an argument although valid in statistical mechanics models (see also MURIEL A 2000), in not valid in continuous fluid mechanics models, where there are not atoms or particles with lower bound of finite mass, but only points with zero dimension, and only mass density. We must notice also here that this argument is not likely to be successful if the fluid is compressible. In fact it has been proved that a blow-up may occur even with smooth compact support initial data, in the case of compressible fluids. One of the reasons is that if there is not lower bound in the density of the fluid, then even without violating the momentum and energy conservation, a density converging to zero may lead to velocities of some points converging to infinite.

Nevertheless if we formulate in the context of continuous fluid mechanics the conservation of particles as a local structure (DEFINITION 5.1) then we can derive a similar argument (see proof of PROPOSITION 5.1) where **if a Blowup occurs in finite time then the kinetic energy of a finite small ball (called in** DEFINITION 5.1 **particle-range) will become unbounded**, which is again impossible, due to the hypotheses if finite initial energy and energy conservation.

The next table compares the hypotheses and conclusions both in continuous fluid mechanics models and statistical mechanics models of the 4^{th} Clay millennium problem in its officially formulation together with the hypothesis of conservation of particles. It would be paradoxical that we would be able to prove the regularity in statistical mechanics and we would not be able to prove it in continuous fluid mechanics.

Table 1

COMPARISON AND MUTUAL SIGNIFICANCE OF DIFFERENT TYPES OF MATHEMATICAL MODELS FOR THE 4^{TH} CLAY PROBLEM (NO EXTERNAL FORCE)	**CONTINUOUS FLUID MECHANICS MODEL**	**STATISTICAL MECHANICS MODEL**
SMOOTH SCHWARTZ INITIAL CONDITIONS	YES	POSSIBLE TO IMPOSE
FINITE INITIAL ENERGY	YES	YES
CONSERVATION OF THE PARTICES	***YES(NON-OBVIOUS FORMULATION)***	***YES (OBVIOUS FORMULATION)***
LOCAL SMOOTH EVOLUTION IN A INITIAL FINITE TIME INTERVAL	YES	POSSIBLE TO DERIVE
EMERGENCE OF A BLOW-UP IN FINITE TIME	IMPOSSIBLE TO OCCUR	IMPOSSIBLE TO OCCUR

3. The official formulation of the Clay Mathematical Institute 4^{th} Clay millennium conjecture of 3D regularity and some definitions.

In this paragraph we highlight the basic parts of the official formulation of the 4^{th} Clay millennium problem, together with some more modern, since 2006, symbolism, by relevant researchers, like T. Tao.

In this paper I consider the conjecture (A) of C. L. FEFFERMAN 2006 official formulation of the 4^{th} Clay millennium problem , which I indentify throughout the paper as <u>the 4^{th} Clay millennium problem</u>.

The Navier-Stokes equations are given by (by R we denote the field of the real numbers, v>0 is the viscosity coefficient)

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + v\Delta u_i \qquad (x\varepsilon R^3 \text{ , t>=0 , n=3}) \qquad \text{(eq.1)}$$

$$divu = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0$$ ($x \varepsilon R^3$, t>=0 , n=3) (eq.2)

with initial conditions u(x,0)=u^0(x) **x$\varepsilon$$\mathbf{R}^3$**
and u_0 (x) C∞ divergence-free vector field on **$\mathbf{R}^3$**
(eq.3)

$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator .The Euler equations are when ν=0

For physically meaningful solutions we want to make sure that u^0(x) does not grow large as |x|→∞. This is set by defining u^0(x) and called in this paper **Schwartz initial conditions** , in other words

$$\left|\partial_x^a u^0(x)\right| \le C_{a,K}(1+|x|)^{-K}$$ on $\mathbf{R}^3$ for any α and K (eq.4)

(Schwartz used such functions to define the space of Schwartz distributions)

We accept as physical meaningful solutions only if it satisfies

p, u $\in C^\infty(R^3 \times [0,\infty))$ (eq.5)

and

$$\int_{\Re^3} |u(x,t)|^2 dx < C$$ for all t>=0 (**Bounded or finite energy**) (eq.6)

The conjecture (A) of he Clay Millennium problem (case of no external force, but homogeneous and regular velocities) claims that for the Navier-Stokes equations , v>0, n=3 , with divergence free , Schwartz initial velocities , there are for all times t>0 , smooth velocity field and pressure, that are solutions of the Navier-Stokes equations with bounded energy**, in other words satisfying the equations eq.1 , eq.2 , eq. 3, eq.4 , eq.5 eq.6** . It is stated in the same formal formulation of the Clay millennium problem by C. L. Fefferman see C. L. FEFFERMAN 2006 (see page 2nd line 5 from below) that the conjecture (A) has been proved to holds locally. "..if the time internal [0,∞), is replaced by a small time interval [0,T), with T depending on the initial data....". In other words there is ∞>T>0, such that there is continuous and smooth solution u(x,t)$\in C^\infty(R^3 \times [0,T))$. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^∞

Following TAO, T 2013, we define some specific terminology , about the hypotheses of the Clay millennium problem, that will be used in the next.

We must notice that the definitions below can apply also to the case of inviscid flows, satisfying the ***Euler*** *equations.*

DEFINITION 3.1 (Smooth solutions to the Navier-Stokes system). *A smooth set of data* for the Navier-Stokes system up to time T is a triplet (u_0, f, T), where $0 < T < \infty$ is a time, the initial velocity vector field $u_0 : R^3 \to R^3$ and the forcing term $f : [0, T] \times R^3 \to R^3$ are assumed to be smooth on R^3 and $[0, T] \times R^3$ respectively (thus, u_0 is infinitely differentiable in space, and f is infinitely differentiable in space time), and u_0 is furthermore required to be divergence-free:

$\nabla \cdot u_0 = 0$.

If f = 0, we say that the data is *homogeneous.*

In the proofs of the main conjecture we will not consider any external force, thus the data will always be homogeneous. But we will state intermediate propositions with external forcing. Next we are defining simple diffentiability of the data by Sobolev spaces.

DEFINITION 3.2 We define the H^1 norm (or enstrophy norm) H^1 (u_0, f, T) of the data to be the quantity

$H^1 (u_0, f, T) := \|u_0\|_{H^1_X(R^3)} + \|f\|_{L^\infty_t H^1_X(R^3)} < \infty$ and say that (u_0, f, T) *is H^1* if

$H^1 (u_0, f, T) < \infty$.

DEFINITION 3.3 We say that a *smooth set of data* (u_0, f, T) is *Schwartz* if, for all integers α, m, k ≥ 0, one has

$$\sup_{x \in R^3} (1+|x|)^k \left|\nabla_x^a u_0(x)\right| < \infty$$

and $$\sup_{(t,x)\in[0,T]\times R^3} (1+|x|)^k \left|\nabla_x^a \partial_t^m f\ (x)\right| < \infty$$

Thus, for instance, the solution or initial data having Schwartz property implies having the H^1 property.

DEFINITION 3.4 A *smooth solution* to the Navier-Stokes system, or a *smooth solution* for short, is a quintuplet (u, p, u_0 , f, T), where (u_0, f, T) is a *smooth set of data*, and the velocity vector field u : [0, T] × R^3 → R^3 and pressure field p : [0, T]× R^3 → R are smooth functions on [0, T]× R^3 that obey the Navier-Stokes equation (eq. 1) but with external forcing term f,

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i + f_i \quad (x \in R^3,\ t>=0,\ n=3)$$

and also the incompressibility property (eq.2) on all of [0, T] × R^3 , but also the initial

condition u(0, x) = u_0(x) for all x ∈ R^3

DEFINITION 3.5 Similarly, we say that (u, p, u_0, f, T*) is H^1* if the associated data (u_0, f, T) *is H^1* , and in addition one has

$$\|u\|_{L_t^\infty H_X^1([0,T]\times R^3)} + \|u\|_{L_t^2 H_X^2([0,T]\times R^3)} < \infty$$

We say that the solution is *incomplete in [0,T),* if it is defined only in [0,t] for every t<T.

We use here the notation of *mixed norms* (as e.g. in TAO, T 2013). That is if $\|u\|_{H_x^k(\Omega)}$ is the classical Sobolev norm ,of smooth function of a spatial domain Ω, $u:\Omega \to R$, I is a time interval and $\|u\|_{L_t^p(I)}$ is the classical L^p-norm, then the mixed norm is defined by

$$\|u\|_{L_t^p H_x^k(I\times\Omega)} := (\int_I \|u(t)\|_{H_x^k(\Omega)}^p dt)^{1/p}$$

and

$$\|u\|_{L_t^\infty H_x^k(I\times\Omega)} := ess\sup_{t\in I}\|u(t)\|_{H_x^k(\Omega)}$$

Similar instead of the Sobolev norm for other norms of function spaces.

We also denote by $C_x^k(\Omega)$, for any natural number $k \geq 0$, the space of all k times continuously differentiable functions $u:\Omega \to R$, with finite the next norm

$$\|u\|_{C_x^k(\Omega)} := \sum_{j=0}^{k} \|\nabla^j u\|_{L_x^\infty(\Omega)}$$

We use also the next notation for *hybrid norms*. Given two normed spaces X, Y on the same domain (in either space or time), we endow their intersection $X \cap Y$ with the norm

$$\|u\|_{X\cap Y} := \|u\|_X + \|u\|_Y .$$

In particular in the we will use the next notation for intersection functions spaces, and their hybrid norms.

$$X^k(I\times\Omega) := L_t^\infty H_x^k(I\times\Omega) \cap L_x^2 H_x^{k+1}(I\times\Omega) .$$

We also use the *big O notation*, in the standard way, that is X=O(Y) means

$X \le CY$ for some constant C. If the constant C depends on a parameter s, we denote it by C_s and we write $X=O_s(Y)$.

We denote the difference of two sets A, B by A\B. And we denote Euclidean balls

by $B(a,r) := \{x \in R^3 : |x-a| \le r\}$, where |x| is the Euclidean norm.

With the above terminology the target Clay millennium conjecture in this paper can be restated as the next proposition

The 4th Clay millennium problem (Conjecture A)

(**Global regularity for homogeneous Schwartz data**). *Let (*u_0*, 0, T) be a homogeneous Schwartz set of data. Then there exists a smooth finite energy solution (u, p,* u_0*, 0, T) with the indicated data (notice it is for any T>0, thus global in time).*

4. Some known or directly derivable, useful results that will be used.

In this paragraph I state ,some known theorems and results, that are to be used in this paper, so that the reader is not searching them in the literature and can have a direct, at a glance, image of what holds and what is proved.

A review of this paragraph is as follows:

Propositions 4.1, 4.2 are mainly about the uniqueness and existence locally of smooth solutions of the Navier-Stokes and Euler equations with smooth Schwartz initial data. Proposition 4.3 are necessary or sufficient or necessary and sufficient conditions of regularity (global in time smoothness) for the Euler equations without viscosity. Equations 8-15 are forms of the energy conservation and finiteness of the energy loss in viscosity or energy dissipation. Equations 16-18 relate quantities for the conditions of regularity. Proposition 4.4 is the equivalence of smooth Schwartz initial data with smooth compact support initial data for the formulation of the 4^{th} Clay millennium problem. Propositions 4.5-4.9 are necessary and sufficient conditions for regularity, either for the Euler or Navier-Stokes equations, while Propositions 4.10 is a necessary and sufficient condition of regularity for only the Navier-Stokes with non-zero viscidity.

In the next I want to use, the basic local existence and uniqueness of smooth solutions to the Navier-Stokes (and Euler) equations , that is usually referred also as the well posedness, as it corresponds to the existence and uniqueness of the physical reality causality of the flow. The theory of well-posedness for smooth solutions is summarized in an adequate form for this paper by the Theorem 5.4 in TAO, T. 2013.

I give first the definition of **mild solution** as in TAO, T. 2013 page 9. Mild solutions must satisfy a condition on the pressure given by the velocities. Solutions of smooth initial Schwartz data are always mild, but the concept of mild solutions is a generalization to apply for non-fast decaying in space initial data , as the Schwartz data, but for which data we may want also to have local existence and uniqueness of solutions.

DEFINITION 4.1

We define a *H^1 mild solution* (u, p, u_0, f, T) to be fields u, f :[0, T] × R^3 → R^3,

p : :[0, T] × R^3 → R, u_0 : R^3 → R^3, with 0 < T < ∞ , obeying the regularity hypotheses

$$u_0 \in H^1_x(R^3)$$

$$f \in L^\infty_t H^1_x([0,T]\times R^3)$$

$$u \in L^\infty_t H^1_x \cap L^2_t H^2_x([0,T]\times R^3)$$

with the pressure p being given by (Poisson)

$$p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla\cdot f \qquad \text{(eq. 7)}$$

(Here the summation conventions is used , to not write the Greek big Sigma).

which obey the incompressibility conditions (eq. 2), (eq. 3) and satisfy the integral form of the Navier-Stokes equations

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}(-(u\cdot\nabla)u - \nabla p + f)(t')dt'$$

with initial conditions $u(x,0)=u^0(x)$ **.**

We notice that the definition holds also for the in viscid flows, satisfying the Euler equations. The viscosity coefficient here has been normalized to **v=1.**

In reviewing the local well-posedness theory of H^1 mild solutions, the next can be said. The content of the theorem 5.4 in TAO, T. 2013 (that I also state here for the convenience of the reader and from which derive our PROPOSITION 4.2) is largely standard (and in many cases it has been improved by more powerful current well-posedness theory). I mention here for example the relevant research by PRODI G 1959 and SERRIN,J 1963, The local existence theory follows from the work of KATO, T. PONCE, G. 1988 , the regularity of mild solutions follows from the work of LADYZHENSKAYA, O. A. 1967 . There are now a number of advanced local well-posedness results at regularity, especially that of KOCH, H., TATARU, D.2001.

There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions with different methods. As it is referred in C. L. FEFFERMAN 2006 I refer too the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4,

I state here for the convenience of the reader the summarizing theorem 5.4 as in TAO T. 2013. I omit the part (v) of Lipchitz stability of the solutions from the statement of the theorem. I use the standard O() notation here, x=O(y) meaning x<=cy for some absolute constant c. If the constant c depends on a parameter k, we set it as index of $O_k()$.

It is important to remark here that the existence and uniqueness results locally in time (well-posedness) , hold also not only for the case of viscous flows following the Navier-Stokes equations, but also for the case of inviscid flows under the Euler equations. There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions both for the Navier-Stokes and the Euler equation with the same methodology , where the value of the viscosity coefficient v=0, can as well be included. I refer e.g. the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138.

PROPOSITION 4.1 *(Local well-posedness in H^1). Let (u_0, f, T) be H^1 data.*

(i) *(Strong solution) If (u, p, u_0, f, T) is an H^1 mild solution, then*

$$u \in C_t^0 H_x^1([0,T] \times R^3)$$

(ii) (Local existence and regularity) If

$$(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})^4 T < c$$

for a sufficiently small absolute constant c > 0, then there exists

a H^1 mild solution (u, p, u_0, f, T) with the indicated data, with

$$\|u\|_{X^k([0,T]\times R^3)} = O(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})$$

and more generally

$$\|u\|_{X^k([0,T]\times R^3)} = O_k(\|u_0\|_{H_X^k(R^3)}, \|f\|_{L_t^1 H_X^1(R^3)}, 1)$$

for each k>=1 . In particular, one has local existence whenever

T is sufficiently small, depending on the norm H^1(u_0, f, T).

(iii) (Uniqueness) There is at most one H^1 mild solution (u, p, u_0, f, T)

with the indicated data.

*(iv**) (Regularity**) If (u, p, u_0, f, T) is a H^1 mild solution, and (u_0, f, T)*

is (smooth) Schwartz data, then u and p is smooth solution; in fact, one has

$\partial_t^j u, \partial_t^j p \in L_t^\infty H^k([0,T] \times R^3)$ *for all j, K >=0.*

For the proof of the above theorem, the reader is referred to the TAO, T. 2013 theorem 5.4, but also to the papers and books , of the above mentioned other authors**.**

Next I state the local existence and uniqueness of smooth solutions of the Navier-Stokes (and Euler) equations with smooth Schwartz initial conditions , that I will use in this paper , explicitly as a PROPOSITION 4.2 here.

PROPOSITION 4.2 **Local existence and uniqueness of smooth solutions or smooth well posedness**. *Let* $\boldsymbol{u_0(x)}$ *,* $\boldsymbol{p_0(x)}$ *be smooth and Schwartz initial data at t=0 of the Navier-Stokes (or Euler) equations, then there is a finite time interval [0,T] (in general depending on the above initial conditions) so that there is a unique smooth local in time solution of the Navier-Stokes (or Euler) equations*

$$\boldsymbol{u(x)}, \boldsymbol{p(x)} \in C^\infty(R^3 \times [0,T])$$

Proof: We simply apply the PROPOSITION 4.1 above and in particular , from the part (ii) and the assumption in the PROPOSITION 4.2, that the initial data are smooth Schwartz , we get the local existence of H^1 mild solution (u, p, u_0, 0, T). From the part (iv) we get that it is also a smooth solution. From the part (iii), we get that it is unique.

As an alternative we may apply the theorems in MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138, and getthe local in time solution, then derive from the part (iv) of the PROPOSITION 4.1 above, that they are also in the classical sense smooth. QED.

Remark 4.1 We remark here that the property of smooth Schwartz initial data, is not in general conserved in later times than t=0, of the smooth solution in the Navier-Stokes equations, because it is a very strong fast decaying property at spatially infinity. But for lower rank derivatives of the velocities (and vorticity) we have the **(global and) local energy estimate** , and **(global and) local enstrophy estimate** theorems that reduce the decaying of the solutions at later times than t=0, at spatially infinite to the decaying of the initial data at spatially infinite. See e.g. TAO, T. 2013, Theorem 8.2 (Remark 8.7) and Theorem 10.1 (Remark 10.6).

Furthermore in the same paper of formal formulation of the Clay millennium conjecture , L. FEFFERMAN 2006 (see page 3rd line 6 from above), it is stated that the 3D global regularity of such smooth solutions is controlled by the **bounded accumulation in finite time intervals** of the vorticity (Beale-Kato-Majda). I state this also explicitly for the convenience of the reader, for smooth solutions of the Navier-Stokes equations with smooth Schwartz initial conditions, as the PROPOSITION 4.6 **When we say here bounded accumulation** e.g. of the deformations D, **on finite internals**, we mean in the sense e.g. of the proposition 5.1 page 171 in the book MAJDA A.J-BERTOZZI A. L. 2002 , which is a definition designed to control the existence or not of finite blowup times. In other words for any finite time interval

[0, T], there is a constant M such that

$$\int_0^t |D|_{L^\infty}(s)ds <= M$$

I state here for the convenience of the reader, a well known proposition of equivalent necessary and sufficient conditions of existence globally in time of solutions of the Euler equations, as inviscid smooth flows. It is the proposition 5.1 in MAJDA A.J-BERTOZZI A. L. 2002 page 171.

The *stretching* is defined by

$S(x,t) =: D\xi \cdot \xi$ if $\xi \neq 0$ and $S(x,t) =: 0$ if $\xi = 0$ where $\xi =: \dfrac{\omega}{|\omega|}$, ω being the vortcity.

PROPOSITION 4.3 *Equivalent Physical Conditions for Potential Singular Solutions of the Euler equations . The following conditions are equivalent for smooth Schwartz initial data:*

(1) The time interval, [0, T^*) *with T* < ∞ is a maximal interval of smooth* H^s

existence of solutions for the 3D Euler equations.

(2) The vorticity ω accumulates so rapidly in time that

$\int_0^t |\omega|_{L^\infty}(s)ds \to +\infty$ *as t tends to T**

(3) The deformation matrix D accumulates so rapidly in time that

$\int_0^t |D|_{L^\infty}(s)ds \to +\infty$ *as t tends to T**

*(4) The stretching factor S(***x***, t) accumulates so rapidly in time that*

$$\int_0^t [\max_{x \in R^3} S(x,s)]ds \to +\infty$$ *as t tends to T**

The next theorem establishes the equivalence of smooth connected compact support initial data with the smooth Schwartz initial data, for the homogeneous version of the 4th Clay Millennium problem. It can be stated either for local in time smooth solutions or global in time smooth solutions. The advantage assuming connected compact support smooth initial data, is obvious, as this is preserved in time by smooth functions and also integrations are easier when done on compact connected sets.

PROPOSITION 4.4. **(3D global smooth compact support non-homogeneous regularity implies 3D global smooth Schwartz homogeneous regularity)** *If it holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth compact support (connected with smooth boundary) initial data of velocities and pressures (thus finite initial energy) and smooth compact support (the same connected support with smooth boundary) external forcing for all times t>0, exist also globally in time t>0 (are globally regular) then it also holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth Schwartz initial data of velocities and pressures (thus finite initial energy) , exist also globally in time for all t>0 (are regular globally in time).*

(for a proof see KYRITSIS, K. 2017, PROPOSITION 6.4)

Remark 4.2 Finite initial energy and energy conservation equations:

When we want to prove that the smoothness in the local in time solutions of the Euler or Navier-Stokes equations is conserved, and that they can be extended indefinitely in time, we usually apply a "reduction ad absurdum" argument: Let the maximum finite time T* and interval [0,T*) so that the local solution can be extended smooth in it.. Then the time T* will be a blow-up time, and if we manage to extend smoothly the solutions on [0,T*]. Then there is no finite Blow-up time T* and the solutions holds in [0,+∞). Below are listed necessary and sufficient conditions for this extension to be possible. Obviously not smoothness assumption can be made for the time T*, as this is what must be proved. But we still can assume that at T* the energy conservation and momentum conservation will hold even for a singularity at T*, as these are universal laws of nature, and the integrals that calculate them, do not require smooth functions but only integrable functions, that may have points of discontinuity.

A very well known form of the energy conservation equation and accumulative energy dissipation is the next:

$$\frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx + \int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt = \frac{1}{2}\int_{R^3}\|u(x,0\|^2 dx \quad \text{(eq. 8)}$$

where

$$E(0) = \frac{1}{2}\int_{R^3}\|u(x,0)\|^2 dx \quad \text{(eq. 9)}$$

is the initial finite energy

$$E(T) = \frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx \quad \text{(eq. 10)}$$

is the final finite energy

and $\Delta E = \int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt$ (eq. 11)

is the accumulative finite energy dissipation from time 0 to time T , because of viscosity in to internal heat of the fluid. For the Euler equations it is zero. Obviously

ΔE<=E(0)>=E(T) (eq. 12)

The rate of energy dissipation is given by

$$\frac{dE}{dt}(t) = -v\int_{R^3}\|\nabla u\|^2 dx < 0 \quad \text{(eq. 13)}$$

(v, is the viscosity coefficient. See e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 Proposition 1.13, equation (1.80) pp. 28)

At this point we may discuss, that for the smooth local in time solutions of the Euler equations, in other words for flows without viscosity, it is paradoxical from the physical point of view to assume, that the total accumulative in time energy dissipation is zero while the time or space-point density of the energy dissipation (the former is the $\|\nabla u(x,t)\|^2{}_{L_\infty}$), is not zero! It is indeed from the physical meaningful point of view unnatural, as we cannot assume that there is a loss of energy from to viscosity at a point and a gain from “antiviscosity" at another point making the total zero. Neither to assume that the time and point density of energy dissipation is non-zero or even infinite at a space point, at a time, or in general at a set of time and space points of measure zero and zero at all other points, which would still make the total accumulative energy dissipation zero. **The reason is of course that the absence of viscosity, occurs at every point and every time, and not only in an accumulative energy level.** If a physical researcher does not accept such inviscid solutions of the Euler equation as having physical meaning, then for all other solutions that have physical meaning and the $\|\nabla u(x,t)\|^2_{L_\infty}$ is zero (and come so from appropriate initial data), we may apply the PROPOSITION 4.7 below and **deduce directly, that the local in time smooth solutions of the Euler equations, with smooth Schwartz initial data, and finite initial energy, and zero time and space point energy dissipation density due to viscosity, are also regular (global in time smooth).** For such regular inviscid solutions, we may see from the inequality in (eq. 15) below, that the total L2-norm of the vorticity is not increasing by time. We capture this remark in PROPOSITION 4.11 below.

Remark 4.3 The next are 3 very useful inequalities for the unique local in time [0,T], smooth solutions u of the Euler and Navier-Stokes equations with smooth Schwartz initial data and finite initial energy (they hold for more general conditions on initial data, but we will not use that):

By $||.||_m$ we denote the Sobolev norm of order m. So if m=0 itis essentially the L_2-norm. By $||.||_{L\infty}$ we denote the supremum norm, u is the velocity, ω is the vorticity, and cm, c are constants.

1) $$\|u(x,T)\|_m \le \|u(x,0)\|_m \exp\Big(\int_0^T c_m \|\nabla(u(x,t)\|_{L_\infty} dt\Big) \qquad \text{(eq. 14)}$$

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp117, equation (3.79))

2) $$\|\omega(x,t)\|_0 \le \|\omega(x,0)\|_0 \exp\Big(c\int_0^T \|\nabla u(x,t)\|_{L_\infty} dt\Big) \qquad \text{(eq. 15)}$$

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp117, equation (3.80))

3) $$\|\nabla u(x,t)\|_{L_\infty} \le \|\nabla u(x,0)\|_0 \exp\Big(\int_0^t \|\omega(x,s)\|_{L_\infty} ds\Big) \qquad \text{(eq. 16)}$$

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp118, last equation of the proof)

The next are a list of well know necessary and sufficient conditions , for regularity (global in time existence and smoothness) of the solutions of Euler and Navier-Stokes equations, under the standard assumption in the 4^{th} Clay Millennium problem of smooth Schwartz initial data, that after theorem Proposition 4.4 above can be formulated equivalently with smooth compact connected support data. We denote by T* be the maximum Blow-up time (if it exists) that the local solution u(x,t) is smooth in [0,T*).

1) PROPOSITION 4.5 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth Schwartz initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the* ***Sobolev norm || u(x,t)||$_m$*** *, m>=3/2+2 , remains bounded , by the same bound in all of [0,T*), then , there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)*

Remark 4.4 See e.g. . MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, line 10 from below)

2) PROPOSITION 4.6 **(Necessary and sufficient condition for regularity.** Beale-Kato-Majda**)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the* ***vorticity has bounded by M, accumulation*** *in [0,T*]:*

$$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M \qquad \textit{(eq17)}$$

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.5 See e.g. . MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, Theorem 3.6. Also page 171 theorem 5.1 for the case of inviscid flows. . See also LEMARIE-RIEUSSET P.G. 2002 . Conversely if regularity holds, then in any interval from the smoothness in a compact connected set, the vorticity is supremum bounded. The above theorems in the book MAJDA A.J-BERTOZZI A. L. 2002 guarantee that the above conditions extent the local in time solution to global in time , that is to solutions (u, p, u_0, f, T) which is H^1 mild solution, **for any T**. Then applying the part (iv) of the PROPOSITION 4.1 above, we get that this solutions is also smooth in the classical sense, for all T>0, thus globally in time smooth.

3) PROPOSITION 4.7 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the **vorticity is bounded by M, supremum norm** L∞ in [0,T*]:*

$\|\omega(x,t)\|_{L_\infty} \leq M$ *for all t in [0,T*)* *(eq. 18)*

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.6 Obviously if $\|\omega(x,t)\|_{L_\infty} \leq M$, then also the integral exists and is bounded: $\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M_1$ and the previous proposition 4.6 applies. Conversely if regularity holds, then in any interval from smoothness in a compact connected set, the vorticity is supremum bounded.

4) PROPOSITION 4.8 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the space accelerations are bounded by M, in the supremum norm L∞ in [0,T*]:*

$\|\nabla u(x,t)\|_{L_\infty} \leq M$ *for all t in [0,T*)* *(eq. 19)*

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.7 Direct from the inequality (eq.14) and the application of the proposition 4.5. Conversely if regularity holds, then in any finite time interval from smoothness, the accelerations are supremum bounded.

5) PROPOSITION 4.9 **(**FEFFERMAN C. L. 2006. **Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if*

the velocities ||u(x,t)|| do not get unbounded as t->T.*

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞).*

Remark 4.8.This is mentioned in the Official formulation of the 4th Clay Millennium problem FEFFERMAN C. L. 2006 pp.2 , line 1 from below: quote "...For the Navier-Stokes equations (v>0) , if there is a solution with a finite blowup time T, then the velocities $u_i(x,t)$, 1<=i<=3 become unbounded near the blowup time." The converse-negation of this is that if the velocities remain bounded near the T*, then there is no Blowup at T* and the solution is regular or global in time smooth. Conversely of course , if regularity holds, then in any finite time interval, because of the smoothness, the velocities, in a compact set are supremum bounded.

I did not find a dedicated such theorem in the books or papers that I studied, but since prof. C.L Fefferman , who wrote the official formulation of the 4th Clay Millennium problem, was careful to specify that is in the case of non-zero viscosity v>0, and not of the Euler equations as the other conditions, I assume that he is aware of a proof of it.

6) PROPOSITION 4.10. **(Necessary condition for regularity)**
Let us assume that the local solution u(x,t) , t in [0,T) of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), in other words that are regular, then the trajectories-paths length l(a,t) does not get unbounded as*

t->T.*

Proof: Let us assume that the solutions is regular. Then also for all finite time intervals [0,T] , the velocities and the accelerations are bounded in the L_∞ , supremum norm, and this holds along all trajectory-paths too. Then also the length of the trajectories , as they are given by the formula

$$l(a_0,T)=\int_0^T \|u(x(a_0,t)\|dt \qquad \text{(eq. 20)}$$

are also bounded and finite (see e.g. APOSTOL T. 1974 , theorem 6.6 p128 and theorem 6.17 p 135). Thus if at a trajectory the lengths becomes unbounded as t goes to T*, then there is a blow-up. QED.

7) PROPOSITION 4.11.**(Physical meaningful inviscid solutions of the Euler equations are regular)**
Let us consider the local solution u(x,t) , t in [0,T) of the Euler equations with zero viscosity, and with smooth compact connected support initial data. If we conside,r because of zero-viscosity at every space point and at every time, as physical meaningful solutions those that also the time and space points energy dissipation density, due to viscosity, is zero or* $\|\nabla u(x,t)\|^2_{L_\infty}=0$ *, then , they can be extended smooth to all times [0,+∞), in other words they are regular.*

Proof: Direct from the PROPOSITION 4.8. QED.

Remark 4.9.

Similar results about the local smooth solutions, hold also for the non-homogeneous case with external forcing which is nevertheless space-time smooth of bounded accumulation in finite time intervals. Thus an alternative formulation to see that the velocities and their gradient , or in other words up to their 1^{st} derivatives and the external forcing also up to the 1^{st} derivatives , control the global in time existence is the next proposition. See TAO. T. 2013 Corollary 5.8

PROPOSITION 4.12 (**Maximum Cauchy development**)

Let (u_0, f, T) be H^1 data. Then at least one of the following two statements hold:

1) There exists a mild H^1 solution (u, p, u_0, f, T) in [0,T] ,with the given data.

2)There exists a blowup time $0 < T^* < T$ and an incomplete mild H^1 solution

(u, p, u_0, f, T^*) up to time T^* in [0, T^*), defined as complete on every [0,t], $t<T^*$ which blows up in the enstrophy H^1 norm in the sense that

$$\lim_{t\to T^*, t<T^*} \|u(x,t)\|_{H^1_x(R^3)} = +\infty$$

Remark 4.10 The term “almost smooth” is defined in TAO, T. 2013, before Conjecture 1.13. The only thing that almost smooth solutions lack when compared to smooth solutions is a limited amount of time differentiability at the starting time t = 0;

The term *normalized pressure*, refers to the symmetry of the Euler and Navier-Stokes equations to substitute the pressure, with another that differs at, a constant in space but variable in time measureable function. In particular normalized pressure is one that satisfies the (eq. 7) except for a measurable at a, constant in space but variable in time measureable function. It is proved in TAO, T. 2013, at Lemma 4.1, that the pressure is normalizable (exists a normalized pressure) in almost smooth finite energy solutions, for almost all times. The viscosity coefficient in these theorems of the above TAO paper has been normalized to ν=1**.**

5. Conservation of the particles as a local structure of fluids in the context of continuous fluid mechanics. Proof of the regularity for fluids with conservation of particles as a local structure, and the

hypotheses of the official formulation of the 4th Clay millennium problem, for the Euler and Navier-Stokes equations.

Remark 5.1 (**<u>Physical interpretation of the definition 5.1</u>**) The smoothness of the particle-trajectory mapping (or displacement transformation of the points), the smoothness of the velocity field and vorticity field, is a condition that involves statements in the orders of micro scales of the fluid, larger, equal and also by far smaller that the size of material molecules, atoms and particles, from which it consists. This is something that we tend to forget in continuous mechanics, because continuous mechanics was formulated before the discovery of the existence of material atoms. On the other-hand it is traditional to involve the atoms and particles of the fluid, mainly in mathematical models of statistical mechanics. Nevertheless we may formulate properties of material fluids in the context of continuous fluid mechanics, that reflect approximately properties and behavior in the flow of the material atoms. This is in particular the DEFINITION 5.1. **For every atom or material particle of a material fluid, we may assume around it a ball of fixed radius, called *<u>particle range</u>* depending on the size of the atom or particle, that covers the particle and a little bit of the electromagnetic, gravitational or quantum vacuum field around it, that their velocities and space-time accelerations are affected by the motion of the molecule or particle.** E.g. for the case water, we are speaking here for molecules of H_2O, that are estimated to have a diameter of 2.75 angstroms or 2r= 2.75*10^(-10) meters, we may define as water molecule **particle range** the balls $B(r_0)$ of radius $r_{0=}$3*10^(-10) meters around the water molecule. As the fluid flows, especially in our case here of incompressible fluids, the shape and size of the molecules do not change much, neither there are significant differences of the velocities and space-time accelerations of parts of the molecule. Bounds δ_u δ_ω of such differences remain constant as the fluid flows. **We may call this effect as the *<u>principle of conservation of particles</u>* as a local structure.** This principle must be posed in equal setting as the energy conservation and incompressibility together with the Navier-Stokes or Euler equations. Of course if the fluid is say of solar plasma matter, such a description would not apply. Nevertheless then incompressibility is hardly a property of it. But if we are talking about incompressible fluids that the molecule is conserved as well as the atoms and do not change atomic number (as e.g. in fusion or fission) then this principle is physically valid. The principle of conservation of particles as a local structure, blocks the self-similarity effects of concentrating the energy and turbulence in very small areas and creating thus a Blow-up. It is the missing invariant in the discussion of many researchers about superctitical, critical and subcritical invariants in scale transformations of the solutions.

The next DEFINITION 5.1 formulates precisely mathematically this principle for the case of incompressible fluids.

DEFINITION 5.1. (**<u>Conservation of particles as local structure in a fluid</u>**)

Let a smooth solution of the Euler or Navier-Stokes equations for incompressible fluids, that exists in the time interval [0,T). We may assume initial data on all of R^3 or *only on a connected compact support V_0. For simplicity let us concentrate only on the latter simpler case. Let us denote by F the displacement transformation of the flow Let us also denote by g the partial derivatives of 1st order in space and time , that is* $\left|\partial_x^a \partial_t^b u\ (x)\right|$, *|α|=1, |b|<=1,and call then space-time accelerations . We say that there is* ***conservation of the particles*** ***in the interval [0,T) in*** *a derivatives homogenous setting, as a local structure of the solution if and only if:*

There is a small radius r, and small constants δ_x , δ_u , δ_ω , >0 so that for all t in [0,T) there is a finite cover C_t (in the case of initial data on R^3 , it is infinite cover, but finite on any compact subset) of V_t , from balls B(r) of radius r, called ***ranges of the particles*** *, such that:*

4) *For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||F(x_1)-F(x_2)||<=r+ \delta_x$ for all t>=s in [0,T).*
5) *For an x_1 and x_2 in a ball B(r) of V_s,s in [0,T), $||u(F(x_1))-u(F(x_2))||<= \delta_u$ for all t >=s in [0,T).*
6) *For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||g(F(x_1))-g(F(x_2))||<= \delta_\omega$ for all t >=s in [0,T).*
   If we state the same conditions 1) 2) 3) for all times t in [0,+∞) , then we say that we have the ***strong version*** *of the conservation of particles as local structure.*

PROPOSITION 5.1 (**Velocities on trajectories in finite time intervals with finite total variation, and bounded in the supremun norm uniformly in time.)**

Let u_t : V(t) -> R^3 be smooth local in time in [0,T) ,velocity fields solutions of the Navier-Stokes or Euler equations, with compact connected support V(0) initial data, finite initial energy E(0) and conservation of particles in [0,T*) as a local structure . The [0,T*) is the maximal interval that the solutions are smooth. Then for t in [0,T*) and x in V(t), the velocities are uniformly in time bounded in the supremum norm by a bound M independent of time t.*

$\|u(x,t)\|_{L_\infty} = \sup_{x\in V(t)} \|u(x,t)\| \leq M$ *for all t in [0,T*).*

Therefore the velocities on the trajectory paths, in finite time intervals are of bounded variation and the trajectories in finite time interval, have finite length.

1st Proof (Only for the Navier-Stokes Equations): Let us assume, that the velocities are unbounded in the supremum norm, as t converges to T*. Then there is a sequence of times t_n with t_n converging to time T* , and sequence of corresponding points x_n (t_n), for which the norms of the velocities $||u(x_n (t_n), t_n)||$ converge to infinite.

$$Lim_{n\to+\infty} \| u(x(x_n,t_n),t_n) \| = +\infty \qquad \text{(eq.21)}$$

From the hypothesis of the conservation of particles as a local structure of the smooth solution in [0,T*), for every t_n There is a finite cover C_{tn} of particle ranges, of V_{tn} so that x_n (t_n) belongs to one such ball or particle-range $B_n(r)$ and for any other point $y(t_n)$ of $B_n(r)$, it holds that $||u(x_n (t_n), t_n)-u(y(t_n),t_n)||<= \delta_u$. Therefore

$$||u(x_n (t_n), t_n)||- \delta_u <= ||u(y(t_n),t_n)||<= ||u(x_n (t_n), t_n)||+ \delta_u \qquad \text{(eq.22)}$$

for all times t_n in [0,T*) .

By integrating spatially on the ball $B_n(r)$, and taking the limit as n->+∞ we deduce that

$$Lim_{n\to+\infty} \int_{B_n} \|u\| dx = +\infty$$

But this also means as we realize easily, that also

$$Lim_{n\to+\infty} \int_{B_n} \|u\|^2 dx = +\infty \qquad \text{(eq. 23)}$$

Which nevertheless means that the total kinetic energy of this small, but finite and of constant radius, ball, converges to infinite, as t_n converges to T*. This is impossible by the finiteness of the initial energy, and the conservation of energy. Therefore the velocities are bounded uniformly ,in the supremum norm, in the time interval [0,T*).

Therefore the velocities on the trajectory paths, are also bounded in the supremum norm , uniformly in the time interval [0,T*). But this means by PROPOSITION 4.9 that the local smooth solution is regular , and globally in time smooth, which from PROPOSITION 4.8 means that the Jacobian of the 1st order derivatives of the velocities are also bounded in the supremum norm uniformly in time bounded in [0,T*). Which in its turn gives that the velocities are of bounded variation on the trajectory paths (see e.g. APOSTOL T. 1974 , theorem 6.6 p128 and theorem 6.17 p 135) and that the trajectories in have also finite length in [0,T*), because the trajectory length is given by the formula $l(a_0,T)=\int_0^T \|u(x(a_0,t)\|dt$.

QED.

2nd Proof (Both for the Euler and Navier-Stokes equations): Instead of utilizing the condition 2) of the definition 5.1, we may utilize the condition 3). And we start assuming that the Jacobian of the velocities is unbounded in the supremum norm (instead of the velocities), as time goes to the Blow-up time T*. Similarly we conclude that the energy dissipation density at a time on balls that are particle-ranges goes to infinite, giving the same for the total accumulative in time energy dissipation (see (eq. 11), which again is impossible from the finiteness of the initial energy and energy conservation. Then by PROPOSITION 4.8 we conclude that the solution is regular, and thus also that the velocities are bounded in the supremum norm, in all finite time intervals. Again we deduce in the same way, that the total variation of the velocities is finite in finite time intervals and so are the lengths of the trajectories too.

QED.

PROPOSITION 5.2 (**Global regularity as in the 4th Clay Millennium problem).**

Let the Navier-Stokes or Euler equations with smooth compact connected initial data , finite initial energy and conservation of particles as local structure. Then the unique local in time solutions are also regular (are smooth globally in time).

Proof: We apply the PROPSOITION 5.1 above and the necessary and sufficient condition for regularity in PROPSOITION 4.9 (which is only for the Navier-Stokes equations). Furthermore we apply the part of the 2d proof of the PROPOSITION 5.1 , that concludes regularity from PROPSOITION 4.8 which holds for both the Euler and Navier-Stokes equations QED.

6. Bounds of measures of the turbulence from length of the trajectory paths, and the total variation of the velocities, space acceleration and vorticity. The concept of homogeneous smoothness.

Remark 6.1 In the next we define **a measure of the turbulence** of the trajectories, of the velocities, of space-time accelerations and of the vorticity, through the **total variation** of the component functions in finite time intervals. This is in the context of deterministic fluid dynamics and not stochastic fluid dynamics. We remark that in the case of a blowup the measures of turbulence below will become infinite.

DEFINITION 6.1 **(The variation measure of turbulence)**

Let smooth local in time in [0,T] solutions of the Euler or Navier-Stokes equations. The total length L(P) of a trajectory path P, in the time interval [0,T] is defined as **the variation measure of turbulence of the displacements** on the trajectory P, in [0,T].The total variation TV(||u||) of the norm of the velocity ||u|| on the trajectory P in [0,T] is defined as **the variation measure of turbulence of the velocity** on the trajectory P in [0,T]. The total variation TV(g) of the space-accelerations g (as in DEFINITION 5.1) on the trajectory P in [0,T] is defined as **the variation measure of turbulence of the space-time accelerations** on the trajectory P in [0,T]. The total variation TV(||ω||) of the norm of the vorticity ||ω|| on the trajectory P in [0,T] is defined as **the variation measure of turbulence of the vorticity** on the trajectory P in [0,T].

PROPOSITION 6.1 **Conservation in time of the boundedness of the maximum turbulence, that depend only on the initial data and time lapsed.**

Let the Euler or Navier-Stokes equations with smooth compact connected initial data finite initial energy and conservation of the particles as a local structure. Then for all times t, there are bounds $M_1(t)$, $M_2(t)$, $M_3(t)$, so that the maximum turbulence of the trajectory paths , of the velocities and of the space accelerations are bounded respectively by the above universal bounds, that depend only on the initial data and the time lapsed.

Proof: From the PROPOSITIONS 5.1, 5.2 we deduce that the local in time smooth solutions are smooth for all times as they are regular. Then in any time interval [0,T], the solutions are smooth, and thus from the PROPOSITION 4.8, the space acceleration g, are bounded in [0,T], thus also as smooth functions their total variation TV(g) is finite, and bounded. (see e.g. APOSTOL T. 1974 , theorem 6.6 p128 and theorem 6.17 p 135). From the PROPOSITION 4.7, the vorticity is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation TV(||ω||) is finite, and bounded on the trajectories. From the PROPOSITION 4.9, the velocity is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation TV(||u||) is finite, and bounded on the trajectories. From the PROPOSITION 4.10, the motion on trajectories is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation which is the length of the trajectory path L(P) is finite, and bounded in [0,T].In the previous theorems the bounds that we may denote them here by $M_1(t)$, $M_2(t)$, $M_3(t)$, respectively as in the statement of the current theorem, depend on the initial data, and the time interval [0,T].

QED.

Remark 6.2. (Homogeneity of smoothness relative to a property P.) There are many researchers that they consider that the local smooth solutions of the Euler or Navier-Stokes equations with smooth Schwartz initial data and finite initial energy,(even without the hypothesis of conservation of particles as a local structure) are general smooth functions. But it is not so! They are special smooth functions with the remarkable property that there are some critical properties P_i that if such a property holds in the time interval [0,T) for the coordinate partial space-derivatives of 0, 1, or 2 order , then this property holds also for the other two orders of derivatives. In other words if it holds for the 2 order then it holds for the orders 0, 1 in [0,T) . If it holds for the order 1, then it holds for the orders 0, 2 in [0,T]. If it holds for the order 0, them it holds also for the orders 1,2 in [0,T]. This pattern e.g. can be observed for the property P_1 of uniform boundedness in the supremum norm, in the interval [0,T*) in the PROPOSITIONS 4.5-4.10 . But one might to try to prove it also for a second property P_2 which is the **finitness of the total variation** of the coordinates of the partial derivatives, or even other properties P3 like **local in time Lipchitz conditions**. This creates a strong bond or coherence among the derivatives and might be called ***homogeneous smoothness relative to a property P.*** We may also notice that the formulation of the conservation of particles as local structure is in such a way, that as a property, it shows the same pattern of homogeneity of smoothness relative to the property of uniform in time bounds $P_{4\ ,}$1), 2), 3) in the DEFINITION 5.1. It seem to me though that even this strong type of smoothness is not enough to derive the regularity, unless the homogeneity of smoothness is relative to the property P_4, in other words the conservation of particles as a local structure.

7. Epilogue

I believe that the main reasons of the failure so far in proving of the 3D global regularity of incompressible flows, with reasonably smooth initial conditions like smooth Schwartz initial data, and finite initial energy, is hidden in the difference of the physical reality ontology that is closer to the ontology of statistical mechanics models and the ontology of the mathematical models of continuous fluid dynamics.

Although energy and momentum conservation and finiteness of the initial energy are easy to formulate in both types of models, the conservation of particles as type and size is traditionally formulated only in the context of statistical mechanics. By succeeding in formulating approximately in the context of the ontology of continuous fluid mechanics the conservation of particles during the flow, as local structure, we result in being able to prove the regularity in the case of 3 dimensions which is what most mathematicians were hoping that it holds.

References

APOSTOL T. 1974 "Mathematical Analysis" *2nd edition Addison Wesley Publishing Company. 1974*

BEALE, J. T., KATO, T., AND MAJDA, A 1984., "Remarks on the breakdown of smooth solutions for the 3-D Euler equations," *Commun. Math. Phys.* **94**, 61–66, 1984.

BOGOVSKII M. E. 1980, Solution of some vector analysis problems connected with operators div and grad, Trudy *Sem. S. L. Sobolev 80 (1) (1980) 5-40.*

CANNON, J.R., KNIGHTLY, G.H. 1970 : A note on the Cauchy problem for the Navier-Stokes equations. *SIAM J. Appl. Math. 18, 641-644(1970)*

CHORIN A. J. 1968 Numerical solutions of the Navier-Stokes Equations *Mathematics of Computation oct. 1968, vol 22, no 104 pp 745-762*

CONSTANTIN P. 1995, A few results and open problems regarding incompressible fluids, *Notices of the AMS, 42 (1995), 658-663.*

CONSTANTIN P. 2001 , Some open problems and research directions in the mathematical study of fluid dynamics, in Mathematics Unlimited–2001 and Beyond, *Springer Verlag, Berlin, 2001, 353–360.*

CONSTANTIN P. 2007 On the Euler equations of incompressible fluids. *Bulletin of the AMS Vol 44, no 4, October 2007, pp 603-621.*

GATSKI T. B. GROSCH C. E.ROSE M. E.1989 The Numerical Solution of the Navier-Stokes equations for 3-Dimensional, Unsteady, Incompressible Flows by Compact Support Schemes

Journal of computation Physics 82 , 298-329 (1989) .

GERMAIN P. 2008 Strong solutions and weak-strong uniqueness for the

nonhomogeneous Navier-Stokes system *Journal d'Analyse Mathématique January 2008, Volume 105, Issue 1, pp 169-196.*

GIGA Y. 1983 Weak and Strong Solutions of the Navier-Stokes Initial Value *Problem PubL RIMS, Kyoto Univ. 19 (1983), 887-910*

GIGA, Y., INUI, K., MATSUI, S. 1999 : On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data. *Advances in fluid dynamics 4, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, 27-68(1999).*

GIGA, Y., MAHALOV, A., NICOLAENKO, B. 2004 : The Cauchy problem for the NavierStokes equations with spatially almost periodic initial data. *Hokkaido University preprint series in mathematics 683, (2004)*

GIGA, Y., MATSUI, S., SAWADA, O. 2001 : Global existence of two-dimensional NavierStokes flow with nondecaying initial velocity. *J. Math. Fluid Mech. 3, no.3, 302-315(2001)*

GIGA, Y., MIYAKAWA, T.1985 : Solutions in Lr of the Navier-Stokes initial value problem. *Arch. Rational Mech. Anal. 89, no.3, 267-281(1985)*

GIGA Y. SAWADA, O. 2003 On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem, *Nonlinear Analysis and Applications, To V. Lakshmikantham on his 80th Birthday vol. I (2003), 549-562.*

FEFFERMAN C. L. 2006 Existence and smoothness of the Navier-Stokes equation, *Millennium Prize Problems, Clay Math. Inst., Cambridge, MA, 2006, 57-67.*

FRIEDLANDER S., DENIS SERRE 2002 *Handbook of Mathematical Fluid Dynamics Vol 1,2, Gulf Professional Publishing, 2002.*

KATO, J. 2003 : The uniqueness of nondecaying solutions for the Navier-Stokes equations. *Arch. Ration. Mech. Anal. 169, no.2, 159-175(2003)*

KATO, T. 1983 : Strong L p -solutions of the Navier-Stokes equation in R m, with applications to weak solutions. *Math. Z. 187, no.4, 471-480(1984)*

KATO, T., FUJITA, H.1962 : On the nonstationary Navier-Stokes system. *Rend. Sem. Mat. Univ. Padova 32, 243-260(1962).*

KATO, T. PONCE, G. 1988 Commutator estimates and the Euler and Navier-Stokes

equations, *CPAM 41 (1988), 891-907.*

KOCH, H., TATARU, D.2001 Well-posedness for the Navier-Stokes equations, *Adv. Math. 157 (2001), no. 1, 22-35.*

KYRITSIS, K. 2017 "On the solution of the 4th Clay Millennium problem about the Navier-Stokes equations. Equivalence of smooth compact support and smooth Schwartz initial conditions". Proceedings of *the 1st international conference on quantitative, social, biomedical and economic issues ICQSBEI 2017 pp 147-172*

LADYZHENSKAYA, O. A. 1967 Uniqueness and smoothness of generalized solutions of Navier-Stokes equations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 5 (1967), 169-185; English transl., Sem. Math. Steklov Math. Inst., Leningrad 5 (1969), 60-67.*

LADYZHENSKAYA, O. 1969, The Mathematical Theory of Viscous Incompressible Flows (2nd edition), *Gordon and Breach, New York, 1969.*

LEMARIE-RIEUSSET, P.G. 2002 Recent developments in the Navier-Stokes problem. *Chapman & All/CRC 2002*

MAEKAWA, Y. TERASAWA, Y. 2006 **:** The Navier-Stokes equations with initial data in uniformly local Lp spaces. *Differential and integral equations: an international journal for theory & applications (Athens, Ohio : Khayyam Publ. Co.), Vol. 19, No. 4 (2006), p. 369-400*

MAJDA, A.J-BERTOZZI, A. L. 2002 Vorticity and incompressible flow *Cambridge University texts in applied mathematics 2002*

MURIEL, A 2000 An Exact Solution of the 3-D Navier-Stokes Equation

Department of Electrical Engineering Columbia University

And Department of Philosophy Harvard University.

Also A. Muriel and M. Dresden, *Physica D 101, 299, 1997.*

POLYANIN, A. D. ZAITSEV, V.F., MOUSSIAUX 2002. *Handbook of First Order Partial Differential Equations,* London : Taylor&Francis , ISBN0-415-27267-x

POLYANIN, A. D 2002. *Handbook of Linear Partial Differential Equations for Engineers and Scientist ,* Boca Raton : Chapman &Hall /CRC press, ISBN 1-58488-299-9

PRODI, G 1959 Un teorema di unicit_a per le equazioni di Navier-Stokes, *Ann. Mat. Pura Appl. 48 (1959), 173-182.*

RUDIN, W. 1976 "Principles of Mathematical Analysis " *3rd edition McGraw-Hill Inc. 1976*

SARRA, SCOTT 2003 "The method of characteristics with applications to Conservation Laws", *Journal of Online mathematics and its Applications.*

SERRIN,J 1963 The initial value problem for the Navier-Stokes equations, *In Nonlinear Problems (Proc. Sympos., Madison, Wis., pages 69{98. Univ. of Wisconsin Press, Madison, Wis., 1963*.

SAWADA, O.2003 : On time-local solvability of the Navier-Stokes equations in Besov spaces. *Adv. Differential Equations 8 no.4, 385-412(2003)*

SPIVAK, M. 1965 "Calculus on Manifolds" *The Benjamin/Cumming publishing company 1965.*

TAO, T. 2013 Localisation and compactness properties of the Navier-Stokes global regularity problem. *Analysis & PDE 6 (2013), 25-107*

TAO, T. 2014 Finite time blowup for an averaged three-dimensional Navier-Stokes equation. *Cornell University Library arxiv.org*

So once my confidence was in strength that the correct solution is that there is no Blow-up in finite time I started attacking the problem for a proof in the classical fluid dynamics only with the hypotheses of the standard formulation of the 4th Clay Millennium problem.

The first thing to do was to get rid of the infinite space in the initial conditions of the fluid, and substitute them with smooth compact support initial conditions. In many books of fluid dynamics where most of the results are stated for smooth Schwartz initial conditions and infinite space , the authors often make arguments that as they say "for simplicity we assume compact support initial conditions" It is therefore a common expectation in fluid dynamics although I found no proof for this anywhere. Happily, a rather recent work by Terence Tao gave to me the idea of how this could be proved by arguments that are used in wavelet theory and in particular here in the theorem 12.2 of Tao's paper (TAO, T. 2013 Localisation and compactness properties of the Navier-Stokes global regularity problem. *Analysis & PDE 6 (2013), 25-107*)

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2nd paper

ON THE SOLUTION OF THE 4TH CLAY MILLENNIUM PROBLEM ABOUT THE NAVIER-STOKES EQUATIONS. EQUIVALENCE OF SMOOTH COMPACT SUPPORT AND SMOOTH SCHWARTZ, INITIAL CONDITIONS.

Constantine E. Kyritsis*

Abstract

In this paper I prove, using some relatively recent ideas suggested by T. Tao , that the Schwartz initial conditions of the its official formulation of the problem in the direction of regularity are equivalent to the simpler compact support initial conditions. I prove also, using the Helmholtz-Hodge orthogonal decomposition of vector fields, a powerful fundamental decomposition of the Euler and Navier-Stokes equations which is significant for the internal symmetries of the equations.

Key words: *Incompressible flows*, *regularity, Navier-Stokes equations, 4th Clay millennium problem*

Mathematical Subject Classification: 76A02

6. Introduction

A short outline of the logical structure of the paper is the next.

4) The paragraph 3, contains the official formulation of the 4^{th} Cay millennium problem as in FEFFERMAN C. L. 2006. The official formulation is any one of 4 different conjectures, that two of them, assert the existence of blow-up in the periodic and non-periodic case, and two of them the non-existence of blow-up , that is the global in time regularity in the periodic and non-periodic case. We concentrate on the regularity in the non-periodic case or conjecture (A) with is described by equations 1-6. The paragraph 3 contains definitions like that of deformations, "shape" of a flow (see DEFINITION 3.6) , and more modern symbolism introduced by T, Tao in TAO T. 2013. The current paper follows the formal and mathematical austerity standards that the official formulation has set, together with the suggested by the official formulation relevant results in the literature like in the book MAJDA A.J-BERTOZZI A.

L. 2002 . But we try also not to lose the intuition of the physical interpretation, as we are in the area of mathematical physics rather than pure mathematics. The goal is that reader after reading a dozen of mathematical propositions and their proofs , he must be able at the end to have simple physical intuition , why the proposition holds.

5) The paragraph 4 contains some known theorems and results, that are to be used in this paper, so that the reader is not searching them in the literature and can have a direct, at a glance, image of what holds and what is proved. The same paragraph contains also some well known and very relevant results that

are not used directly but are there for a better understanding of the physics.

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There are also some directly derivable results that will be used.

6) The paragraph 5 contains the fundamental decomposition of the Euler-Navier-stokes equations to a symmetric (potential flow) and anti-symmetric set of equations. (THEOREM 5.1)
7) The paragraph 6 contains:

The proof (following similar proofs as in TAO T. 2013) that also smooth compact support initial data in the official formulation are equivalent with smooth Schwartz initial data (THEOREM 6.4).

In particular the significant results in this paper are the next.

8) For every flow-solution of the Navier-Stokes equations (viscous case) or Euler equations (inviscid case), with smooth compact support initial data and hypotheses that are covered by the hypotheses of the official formulation of the 4^{th} Clay millennium problem, I prove that it holds a fundamental decomposition of the (Euler or) Navier-Stokes equations THEOREM 5.1. It is proved by applying the Helmholtz-Hodge decomposition both on velocities and accelerations-forces . But unlike other known decompositions so far, we do not only decompose the solution but also the equations.

9) That the 3D global in time regularity of flow-solutions with smooth compact support initial data implies the 3D global in time, regularity of flow-solutions with smooth Schwartz initial data (THEOREM 6.4). Therefore the initial smooth Schwartz conditions of the official formulation of the 4th Clay millennium problem are equivalent with the more restrictive smooth compact support initial conditions.

In particular the 2) or theorem of the fundamental decomposition of the Euler and Navier-Stokes equations (THEOREM 5.1), is a significant internal symmetry of the above equations with meaningful physical interpretation, which may be used to design new simpler and more efficient methods of solving numerically, the Euler and Navier-Stokes equations. But this is not the subject of the current paper.

The famous problem of the 4th Clay mathematical Institute as formulated in FEFFERMAN C. L. 2006 , is considered a significant challenge to the science of mathematical physics of fluids, not only because it has withstand the efforts of the scientific community for decades to prove it (or types of converses to it) but also because it is supposed to hide a significant missing perception about the nature of the Navier-Stokes equations.

7. The discrimination between the physical reality ontology and mathematical physics models ontology. A physical reality hint for the global in time ,3D regularity of flows (Conjecture (A) in the 4th Clay millennium problem).

We must discriminate between the physical reality with its natural physical ontology (e.g. atoms, fluids etc) from the mathematical ontology (e.g. sets, numbers, vector fields etc). If we do not do that much confusion will arise. The main difference of the physical reality ontology, from the mathematical reality ontology, is what the mathematician D. Hilbert had remarked in his writings about the infinite. He remarked that nowhere in the physical reality there is anything infinite, while the mathematical infinite, as formulated in a special axiom of the infinite in G. Cantor's theory of sets, is simply a convenient phenomenological abstraction, at a time that the atomic theory of matter was not well established yet in the mathematical community.

In the physical reality ontology, the problem of the global 3-dimensional regularity seems easier to solve . For example it is known (See maximum Cauchy development PROPOSITION 4.9 , and it is referred also in the official formulation of the Clay millennium problem in C. L. FEFFERMAN 2006) that if the global 3D regularity does not hold then the velocities become unbounded or tend in absolute value to infinite as time gets close to the finite Blow-up time. Now knowing that a fluid consists from a finite number of atoms and molecules , which also have finite mass and with a lower bound in their size. If such a phenomenon would occur, it would mean that their kinetic energy, is increasing in an unbounded way. But from the assumptions (see paragraph 3) the initial energy is finite, so this could never happen. We conclude that the fluid is 3D globally in time regular. Unfortunately such an argument although valid from the ontology of the physical reality, or within the context of statistical mechanics of fluids (see also MURIEL A 2000) , but not within the context classical fluid dynamics, is not of direct use for the mathematical ontology of the fluids, where there are not atoms with lower bound of finite mass, but only points with zero dimension, and only mass density. Besides in the case of compressible flows, it has been proved for the formulation of classical fluid dynamics, that a blow-up may occur. But in the case of incompressible blows is different as the desnity of the fluid remains constant.

The next table compares the classical fluid mechanics model where the 4th Clay millennium problem is officially formulated and a more realistic statistical mechanics virtual formulation. A solution in the context of statistical mechanics is obviously not accepted as a solution of the 4th Clay millennium problem because part of the problem is not the physical interpretation but also to deal with the difficulties of the mathematics of the classical fluid dynamics. We notice that the ability to solve it is totally different. But if we accept that it is possible to have somehow "equivalent" in some sense formulations of the problem, then a solution to any of the two models should imply the solution to the other model.

Table 1

COMPARISON AND MUTUAL SIGNIFICANCE OF DIFFERENT TYPES OF MATHEMATICAL MODEL THE 4TH CLAY PROBLEM (NO EXTERNAL FORCE)	**CLASSICAL FLUID DYNAMICS MODEL**	**STATISTICAL MECHANICS MODEL**
SMOOTH SCHWARTZ INITIAL CONDITIONS	YES	POSSIBLE TO IMPOSE
FINITE INITIAL ENERGY	YES	YES
SMOOTH EVOLUTION IN A INITIAL FINITE TIME INTERVAL	YES	POSSIBLE TO DERIVE
EMERGENCE OF A BLOW-UP	?	IMPOSSIBLE TO OCCUR

8. The standard formal assumptions of the Clay Mathematical Institute 4th millennium conjecture of 3D regularity of flows and some definitions.

In this paper we will be concerned with the conjecture (A) of the official formulation of the 4th Clay millennium problem in C. L. FEFFERMAN 2006 . The Navier-Stokes equations are given by (by R we denote the field of the real numbers, ν>0 is the viscosity coefficient)

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i \qquad (x \varepsilon R^3 ,\ t>=0 ,\ n=3) \qquad \text{(eq.1)}$$

$$divu = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x\varepsilon R^3 \text{ , t>=0 , n=3}) \qquad \text{(eq.2)}$$

with initial conditions u(x,0)=u^0(x) **x$\varepsilon$$\mathbf{R}^3$**
and u_0 (x) belongs to C∞ ,a divergence-free vector field on $\mathbf{R}^3$ (eq.3)

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$ is the Laplacian operator .The Euler equations are when ν=0

For physically meaningful solutions we want to make sure that u^0(x) does not grow fast large as |x|→∞. This is set by defining u^0(x) and called in this paper **Schwartz initial conditions** , in other words

$$\left|\partial_x^a u^0(x)\right| \le C_{a,K}(1+|x|)^{-K}$$ on $\mathbf{R}^3$ for any α and K (eq.4)

(Schwartz used such functions to define the space of Schwartz distributions)

We accept as physical meaningful solutions only if it satisfies

p, u $\in C^{\infty}(R^3 \times [0,\infty))$ (eq.5)

and

$$\int_{\Re^3} |u(x,t)|^2 \, dx < C$$ for all t>=0 (**Bounded or finite energy**) (eq.6)

The conjecture (A) of he Clay Millennium problem claims that for the Navier-Stokes equations , v>0, n=3 , with divergence free , Schwartz initial velocities as above , **there are for all times t>0 , smooth velocity field and pressure, that are solutions of the Navier-Stokes equations with bounded energy, in other words satisfying the equations eq.1 , eq.2 , eq. 3, eq.4 , eq.5 eq.6 .** This is what is considered as a solution of the 4th Clay millennium problem in the direction of the global in time regularity. **Global in time regularity** here means that the solution exists and is smooth in all the derivatives , that is , it satisfies **eq.1 , eq.2 , eq. 3, eq.4 , eq.5 eq.6 and is defined for all t>0. If the viscosity coefficient v=0 ,is zero** this a corresponding **conjecture for the Euler equations**, which is not included though in the official formulation of the 4th Clay millennium problem. It is stated in the same formal formulation of the Clay millennium problem by C. L. Fefferman see C. L. FEFFERMAN 2006 (see page 2nd line 5 from below) that the conjecture (A) has been proved to holds locally. "..if the time internal [0,∞), is replaced by a small time interval [0,T), with T depending on the initial data....". In other words there is $\infty>T>0$, such that there is continuous and smooth solution $u(x,t)\in C^{\infty}(R^3\times[0,T))$. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^{∞}

It is stated in the official formulation FEFFERMAN 2006 (see page 2nd line 16 from below) that

"....These problems are also open and very important for the Euler equations (ν= 0),

although the Euler equation is not on the Clay Institute's list of prize problems...."

Following TAO, T 2013, we define some specific terminology , about the hypotheses of the Clay millennium problem, that will be used in the next.

*We must notice that the definitions below can apply also to the case of inviscid flows, satisfying the **Euler** equations.*

DEFINITION 3.1 (Smooth solutions to the Navier-Stokes system). *A smooth set of data* for the Navier-Stokes system up to time T is a triplet (u_0, f, T), where $0 < T < \infty$ is a time, the initial velocity vector field $u_0 : R^3 \to R^3$ and the forcing term $f : [0, T] \times R^3 \to R^3$ are assumed to be smooth on R^3 and $[0, T] \times R^3$ respectively (thus, u_0 is infinitely differentiable in space, and f is infinitely differentiable in space time), and u_0 is furthermore required to be divergence-free:

$\nabla \cdot u_0 = 0$.

If f = 0, we say that the data is *homogeneous.*

In the proofs of the main conjecture we will not consider any external force, thus the data will always be homogeneous. But we will state intermediate propositions with external forcing. Next we are defining simple diffentiability of the data by Sobolev spaces.

DEFINITION 3.2 We define the H^1 norm (or enstrophy norm) H^1 (u_0, f, T) of the data to be the quantity

$H^1 (u_0, f, T) := \|u_0\|_{H^1_X(R^3)} + \|f\|_{L^\infty_t H^1_X(R^3)} < \infty$ and say that (u_0, f, T) *is H^1* if

$H^1 (u_0, f, T) < \infty$.

DEFINITION 3.3 We say that a *smooth set of data* (u_0, f, T) is *Schwartz* if, for all integers α, m, k ≥ 0, one has

$$\sup_{x \in R^3} (1+|x|)^k \left|\nabla_x^a u_0(x)\right| < \infty$$

and $\sup_{(t,x)\in[0,T]\times R^3} (1+|x|)^k \left|\nabla_x^a \partial_t^m f\ (x)\right| < \infty$

Thus, for instance, the solution or initial data having Schwartz property implies having the H^1 property.

DEFINITION 3.4 A *smooth solution* to the Navier-Stokes system, or a *smooth solution* for short, is a quintuplet (u, p, u_0 , f, T), where (u_0, f, T) is a *smooth set of data*, and the velocity vector field u : [0, T] × R^3 → R^3 and pressure field p : [0, T]× R^3 → R are smooth functions on [0, T]× R^3 that obey the Navier-Stokes equation (eq. 1) but with external forcing term f,

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i + f_i \quad (x\epsilon R^3 \text{ , } t>=0 \text{ , } n=3)$$

and also the incompressibility property (eq.2) on all of [0, T] × R^3 , but also the initial

condition u(0, x) = u_0(x) for all x ∈ R^3

DEFINITION 3.5 Similarly, we say that (u, p, u_0, f, T*) is H^1* if the associated data (u_0, f, T) *is H^1* , and in addition one has

$$\|u\|_{L_t^\infty H_X^1([0,T]\times R^3)} + \|u\|_{L_t^2 H_X^2([0,T]\times R^3)} < \infty$$

We say that the solution is *incomplete in [0,T),* if it is defined only in [0,t] for every t<T.

We use here the notation of *mixed norms* (as e.g. in TAO, T 2013). That is if $\|u\|_{H_x^k(\Omega)}$ is the classical Sobolev norm ,of smooth function of a spatial domain Ω, $u:\Omega \to R$, I is a time interval and $\|u\|_{L_t^p(I)}$ is the classical L^p-norm, then the mixed norm is defined by

$$\|u\|_{L_t^p H_x^k(I\times\Omega)} := \left(\int_I \|u(t)\|_{H_x^k(\Omega)}^p dt\right)^{1/p}$$

and

$$\|u\|_{L_t^\infty H_x^k(I\times\Omega)} := ess\sup_{t\in I}\|u(t)\|_{H_x^k(\Omega)}$$

Similar instead of the Sobolev norm for other norms of function spaces.

We also denote by $C_x^k(\Omega)$, for any natural number $k \geq 0$, the space of all k times continuously differentiable functions $u:\Omega \to R$, with finite the next norm

$$\|u\|_{C_x^k(\Omega)} := \sum_{j=0}^{k}\|\nabla^j u\|_{L_x^\infty(\Omega)}$$

We use also the next notation for *hybrid norms*. Given two normed spaces X, Y on the same domain (in either space or time), we endow their intersection $X \cap Y$ with the norm

$$\|u\|_{X\cap Y} := \|u\|_X + \|u\|_Y .$$

In particular in the we will use the next notation for intersection functions spaces, and their hybrid norms.

$$X^k(I\times\Omega) := L_t^\infty H_x^k(I\times\Omega) \cap L_x^2 H_x^{k+1}(I\times\Omega) .$$

We also use the *big O notation*, in the standard way, that is X=O(Y) means

$X \leq CY$ for some constant C. If the constant C depends on a parameter s, we denote it by C_s and we write $X=O_s(Y)$.

We denote the difference of two sets A, B by A\B. And we denote Euclidean balls

by $B(a,r) := \{x \in R^3 : |x-a| \leq r\}$, where |x| is the Euclidean norm.

With the above terminology the 4th Clay millennium conjecture in the direction of regularity of interest in this paper can be restated as the next proposition

DEFINITION 3.6 Let a solution in a time interval [0,T) to the Navier-Stokes or Euler equations as above in eq1-eq6 with smooth Schwartz initial data. It is known that local in time solutions exist. Let us denote the Jacobian matrix of the velocities by V, and the symmetric matrix $D = \frac{1}{2}(V + V^t)$ which is standard in the literature to call the ***matrix of deformations*** of the flow. The space-time evolution of the deformations as a time-varying matrix field, we call also **"the shape"** of the flow.

With the above terminology the relevant **4th Clay millennium conjecture** in this paper can be restated as the next proposition

The 4th Clay millennium problem (Conjecture A)

(**Global in time regularity for homogeneous Schwartz data**). *Let (u_0, 0, T) be a homogeneous Schwartz set of data. Then there exists a smooth finite energy solution (u, p, u_0, 0, T) with the indicated data (notice it is for any T>0, thus* ***global in time regular****).*

9. Some known and useful results that will be used.

In this paragraph I state ,some known theorems and results, that are to be used in this paper, so that the reader is not searching them in the literature and can have a direct, at a glance, image of what holds and what is proved. This paragraph contains also some well known and very relevant results that are not used directly (like PROPOSITION 4.3, 4.4, 4.8) but are there for a better understanding of the physics. There are also directly derivable results (like PROPOSITION 4.7, 4.10,4.11) that will be used.

There are various forms of Helmholtz decomposition, that can be found e.g. in many books, but also in the book MAJDA A.J-BERTOZZI A. L. 2002, as the more modern form of Hodge decomposition in COROLLARY 5.1.16 page 32, and also in proposition 2.16 page 71. The two classical versions of the Helmholtz decomposition (before Hodge) state the next

PROPOSITION 4.1 **The Helmholtz fundamental theorem and Helmholtz decompositions of vector fields.**

Let F be a vector field on a $\boldsymbol{R^3}$ *, which is twice continuously differentiable, and F vanishes faster than 1/r as* $r \to \infty$ *. Then F can be decomposed into a curl-free component and a divergence-free component:*

$$F = -\nabla\Phi + \nabla \times A \qquad \text{(eq.7)}$$

where Φ is a scalar field and A is a vector field given by the well known appropriate singular integral formulae of F on all of R^3

2nd Helmholtz decomposition*. A second type of converse of the Helmholtz decomposition (2nd Helmhotz decompositio.) is the next. Let C be a solenoidal vector field (in other words with zero divergence) and d a scalar field on* R^3 *,which are sufficiently smooth and which vanish faster than* $1/r^2$ *at infinity. Then there exists a vector field F such that*

$$\nabla F = d \qquad \text{(eq.8)}$$

and

$\nabla \times F = C$ (eq.9)

If additionally the vector field F vanishes as r → ∞, then F is unique.

I state also two more theorems from MAJDA A.J-BERTOZZI A. L. 2002 Proposition 1.16 page 32 and proposition 2.16 page 71,that reflect the more modern form of the Helmholtz decomposition as the Hodge decomposition.

PROPOSITION 4.2 ***Hodge's Decomposition in* R*ⁿ*** . *Every vector field*

$u \in L^2(R^n) \cap C^\infty(R^n)$

has a unique orthogonal in L^2 decomposition:

u = w +∇q, div *w* = 0, (eq. 10)

and furthermore with the following properties:

(i) $w, \nabla q \in L^2(R^n) \cap C^\infty(R^n)$, *q a scalar field,*

(ii) w ⊥ ∇q in L^2, i.e., $(w,\nabla q)_L^2 = 0$, (eq. 11)

(iii) for any multi-index β of the derivates D^β , |β| ≥ 0,

$$\left\|D^\beta u\right\|_{L^2}^2 = \left\|D^\beta w\right\|_{L^2}^2 + \left\|\nabla D^\beta q\right\|_{L^2}^2$$ (eq. 12)

.

The next proposition reflects the second or inverse form of the Helmholtz theorems in PROPOSITION 4.1, but in the more modern form as 2nd Hodge decomposition theorem

PROPOSITION 4.3 ***2nd Hodge decomposition in* R³**. *Let* $\omega \in L^2(R^3) \cap C^\infty(R^3)$ *be a smooth vector field in* R³ *,vanishing sufficiently rapidly as* $|x| \to \infty$.

And to determine a velocity u in terms of the ω as its vorticity, we solve the overdetermined (four equations for the three unknown functions u_1 , u_2, u_3) elliptic system

curl u = ω,

div u = 0, *u, a vector field of* R³ . (eq. 13)

The solution to this problem is summarized in the following statements.

(i) Eqs. (13) have a smooth solution u that vanishes as $|x| \to \infty$ *if and only if*

div $\omega = 0$; (eq. 14)

(ii) if div $\omega = 0$*, then the solution u is determined constructively by*

$u = -\mathrm{curl}\psi$, (eq. 15)

where the vector-stream function ψ *solves the Poisson equation*

$\Delta\psi = \omega$. (eq. 16)

The explicit formula for u is a well known kernel integral.

The Hodge decomposition is based on the next proposition-criterion of L^2 -orthogonal vector fields , that we will also utilize, later.
So I state it here for the convenience of the reader. It is the Lemma 1.5 in MAJDA A.J-BERTOZZI A. L. 2002.

PROPOSITION 4.4 *Let w be a smooth, divergence-free vector field in* R^n *and let q be a smooth scalar field of* R^n *such that*

$|w| \cdot |q| = O[(|x|^{1-n})]$ as $x \to +\infty$ (eq. 17)

Then w and ∇q *are* L^2 *orthogonal:*

$$\int_{R^n} (w \cdot \nabla q)dx = 0$$

In the next I want to use, the basic local existence and uniqueness of smooth solutions to the Navier-Stokes (and Euler) equations , that is usually referred also as the well posedness, as it corresponds to the existence and uniqueness of the physical reality causality of the flow. The theory of well-posedness for smooth solutions is summarized in an adequate form for this paper by the Theorem 5.4 in TAO, T. 2013.

I give first the definition of **mild solution** as in TAO, T. 2013 page 9. Mild solutions must satisfy a condition on the pressure given by the velocities. Solutions of smooth initial Schwartz data are always mild, but the concept of mild solutions is a generalization to apply for non-fast decaying in space initial data , as the Schwartz data, but for which data we may want also to have local existence and uniqueness of solutions.

DEFINITION 4.1

We define a *H^1 mild solution* (u, p, u_0, f, T) to be fields u, f :[0, T] × R^3 → R^3,

p : :[0, T] × R^3 → R, u_0 : R^3 → R^3, with 0 < T < ∞ , obeying the regularity hypotheses

$u_0 \in H_x^1(R^3)$

$f \in L_t^\infty H_x^1([0,T]\times R^3)$

$u \in L_t^\infty H_x^1 \cap L_t^2 H_x^2([0,T]\times R^3)$

with the pressure p being given by (Poisson)

$$p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla\cdot f \qquad \text{(eq. 18)}$$

(Here the summation conventions is used , to not write the Greek big Sigma).

which obey the incompressibility conditions (eq. 2), (eq. 3) and satisfy the integral form of the Navier-Stokes equations

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}(-(u\cdot\nabla)u - \nabla p + f)(t')dt'$$

with initial conditions $u(x,0)=u^0(x)$ **.**

We notice that the definition holds also for the in viscid flows, satisfying the Euler equations. The viscosity coefficient here has been normalized to $v=1$**.**

In reviewing the local well-posedness theory of H^1 mild solutions, the next can be said. The content of the theorem 5.4 in TAO, T. 2013 (that I also state here for the convenience of the reader and from which derive our PROPOSITION 4.6) is largely standard (and in many cases it has been improved by more powerful current well-posedness theory). I mention here for example the relevant research by PRODI G 1959 and SERRIN,J 1963, The local existence theory follows from the work of KATO, T. PONCE, G. 1988 , the regularity of mild solutions follows from the work of LADYZHENSKAYA, O. A. 1967 . There are now a number of advanced local well-posedness results at regularity, especially that of KOCH, H., TATARU, D.2001 .

There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions with different methods. As it is referred in C. L. FEFFERMAN 2006 I refer too the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4,

I state here for the convenience of the reader the summarizing theorem 5.4 as in TAO T. 2013. I omit the part (v) of Lipschitz stability of the solutions from the statement of the theorem. I use the standard O() notation here, x=O(y) meaning x<=cy for some absolute constant c. If the constant c depends on a parameter k, we set it as index of $O_k()$.

It is important to remark here that the existence and uniqueness results locally in time (well-posedness) , hold also not only for the case of viscous flows following the Navier-Stokes equations, but also for the case of inviscid flows under the Euler equations. There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions both for the Navier-Stokes and the Euler equation with the same methodology , where the value of the viscosity coefficient v=0, can as well be included. I refer e.g. the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138.

PROPOSITION 4.5 *(Local well-posedness in H^1). Let (u_0, f, T) be H^1 data.*

13) *(Strong solution) If (u, p, u_0, f, T) is an H^1 mild solution, then*

$$u \in C_t^0 H_x^1([0,T]\times R^3)$$

14) *(Local existence and regularity) If*

$$(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})^4 T < c$$

for a sufficiently small absolute constant c > 0, then there exists

a H^1 mild solution (u, p, u_0, f, T) with the indicated data, with

$$\|u\|_{X^k([0,T]\times R^3)} = O(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})$$

and more generally

$$\|u\|_{X^k([0,T]\times R^3)} = O_k(\|u_0\|_{H_X^k(R^3)}, \|f\|_{L_t^1 H_X^1(R^3)}, 1)$$

for each k>=1 . In particular, one has local existence whenever

T is sufficiently small, depending on the norm $H^1(u_0, f, T)$.

(iii) (Uniqueness) There is at most one H^1 mild solution (u, p, u_0, f, T)

with the indicated data.

*(iv) **(Regularity)** If (u, p, u_0, f, T) is a H^1 mild solution, and (u_0, f, T) is (smooth) Schwartz data, then u and p is smooth solution; in fact, one has*

$\partial_t^j u, \partial_t^j p \in L_t^\infty H^k([0,T]\times R^3)$ *for all j, K >=0.*

For the proof of the above theorem, the reader is referred to the TAO, T. 2013 theorem 5.4, but also to the papers and books , of the above mentioned other authors**.**

Next I state the local existence and uniqueness of smooth solutions of the Navier-Stokes (and Euler) equations with smooth Schwartz initial conditions , that I will use in this paper , explicitly as a PROPOSITION 4.6 here.

PROPOSITION 4.6 **Local existence and uniqueness of smooth solutions or smooth well posedness** *. Let **$u_0(x)$** , **$p_0(x)$** be smooth and Schwartz initial data at t=0 of the Navier-Stokes (or Euler) equations, then there is a finite time interval [0,T] (in general depending on the above initial conditions) so that there is a unique smooth local in time solution of the Navier-Stokes (or Euler) equations*

***u(x)** , **p(x)** $\in C^\infty(R^3 \times [0,T])$*

Proof: We simply apply the PROPOSITION 4.5 above and in particular , from the part (ii) and the assumption in the PROPOSITION 4.6, that the initial data are smooth Schwartz , we get the local existence of H^1 mild solution (u, p, u_0, 0, T). From the part (iv) we get that it is also a smooth solution. From the part (iii), we get that it is unique.

As an alternative we may apply the theorems in MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138, and get the local in time solution, then derive from the part (iv) of the PROPOSITION 4.5 above, that they are also in the classical sense smooth. QED.

We remark here that the property of smooth Schwartz initial data, is not in general conserved in later times than t=0, of the smooth solution in the Navier-Stokes equations, because it is a very strong fast decaying property at spatially infinity. But for lower rank derivatives of the velocities (and vorticity) we have the **(global and) local energy estimate** , and **(global and) local enstrophy estimate** theorems that reduce the decaying of the solutions at later times than t=0, at spatially infinite to the decaying of the initial data at spatially infinite. See e.g. TAO, T. 2013, Theorem 8.2 (Remark 8.7) and Theorem 10.1 (Remark 10.6).

Furthermore in the same paper of formal formulation of the 4th Clay millennium conjecture , L. FEFFERMAN 2006 (see page 3rd line 6 from above), it is stated that the 3D global regularity of such smooth solutions is controlled by the bounded accumulation in finite time intervals of the vorticity (Beale-Kato-Majda). I state this also explicitly for the convenience of the reader, for smooth solutions of the Navier-Stokes equations with smooth Schwartz initial conditions, as the PROPOSITION 4.8.

When we say here bounded accumulation e.g. of the deformations D, on finite internals, we mean in the sense e.g. of the proposition 5.1 page 171 in the book MAJDA A.J-BERTOZZI A. L. 2002 , which is a definition designed to control the existence or not of finite blowup times. In other words for any finite time interval

[0, T], there is a constant M such that

$$\int_0^t \left|D\right|_{L^\infty}(s)ds <= M$$

I state here for the convenience of the reader, a well known proposition of equivalent necessary and sufficient conditions of existence globally in time of solutions of the Euler equations, as inviscid smooth flows. It is the proposition 5.1 in MAJDA A.J-BERTOZZI A. L. 2002 page 171.

Obviously after PROPOSITION 4.6, and 4.5, that apply also for smooth solutions of the Euler equations, with smooth Schwartz initial data.

The *stretching* is defined by

$S(x,t) =: D\xi \cdot \xi$ if $\xi \neq 0$ and $S(x,t) =: 0$ if $\xi = 0$ where $\xi =: \dfrac{\omega}{|\omega|}$, ω being the vortcity.

PROPOSITION 4.7 *Equivalent Physical Conditions for Potential Singular Solutions. The following conditions are equivalent for smooth Schwartz initial data :*

(1) The time interval, [0, T^*) *with* $T^* < \infty$ *is a maximal interval of smooth* H^s

existence of solutions for the 3D Euler equations.

(2) The vorticity ω accumulates so rapidly in time that

$$\int_0^t |\omega|_{L^\infty}(s)ds \to +\infty$$ *as t tends to T**

(3) The deformation matrix D accumulates so rapidly in time that

$$\int_0^t |D|_{L^\infty}(s)ds \to +\infty$$ *as t tends to T**

*(4) The stretching factor S(**x**, t) accumulates so rapidly in time that*

$$\int_0^t [\max_{x \in R^3} S(x,s)]ds \to +\infty$$ *as t tends to T**

PROPOSITION 4.8 **,** *A smooth solution viscous (or inviscid) 3D flow which is a solution of the Navier-Stokes (or Euler) equations with smooth Schwartz initial conditions, is globally in time smooth, if and only if its vorticity (Beale-Kato-Majda) or its gradient of velocities has bounded accumulation in finite time intervals or the supremum norm* $||u(.,t)||_\infty$ *of the velocity remains bounded for each finite time interval.*

Proof: See the book MAJDA A.J-BERTOZZI A. L. 2002 page 114 line 12 from below for the Sobolev norm of the velocity , and also page 116 theorem 3.6 and page 117 equations 3.79, 3.80 and the last two equations of the proof of theorem 3.6 in page 118, for the gradient of the velocities and the vorticity (Beale-Kato-Majda). The same result about the vorticity is proved again in section 4.2 of the same book. And in page 171 theorem 5.1 for the case of inviscid flows. See also LEMARIE-RIEUSSET P.G. 2002 . It is directly derived therefore from the above references there that global regularity for smooth solutions is equivalent to that for any finite time interval [0,T] , there is a constant M such that

$$\int_0^t |\nabla u|_{L^\infty}(s)ds <= M \qquad \text{(eq.19)}$$

And also that the 3D global regularity is also equivalent to that for any finite time interval [0,T] there is a constant M1 for such that

$$\int_0^t |\omega|_{L^\infty}(s)ds <= M1$$

For the case of inviscid flows , the reader is referred again to MAJDA A.J-BERTOZZI A. L. 2002, in page 171 in the same book , Theorem 5.1.

The above theorems in the book MAJDA A.J-BERTOZZI A. L. 2002 guarantee that the above conditions extent the local in time solution to global in time , that is to solutions (u, p, u_0, f, T) which is H^1 mild solution, **for any T**. Then applying the part (iv) of the PROPOSITION 4.5 above, we get that this solutions is also smooth in the classical sense, for all T>0, thus globally in time. Conversely a global in time smooth solution , will have all the above conditions of boundedness in finite time intervals, otherwise, we would have the case of a blow-up , for some maximal time T^* which would spoil the C^∞ smoothness in *($R^3 \times [0,T^*]$).*

QED.

Similar results hold also for the non-homogeneous case with external forcing which is nevertheless space-time smooth of bounded accumulation in finite time intervals. Thus an alternative formulation to see that the velocities and their gradient , or in other words up to their 1st derivatives and the external forcing also up to the 1st derivatives , control the global in time existence is the next proposition. See TAO. T. 2013 Corollary 5.8

PROPOSITION 4.9 (**Maximum Cauchy development**)

Let (u_0, f, T) be H^1 data. Then at least one of the following two statements hold:

1) There exists a mild H^1 solution (u, p, u_0, f, T) in [0,T] ,with the given data.

2)There exists a blowup time $0 < T^* < T$ and an incomplete mild H^1 solution

(u, p, u_0, f, T^*) up to time T^* in [0, T^*), defined as complete on every [0,t], $t<T^*$ which blows up in the enstrophy H^1 norm in the sense that

$$\lim_{t\to T^*, t<T^*} \|u(x,t)\|_{H^1_x(R^3)} = +\infty$$

PROPOSITION 4.10 *A smooth viscous flow which is a solution of the Navier-Stokes equations with smooth Schwartz initial conditions and which is globally in time smooth, has deformations of bounded accumulation in finite time intervals.*

Proof: The inequalities 5.12 in page 171 in the MAJDA A.J-BERTOZZI A. L. 2002, hold for all smooth flows, and show as in the proof of Theorem 5.1 in the same book , that a finite time blowup of the deformation matrix (unbounded accumulation of deformation in finite time interval) means a finite time blowup of the flow. We may apply also if necessary PROPOSITION 4.5 , (iv) to shift from H^1 mild solutions in the above theorems of MAJDA A.J-BERTOZZI A. L. 2002 , to smooth solutions. So that globall in time smooth viscous flows have bounded accumulation of deformation in finite-time intervals. QED

,PROPOSITION 4.11 *A smooth inviscid Euler 3D flow is not globally in time smooth (3D regular), and it has finite time blow-up if and only if its deformation matrix field has also a finite time blow-up. Similarly if and only if its vorticity has unbounded accumulation in finite time intervals, therefore a finite time blow-up.*

Proof. (See proposition 5.1 in MAJDA A.J-BERTOZZI A. L. 2002 page 171 for Euler inviscid flows equivalent criteria for blowup. Also Theorem 3.6 Vorticity Control and Global Existence page 115 in the same book for the vorticity accumulation in Euler and Navier-Stokes equations. We may apply also if necessary PROPOSITION 4.5 (iv), to shift from H^1 mild solutions in the above theorems of MAJDA A.J-BERTOZZI A. L. 2002 , to smooth solutions. QED.

The next lemma is technical, is the lemma 12.1 in TAO, T. 2013 and appears also in the work of BOGOVSKII M. E. 1980 . We need it so as to have the proof of our THEOREM 7.3, which is slightly only different from the proof of the Theorem 12.2 in TAO, T. 2013.

PROPOSITION 4.12 (Localisation of divergence-free vector fields). *Let T > 0, 0 < R1 < R2 < R3 < R4, and let u : [0, T)×(B(0, R4)\B(0, R1)) → R3 be spatially smooth and divergence-free, and such that*

$$u, \partial_t u \in L_t^\infty C_x^k([0,T) \times (B(0,R4) \setminus B(0,R1)))$$

for all k ≥ 0 and

$$\int_{|x|=r} u(x,t) \cdot n da(x) = 0 \qquad \text{(eq20.)}$$

for all R1 < r < R4 and t ∈ [0, T), where n is the outward normal and dα is surface measure.

Then there exists a spatially smooth and divergence-free vector field

$\tilde{u}$ *: [0, T)×(B(0, R4)\B(0, R1)) → R^3 which agrees with u on [0, T) × (B(0, R2)\B(0, R1)), but vanishes on [0, T) × (B(0, R4)\B(0, R3)). Furthermore, we have*

$$\tilde{u}, \partial_t \tilde{u} \in L_t^\infty C_x^k([0,T) \times (B(0,R4) \setminus B(0,R1)))$$

for all k ≥ 0.

The next propositions is also technical , it is the Proposition 11.6 in TAO. T 2013 and is necessary so as to have the proof of our THEOREM 6.4.

PROPOSITION 4.13 (Uniform smoothness outside a ball*). Let (u, p, u0, f, T^*) be an incomplete*

almost smooth H^1 solution with normalised pressure for all times 0 < T < T^*. Then there exists

a ball B(0, R) such that

$$u, p, f, \partial_t u \in L_t^\infty C_x^k([0,T^*) \times K) \qquad \textit{(eq.21)}$$

for all k ≥ 0 and all compact subsets K of R^3 \B(0, R).

The term "almost smooth" here is defined in TAO, T. 2013, before Conjecture 1.13. The only thing that almost smooth solutions lack when compared to smooth solutions is a limited amount of time differentiability at the starting time t = 0;

The term *normalized pressure*, refers to the symmetry of the Euler and Navier-Stokes equations to substitute the pressure, with another that differs at constant in space but variable in time measureable function. In particular normalized pressure is one that satisfies the (eq. 18) except for a measurable at constant in space but variable in time measureable function. It is proved in TEO, T. 2013, at Lemma 4.1, that the pressure is normalizeable (exists a normalized pressure) in almost smooth finite energy solutions, for almost all times. The viscosity coefficient here has been normalized to ν=1**.**

10. The theorem of the fundamental decomposition of the Euler and Navier-Stokes equations.

The next theorem, can possibly be utilized to design new and more efficient numerical methods of solving the Navier-Stokes and Euler equations. But this issue is not the subject of this paper.

THEOREM 5.1 **The theorem of the fundamental decomposition of the Euler and Navier-Stokes equations.**

Let a smooth flow-solution of the Euler (inviscid case) or the Navier-Stokes (viscous case) equations in a finite time interval [0,T] , which has velocities , pressure and external forcing with first and second order partial derivatives of them that decay at spatial infinite at least faster than the $1/r^2$

Then the Euler or the Navier-Stokes equations are equivalent to the next pair of vector equations

$$\frac{\partial u_s}{\partial t} + (u_s \cdot \nabla) u_s = -\nabla p^{**} + \nu \Delta(u_s) \text{ (symmetric or ,potential or irrotational part)}$$

(eq.23)

$$\frac{\partial u_a}{\partial t} + (u_a \cdot \nabla) u_a - u_s \times (\nabla \times u_a) = \nu \Delta(u_a) + f_a \text{ (antisymmetric or rotational part)}$$

(eq. 24).

Where the $u_s(x,t) = \nabla \phi(x,t)$ *and* $u_a(x,t) = \nabla \times a(x,t)$ *are the* L^2*-orthogonal Helmholtz-Hodge decomposition of the velocity* $u(x,t) = u_s(x,t) + u_a(x,t)$ *, and* $f_s(x,t) = \nabla \phi_f(x,t)$, $f_a(x,t) = \nabla \times A(x,t)$, $f(x,t) = f_s(x,t) + f_a(x,t)$ *the* L^2*-orthogonal Helmholtz-Hodge decomposition of the external forcing terms and* $(u_s \cdot u_a) + p + \phi_f = p^{**}$.

REMARK 5.1 **.** We must notice here that the fundamental decomposition of the Euler and Navier-Stokes equations, holds also for the case of compressible flows.

Variations of the idea of utilizing the Helmholtz-Hodge decompostion of a flow, has occurred to many researchers of numerical analysis when solving the Navier-Stokes equations (e.g. see references CHORIN A. J. 1968 and GATSKI T. B. GROSCH C. E.ROSE M. E.1989). But here I present a significant twist of the idea, which is not simply to approximate numerically irrotational flows with rotational flows , or to decompose the solution by the Helmholtz-Hodge theorem, but also to decompose the equations themselves

Proof: I write here as equations that the solution-flow it satisfies the Navier-Stokes equations (including the case of Euler equations for v=0) ,

$$\frac{\partial(u)}{\partial t}+((u)\cdot\nabla)(u)=-\nabla p+v\Delta(u)+f \qquad \text{(eq.25)}$$

We apply the 1st Helmholtz-Hodge fundamental decomposition theorem (PROPOSITION 4.1) and the 1st Hodge decomposition theorem (PROPOSITION 4.2) as in paragraph 4, which decomposes in a unique way the vector field of the velocities u(**x**,t) of this flow to the L^2 –orthogonal sum

$$u(x,t)=u_s(x,t)+u_a(x,t) \qquad \text{(eq.26)}$$

where the $u_s(x,t)=\nabla\phi(x,t)$ is the (symmetric) curl-free component which is the grad of scalar field (scalar stream function) $\phi(x,t)$, and the $u_a(x,t)=\nabla\times a(x,t)$ is the (antisymmetric) divergence free component, which the curl of vector field $a(x,t)$.

As from the hypotheses of the present theorem the decay at spatial infinite of the velocities filed u(x,t) is at least as fast as $1/r^2$ and the divu=0, this is possible.

And we apply also the L^2-orthogonal Helholtz-Hodge decomposition for the forcing term, $f_s(x,t)=\nabla\phi_f(x,t)$, $f_a(x,t)=\nabla\times A(x,t)$, $f(x,t)=f_s(x,t)+f_a(x,t)$, with the fits being the curl-free and the second the div-free.

From (eq 26) since divu=0 and $divu_\alpha$ =0 then also

$$divu_s=0 \ ! \qquad \text{(eq.27)}$$

The original Navier-Stokes would be now

$$\frac{D(u_s+u_a)}{Dt}=-\nabla p+v\Delta(u_s+u_a)+f_s+f_a \qquad \text{(eq.28)}$$

Let make here a remark about the case of incompressible flows-solutions with divu=0.

From the identity $\nabla\cdot(\nabla u)-(\nabla\cdot\nabla)u=\nabla\times(\nabla\times u)$, (eq.29)

and from divu=0 in the case of incompressible flows , $\nabla\times u=curlu=\omega$, it is known that the vicsosity term can be expressed also through as minus the rotation of the vorticity as

$$\nu\Delta u=-\nu\nabla\times(\nabla\times u) \qquad \text{(eq.30)}$$

As (eq.28) can be rewritten as

$$\frac{D(u_s+u_a)}{Dt}=-\nabla p+\nu\Delta(u_s)+\nu\Delta(u_a)+f_s+f_a \qquad \text{(eq.31)}$$

and since div(u_s)=div(u_a)=0 and $\nabla\times u_s=$curl(u_s)=0 (eq.32)

and from (eq.29)

$$\frac{D(u_s+u_a)}{Dt}=-\nabla p+\nu\Delta(u_a)\quad+f_s+f_a \qquad \text{(eq.33)}$$

Nevertheless if the flow-solution is not assumed incompressible, the viscosity term for the symmetric component u_s should be included.

We start again with equations (eq. 26), and (eq. 28). We apply direct calculation on the (eq. 28) which becomes

$$\frac{\partial(u_s+u_a)}{\partial t}+((u_s+u_a)\cdot\nabla)(u_s+u_a)=-\nabla p+\nu\Delta(u_s+u_a)$$

+ f_s+f_a (eq. 34)

And after distribution

$$\frac{\partial u_s}{\partial t}+\frac{\partial u_a}{\partial t}+(u_s\cdot\nabla)u_s+(u_a\cdot\nabla)u_a+(u_a\cdot\nabla)u_s+(u_s\cdot\nabla)u_a=-\nabla p+v\Delta u_s+v\Delta u_a+$$
$$f_s+f_a$$

(eq. 35)

In addition we apply the vector identity,

$$(A\cdot\nabla)B+(B\cdot\nabla)A=\nabla(A\cdot B)-A\times(\nabla\times B)-B\times(\nabla\times A)$$ (eq. 36)

to get (because $\nabla\times u_s$ =0) ,

$$(u_s\cdot\nabla)u_a+(u_a\cdot\nabla)u_s=\nabla(u_s\cdot u_a)-u_s\times(\nabla\times u_a)$$ (eq. 37)

so that the (eq. 35) becomes,

$$\frac{\partial u_s}{\partial t}+(u_s\cdot\nabla)u_s+\frac{\partial u_a}{\partial t}+(u_a\cdot\nabla)u_a+\nabla(u_s\cdot u_a)-u_s\times(\nabla\times u_a)=-\nabla p-v\nabla\times(\nabla\times u_a)+$$
$$v\Delta u_s+f_s+f_a$$

(eq. 38)

Now, we observe that the terms split to curl-free and div-free. We denote again the $(u_s\cdot u_a)+\nabla\phi_f=p^*$, where $f_s(x,t)=\nabla\phi_f(x,t)$ and we move it to the right hand, together with the ∇p, so after setting $p^*+p=p^{**}$, the (eq. 38) becomes,

$$\frac{\partial u_s}{\partial t}+(u_s\cdot\nabla)u_s+\frac{\partial u_a}{\partial t}+(u_a\cdot\nabla)u_a-u_s\times(\nabla\times u_a)=-\nabla p^{**}-v\nabla\times(\nabla\times u_a)+v\Delta u_s+f_a$$

(eq. 39)

Here we want to split this equation in to two, equivalent, by noticing that the left and right hand sides split to tow groups of curl-free and div-free, L^2-orthogonal components (as in the unique Hodge-decomposition). For the right hand side the curl-free is the ∇p^{**} and the div-free is the $v\nabla\times(\nabla\times u_a)+\nabla\times A_f$ (where $f_a(x,t)=\nabla\times A_f(x,t)$) because $div v\nabla\times(\nabla\times u_a)=v div\Delta(u_a)=v\Delta(div u_a)=0$, and $div\nabla\times A_f=0$. And since by the hypothesis of the present theorem decay fast enough at spatial infinity (from the L^2 -orthogonality of the Hodge decomposition , we have also $\|u_s\|,\|u_a\| <= \|u\|$), we can apply the PROPOSITION 4.4, to get that they are also L^2-orthogonal.

Similarly for the left side , the group of terms which are curl-free is the $\frac{\partial u_s}{\partial t} + (u_s \cdot \nabla)u_s$, while the group of terms which is div-free is the $\frac{\partial u_a}{\partial t} + (u_a \cdot \nabla)u_a - u_s \times (\nabla \times u_a)$. In particular the curl-free group of terms can be written as the gradient of a scalar function, $\frac{\partial u_s}{\partial t} + (u_s \cdot \nabla)u_s = \nabla(\frac{\partial \phi}{\partial t} + (1/2)|\nabla \phi|^2)$, from the fact that $u_s(x,t) = \nabla \phi(x,t)$, from equations (eq. 26), of the Hodge decomposition, so it is direct that it is curl-free.

From $divu_a = 0$, it is easily seen that $div(\frac{\partial u_a}{\partial t} + (u_a \cdot \nabla)u_a) = \frac{\partial divu_a}{\partial t} + (u_a \cdot \nabla)divu_a = 0$.

From the vector calculus identities

$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B)$, $\Delta u_a = -\nabla \times (\nabla \times u_a)$ $curlu_s$=0, $divu_a = 0$ we derive that $div(u_s \times (\nabla \times u_a) = (curlu_s) \cdot \nabla \times u_a - u_s \cdot (\nabla \times (\nabla \times u_a)) = 0 + \Delta(divu_a) = 0$

Again as before by the hypothesis of the present theorem they decay fast enough at spatial infinity (from the L^2 -orthogonality of the unique Hodge decomposition we have also, $\|u_s\|,\|u_a\| <= \|u\|$), and we can apply the PROPOSITION 4.4, to get that they are also L^2-orthogonal. Therefore from the uniqueness of the orthogonal decomposition the equation (eq. 39) splits in an equivalent way to a pair of equations

$\frac{\partial u_s}{\partial t} + (u_s \cdot \nabla)u_s = -\nabla p^{**} + v\Delta(u_s)$ (symmetric, irrotational part) (eq. 40)

$\frac{\partial u_a}{\partial t} + (u_a \cdot \nabla)u_a - u_s \times (\nabla \times u_a) = -v\nabla \times (\nabla \times u_a) + \nabla \times A_f$ (antisymmetric rotational part)

(eq. 41)

We also notice that the (eq. 40) is the Euler equation of a potential flow.

If the flow-solution is also incompressible, then the viscosity term in (eq. 40) vanishes. We have completed the proof. QED.

COROLLARY 5.1 For *a smooth flow-solution of the Euler (invisid case) or Navier-Stokes (viscous case), with smooth compact support initial data , which exists as smooth solution ,locally in time in a finite time interval [0,T] and with smooth compact support (the same support) external forcing for all times t in [0,T] ,applies the previous fundamental decomposition.*

REMARK 5.2 .We may assume that the compact support is an image by an smooth 1-1 onto Diffeomorphism of the 3D spherical ball, thus simply connected and with smooth boundary, if we want to have a clear mental image.

Proof: Since the solutions and external forcing is also of smooth compact support as it is preserved, after the initial data, the velocities, pressures and external forcing have up to 2^{nd} derivatives that decay spatially faster than $1/r^2$, thus the above fundamental decomposition applies. QED.

REMARK 5.3 **On the physical meaning of the theorem of the fundamental decomposition of the Euler and Navier-Stokes equations.**

The idea, that we can , extract from a massive motion of particles or bodies , a simpler massive motion where, no particle or body, has self-rotation (spin or vorticity), is a fundamental concept in mathematical physics, and is responsible for the split of the concept of momentum to linear and angular. For example , let us imagine the massive motion of the stars and planets in the galaxies , where no collision is assumed possible but only interactions mainly through gravitation. It is direct, that the shape of the massive motion , and therefore the shape of the galaxy, will not change, if we eliminate all self-rotations of the stars a, planets and black holes. The situation is not an exact analogue of the case of incompressible fluids, but one gets the idea.

11. **Equivalence of smooth compact support and smooth Schwartz, initial conditions.**

I use here some arguments that exist in the work of TAO. T. 2013.

THEOREM 6.4. (3D global smooth compact support non-homogeneous regularity implies 3D global smooth Schwartz homogeneous regularity) *If it holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth compact support (connected with smooth boundary) initial data of velocities and pressures (thus finite initial energy) and smooth compact support (the same connected support with smooth boundary) external forcing for all times t>0, exist also globally in time t>0 (are globally regular) then it also holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth Schwartz initial data of velocities and pressures (thus finite initial energy) , exist also globally in time for all t>0 (are globally regular).*

REMARK 6.3 .We may as in the previous theorems, assume that the compact support is an image by an smooth 1-1 onto Diffeomorphism of the 3D spherical ball, thus simply connected and with smooth boundary, if we want to have a clear mental image.

Proof: We repeat here the proof of theorem 12.2 in TAO,T. 2013, except that we have the luxury to utilize stronger hypotheses, and we stop when the solution with compact support is created, rather than proceeding to embed it to a periodic solution. The reader is advised to read the proof of Theorem, 12.2 in TAO, T. 2013.

Given the Cauchy maximum development as in TAO. T. 2013 Corollary 5.8

or PRPOSITION 4.11 in this paper, it suffices to show that if there exists a blowup time and (u,p, u_0, f, T^*) , f=0, is an smooth incomplete , thus H^1 solution , up to the blowup time T^* with (u_0, f, T) , $T< T^*$ with smooth Schwartz initial data , then u does not blowup in the H^1 (enstrophy) norm, thus

$$\lim_{t\to T^*, t<T^*} \sup\|u(t)\|_{H^1(R^3)} < +\infty \qquad \text{(eq.45)}$$

(For the terminology and notations in this proof , see again DEFINITIONS 3.1-3.5.)

So we start by assuming the existence of a blowup time T^* .

Let R>0 be a sufficiently large radius. By arguing as in Corollary 11.1 in TAO, T. 2013, by the monotone convergence theorem, we have that the next mixed norm of u is finite at the large for all the interval [0, T^*) , that is

$$u \in L_t^\infty H_x^1(R^3 \setminus B(0,R)) \text{ for } t \in [0,T^*) \,, \qquad \text{(eq.46)}$$

where R is independed of t, depending only on T^*. To have this we may take the limit as t converges to T^* in the inequality (78) of δ in TAO,T. 2013. This fact anyway is proved in the proof of theorem 12.2 in TAO,T. 2013.

This means that even if t convergences from the left to T^* the norm will remain bounded in the large outside the ball B(0, R). So the blowup , might occur only locally inside the ball B(0, R), in which case we have

$$\lim_{t \to T^*, t < T^*} \sup \|u(t)\|_{H_x^1(B(0,R))} = +\infty \qquad \text{(eq.47)}$$

By proposition 11.6 and remark 11.8 in TAO, T. 2013 (and increasing R if necessary) we also have

$$u, p, f, \partial_t u \in L_t^\infty C_x^k([0,T^*) \times (B(0,5R) \setminus B(0,2R))) \text{ for all k>=0} \qquad \text{(eq. 48)}$$

From the Stokes theorem, and the divergence-free nature of the field u, we also have

$\int_{|x|=r} u(x,t) \cdot n da(x) = 0$ for all r>0, and t in [0,T^*). Thus we can apply Lemma 12.1 in TAO, T. 2013 or PROPOSITION 4.12 in this paper. And by applying it we can find a smooth divergence-free field $\tilde{u}$: $[0,T^*) \times (B(0,5R) \setminus B(0,2R)) \to B(0,4R)$, which agrees with u on B(0,3R)\B(0,2R) and vanishes outside B(0,5R), with

$$\tilde{u}, \partial_t \tilde{u} \in L_x^\infty C_x^k (B(0,5R \setminus B(0,2R))$$ for all k>=0 (eq. 49)

We then extend $\tilde{u}$, by zero outside of B(0,5R) and by u inside of B(0,2R). Then

$\tilde{u}$ is now smooth on all of $[0,T^*)\times R^3$.

Let *η* be a smooth function supported on B(0,5R) and equals 1 on B(0,4R). We define a new forcing term $\tilde{f}:[0,T^*)\times R^3 \to R$ by the formula (Navier-Stokes)

$$\tilde{f} := \partial_t \tilde{u} + (\tilde{u}\cdot\nabla)\tilde{u} - \nu\Delta\tilde{u} + \nabla(p\eta) \qquad \text{(eq. 50)}$$

Then $\tilde{f}$ is smooth, supported on B(0,5R) and agrees with f =0 , on B(0,3R). From this and (eq, 48), (eq. 49) we easily verify that

$$\tilde{f} \in L_t^\infty H_x^1([0,T^*)\times R^3)\cdot$$

Note also that by taking divergences in (eq. 50) and using the compact support of *pη,* $\tilde{u}$, $\tilde{f}$ that

$$p\eta = -\Delta^{-1}((\tilde{u}\cdot\nabla)\tilde{u}) + \Delta^{-1}\nabla\cdot\tilde{f}\cdot$$

Thus the $(\tilde{u}, p\eta, \tilde{u}(0), \tilde{f}, T^*)$ is an smooth incomplete H^1, pressure normalized, (and hence mild) solution with all components supported in B(0,5R).

Then by the hypotheses of the theorem, it is also a complete solution in $[0,T^*]$, and in fact existing globally in time. This implies (since $\tilde{u}$ and u agree on B(0, R)) that

$$u \in L_t^\infty H_x^1([0,T^*] \times B(0,R)) \quad \text{(eq.51)}$$

which, contradicts the (eq. 47) in other words the existence of blowup. Therefore the theorem holds. QED.

12. Epilogue

In this paper I proved , using some relatively recent ideas suggested by T. Tao , that the Schwartz initial conditions of the its official formulation of the problem in the direction of regularity are equivalent to the simpler compact support initial conditions (THEOREM 6.4.). Finally I proved using the Helmholtz-Hodge orthogonal decomposition of vector fields, a powerful fundamental decomposition of the Euler and Navier-Stokes equations which is significant for the internal symmetries of the equations (THEOREM 5.1).

References

BEALE, J. T., KATO, T., AND MAJDA, A 1984., "Remarks on the breakdown of smooth solutions for the 3-D Euler equations," *Commun. Math. Phys.* **94**, 61–66, 1984.

BOGOVSKII M. E. 1980, Solution of some vector analysis problems connected with operators div and grad, Trudy Sem. S. L. Sobolev 80 (1) (1980) 5-40.

CANNON, J.R., KNIGHTLY, G.H. 1970 : A note on the Cauchy problem for the Navier-Stokes equations. SIAM J. Appl. Math. 18, 641-644(1970)

CHORIN A. J. 1968 Numerical solutions of the Navier-Stokes Equations *Mathematics of Computation oct. 1968, vol 22, no 104 pp 745-762*

CONSTANTIN P. 1995, A few results and open problems regarding incompressible fluids, Notices of the AMS, 42 (1995), 658-663.

CONSTANTIN P. 2001 , Some open problems and research directions in the mathematical study of fluid dynamics, in Mathematics Unlimited–2001 and Beyond, Springer Verlag, Berlin, 2001, 353–360.

CONSTANTIN P. 2007 On the Euler equations of incompressible fluids. Bulletin of the AMS Vol 44, no 4, October 2007, pp603-621.

GATSKI T. B. GROSCH C. E.ROSE M. E.1989 The Numerical Solution of the Navier-Stokes equations for 3-Dimensional, Unsteady, Incompressible Flows by Compact Support Schemes

Journal of computation Physics 82 , 298-329 (1989) .

GERMAIN P. 2008 Strong solutions and weak-strong uniqueness for the

nonhomogeneous Navier-Stokes system Journal d'Analyse Mathématique January 2008, Volume 105, Issue 1, pp 169-196.

GIGA Y. 1983 Weak and Strong Solutions of the Navier-Stokes Initial Value Problem PubL RIMS, Kyoto Univ. 19 (1983), 887-910

GIGA, Y., INUI, K., MATSUI, S. 1999 : On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data. Advances in fluid dynamics 4, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, 27-68(1999).

GIGA, Y., MAHALOV, A., NICOLAENKO, B. 2004 : The Cauchy problem for the NavierStokes equations with spatially almost periodic initial data. Hokkaido University preprint series in mathematics 683, (2004)

GIGA, Y., MATSUI, S., SAWADA, O. 2001 : Global existence of two-dimensional NavierStokes flow with nondecaying initial velocity. J. Math. Fluid Mech. 3, no.3, 302-315(2001)

GIGA, Y., MIYAKAWA, T.1985 : Solutions in Lr of the Navier-Stokes initial value problem. Arch. Rational Mech. Anal. 89, no.3, 267-281(1985)

GIGA Y. SAWADA, O. 2003 On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem, Nonlinear Analysis and Applications, To V. Lakshmikantham on his 80th Birthday vol. I (2003), 549-562.

FEFFERMAN C. L. 2006 Existence and smoothness of the Navier-Stokes equation, Millennium Prize Problems, Clay Math. Inst., Cambridge, MA, 2006, 57-67.

FRIEDLANDER S., DENIS SERRE 2002 Handbook of Mathematical Fluid Dynamics Vol 1,2, Gulf Professional Publishing, 2002.

KATO, J. 2003 : The uniqueness of nondecaying solutions for the Navier-Stokes equations. Arch. Ration. Mech. Anal. 169, no.2, 159-175(2003)

KATO, T. 1983 : Strong L p -solutions of the Navier-Stokes equation in R m, with applications to weak solutions. Math. Z. 187, no.4, 471-480(1984)

KATO, T., FUJITA, H.1962 : On the nonstationary Navier-Stokes system. Rend. Sem. Mat. Univ. Padova 32, 243-260(1962).

KATO, T. PONCE, G. 1988 Commutator estimates and the Euler and Navier-Stokes equations, CPAM 41 (1988), 891-907.

KOCH, H., TATARU, D.2001 Well-posedness for the Navier-Stokes equations, Adv. Math. 157 (2001), no. 1, 22-35.

LADYZHENSKAYA, O. A. 1967 Uniqueness and smoothness of generalized solutions of Navier-Stokes equations, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 5 (1967), 169-185; English transl., Sem. Math. Steklov Math. Inst., Leningrad 5 (1969), 60-67.

LADYZHENSKAYA O. 1969, The Mathematical Theory of Viscous Incompressible Flows (2nd edition), Gordon and Breach, New York, 1969.

LEMARIE-RIEUSSET P.G. 2002 Recent developments in the Navier-Stokes problem. Chapman & All/CRC 2002

MAEKAWA, Y. TERASAWA, Y. 2006 **:** The Navier-Stokes equations with initial data in uniformly local Lp spaces. Differential and integral equations: an international journal for theory & applications (Athens, Ohio : Khayyam Publ. Co.), Vol. 19, No. 4 (2006), p. 369-400

MAJDA A.J-BERTOZZI A. L. 2002 Vorticity and incompressible flow Cambridge University texts in applied mathematics *2002*

MURIEL A 2000 An Exact Solution of the 3-D Navier-Stokes Equation

Department of Electrical Engineering Columbia University

And Department of Philosophy Harvard University.

Also A. Muriel and M. Dresden, *Physica D 101, 299, 1997.*

PRODI, G 1959 Un teorema di unicit_a per le equazioni di Navier-Stokes, Ann. Mat. Pura Appl. 48 (1959), 173-182.

SERRIN,J 1963 The initial value problem for the Navier-Stokes equations, In Nonlinear Problems (Proc. Sympos., Madison, Wis., pages 69{98. Univ. of Wisconsin Press, Madison, Wis., 1963.

SAWADA, O.2003 : On time-local solvability of the Navier-Stokes equations in Besov spaces. Adv. Differential Equations 8 no.4, 385-412(2003)

TAO T. 2013 Localisation and compactness properties of the Navier-Stokes global regularity problem. Analysis & PDE 6 (2013), 25-107

TAO T. 2014 Finite time blowup for an averaged three-dimensional Navier-Stokes equation. Cornell University Library arxiv.org

Having reduced the 4th Clay Millennium problem to an equivalent with the same hypotheses of finite initial energy but on compact support initial conditions too, made all sorts of arguments easier or possible to do.
It was not obvious how the finite initial energy and the energy conservation could be used to prove the non-existence of a Blow-up in finite time. To surround carefully the problem I proved more than 8 different necessary and sufficient conditions of non-existence of a Blow-up in finite time. Finally, it was that the pressures must remain bounded in finite time intervals which proved that there cannot be a Blow-up in finite time. Anthe pressures must remain bounded because of the conservation of energy , the initial finite energy and that pressures as it known define a conservative field of forces in the fluid all the times.

The next 3rd paper which was completed and uploaded in the internet during 25 February 2018 is I believe the final solution of the 4th Clay Millennium problem, and it has been published in the *World Journal of Research and Review* https://www.wjrr.org/ during August 2021.

3rd paper

A SOLUTION OF THE 4^{TH} CLAY MILLENNIUM PROBLEM ABOUT THE NAVIER-STOKES EQUATIONS.

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ABSTRACT

In this paper it is solved the 4^{th} Clay Millennium problem about the Navier-Stokes equations, in the direction of regularity. It is done so by utilizing the hypothesis of finite initial energy The final key result to derive the regularity is that the pressures are bounded in finite time intervals, as proved after projecting the bounded by the conservation of energy , virtual work of the pressures forces on specially chosen bundles of instantaneous paths.

Key words: *Incompressible flows*, *regularity, Navier-Stokes equations, 4^{th} Clay millennium problem*

Mathematical Subject Classification: 76A02

1. Introduction

The famous problem of the 4th Clay Mathematical Institute as formulated in FEFFERMAN CL 2006 is considered a significant challenge to the science of mathematical physics of fluids, not only because it has lasted the efforts of the scientific community for decades to prove it (or converses to it) but also because it is supposed to hide a significant missing perception about the nature of our mathematical formulations of physical flows through the Euler and Navier-Stokes equations.

When the 4th Clay Millennium Problem was formulated in the standard way, the majority was hoping that the regularity was also valid in 3 dimensions as it had been proven to hold in 2 dimensions.

The main objective of this paper is to prove the regularity of the Navier-Stokes equations with initial data as in the standard formulation of the 4^{th} Clay Millennium Problem. (see PROPOSITION 5.2 (**The solution of the 4^{th} Clay Millennium problem).**

The problem was solved in its present form during the spring 2017 and was uploaded as a preprint in February 2018 (see KYRITSIS. K. Feb 2018).

The main core of the solution is the paragraph 4, where a new sufficient conditions of regularity is proved based on the pressures and paragraph 5, where it is proven that the pressures are bounded in finite time intervals, which leads after the previous sufficient conditions to the proof of the regularity of the Navier-Stokes equations. The paragraph 2 is devoted to reviewing the standard formulation of the 4^{th} Clay Millennium problem, while the paragraph 3 is devoted in to collecting some well-known results that are good for the reader to have readily available to follow the later arguments.

According to CONSTANTIN P. 2007 "..The blow-up problem for the Euler equations is a major open problem of PDE, theory of far greater physical importance that the blow-up problem of the Navier-Stokes equation, which is of course known to non-specialists because of the Clay Millennium problem…" For this reason, many of the propositions of this paper are stated for the Euler equations of inviscid flows as well.

2. The standard formulation of the Clay Mathematical Institute 4^{th} Clay millennium conjecture of 3D regularity and some definitions.

In this paragraph we highlight the basic parts of the standard formulation of the 4^{th} Clay millennium problem, together with some more modern, since 2006, symbolism, by relevant researchers, like T. Tao.

In this paper I consider the conjecture (A) of C. L. FEFFERMAN 2006 standard formulation of the 4th Clay millennium problem , which I identify throughout the paper as <u>the 4th Clay millennium problem</u>.

The Navier-Stokes equations are given by (by R we denote the field of the real numbers, v>0 is the density normalized viscosity coefficient)

$$\frac{\partial}{\partial t}u_i+\sum_{j=1}^{n}u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\Delta u_i \qquad (x\varepsilon R^3,\ t>=0,\ n=3) \qquad \text{(eq.1)}$$

$$divu=\sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i}=0 \qquad (x\varepsilon R^3,\ t>=0,\ n=3) \qquad \text{(eq.2)}$$

with initial conditions $u(x,0)=u^0(x)$ $\quad$ $\mathbf{x\varepsilon R^3}$ and u^0 (x) C^∞ divergence-free vector field on $\mathbf{R^3}$ (eq.3)

$\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator .The Euler equations are (eq1), (eq2), (eq3) when $\nu=0$.

It is reminded to the reader, that in the equations of Navier-Stokes, as in (eq. 1) the as the density, is constant, it is custom to either normalised to 1, or it is divided out from the left side and it is included in the pressures and viscosity coefficient.

For physically meaningful solutions we want to make sure that $u^0(x)$ does not grow large as $|x|\to\infty$. This is set by defining $u^0(x)$ and called in this paper **Schwartz initial conditions** , in other words

$$\left|\partial_x^a u^0(x)\right|\le C_{a,K}(1+|x|)^{-K} \text{ on } \mathbf{R^3} \text{ for any } \alpha \text{ and K} \qquad \text{(eq.4)}$$

(Schwartz used such functions to define the space of Schwartz distributions)

We accept as physical meaningful solutions only if it satisfies

$$p,\ u\in C^\infty(R^3\times[0,\infty)) \qquad \text{(eq.5)}$$

and

$$\int_{\Re^3}\left|u(x,t)\right|^2 dx<C \text{ for all } t>=0 \ (\textbf{Bounded or finite energy}) \qquad \text{(eq.6)}$$

The conjecture (A) of he Clay Millennium problem (case of no external force, but homogeneous and regular velocities) claims that for the Navier-Stokes equations , v>0, n=3 , with divergence free , Schwartz initial velocities , there are for all times t>0 , smooth velocity field and pressure, that are solutions of the Navier-Stokes equations with bounded energy**, in other words satisfying the equations eq.1 , eq.2 , eq. 3, eq.4 , eq.5 eq.6** . It is stated in the same formal formulation of the Clay millennium problem by C. L. Fefferman see C. L. FEFFERMAN 2006 (see page 2nd line 5 from below) that the conjecture (A) has been proved to holds locally. "..if the time internal $[0,\infty)$, is replaced by a small time interval [0,T), with T depending on the initial data....". In other words there is $\infty > T > 0$, such that there is continuous and smooth solution $u(x,t) \in C^{\infty}(R^3 \times [0,T))$. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^{∞}

Following TAO, T 2013, we define some specific terminology, about the hypotheses of the Clay millennium problem, that **will not be used** in the next in the main solution of the 4th Clay Millennium problem, but we include it just for the sake of the state of the art that TAO, T, in 2013 (see references) has created in studying this 4th Clay Millennium problem. For more details about the involved functional analysis norms, the reader should look in the above paper TAO, T, in 2013 in the references.

We must notice that the definitions below can apply also to the case of inviscid flows, satisfying the ***Euler*** *equations.*

DEFINITION 2.1 (Smooth solutions to the Navier-Stokes system). *A smooth set of data* for the Navier-Stokes system up to time T is a triplet (u_0, f, T), where $0 < T < \infty$ is a time, the initial velocity vector field $u_0 : R^3 \to R^3$ and the forcing term $f : [0, T] \times R^3 \to R^3$ are assumed to be smooth on R^3 and $[0, T] \times R^3$ respectively (thus, u_0 is infinitely differentiable in space, and f is infinitely differentiable in space time), and u_0 is furthermore required to be divergence-free:

$$\nabla \cdot u_0 = 0.$$

If f = 0, we say that the data is *homogeneous.*

In the proofs of the main conjecture we will not consider any external force, thus the data will always be homogeneous. But we will state intermediate propositions with external forcing. Next we are defining simple diffentiability of the data by Sobolev spaces.

DEFINITION 2.2 We define the H^1 norm (or enstrophy norm) $H^1 (u_0, f, T)$ of the data to be the quantity

$H^1 (u_0, f, T) := \|u_0\|_{H^1_X(R^3)} + \|f\|_{L^\infty_t H^1_x(R^3)} < \infty$ and say that (u_0, f, T) *is H^1* if

$H^1 (u_0, f, T) < \infty$.

DEFINITION 2.3 We say that a *smooth set of data* (u_0, f, T) is *Schwartz* if, for all integers α, m, k ≥ 0, one has

$$\sup_{x \in R^3} (1+|x|)^k \left|\nabla_x^a u_0(x)\right| < \infty$$

and $$\sup_{(t,x)\in[0,T]\times R^3} (1+|x|)^k \left|\nabla_x^a \partial_t^m f\ (x)\right| < \infty$$

Thus, for instance, the solution or initial data having Schwartz property implies having the H^1 property.

DEFINITION 2.4 A *smooth solution* to the Navier-Stokes system, or a *smooth solution* for short, is a quintuplet (u, p, u_0 , f, T), where (u_0, f, T) is a *smooth set of data*, and the velocity vector field u : [0, T] × R^3 → R^3 and pressure field p : [0, T]× R^3 → R are smooth functions on [0, T]× R^3 that obey the Navier-Stokes equation (eq. 1) but with external forcing term f,

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu\Delta u_i + f_i \quad (x\epsilon R^3 , t>=0 , n=3)$$

and also the incompressibility property (eq.2) on all of [0, T] × R^3 , but also the initial

condition u(0, x) = u_0(x) for all x ∈ R^3

DEFINITION 2.5 Similarly, we say that (u, p, u_0, f, T*) is H^1* if the associated data (u_0, f, T) *is H^1* , and in addition one has

$$\|u\|_{L_t^\infty H_X^1([0,T]\times R^3)} + \|u\|_{L_t^2 H_X^2([0,T]\times R^3)} < \infty$$

We say that the solution is *incomplete in [0,T),* if it is defined only in [0,t] for every t<T.

We use here the notation of *mixed norms* (as e.g. in TAO, T 2013). That is if $\|u\|_{H_x^k(\Omega)}$ is the classical Sobolev norm ,of smooth function of a spatial domain Ω, $u:\Omega\to R$, I is a time interval and $\|u\|_{L_t^p(I)}$ is the classical L^p-norm, then the mixed norm is defined by

$$\|u\|_{L_t^p H_x^k(I\times\Omega)} := (\int_I \|u(t)\|_{H_x^k(\Omega)}^p dt)^{1/p}$$

and

$$\|u\|_{L_t^\infty H_x^k(I\times\Omega)} := ess\sup_{t\in I}\|u(t)\|_{H_x^k(\Omega)}$$

Similar instead of the Sobolev norm for other norms of function spaces.

We also denote by $C_x^k(\Omega)$, for any natural number $k\geq 0$, the space of all k times continuously differentiable functions $u:\Omega\to R$, with finite the next norm

$$\|u\|_{C_x^k(\Omega)} := \sum_{j=0}^{k}\|\nabla^j u\|_{L_x^\infty(\Omega)}$$

We use also the next notation for *hybrid norms*. Given two normed spaces X, Y on the same domain (in either space or time), we endow their intersection $X \cap Y$ with the norm

$\|u\|_{X \cap Y} := \|u\|_X + \|u\|_Y$.

In particular in the we will use the next notation for intersection functions spaces, and their hybrid norms.

$X^k(I \times \Omega) := L_t^\infty H_x^k(I \times \Omega) \cap L_x^2 H_x^{k+1}(I \times \Omega)$.

We also use the *big O notation*, in the standard way, that is X=O(Y) means

$X \leq CY$ for some constant C. If the constant C depends on a parameter s, we denote it by C_s and we write $X=O_s(Y)$.

We denote the difference of two sets A, B by A\B. And we denote Euclidean balls

by $B(a,r) := \{x \in R^3 : |x-a| \leq r\}$, where |x| is the Euclidean norm.

With the above terminology the target Clay millennium conjecture in this paper can be restated as the next proposition

The 4th Clay millennium problem (Conjecture A)

(**Global regularity for homogeneous Schwartz data**). *Let (*u_0*, 0, T) be a homogeneous Schwartz set of data. Then there exists a smooth finite energy solution (u, p,* u_0*, 0, T) with the indicated data (notice it is for any T>0, thus global in time).*

3. Some known or directly derivable, useful results that will be used.

In this paragraph I state, some known theorems and results, that are to be used in this paper, or is convenient for the reader to know , so that the reader is not searching them in the literature and can have a direct, at a glance, image of what already holds and what is proved.

A review of this paragraph is as follows:

Propositions 3.1, 3.2 are mainly about the uniqueness and existence locally of smooth solutions of the Navier-Stokes and Euler equations with smooth Schwartz initial data. Proposition 3.3 are necessary or sufficient or necessary and sufficient conditions of regularity (global in time smoothness) for the Euler equations without viscosity. Equations 8-13 are forms of the energy conservation and finiteness of the energy loss in viscosity or energy dissipation. Equations 14-16 relate quantities for the conditions of regularity. Proposition 3.4 is the equivalence of smooth Schwartz initial data with smooth compact support initial data for the formulation of the 4th Clay millennium problem. Propositions 3.5-3.9 are necessary and sufficient conditions for regularity, either for the Euler or Navier-

Stokes equations, while Propositions 4.10 is a necessary and sufficient condition of regularity for only the Navier-Stokes with non-zero viscoidity.

In the next I want to use, the basic local existence and uniqueness of smooth solutions to the Navier-Stokes (and Euler) equations , that is usually referred also as the well posedness, as it corresponds to the existence and uniqueness of the physical reality causality of the flow. The theory of well-posedness for smooth solutions is summarized in an adequate form for this paper by the Theorem 5.4 in TAO, T. 2013.

I give first the definition of **mild solution** as in TAO, T. 2013 page 9. Mild solutions must satisfy a condition on the pressure given by the velocities. Solutions of smooth initial Schwartz data are always mild, but the concept of mild solutions is a generalization to apply for non-fast decaying in space initial data , as the Schwartz data, but for which data we may want also to have local existence and uniqueness of solutions.

DEFINITION 3.1

We define a H^1 *mild solution* (u, p, u_0, f, T) to be fields $u, f : [0, T] \times R^3 \to R^3$,

$p : [0, T] \times R^3 \to R$, $u_0 : R^3 \to R^3$, with $0 < T < \infty$, obeying the regularity hypotheses

$$u_0 \in H_x^1(R^3)$$

$$f \in L_t^\infty H_x^1([0,T] \times R^3)$$

$$u \in L_t^\infty H_x^1 \cap L_t^2 H_x^2([0,T] \times R^3)$$

with the pressure p being given by (Poisson)

$$p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla \cdot f \qquad \text{(eq. 7)}$$

(Here the summation conventions is used , to not write the Greek big Sigma).

which obey the incompressibility conditions (eq. 2), (eq. 3) and satisfy the integral form of the Navier-Stokes equations

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}(-(u \cdot \nabla)u - \nabla p + f)(t')dt'$$

with initial conditions $u(x,0)=u^0(x)$ **.**

We notice that the definition holds also for the in viscid flows, satisfying the Euler equations. The viscosity coefficient here has been normalized to $\nu=1$**.**

In reviewing the local well-posedness theory of H^1 mild solutions, the next can be said. The content of the theorem 5.4 in TAO, T. 2013 (that I also state here for the convenience of the reader and from which derive our PROPOSITION 3.2) is largely standard (and in many cases it has been improved by more powerful current well-posedness theory). I mention here for example the relevant research by PRODI G 1959 and SERRIN,J 1963, The local existence theory follows from the work of KATO, T. PONCE, G. 1988 , the regularity of mild solutions follows from the work of LADYZHENSKAYA, O. A. 1967 . There are now a number of advanced local well-posedness results at regularity, especially that of KOCH, H., TATARU, D.2001.

There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions with different methods. As it is referred in C. L. FEFFERMAN 2006 I refer too the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4,

I state here for the convenience of the reader the summarizing theorem 5.4 as in TAO T. 2013. I omit the part (v) of Lipchitz stability of the solutions from the statement of the theorem. I use the standard O() notation here, x=O(y) meaning x<=cy for some absolute constant c. If the constant c depends on a parameter k, we set it as index of $O_k()$.

It is important to remark here that the existence and uniqueness results locally in time (well-posedness) , hold also not only for the case of viscous flows following the Navier-Stokes equations, but also for the case of inviscid flows under the Euler equations. There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions both for the Navier-Stokes and the Euler equation with the same methodology , where the value of the viscosity coefficient v=0, can as well be included. I refer e.g. the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138.

PROPOSITION 3.1 *(Local well-posedness in H^1). Let (u_0, f, T) be H^1 data.*

(iii) *(Strong solution) If (u, p, u_0, f, T) is an H^1 mild solution, then*

$$u \in C_t^0 H_x^1([0,T] \times R^3)$$

(iv) *(Local existence and regularity) If*

$$(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})^4 T < c$$

for a sufficiently small absolute constant c > 0, then there exists

a H^1 mild solution (u, p, u_0, f, T) with the indicated data, with

$$\|u\|_{X^k([0,T]\times R^3)} = O(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})$$

and more generally

$$\|u\|_{X^k([0,T]\times R^3)} = O_k(\|u_0\|_{H_X^k(R^3)}, \|f\|_{L_t^1 H_X^1(R^3)}, 1)$$

for each k>=1 . In particular, one has local existence whenever

T is sufficiently small, depending on the norm H^1(u_0, f, T).

(iii) (Uniqueness) There is at most one H^1 mild solution (u, p, u_0, f, T)

with the indicated data.

(iv) ***(Regularity)*** *If (u, p, u_0, f, T) is a H^1 mild solution, and (u_0, f, T)*

is (smooth) Schwartz data, then u and p is smooth solution; in fact, one has

$\partial_t^j u, \partial_t^j p \in L_t^\infty H^k([0,T]\times R^3)$ *for all j, K >=0.*

For the proof of the above theorem, the reader is referred to the TAO, T. 2013 theorem 5.4, but also to the papers and books , of the above mentioned other authors**.**

Next I state the local existence and uniqueness of smooth solutions of the Navier-Stokes (and Euler) equations with smooth Schwartz initial conditions , that I will use in this paper , explicitly as a PROPOSITION 4.2 here.

PROPOSITION 3.2 **Local existence and uniqueness of smooth solutions or smooth well posedness**. *Let* ***$u_0(x)$*** *,* ***$p_0(x)$*** *be smooth and Schwartz initial data at t=0 of the Navier-Stokes (or Euler) equations, then there is a finite time interval [0,T] (in general depending on the above initial conditions) so that there is a unique smooth local in time solution of the Navier-Stokes (or Euler) equations*

u(x) *,* ***p(x)*** $\in C^\infty(R^3 \times [0,T])$

Proof: We simply apply the PROPOSITION 3.1 above and in particular , from the part (ii) and the assumption in the PROPOSITION 3.2, that the initial data are smooth Schwartz , we get the local existence of H^1 mild solution $(u, p, u_0, 0, T)$. From the part (iv) we get that it is also a smooth solution. From the part (iii), we get that it is unique.

As an alternative we may apply the theorems in MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138, and getthe local in time solution, then derive from the part (iv) of the PROPOSITION 4.1 above, that they are also in the classical sense smooth. QED.

Remark 3.1 We remark here that the property of smooth Schwartz initial data, is not known in general if is conserved in later times than t=0, of the smooth solution in the Navier-Stokes equations, because it is a very strong fast decaying property at spatially infinity. But for lower rank derivatives of the velocities (and vorticity) we have the **(global and) local energy estimate** , and **(global and) local enstrophy estimate** theorems that reduce the decaying of the solutions at later times than t=0, at spatially infinite to the decaying of the initial data at spatially infinite. See e.g. TAO, T. 2013, Theorem 8.2 (Remark 8.7) and Theorem 10.1 (Remark 10.6).

Furthermore in the same paper of formal formulation of the Clay millennium conjecture , L. FEFFERMAN 2006 (see page 3rd line 6 from above), it is stated that the 3D global regularity of such smooth solutions is controlled by the **bounded accumulation in finite time intervals** of the vorticity (Beale-Kato-Majda). I state this also explicitly for the convenience of the reader, for smooth solutions of the Navier-Stokes equations with smooth Schwartz initial conditions, as the PROPOSITION 3.6 **When we say here bounded accumulation** e.g. of the deformations D, **on finite internals**, we mean in the sense e.g. of the proposition 5.1 page 171 in the book MAJDA A.J-BERTOZZI A. L. 2002 , which is a definition designed to control the existence or not of finite blowup times. In other words for any finite time interval

[0, T], there is a constant M such that

$$\int_0^t |D|_{L^\infty}(s)ds <= M$$

I state here for the convenience of the reader, a well known proposition of equivalent necessary and sufficient conditions of existence globally in time of solutions of the Euler equations, as inviscid smooth flows. It is the proposition 5.1 in MAJDA A.J-BERTOZZI A. L. 2002 page 171.

The *stretching* is defined by

$S(x,t) =: D\xi \cdot \xi$ if $\xi \neq 0$ and $S(x,t) =: 0$ if $\xi = 0$ where $\xi =: \dfrac{\omega}{|\omega|}$, ω being the vortcity.

PROPOSITION 3.3 *Equivalent Physical Conditions for Potential Singular Solutions of the Euler equations . The following conditions are equivalent for smooth Schwartz initial data:*

(1) The time interval, [0, T^*) *with* $T^* < \infty$ *is a maximal interval of smooth* H^s

existence of solutions for the 3D Euler equations.

(2) The vorticity ω accumulates so rapidly in time that

$$\int_0^t |\omega|_{L^\infty}(s)ds \to +\infty$$ *as t tends to T**

(3) The deformation matrix D accumulates so rapidly in time that

$$\int_0^t |D|_{L^\infty}(s)ds \to +\infty$$ *as t tends to T**

*(4) The stretching factor S(***x***, t) accumulates so rapidly in time that*

$$\int_0^t [\max_{x \in R^3} S(x,s)]ds \to +\infty$$ *as t tends to T**

<u>The next theorem establishes the equivalence of smooth connected compact support initial data with the smooth Schwartz initial data, for the homogeneous version of the 4th Clay Millennium problem. It can be stated either for local in time smooth solutions or global in time smooth solutions. The advantage assuming connected compact support smooth initial data, is obvious, as this is preserved in time by smooth functions and also integrations are easier when done on compact connected sets.</u>

PROPOSITION 3.4. **(3D global smooth compact support non-homogeneous regularity implies 3D global smooth Schwartz homogeneous regularity)** *If it holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth compact support (connected with smooth boundary) initial data of velocities and pressures (thus finite initial energy) and smooth compact support (the same connected support with smooth boundary) external forcing for all times t>0, exist also globally in time t>0 (are globally regular) then it also holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth Schwartz initial data of velocities and pressures (thus finite initial energy) , exist also globally in time for all t>0 (are regular globally in time).*

Proof: see KYRITSIS, K. June 2017, or KYRITSIS, K. February 2019, PROPOSITION 6.4)

Remark 3.2 Finite initial energy and energy conservation equations:

When we want to prove that the smoothness in the local in time solutions of the Euler or Navier-Stokes equations is conserved, and that they can be extended indefinitely in time, we usually apply a "reduction ad absurdum" argument: Let the maximum finite time T* and interval [0,T*) so that the local solution can be extended smooth in it.. Then the time T* will be a blow-up time, and if we manage to extend smoothly the solutions on [0,T*]. Then there is no finite Blow-up time T* and the solutions holds in [0,+∞). Below are listed necessary and sufficient conditions for this extension to be possible. Obviously not smoothness assumption can be made for the time T*, as this is what must be proved. But we still can assume that at T* the energy conservation and momentum conservation will hold even for a singularity at T*, as these are universal laws of nature, and the integrals that calculate them, do not require smooth functions but only integrable functions, that may have points of discontinuity.

A very well known form of the energy conservation equation and accumulative energy dissipation is the next:

$$\frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx + v\int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt = \frac{1}{2}\int_{R^3}\|u(x,0\|^2 dx \qquad \text{(eq. 8)}$$

where

$$E(0) = \frac{1}{2}\int_{R^3}\|u(x,0)\|^2 dx \qquad \text{(eq. 9)}$$

is the initial finite energy

$$E(T) = \frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx \qquad \text{(eq. 10)}$$

is the final finite energy

and $\Delta E = v\int_0^T \int_{R^3} \left\| \nabla u(x,t) \right\|^2 dxdt$ (eq. 11)

is the accumulative finite energy dissipation from time 0 to time T , because of viscosity in to internal heat of the fluid. For the Euler equations it is zero. Obviously

ΔE<=E(0)>=E(T) (eq. 12)

The rate of energy dissipation is given by

$$\frac{dE}{dt}(t) = -v\int_{R^3} \left\| \nabla u \right\|^2 dx < 0 \quad \text{(eq. 13)}$$

(v, is the density normalized viscosity coefficient. See e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 Proposition 1.13, equation (1.80) pp. 28)

Remark 3.3 The next are 3 very useful inequalities for the unique local in time [0,T], smooth solutions u of the Euler and Navier-Stokes equations with smooth Schwartz initial data and finite initial energy (they hold for more general conditions on initial data, but we will not use that):

By $||.||_m$ we denote the Sobolev norm of order m. So if m=0 itis essentially the L_2-norm. By $||.||_{L\infty}$ we denote the supremum norm, u is the velocity, ω is the vorticity, and cm, c are constants.

1) $\left\| u(x,T) \right\|_m \leq \left\| u(x,0) \right\|_m \exp(\int_0^T c_m \left\| \nabla (u(x,t) \right\|_{L_\infty} dt)$ (eq. 14)

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp117, equation (3.79))

2) $\left\| \omega(x,t) \right\|_0 \leq \left\| \omega(x,0) \right\|_0 \exp(c\int_0^T \left\| \nabla u(x,t) \right\|_{L_\infty} dt)$ (eq. 15)

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp117, equation (3.80))

3) $\left\| \nabla u(x,t) \right\|_{L_\infty} \leq \left\| \nabla u(x,0) \right\|_0 \exp(\int_0^t \left\| \omega(x,s) \right\|_{L_\infty} ds)$ (eq. 16)

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp118, last equation of the proof)

The next are a list of well know necessary and sufficient conditions , for regularity (global in time existence and smoothness) of the solutions of Euler and Navier-Stokes equations, under the standard assumption in the 4th Clay Millennium problem of smooth Schwartz initial data, that after theorem Proposition 4.4 above can be formulated equivalently with smooth compact connected support data. We denote by T* be the maximum Blow-up time (if it exists) that the local solution u(x,t) is smooth in [0,T*).

DEFINITION 3.2

*When we write that **a quantity Q(t) of the flow ,in general depending on time, is uniformly in time bounded during the flow**, we mean that there is a bound M independent from time , such that Q(t)<=M for all t in [0, T*).*

8) PROPOSITION 3.5 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth Schwartz initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the **Sobolev norm || u(x,t)||$_m$** , m>=3/2+2 , remains bounded , by the same bound in all of [0,T*), then , there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)*

Remark 3.4 See for a proof e.g. . MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, line 10 from below)

9) PROPOSITION 3.6 **(Necessary and sufficient condition for regularity.** Beale-Kato-Majda**)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the **vorticity has bounded by M, accumulation** in [0,T*]:*

$$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M \qquad \textit{(eq17)}$$

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 3.5 See e.g. . MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, Theorem 3.6. Also page 171 theorem 5.1 for the case of inviscid flows. . See also LEMARIE-RIEUSSET P.G. 2002 . Conversely if regularity holds, then in any interval from the smoothness in a compact connected set, the vorticity is supremum bounded. The above theorems in the book MAJDA A.J-BERTOZZI A. L. 2002 guarantee that the above conditions extent the local in time solution to global in time , that is to solutions (u, p, u_0, f, T) which is H^1 mild solution, **for any T**. Then applying the part (iv) of the PROPOSITION 4.1 above, we get that this solutions is also smooth in the classical sense, for all T>0, thus globally in time smooth.

10) PROPOSITION 3.7 **(Necessary and sufficient condition of vorticity for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the **vorticity is bounded by M, in the supremum norm** L∞ in [0,T*] and on any compact set:*

$\|\omega(x,t)\|_{L_\infty} \le M$ *for all t in [0,T*)* *(eq. 18)*

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 3.6 Obviously if $\|\omega(x,t)\|_{L_\infty} \le M$, then also the integral exists and is bounded: $\int_0^{T*} \|\omega(x,t)\|_{L_\infty} dt \le M_1$ and the previous proposition 3.6 applies. Conversely if regularity holds, then in any interval from smoothness in a compact connected set, the vorticity is supremum bounded.

11) PROPOSITION 3.8 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the space partial derivatives or Jacobean is bounded by M, in the supremum norm L∞ in [0,T*]:*

$\|\nabla u(x,t)\|_{L_\infty} \le M$ *for all t in [0,T*)* *(eq. 19)*

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 3.7 Direct from the inequality (eq.14) and the application of the proposition 3.5. Conversely if regularity holds, then in any finite time interval from smoothness, the space derivatives are supremum bounded.

12) PROPOSITION 3.9 **(**FEFFERMAN C. L. 2006. **Velocities necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if*

the velocities ||u(x,t)|| do not get unbounded as t->T.*

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞).*

Remark 3.8.This is mentioned in the Standard formulation of the 4^{th} Clay Millennium problem FEFFERMAN C. L. 2006 pp.2 , line 1 from below: quote "...For the Navier-Stokes equations (v>0) , if there is a solution with a finite blowup time T, then the velocities $u_i(x,t)$, 1<=i<=3 become unbounded near the blowup time." The converse-negation of this is that if the velocities remain bounded near the T*, then there is no Blowup at T* and the solution is regular or global in time smooth. Conversely of course , if regularity holds, then in any finite time interval, because of the smoothness, the velocities, in a compact set are supremum bounded.

I did not find a dedicated such theorem in the books or papers that I studied, but I take it for granted as the official formulation of the problem too. . **Furthermore, we shall interpret the velocities either as coordinate or material fluid velocities.**

We notice that Fefferman C.L. states this condition only for the viscous flows, but since PROPOSITION 3.7 holds for the inviscid flows under the Euler equations, this necessary and sufficient condition holds also for the inviscid flows too.

Remark 3.9.

Similar results about the local smooth solutions, hold also for the non-homogeneous case with external forcing which is nevertheless space-time smooth of bounded accumulation in finite time intervals. Thus an alternative formulation to see that the velocities and their gradient , or in other words up to their 1^{st} derivatives and the external forcing also up to the 1^{st} derivatives , control the global in time existence is the next proposition. See TAO. T. 2013 Corollary 5.8

PROPOSITION 3.10 (**Maximum Cauchy development**)

Let (u_0, f, T) be H^1 data. Then at least one of the following two statements hold:

1) There exists a mild H^1 solution (u, p, u_0, f, T) in $[0,T]$,with the given data.

2)There exists a blowup time $0 < T^* < T$ and an incomplete mild H^1 solution

(u, p, u_0, f, T^*) up to time T^* in $[0, T^*)$, defined as complete on every $[0,t]$, $t<T^*$ which blows up in the enstrophy H^1 norm in the sense that

$$\lim_{t\to T^*, t<T^*} \|u(x,t)\|_{H^1_x(R^3)} = +\infty$$

Remark 3.10 The term “almost smooth” is defined in TAO, T. 2013, before Conjecture 1.13. The only thing that almost smooth solutions lack when compared to smooth solutions is a limited amount of time differentiability at the starting time t = 0;

The term *normalized pressure*, refers to the symmetry of the Euler and Navier-Stokes equations to substitute the pressure, with another that differs at, a constant in space but variable in time measureable function. In particular normalized pressure is one that satisfies the (eq. 7) except for a measurable at a, constant in space but variable in time measureable function. It is proved in TAO, T. 2013, at Lemma 4.1, that the pressure is normalizable (exists a normalized pressure) in almost smooth finite energy solutions, for almost all times. The viscosity coefficient in these theorems of the above TAO paper has been normalized to **ν=1.**

PROPOSITION 3.11 (**Differentiation of a potential**)

Let a sub-Newtonian kernel $K(x,y)$, and f a bounded and integral function on the open set Ω, of R^n ,then for all x in $\Omega\cup\Sigma$, where Σ is a relatively open subset of $\partial\Omega$,

the $\int_\Omega K(x,y)F(y)dy$ is in $C^1(\Omega\cup\Sigma)$ and

$$D_{xi} \int_\Omega K(x,y)F(y)dy=\int_\Omega D_{xi} K(x,y)F(y)dy$$

Proof: By D_{xi} we denote the partial derivative relative to x_i . For the definition of sub-Newtonian kernel and a proof of the above theorem, see HELMS L.L. (2009) paragraph 8.2 pp 303 and Theorem 8.2.7 pp 306. QED.

PROPOSITION 3.12 (**Estimates of partial derivatives of harmonic functions**)

Assume u is harmonic function in the open set Ω of R^n. Then

$$|D^a u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

For each ball $B(x_0, r) \subset \Omega$ and each multi-index a of order |a|=k.

Here $C_0 = \frac{1}{a(n)}$, $C_k = \frac{(2^{n+1}nk)^k}{a(n)}$, *k=1,….*

In particular, for the fundamental harmonic function u=1/(||x-y||) the next estimates for the partial derivatives hold:

$$\frac{\partial}{\partial x_i}\frac{1}{\|x-y\|} \leq \frac{1}{\|x-y\|^2} \qquad \text{(eq. 20)}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{\|x-y\|} \leq \frac{4}{\|x-y\|^3} \qquad \text{(eq.21)}$$

And in general there is constant C(n, β) such that

$$\left|D^{\beta}u\right| \leq C(n,\beta)\frac{1}{\|x-y\|^{n-2+k}} \quad \text{if } |\beta|=\kappa>=1 \qquad \text{(eq.22)}$$

Proof: See EVANS L. C. (2010) chapter 2, Theorem 7, pp 29. And for the fundamental harmonic function also see HELMS L.L. (2009), pp 317 equations 8.18, 8.19, 8.20

QED.

PROPOSITION 3.13 (**The well-known divergence theorem in vector calculus**)

Let an non-empty bounded open set Ω, of R^n with C^1 boundary ∂Ω, and let

F: $\bar{\Omega} \to R^n$ be vector field that is continuously differentiable in Ω and continuous up to the boundary. Then the divergence theorem asserts that

$$\int_{\Omega} \nabla \cdot F = \int_{\partial\Omega} F \cdot v \qquad \text{(eq. 23)}$$

where v is the outward pointing unit normal to the boundary ∂Ω.

PROPOSITION 3.14 **(Representation formula of the bounded solutions of the Poisson equation.)** *Let* $f \in C_c^2(R^n), n \geq 3.$ *In other words f is with continuous second derivatives, and of compact support. Then any bounded solution of the scalar Poisson equation*

$-\Delta u = f$ *in R^n has the form*

$u(x) = \int_{R^n} \Phi(x-y) f(y) dy + C$ $x \in R^n$ *for some constant C, and Φ(x) is the fundamental harmonic function. For n=3 ,* $\Phi(x) = \frac{1}{4\pi\|x\|}$.

Proof: See EVANS L. C. (2010) §2.2 Theorem 1 pp23 and mainly Theorem 9 pp 30. The proof is a direct consequence of the Liouville's theorem of harmonic functions. There is a similar representation described e.g. in as in MAJDA, A.J-BERTOZZI, A. L. 2002 §1.9.2 Lemma 1.12 pp 38, where f is defined on all of *R^n* and not only on a compact support. Notice also that solutions of the above Poisson equation that are also of compact support are included in the representation. More general settings of the Poisson equation with solutions only on bounded regions and with prescribed functions on the boundary of the region, do exist and are unique but require correction terms and Green's functions as described again in EVANS L. C. (2010) §2.2 Theorem 5 pp28 QED.

Remark 3.11

Such a more general form of the solution of the Poisson equations as in MAJDA, A.J-BERTOZZI, A. L. 2002 §1.9.2 Lemma 1.12 pp 38, and in particular when smooth bounded input data functions lead to smooth bounded output solutions, could allow us to state the new necessary and sufficient conditions of the next paragraph 4, with the more general hypothesis of the smooth Schwartz initial data, rather than compact support initial data Nevertheless, it holds the equivalence of the smooth Schwartz initial data with compact support initial data holds after PROPOSITION 3.4. and KYRITSIS, K. June 2017, PROPOSITION 6.4.. **In other words, we could proceed and try to prove the 4th Clay Millennium problem without utilizing the PROPOSITION 3.4. of the equivalence of smooth Schwartz initial data and smooth compact support initial data for the 4th Clay Millennium problem**. Still it is simpler when thinking about the phenomena to have in mind simpler settings like compact support flows and that is the mode in which we state our results in the next in this paper.

4. The pressures sufficient condition of regularity.

In this paragraph we utilize one part of our main strategy to solve the 4th Clay Millennium problem, which is to use the Poisson equation as many times as we can, to express some quantities of the flow by other quantities, and also apply the well-known solutions of the Poisson equations through the special smooth harmonic functions. The second part of the strategy is to integrate over trajectories, and derive integral equations of the velocities and their partial spatial derivatives. Since integrals of velocities may turn out to involve the finite energy which is invariant we hope so to bound the supremum norm of the special partial derivatives of the velocities and use the very well-known necessary and sufficient condition for regularity as in PROPOSITION 3.8

PROPOSITION 4.1. **(The pressures, necessary and sufficient condition for regularity)**

Let the local solution u(x,t) , t in [0,T) of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, then it can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), and thus to all times [0,+∞), in other words the solution is regular, if and only there is a time uniform bound M for the pressures p , in other words such that*

$p \leq M$ for all t in [0,T) Still in other words smoothness and boundedness of the pressures p on the compact support V(t) and in finite time intervals [0,T] is a characteristic condition for regularity.*

Proof: Let us start from this characteristic smoothness and boundedness of the pressures p on the compact support V(t) and in finite time intervals [0,T] to derive regularity.

We notice that in the Navier-Stokes equations of incompressible fluids the pressure forces defines a ***conservative force-field***, as it is the gradient of a scalar-field that of the pressures p, which play the role of scalar potential. And this property, of being a **conservative force-field**, is **an invariant** during the flow. It is an invariant **even for viscous flows**, compared to other classical invariants, the Kelvin circulation invariant and the Helmholtz vorticity-flux invariant which hold only for inviscid flows. That the force-field F_p is a conservative field, means that if we take two points $x_1(0)$, $x_2(0)$, and any one-dimensional path $P(x_1(0), x_2(0))$, starting and ending on them, then for any test particle of mass m, the integral of the work done by the forces is independent from the particular path, and depends only on the two points $x_1=x_1(0)$, $x_2=x_2(0)$, and we denote it here by $W(x_1,x_2)$.

$$W(x_1,x_2) = \int_{P(x1,x2)} F_p ds \qquad \text{(eq. 24)}$$

In particular, it is known by the **gradient theorem**, that this work equals, the difference of the potential at these points, and here, it is the pressures:

$W(x_1, x_2)=(1/c)||p(x_2(0))-p(x_1(0))||$. (eq. 25)

(The constant (1/c) here is set, because of the normalization of the constant density in the equations of the Navier-Stokes, and accounts for the correct dimensions of units of measurements of the pressure, force, and work).

Similarly, if we take a test-flow , of test particles instead of one test particle, in the limit of points, again the work density, depends only on the two points x_1, x_2.

From the hypothesis that the pressure are time-uniformly bounded by the same constant M in [0,T*) we deduce that the integral on a trajectory in (eq24) is also time-uniformly bounded by the same constant M in [0,T*).

$$\int_{P(x0,x1)} F_p ds <= M$$

We may re-write this integral by changing integration parameter to be the time as

$\int_{P(t0,t1)} F_p \frac{ds}{dt} dt <= M$ in [0,T*). Or as it is on a trajectory

$\int_{P(t0,t1)} F_p u_m dt <= M$ in [0,T*). (eq26)

Where u_m is the material velocity on the trajectory.

If the u_m->+∞ blows-up as t->T*, then also the convective acceleration

$\frac{Du}{Dt}$ ->+∞ will blowup as t->T*,

And from the Navier-Stokes equations

$$\frac{Du}{Dt} = -\nabla p + v\Delta u_i \qquad \text{(eq. 27)}$$

The pressure forces $F_p = -\nabla p$ -> +∞ will blowup as t->T*,

as the friction term only subtracts from the pressure forces.

Nevertheless if both F_p and u_m will blow-up, so also it will , the integral in

(eq 15) which is a contradiction. Thus the material velocities do not blowup!

Thus we may apply the necessary and sufficient condition for regularity as in PROPOSITION 3.6 **(**FEFFERMAN C. L. 2006. **Velocities necessary and sufficient condition for regularity)** and we derive the regularity.

QED.

PROPOSITION 4.2. **(Smooth particle Trajectory mapping and Trajectories finite length, necessary condition of regularity)**

Let the local solution u(x,t), t in [0,T) of the Euler or Navier-Stokes equations of inviscid or viscous flows correspondingly, and with smooth compact connected support initial data, then it can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), and thus to all times [0,+∞), in other words the solution is regular, if and only if the particle trajectory mapping is smooth in finite time intervals and the trajectories-paths are smooth and of length l(a,t)<=M that remains bounded by a constant M for all t in [0,T*).*

Proof:

The particle trajectory mapping is the representation of the spatial flow in time of the fluid per trajectories-paths. For a definition see MAJDA, A.J-BERTOZZI, A. L. 2002 § 1.3 Equation 1.13 pp 4. Here we apply this mapping on the compact support V initial data.

Let us assume now that the solutions is regular. Then also for all finite time intervals [0,T] , the velocities and the accelerations are bounded in the L_∞ , supremum norm, and this holds along all trajectory-paths too. Then also the length of the trajectories, as they are given by the formula

$$l(a_0,T) = \int_0^T \|u(x(a_0,t)\| dt \qquad \text{(eq. 28)}$$

are also bounded and finite (see e.g. APOSTOL T. 1974 , theorem 6.6 p128 and theorem 6.17 p 135). Thus if at a trajectory the lengths becomes unbounded as t converges to T*, then there is a blow-up.

QED.

5. The finite energy, bounded pressure variance theorem for inviscid and viscous flows and the solution of the 4^{th} Clay Millennium problem.

Remark 5.1 This paragraph utilizes two simple techniques

a) Energy conservation in various alternative forms and formulae.

b) The property of the pressures forces being conservative in the present situation of incompressible flows (gradient theorem).

The 4th Clay Millennium problem is not just a challenging exercise of mathematical calculations. It is an issue of the standard modelling the physical reality, and therefore we may utilize all our knowledge of the underlying physical reality.

In the strategy that this paper has adopted here to solve the 4th Clay Millennium problem, in a short and elegant way, we will involve as much as possible intuitive physical ideas that may lead us to choose the correct and successful mathematical formulae and techniques, still everything will be within strict and exact mathematics. As T. Tao has remarked in his discussion of the 4thClay Millennium problem, to prove that the velocity remains bounded (regularity) for all times, by following the solution in the general case, seems hopeless due to the vast number of flow-solution cases. And that the energy conservation is not of much help. And it seems that it is so! But we need more smart and shortcut ideas, through invariants of the flow. In particular, we need clever techniques to calculate in alternative ways part of the energy of the flow, with virtual-test flows, and alternative integrals of virtual work of the pressure forces on instantaneous paths, and that still have the physical units' dimensions of energy. We will base our strategy to the next three factors

1) The conservation of energy and the hypothesis of finite initial energy. Then as by proposition 3.8 , we have from this necessary and sufficient condition of regularity that we need to have that the partial derivatives of the Jacobean are bounded: $\|\nabla u(x,t)\|_{L_\infty} \le M$, and are uniformly in time bounded in the maximal time interval [0, T*) that a solution exists, then we need to highlight a formula that computes <u>*the partial derivatives of the velocities from integrals of the velocities*</u> in space and time till then, because the bounded energy invariant is in the form of integrals of velocities.
2) The shortcut of physical magnitude with physical units' dimensions of energy as indeed calculating energy: In other words if we reach in the calculations to an expression which has as physical magnitude the physical dimensions units of energy then the expression calculates indeed energy.
   This is very valuable and discriminates the solving the problem as problem of mathematical physics, compared to solving it as pure mathematical problem in the context of partial differential equations.

3) The technique of virtual-test flows on instantaneous paths, to find special formulas for the calculation of energy from alternative magnitudes. Instead of having to recalculate the energy starting from the classical formulae based on the velocities and transform it as the fluid flows, we may use shortcuts to calculate parts of the energy of the fluid based on alternative perceptions, like virtual test-particles flows, and work of the pressure forces on instantaneous paths. Of course the alternative formulae must always have the physical units' dimensions of energy.
4) Meanwhile one smart idea to start is to think of alternative ways that forms of energy and projections of them on to bundle of paths, can be measured, even at single time moment and state of the fluid and relate it with its total energy which is finite and remains bounded throughout the flow. Such

alternative measurements of parts of the energy as projected on to a bundle of paths, can be done by integrating the conservative pressure forces F_p of the fluid (gradient of the pressures) on paths AB, of space, and relate the resulting theoretical work of them with the pressure differences p(A)-p(B) since the pressures are a potential to such conservative pressure forces.

PROPOSITION 5.1. (**The finite energy, uniformly in time bounded pressure-variance, theorem).**

Let a local in time , t in [0,T) , smooth flow solution with velocities u(x,t) , with pressures p(x,t), of the Navier-Stokes equations of viscous fluids or of Euler equations of inviscid fluids, with smooth Schwartz initial data, and finite initial energy E(0), as in the standard formulation of the 4^{th} Clay Millennium problem,. Then the pressure differences $|p(x_2(t))-p(x_1(t))|$ for any two points $x_1(t)$, $x_2(t)$, for times that the solution exists, remain bounded by kE(0), where k is a constant depending on the initial conditions, and E(0) is the finite initial energy.

Proof: Let us look again at the Navier-Stokes equations as in (eq. 1) that we bring them here

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u_i \qquad \text{(eq. 29)}$$

Where $\frac{Du}{Dt}$ is the material acceleration, along the trajectory path.

(It is reminded to the reader, that in the equations of Navier-Stokes, as in (eq. 1) as the density, is constant, it is custom to either normalised to 1, or it is divided out from the left side and it is included in the pressures and viscosity coefficient).

We may separate the forces (or forces multiplied by a constant mass density) , that act at a point, by the two terms of the right side as

$$F_p = -\nabla p \qquad \text{(eq. 30)}$$

which is the force-field due the pressures and the

$$F_v = \nu \Delta u \qquad \text{(eq. 31)}$$

which is the force-field due to the viscosity.

We notice that (eq. 30) defines a ***conservative force-field***, as it is the gradient of a scalar-field that of the pressures p, which play the role of scalar potential. And this property, of being a **conservative force-field**, is **an invariant** during the flow. It is an invariant **even for viscous flows**, compared to other classical invariants, the Kelvin circulation invariant and the Helmholtz vorticity-flux invariant which hold only for inviscid flows. That the force-field F_p is a conservative field, means that if we take two points $x_1(0)$, $x_2(0)$, and any one-dimensional path $P(x_1(0), x_2(0))$, starting and ending on them, then for any test particle of mass m, the integral of the work done by the forces is independent from the particular path, and depends only on the two points $x_1=x_1(0)$, $x_2=x_2(0)$, and we denote it here by $W(x_1,x_2)$.

$$W(x_1, x_2) = \int_{P(x1,x2)} F_p ds \qquad \text{(eq. 32)}$$

In particular, it is known by the **gradient theorem**, that this work equals, the difference of the potential at these points, and here, it is the pressures:

$W(x_1, x_2)=(1/c)||p(x_2(0))-p(x_1(0))|$. (eq. 33)

(The constant (1/c) here is set, because of the normalization of the constant density in the equations of the Navier-Stokes, and accounts for the correct dimensions of units of measurements of the pressure, force, and work).

Similarly, if we take a test-flow , of test particles instead of one test particle, in the limit of points, again the work density, depends only on the two points x_1, x_2.

Let now again the two points $x_1(0)$, $x_2(0)$, at the initial conditions of the flow then as we assume Schwartz smooth initial conditions (and not connected compact smooth initial conditions) , there is at least one double circular cone denoted by $DC(x_1(0),x_2(0))$, made by two circular cones united at their circular bases C and with vertices $x_1(0)$, $x_2(0)$ opposite to the plane of the common circular base C. And let us take a bundle of paths, that start from $x_1(0)$, and end at $x_2(0)$ and fill all the double cone DC. We may assume now a test-fluid (a flow of test-particles), inside this double cone which has volume V, that flows from $x_1(0)$, to $x_2(0)$ along these paths. Let us now integrate the work-density on paths done by the pressure forces F_p of the original fluid, as they act on the test-fluid, and inside this 3 dimensional double cone $DC(x_1(0),x_2(0))$. **This will give an instance of a spatial distribution of work done by the pressure forces in the fluid as projected to the assumed paths**. This energy is from the instant action of the pressure forces spatially distributed, and **depends not only on the volume of integration but also on the chosen bundle of paths**. It is a double integral, 1 dimensional and 2 dimensional (say on the points of the circular base C), covering all the interior of the double cone DC. Because the work-density per path is constant on each such path, by utilizing the Fubini's theorem (e.g. see SPIVAK, M. 1965 pp 56), the final integral is:

$$W = \int_C \int_{x1}^{x2} F_p dx ds$$

$$W(0) = c \cdot V \cdot || \mathrm{p(x2(0)) - p(x1(0))} | . \qquad \text{(eq. 34)}$$

On the other hand this work that would be done by the pressure forces of the original fluid at any time t, is real energy, it is **an instance of a spatial distribution of work done by the pressure forces in the fluid as projected to the assumed paths** and it would be subtracted from the finite initial energy E(0). **Although this energy is only an instance at fixed time t as distributed in space of the action of pressure forces as projected on the assumed bundle of paths, it still has to be finite as calculated in the 3-dimesional double cone. This therefore translates in to that the instance in time of energy flow due to pressure forces as projected on to the assumed bundle of paths, of the original fluid is uniformly in time bounded or in other words bounded in every finite time interval.** Therefore:

$W<=E(0)$. (eq. 35)

And after combining the (eq. 35), with (eq. 34), we get

$$c \cdot V \cdot \| \mathrm{p(x2(0))} - \mathrm{p(x1(0))} | \leq \mathrm{E(0)} \qquad \text{(eq. 36)}$$

As we remarked that the force field F_d due to pressures is conservative and is an invariant of the flow, and so is the volumes , therefore we can repeat this argument for later times t in [0,T), so that we also have

$$W(t) = c \cdot V \cdot \| \mathrm{p(x2(t))} - \mathrm{p(x1(t))} | \leq \mathrm{E(t)} \qquad \text{(eq. 37)}$$

But since due to energy conservation we have E(t)<=E(0) (for inviscid fluids E(t)=E(0)), then also it holds

$$c \cdot V \cdot \| \mathrm{p(x2(t))} - \mathrm{p(x1(t))} | \leq \mathrm{E(0)} \qquad \text{(eq. 38)}$$

Which is what it is required to prove for x_1 , x_2 and k=1/(cV).

In particular, we notice that if there is a supremum sup(p) and infimum inf(p) of pressures at time t, so that | sup(p)- inf(p) | is a measure of the variance of the pressures at time t, then this variance is bounded up to a constant, by the initial finite energy, justifying the title of the theorem. For the case of fluid with smooth compact connected support initial data, the infimum of the pressures is zero, which occurs at the boundary of the compact support. ***So the pressures, in general, are uniformly bounded by the same constant throughout the time interval [0,T*). (which includes the case T*=**+∞)* QED.

Remark 5.2. It is interesting to analyse if we could prove the same proposition as the above 5.1 , not for smooth Schwartz initial data, on all the 3 dimensional space , but for connected compact Cp region smooth Schwartz initial data. Actually the PROPOSITION 3.4. or in KYRITSIS, K. June 2017, PROPOSITION 6.4. is stated for compact support which is simply connected and of smooth boundary and homeomorphic under a smooth 1-1 correspondence with a 3-dimensional sphere.

The arguments with the finite energy and the pressures is the same , except we must be able to find for any two points $x_1(0)$,$x_2(0)$ in the connect compact smooth region Cp, a double circular cone denoted by DC($x_1(0)$,$x_2(0)$), made by two circular cones united at their circular bases C and with vertices $x_1(0)$, $x_2(0)$ opposite to the plane of the common circular base C. This is actually a matter of geometric topology. If the compact connected region Cp is also simply connected and there is a smooth homeomorphism F that it sends it to a sphere S_3 in the 3 dimensional space, as assumed in the proof of the PROPOSITION 3.4. or in KYRITSIS, K. June 2017, PROPOSITION 6.4. then of course for the points F($x_1(0)$), F($x_2(0)$) , there is such a double circular cone denoted by DC(F($x_1(0)$),F($x_2(0)$)), and the inverse image F^{-1}(DC) is a curvilinear such double cone in the compact connected region Cp, Then we apply the 3-dimensional integrations of eq(30) on it. In general, though the If we want to prove the PROPOSITION 5.1. more generally for compact connected region Cp which is not simply connected it is known that there is a smooth homeomorphism F that it sends it to a sphere S_3, with n-handles H_n in the 3 dimensional space, or in symbols. S_3 U H_1 U ….U H_n . Then again there is a choice of the F (a matter of geometric 3-dimensional topology) such that the images of he initial points F($x_1(0)$), F($x_2(0)$), are in the interior of the sphere S_3. And thus there is again a double circular cone denoted by DC(F(x_1(0)),F(x_2(0))), and the inverse image F^{-1}(DC) is a curvilinear such double cone in the compact connected region Cp, to make the integrations as in eq(30). Therefore we may remark that we may have the PROPOSITION 5.1 to hold not for smooth Schwartz initial data, on all the 3 dimensional space , but for connected compact Cp region smooth Schwartz initial data.

PROPOSITION 5.2 (**The solution of the 4th Clay Millennium problem).** *Let a local in time , t in [0,T) , smooth flow solution with velocities u(x,t) , of the Navier-Stokes equations of viscous fluids with smooth Schwartz initial data, and finite initial energy E(0), as in the standard formulation of the 4th Clay Millennium problem. Then the solution is regular, in other words it can be extended as smooth solution for all times t in [0,+∞).*

Proof: From the previous proposition 5.1, we have that the pressures are smooth and bounded in finite time intervals (with smooth Schwartz initial data either on all 3-space or with smooth compact support initial data on a 3-ball) and therefore we apply the pressures sufficient condition of regularity as in PROPOSITION 4.1. **(The pressures sufficient condition for regularity).** Hence the solution of the Clay Millennium problem in its original formulation. We also have proved that not only there is not a blow-up at finite time but also that there is no blow-up even at time= +∞ QED.

Remark 5.3. As we mentioned above and also in Remark 3.11, it was our choice to prefer to use rather than not use, the PROPOSITION 3.4. and KYRITSIS, K. June 2017, PROPOSITION 6.4. in other words, the equivalence of the smooth Schwartz initial data with smooth compact support initial data for the Clay Millennium problem. But as PROPOSITION 4.4 is stated only for the Navier-Stokes equations and viscous flows, and not for the Euler equations. So we missed to prove the regularity of the Euler equations with the previous method. It will be left for the future the investigation of a different line of statements that might as well prove the regularity of the Euler equations under the standard hypotheses for initial data as in the Clay Millennium problem.

6. Epilogue.

In this paper it is has been proved the regularity of the Navier-Stokes equations and therefore it has been solved the 4th Clay Millennium problem. To do so it was utilized mainly that the initial energy was finite, the conservation of the energy, with alternative ways to compute parts of it, and that many of the magnitudes of the flow are interrelated through the very well-studied and regular Poisson equation through harmonic functions. ***Finite initial energy, conservation of energy and the regularity of the pressures gave finally the regularity of the Navier-Stokes equations with the standard hypotheses for initial data as in the corresponding Clay Millennium problem.***

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References

APOSTOL T. 1974 “Mathematical Analysis” *2nd edition Addison Wesley Publishing Company. 1974*

BEALE, J. T., KATO, T., AND MAJDA, A 1984., “Remarks on the breakdown of smooth solutions for the 3-D Euler equations,” *Commun. Math. Phys.* **94**, 61–66, 1984.

BOGOVSKII M. E. 1980, Solution of some vector analysis problems connected with operators div and grad, Trudy *Sem. S. L. Sobolev 80 (1) (1980) 5-40.*

CANNON, J.R., KNIGHTLY, G.H. 1970 : A note on the Cauchy problem for the Navier-Stokes equations. *SIAM J. Appl. Math. 18, 641-644(1970)*

CHORIN A. J. 1968 Numerical solutions of the Navier-Stokes Equations *Mathematics of Computation oct. 1968, vol 22, no 104 pp 745-762*

CONSTANTIN P. 1995, A few results and open problems regarding incompressible fluids, *Notices of the AMS, 42 (1995), 658-663.*

CONSTANTIN P. 2001 , Some open problems and research directions in the mathematical study of fluid dynamics, in Mathematics Unlimited–2001 and Beyond, *Springer Verlag, Berlin, 2001, 353–360.*

CONSTANTIN P. 2007 On the Euler equations of incompressible fluids. *Bulletin of the AMS Vol 44, no 4, October 2007, pp 603-621.*

DURMAGAMBETOV ASSET et al 2015 “Navier Stokes equations Millennium prize problems” Natural science, 2015, 7, 88-89 https://www.scirp.org/journal/ns/

EVANS L. C. (2010) Partial Differential Equations 2nd edition, Graduate Texts in Mathematics, American mathematic Society, Providence Rhode Island

GATSKI T. B. GROSCH C. E.ROSE M. E.1989 The Numerical Solution of the Navier-Stokes equations for 3-Dimensional, Unsteady, Incompressible Flows by Compact Support Schemes

Journal of computation Physics 82 , 298-329 (1989) .

GERMAIN P. 2008 Strong solutions and weak-strong uniqueness for the

nonhomogeneous Navier-Stokes system *Journal d'Analyse Mathématique January 2008, Volume 105, Issue 1, pp 169-196.*

GIGA Y. 1983 Weak and Strong Solutions of the Navier-Stokes Initial Value *Problem PubL RIMS, Kyoto Univ. 19 (1983), 887-910*

GIGA, Y., INUI, K., MATSUI, S. 1999 : On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data. *Advances in fluid dynamics 4, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, 27-68(1999).*

GIGA, Y., MAHALOV, A., NICOLAENKO, B. 2004 : The Cauchy problem for the NavierStokes equations with spatially almost periodic initial data. *Hokkaido University preprint series in mathematics 683, (2004)*

GIGA, Y., MATSUI, S., SAWADA, O. 2001 : Global existence of two-dimensional NavierStokes flow with nondecaying initial velocity. *J. Math. Fluid Mech. 3, no.3, 302-315(2001)*

GIGA, Y., MIYAKAWA, T.1985 : Solutions in Lr of the Navier-Stokes initial value problem. *Arch. Rational Mech. Anal. 89, no.3, 267-281(1985)*

GIGA Y. SAWADA, O. 2003 On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem, *Nonlinear Analysis and Applications, To V. Lakshmikantham on his 80th Birthday vol. I (2003), 549-562.*

HELMS L.L. (2009) Potential Theory , Universitext , Springer Verlag.

FEFFERMAN C. L. 2006 Existence and smoothness of the Navier-Stokes equation, *Millennium Prize Problems, Clay Math. Inst., Cambridge, MA, 2006, 57-67.*

FRIEDLANDER S., DENIS SERRE 2002 *Handbook of Mathematical Fluid Dynamics Vol 1,2, Gulf Professional Publishing, 2002.*

KATO, J. 2003 : The uniqueness of nondecaying solutions for the Navier-Stokes equations. *Arch. Ration. Mech. Anal. 169, no.2, 159-175(2003)*

KATO, T. 1983 : Strong L p -solutions of the Navier-Stokes equation in R m, with applications to weak solutions*. Math. Z. 187, no.4, 471-480(1984)*

KATO, T., FUJITA, H.1962 : On the nonstationary Navier-Stokes system. *Rend. Sem. Mat. Univ. Padova 32, 243-260(1962).*

KATO, T. PONCE, G. 1988 Commutator estimates and the Euler and Navier-Stokes

equations, *CPAM 41 (1988), 891-907.*

KOCH, H., TATARU, D.2001 Well-posedness for the Navier-Stokes equations, *Adv. Math. 157 (2001), no. 1, 22-35.*

KYRITSIS, K. June 2017 "On the solution of the 4th Clay Millennium problem about the Navier-Stokes equations. Equivalence of smooth compact support and smooth Schwartz initial conditions". Proceedings of *the 1st international conference on quantitative, social, biomedical and economic issues ICQSBEI 2017 pp 147-172*

KYRITSIS, K. November 2017 "On the 4th Clay Millennium problem: Proof of the regularity of the solutions of the Euler and Navier-Stokes equations, based on the conservation of particles" *Journal of Scientific Research and Studies Vol 4 (11) , pp304-317,November 2017.*

KYRITSIS. K. Feb 2018 A solution of the 4th Clay Millennium problem about the Navier-Stokes equations. Preprint 1) http://cris.teiep.gr/jspui/handle/123456789/1628?locale=en 2) https://www.researchgate.net/publication/323389082_A_SOLUTION_OF_THE_4_TH_CLAY_MILLENNIUM_PROBLEM_ABOUT_THE_NAVIER-STOKES_EQUATIONS

KYRITSIS, K. February 2019 "The solutions of the 3rd and 4th Clay Millennium problems" *https://arxiv.org/abs/1902.07265*

LADYZHENSKAYA, O. A. 1967 Uniqueness and smoothness of generalized solutions of Navier-Stokes equations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 5 (1967), 169-185; English transl., Sem. Math. Steklov Math. Inst., Leningrad 5 (1969), 60-67.*

LADYZHENSKAYA, O. 1969, The Mathematical Theory of Viscous Incompressible Flows (2nd edition), *Gordon and Breach, New York, 1969.*

LEMARIE-RIEUSSET, P.G. 2002 Recent developments in the Navier-Stokes problem. *Chapman & All/CRC 2002*

MAEKAWA, Y. TERASAWA, Y. 2006 **:** The Navier-Stokes equations with initial data in uniformly local Lp spaces. *Differential and integral equations: an international journal for theory & applications (Athens, Ohio : Khayyam Publ. Co.), Vol. 19, No. 4 (2006), p. 369-400*

MAJDA, A.J-BERTOZZI, A. L. 2002 Vorticity and incompressible flow *Cambridge University texts in applied mathematics 2002.*

MOSCHANDREOU. T. 2021 "No finite time Blow-up for 3D incompressible Navier-Stokes equations, via scaling invariance" Mathematica and Statistics, 9(3): 386-393

MURIEL, A 2000 An Exact Solution of the 3-D Navier-Stokes Equation

Department of Electrical Engineering Columbia University

And Department of Philosophy Harvard University.

Also A. Muriel and M. Dresden, *Physica D 101, 299, 1997.*

POLYANIN, A. D. ZAITSEV, V.F., MOUSSIAUX 2002. *Handbook of First Order Partial Differential Equations,* London : Taylor&Francis , ISBN0-415-27267-x

POLYANIN, A. D 2002. *Handbook of Linear Partial Differential Equations for Engineers and Scientist ,* Boca Raton : Chapman &Hall /CRC press, ISBN 1-58488-299-9

PRODI, G 1959 Un teorema di unicit_a per le equazioni di Navier-Stokes, *Ann. Mat. Pura Appl. 48 (1959), 173-182.*

RUDIN, W. 1976 "Principles of Mathematical Analysis " *3rd edition McGraw-Hill Inc. 1976*

FAKOUR M. et al 2017 Heat transfer and fluid flow of blood with nanoparticles through porous vessels in a magnetic field: A quasi-one dimensional analytical approach. ***Mathematical Biosciences Volume 283, January 2017, Pages 38-47***

SARRA, SCOTT 2003 "The method of characteristics with applications to Conservation Laws", *Journal of Online mathematics and its Applications.*

SERRIN,J 1963 The initial value problem for the Navier-Stokes equations, *In Nonlinear Problems (Proc. Sympos., Madison, Wis., pages 69{98. Univ. of Wisconsin Press, Madison, Wis., 1963*.

SAWADA, O.2003 : On time-local solvability of the Navier-Stokes equations in Besov spaces. *Adv. Differential Equations 8 no.4, 385-412(2003)*

SPIVAK, M. 1965 "Calculus on Manifolds" *The Benjamin/Cumming publishing company 1965.*

TAO, T. 2013 Localisation and compactness properties of the Navier-Stokes global regularity problem. *Analysis & PDE 6 (2013), 25-107*

TAO, T. 2014 Finite time blowup for an averaged three-dimensional Navier-Stokes equation. *Cornell University Library arxiv.org*