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Abstract

We prove a lower bound for the k-th Steklov eigenvalues in terms of an isoperimetric constant called the k-th Cheeger-Steklov constant in three different situations: finite spaces, measurable spaces, and Riemannian manifolds. These lower bounds can be considered as higher order Cheeger type inequalities for the Steklov eigenvalues. In particular it extends the Cheeger type inequality for the first nonzero Steklov eigenvalue previously studied by Escobar in 1997 and by Jammes in 2015 to higher order Steklov eigenvalues. The technique we develop to get this lower bound is based on considering a family of accelerated Markov operators in the finite and measurable situations and of mass concentration deformations of the Laplace-Beltrami operator in the manifold setting which converges uniformly to the Steklov operator. As an intermediary step in the proof of the higher order Cheeger type inequality, we define the Dirichlet–Steklov connectivity spectrum and show that the Dirichlet connectivity spectra of this family of operators converges to (or is bounded by) the Dirichlet–Steklov spectrum uniformly. Moreover, we obtain bounds for the Steklov eigenvalues in terms of its Dirichlet-Steklov connectivity spectrum which is interesting in its own right and is more robust than the higher order Cheeger type inequalities. The Dirichlet–Steklov spectrum is closely related to the Cheeger–Steklov constants.

Résumé

Pour tout $k \in \mathbb{N}$, une borne inférieure pour la k-ième valeur propre de Steklov en termes d'une constante isopérimétrique, appelée la k-ième constante de Cheeger-Steklov, est obtenue dans trois situations différentes : espaces finis, espaces mesurables et variétés riemanniennes. Ces bornes inférieures peuvent être considérées comme des inégalités de type Cheeger d'ordre supérieur pour les valeurs propres de Steklov. En particulier, elles étendent l'inégalité de type Cheeger pour la première valeur propre non nulle de Steklov étudiée par Escobar en 1997 et par Jammes en 2015. La technique développée pour obtenir ces bornes inférieure utilise une famille d'opérateurs de Markov accélérés dans les situations finies et mesurables et une famille d'opérateurs de Laplace-Beltrami déformés et concentrés près de la frontière. Lors d'une étape intermédiaire de la preuve de l'inégalité de type Cheeger d'ordre supérieur, nous définissons le spectre de connectivité de Dirichlet-Steklov et nous montrons que les spectres de connectivité de Dirichlet de cette famille d'opérateurs convergent uniformément vers (ou sont bornés par) le spectre de Dirichlet-Steklov. De plus, nous obtenons des bornes pour les valeurs propres de Steklov en termes du spectre de connectivité de Dirichlet-Steklov, ce dernier étant intéressant en lui-même. Il est aussi plus robuste que les inégalités de type Cheeger d'ordre supérieur. Le spectre de Dirichlet–Steklov et les constantes de Cheeger–Steklov sont étroitement liés.

Keywords: Dirichlet–to–Neumann operator, Steklov problem, eigenvalues, isoperimetric ratios, higher order Cheeger inequalities, finite Markov processes, jump Markov processes, Brownian motion on Riemannian manifolds, Laplace-Beltrami operator.

Mots clés: opérateur de transfert Dirichlet–Neumann, problème de Steklov, valeurs propres, rapports isopérimétriques, inégalités de Cheeger d'ordre supérieur, processus de Markov finis, processus markoviens de saut, mouvement brownien sur les variétés de Riemann, opérateur de Laplace–Beltrami.

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1 Introduction

Let (M,g) be a compact Riemannian manifold of dimension n with smooth boundary, the Steklov eigenvalue problem is

$$\begin{cases} \Delta f = 0, & \text{in } M \\ \frac{\partial f}{\partial \nu} = \sigma f, & \text{on } \partial M \end{cases} \tag{1}$$

where $\Delta = \operatorname{div} \nabla$ is the Laplace–Beltrami operator on M and ν is the unit outward normal vector along ∂M. Its spectrum consists of a sequence of nonnegative real numbers with accumulation point only at infinity. We denote the sequence of the Steklov eigenvalues by

$$0 = \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_k \leqslant \cdots \nearrow \infty$$

The Steklov eigenvalues can be also considered as the eigenvalues of the Dirichlet–to–Neumann operator

$$S : C^{\infty}(\partial M) \ni f \mapsto \frac{\partial F}{\partial \nu} \in C^{\infty}(\partial M)$$

where F is the harmonic extension of f into the interior of M. The Steklov problem was first introduced by Steklov [34] in 1902 for bounded domains of the plane. Many interesting developments and progress in the study of the Steklov problem have been attained in recent years. We refer the reader to the survey paper [21] and the references therein for recent developments, and to [24] for a historical account. The relationship between the Steklov eigenvalues and geometry of the underlying space, and also its similarity and difference with the Laplace eigenvalues have been a main focus of interest and a source of inspiration, see for example [14, 17, 10, 22, 15, 18, 23].

The focus of this paper is on obtaining lower bounds for the k-th Steklov eigenvalue σ_k in terms of some isoperimetric constants in three different settings. Our results can be viewed as counterparts of the higher order Cheeger inequalities for the Laplace eigenvalues in discrete setting proved by Lee, Oveis Gharan and Trevisan [27], and in manifold setting by the second author [31]. It is also an extension of Escobar's [14, 15] and Jammes' [23] results for σ_2. We first recall previous results known in this direction.
Let $\mathcal{A}$ denote the family of all nonempty open subsets A of M with piecewise smooth boundary. For every $A \in \mathcal{A}$ let $\mu(A)$ denote its Riemannian measure and $\underline{\mu}(\partial A)$ denote the $(n-1)$-dimensional Riemannian measure of ∂A. We define for every $A \in \mathcal{A}$ the isoperimetric ratios

$$\eta(A) := \frac{\underline{\mu}(\partial_i A)}{\mu(A)} \qquad \eta'(A) := \frac{\underline{\mu}(\partial_i A)}{\underline{\mu}(\bar{A} \cap \partial M)} \tag{2}$$

where $\partial_i A := \partial A \cap \operatorname{Int} M$. Here $\operatorname{Int} M$ denotes the interior of M. Consider the following isoperimetric constants

$$h_2(M) := \inf_A \max\{\eta(A), \eta(M\backslash A)\} \qquad h_2'(M) := \inf_A \max\{\eta'(A), \eta'(M\backslash A)\}$$

The constant $h_2(M)$ is the well-known Cheeger constant [8]. Motivated by the celebrated result of Cheeger [8], Escobar [14, 15] introduced the isomerimetric constant $h_2'(M)$ and obtained a lower bound for σ_2 in terms of this isoperimetric constant and the first nonzero eigenvalue of a Robin problem. Recently, Jammes [23] obtained a simpler and more explicit lower bound for σ_2 in terms of an isoperimetric $\tilde{h}_2'(M)$ similar to the one introduced by Escobar, and the Cheeger constant $h_2(M)$:

$$\sigma_2(M) \geqslant \frac{1}{4}\tilde{h}_2'(M) h_2(M) \tag{3}$$

where $\tilde{h}_2'(M) := \inf\left\{\eta'(A) \,:\, A \in \mathcal{A}, \text{ and } \mu(A) \leqslant \frac{\mu(M)}{2}\right\}$. The proof of (3) is simple and only uses the co-area formula. The constants $h_2'(M)$ and $\tilde{h}_2'(M)$ are interesting geometric quantities. It is an intriguing question if similar geometric lower bounds hold for higher order Steklov eigenvalues σ_k. We give an affirmative answer to this question not only in Riemannian setting but also in the setting of finite and measurable spaces.
Let (M,μ) be a measure space and V a proper subset of M, and let L be an operator acting on a functional subspace $\mathcal{H}$ of $\mathbb{L}^2(\mu)$. Throughout the paper we deal with either of three different settings listed below:

(FS) Finite state spaces: M is a finite set, V is a proper subset of cardinality v, L is a reversible irreducible Markov generator and μ is its unique invariant probability measure. Here $\mathcal{H}$ is the space of functions on M denoted by $\mathcal{F}(M)$.

(MS) Measurable state spaces: (M, μ) is a probability measure space with $\sigma-$algebra $\mathcal{M}$, and V is a measurable subset of M such that $0 < \mu[V] < 1$. Here, L is a Markov generator of the form $P - I$, where P is a Markov kernel reversible with respect to μ and I is the identity, and $\mathcal{H} = \mathbb{L}^2(\mu)$.

(RM) Riemannian manifolds: M is a compact Riemannian manifold with smooth boundary ∂M, μ is its Riemannian measure, L is the Laplace-Beltrami operator Δ, and $\mathcal{H}$ is the Sobolev space $H^1(\mu)$. Here V is equal to ∂M.

With the help of L we define an operator S on V and call it the Steklov operator. In setting (RM), the operator S we consider is in fact the Dirichlet–to–Neumann operator discussed above. For the definition of S in (FS) and (MS) settings we refer to definitions (9) in Section 2, and (28) in Section 3, respectively. We denote the eigenvalues of S by $\sigma_k(M)$ or simply σ_k. Let $\mathcal{A}$ be a family of admissible sets in M:

- in (FS) settings, $\mathcal{A}$ is the set of all nonempty subsets of M;
- in (MS) setting, $\mathcal{A}$ is the set of all non-negligible elements of $\mathcal{M}$, i.e. $A \in \mathcal{M}$ such that $0 < \mu[A] \leqslant 1$;
- in (RM) setting, $\mathcal{A}$ is the set of all nonempty open domains A in M such that $\partial_e A := \bar{A} \cap \partial M$ and $\partial_i A := \partial A \cap M$ are smooth manifolds of dimension $n-1$ when they are nonempty.

In (FS) and (MS) settings, we introduce the boundary of any $A \in \mathcal{A}$ via

$$\partial A \quad := \quad \{(x, y) \,:\, x \in A,\, y \in A^{\mathrm{c}}\}$$

and define the following isoperimetric ratios

$$\eta(A) \;:=\; \frac{\underline{\mu}(\partial A)}{\mu(A)} \qquad\qquad \eta'(A) \;:=\; \frac{\underline{\mu}(\partial A)}{\mu(A \cap V)}$$

where $\underline{\mu}$ is a measure on $M \times M$. We refer to (13) and (33) for the definition of $\underline{\mu}$ in (FS) and (MS) settings respectively. In (RM) setting, the isoperimetric rations $\eta(A)$ and $\eta'(A)$ are already defined in the beginning, see (2). We then consider

$$\rho(A) := \min_{\substack{B \in \mathcal{A} \\ B \subseteq A}} \eta(B)\,, \qquad\qquad \rho'(A) := \min_{\substack{B' \in \mathcal{A} \\ B' \subseteq A}} \eta'(B')$$

in (FS) and (MS) settings. And in (RM) setting we take

$$\rho(A) := \inf_{\substack{B \in \mathcal{A} \\ B \subset A \\ \bar{B} \cap \partial_i A = \varnothing}} \eta(B)\,, \qquad\qquad \rho'(A) := \inf_{\substack{B' \in \mathcal{A} \\ B' \subset A \\ \bar{B}' \cap \partial_i A = \varnothing}} \eta'(B')$$

The constant $\rho(A)$ in (RM) setting is the Cheeger constant of A when the Dirichlet boundary condition on $\partial_i A$ is imposed, we refer to [7, 36] for more information on the Cheeger constant on manifolds with Dirichlet and Neumann boundary conditions. We are now ready to define the *higher order Cheeger–Steklov constants.* For any $k \in \mathbb{N}$ and for any of three settings (FS), (MS) and (RM), we define the k-th Cheeger–Steklov constant of M by

$$\iota_k(M) \quad := \quad \inf_{(A_1, \cdots, A_k) \in \mathcal{A}_k} \max_{l \in [\![k]\!]} \rho(A_l) \rho'(A_l)$$

where $[\![k]\!] := \{1, \ldots, k\}$ and $\mathcal{A}_k$ is the set of all k-tuples $(A_1, \cdots, A_k)$ of mutually disjoint elements of $\mathcal{A}$. We recall the definition of the higher order Cheeger constants for the eigenvalues of a Markov generator in settings (FS) and (MS) and for the eigenvalues of the Laplace–Beltrami operator in setting (RM):

$$h_k(M) \quad := \quad \inf_{(A_1, \cdots, A_k) \in \mathcal{A}_k} \max_{l \in [\![k]\!]} \eta(A_l)$$

The sequence of the higher order Cheeger constants is called the *connectivity spectrum.* One can see how closely h_k and ι_k are related. We now state our main theorems.

Theorem A *In setting (*FS*), there exists a universal positive constant* c_0 *such that*

$$\forall\ k \in [\![v]\!], \qquad \sigma_k(M) \ \geqslant\ \frac{c_0}{k^6}\frac{\iota_k(M)}{\|L\|}$$

where $\|L\|$ *is the largest absolute value of the elements of the diagonal of* L*.*

The following theorem is an extension of Theorem A to setting (MS).

Theorem B *In setting (*MS*), there exists a universal positive constant* c_1 *such that*

$$\forall\ k \in \mathbb{N}, \qquad \sigma_k(M) \ \geqslant\ \frac{c_1}{k^6}\iota_k(M)$$

The higher order Cheeger-Steklov inequality in setting (RM) which is an extension of Escobar and Jammes results to higher Steklov eigenvalues states

Theorem C *In setting (*RM*), there exists a universal positive constant* c_2 *such that*

$$\forall\ k \in \mathbb{N}, \qquad \sigma_k(M) \ \geqslant\ \frac{c_2}{k^6}\iota_k(M)$$

We recall that for $k = 2$, the Cheeger inequality in setting (FS) was studied in [1, 2, 13], and in settings (MS) in [26], see also the lecture notes by Saloff-Coste [33] for a review. The higher-order Cheeger inequality in setting (FS) was conjectured by the second author [30], see also [12]. This conjecture was proved by Lee, Oveis Gharan and Trevisan [27]. Later, the second author [31] extended their result to (MS) and (RM) settings; see also [19] for the result on closed manifolds. The higher order Cheeger inequality in (FS) setting for the operator L states (see [27, Theorem 3.8] and [31, Theorem 2])

$$\forall\ k \in [\![v]\!], \qquad \lambda_k(M) \ \geqslant\ \frac{c_3}{k^8}\frac{h_k^2(M)}{\|L\|} \tag{4}$$

and in (MS) and (RM) settings states [31]

$$\forall\ k \in \mathbb{N}, \qquad \lambda_k(M) \ \geqslant\ \frac{c_4}{k^6}h_k^2(M) \tag{5}$$

where c_3 and c_4 are universal positive constants. As we mentioned before, our main results, Theorems A, B and C for Steklov eigenvalues, can be viewed as a counterpart of the higher order Cheeger inequalities for the Laplace spectrum. We remark that even for $k = 2$, Theorem A and Theorem B are new.
We now discuss about an improvement of the dependency on k in Theorems A, B, and C. In [27, Theorem 4.1] and [31, Theorem 13], it is shown that one can obtain a better lower bound when λ_k is replaced by λ_{2k} in (4) and (5)

$$\lambda_{2k}(M) \ \geqslant\ \begin{cases} \frac{\tilde c_3}{\log(k+1)}\frac{h_k^2(M)}{\|L\|} & \text{in setting } (\mathsf{FS}) \\ \frac{\tilde c_4}{\log^2(k+1)}h_k^2(M) & \text{in settings } (\mathsf{MS}) \text{ and } (\mathsf{RM}) \end{cases} \tag{6}$$

For Steklov eigenvalues we obtain analogous results.

Proposition A *There are universal positive constants* $\tilde c_1$ *and* $\tilde c_2$ *such that*

$$\sigma_{2k}(M) \ \geqslant\ \begin{cases} \frac{\tilde c_1}{\log^2(k+1)}\frac{\iota_k(M)}{\|L\|} & \forall\ k \in [\![v]\!], \quad \textit{in setting } (\mathsf{FS}) \\ \frac{\tilde c_2}{\log^2(k+1)}\iota_k(M) & \forall\ k \in \mathbb{N}, \quad \textit{in settings } (\mathsf{MS}) \textit{ and } (\mathsf{RM}) \end{cases} \tag{7}$$

Remark 1 The sharpness of the coefficient of h_k in (6) was investigated in [31] using the noisy hypercube graph, and in [27] using the Ornstein–Uhlenbeck process. Understanding the asymptotic sharpness of the coefficient of ι_k in (7) is an interesting problem which needs a further investigation and remains open.

□

We now briefly discuss the idea of the proof of the main Theorems. To prove the main theorems we first introduce the Dirichlet-Steklov connectivity spectrum of S on M. Second we show that eigenvalues of S can be viewed as a limit of eigenvalues of a family of operators. Then we prove that the Dirichlet connectivity spectrum (introduced in [30] and in [31]) of this family of operators converges to Dirichlet-Steklov connectivity spectrum of S. Moreover, we show that this convergence is uniform in some sense. Then we use the known lower bounds [27, 31] for eigenvalues of this family of operators in terms of their Dirichlet connectivity spectra to show that the Steklov eigenvalues have similar lower bounds in term of the Dirichlet-Steklov connectivity spectrum. The final step is to relate the Dirichlet–Steklov connectivity spectrum to the higher order Cheeger–Steklov constants. This is done using the co-area formula in each setting (FS), (MS) and (RM). Although the main idea of the proof in these three settings are the same, the details and technicalities that we need to deal with in each setting are different. This makes the investigation of each setting interesting in its own and not only as a straightforward consequence of another setting. We aim to explore a deeper underlying connection between these three settings in future studies.

It is also interesting to study the higher order Cheeger-Steklov inequality when L is a diffusion operator and when we also have a density on V. Here the associated Dirichlet–to–Neumann map S (known as the voltage–to–current map) appears in the study of the electrical impedance tomography [5, 35]. The techniques and methods that we develop in this paper can be used to obtain the higher order Cheeger–Steklov inequality in this setting in terms of a weighted version of the higher order Cheeger–Steklov constants. The classical Cheeger inequality for weighted manifolds is studied in [6], see also [9, 31]. We will address this in more details in a forthcoming work.

The paper is organized as follows. Section 2 deals with (FS) setting and the proof of Theorem A and Proposition A. In Section 3 we extends results in (FS) setting to (MS) setting. We also show that under the Dirichlet gap assumption on $M\backslash V$ the proof of Theorem B can be simplified. In Section 4 we prove Theorem C. We also provide examples which show the necessity of both isoperimetric ratios appearing in the definition of ι_k. Although the ideas and techniques in three sections 2, 3, and 4 are related, the reader does not need to read the sections in order.

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2 The finite state space framework

Let $L := L(x,y)_{x,y\in M}$ be an irreducible Markov generator on the finite set M. Recall that L is Markovian if

$$\forall\ x \neq y \in M, \quad L(x,y) \geqslant 0, \qquad \text{and} \qquad \sum_{y\in M} L(x,y) = 0$$

and is called irreducible if for every $x, y \in M$ there exists a sequence $x = x_0, x_1, \ldots, x_l = y$ of elements of M such that $L(x_j, x_{j+1}) > 0$ for any $j \in [\![0, l-1]\!] := \{0, \ldots, l-1\}$. Denote by $\mu := (\mu(x))_{x\in M}$ its unique invariant probability, characterized by

$$\forall\ y \in M, \qquad \sum_{x\in M} \mu(x) L(x,y) \quad = \quad 0$$

Let V be a proper subset of M, i.e. $\varnothing \subsetneq V \subsetneq M$. Define the corresponding Steklov operator S on $\mathcal{F}(V)$, the space of functions on V, via the following procedure. Given $f \in \mathcal{F}(V)$, let F be its harmonic extension on M, namely the unique $F \in \mathcal{F}(M)$ satisfying

$$\begin{cases} L[F](x) = 0\,, & \text{if } x \in M\backslash V \\ F(x) = f(x)\,, & \text{if } x \in V \end{cases} \tag{8}$$

Then we consider

$$\forall\, x \in V, \qquad S[f](x) \;\; := \;\; L[F](x) \tag{9}$$

The following observation should be classical.

Proposition 2 *The operator S is an irreducible Markov generator on V whose invariant measure is ν, the normalized restriction of μ to V.*

Assume that μ is furthermore reversible for L, namely

$$\forall\, x, y \in M, \qquad \mu(x)L(x,y) \;\; = \;\; \mu(y)L(y,x)$$

It follows that S is equally reversible with respect to ν, and the spectra of $-S$ and $-L$ are non-negative. Denote by $0 = \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_v$, with $v := \mathrm{card}(V)$, the eigenvalues of $-S$ in $\mathbb{R}$ with multiplicities, indexed so that $0 = \sigma_1 < \sigma_2 \leqslant \sigma_3 \leqslant \cdots \leqslant \sigma_v$. Our goal is to investigate these eigenvalues.

For any $r > 0$, consider the Markov generator defined by

$$\forall\, x \neq y \in M, \qquad L^{(r)}(x,y) \;\; := \;\; \begin{cases} rL(x,y)\,, & \text{if } x \in M\backslash V \\ L(x,y)\,, & \text{if } x \in V \end{cases}$$

Since μ is reversible for L, we will see (in Lemma 11) that $L^{(r)}$ is reversible with respect to its invariant measure $\mu^{(r)}$. Hence the eigenvalues of $-L^{(r)}$ are non-negative. Let $0 = \lambda_1^{(r)}, \lambda_2^{(r)}, \lambda_3^{(r)}, \ldots, \lambda_m^{(r)}$, with $m := \mathrm{card}(M)$, be the eigenvalues of $-L^{(r)}$ in $\mathbb{R}$ with multiplicities, indexed so that $0 = \lambda_1^{(r)} < \lambda_2^{(r)} \leqslant \lambda_3^{(r)} \leqslant \cdots \leqslant \lambda_m^{(r)}$.

Proposition 3 *Assume that L is reversible. For any $k \in [\![v]\!] := \{1, ..., v\}$, we have*

$$\lim_{r \to +\infty} \lambda_k^{(r)} \;\; = \;\; \sigma_k$$

and for any $k \in [\![m]\!]\backslash[\![v]\!]$,

$$\lim_{r \to +\infty} \lambda_k^{(r)} \;\; = \;\; +\infty$$

Remark 4 We believe that the above proposition should be true in the non-reversible case (where in the last convergence, $\lambda_k^{(r)}$ is replaced by its real part).

□

We would like to estimate these eigenvalues via Cheeger type inequalities. Denote by $\mathcal{A}$ the set of nonempty subsets from M. We associate to any $A \in \mathcal{A}$ a Dirichlet-Steklov operator S_A on $\mathcal{F}(A \cap V)$ in the following way: given $f \in \mathcal{F}(A \cap V)$, consider $F \in \mathcal{F}(M)$ such that

$$\begin{cases} L[F](x) = 0\,, & \text{if } x \in A\backslash V \\ F(x) = 0\,, & \text{if } x \in M\backslash A \\ F(x) = f(x)\,, & \text{if } x \in A \cap V \end{cases} \tag{10}$$

The existence and uniqueness of such a F are similar to those of the solution of (8), see e.g. the proof of Proposition 2. Indeed, one is brought back to this situation by replacing V by $V \cup (M\backslash A)$ and by extending f to this set by making it vanish on $M\backslash A$.

Next define

$$\forall\, x \in A \cap V, \qquad S_A[f](x) \;\; := \;\; L[F](x)$$

When $A \cap V \neq \varnothing$, we will check that S_A is always a subMarkovian generator (i.e. $S_A(x,y) \geqslant 0$, for any $x \neq y$, and $\sum_{y \in V} S_A(x,y) \leqslant 0$) maybe not irreducible, but Perron-Frobenius' theorem enables to consider the smallest eigenvalue $\sigma_1(A)$ of $-S_A$. By convention, when $A \cap V = \varnothing$, $\mathcal{F}(\varnothing) := \{0\}$ and $\sigma_1(A) = +\infty$. Next we introduce the Dirichlet–Steklov connectivity spectrum $(\kappa_1, \kappa_2, ..., \kappa_v)$ of S via

$$\forall\, k \in [\![v]\!], \qquad \kappa_k \;\; := \;\; \min_{(A_1, \ldots, A_k) \in \mathcal{A}_k} \max_{l \in [\![k]\!]} \sigma_1(A_l) \tag{11}$$

where $\mathcal{A}_k$ is the set of k-tuples $(A_1, A_2, ..., A_k)$ of disjoints elements from $\mathcal{A}$. Notice that definition (11) can be written as

$$\forall\ k \in [\![v]\!], \qquad \kappa_k \ := \ \min_{(A_1,...,A_k)\in\mathcal{A}_k(V)} \max_{l\in[\![k]\!]} \sigma_1(A_l) \tag{12}$$

where $\mathcal{A}_k(V)$ is the set of all disjoint k-tuple in $\mathcal{A}(V) := \{A \in \mathcal{A} : A \cap V \in \mathcal{A}\}$. The above definitions are valid in all generality, but (for the moment) they are mainly useful under the reversibility assumption:

Theorem 5 *Assume that L is reversible. There exists a universal constant $c > 0$ such that*

$$\forall\ k \in [\![v]\!], \qquad \frac{c}{k^6}\kappa_k \leqslant \sigma_k \leqslant \kappa_k$$

The interest of the Dirichlet–Steklov connectivity spectrum is that it is strongly related to higher order inequalities. We need further definitions. Introduce the boundary of any $A \in \mathcal{A}$ via

$$\partial A \ := \ \{(x,y) : x \in A,\, y \in A^{\mathrm{c}}\}$$

Consider the measure $\underline{\mu}$ defined on $M \times M$ by

$$\forall\ x, y \in M, \qquad \underline{\mu}(x,y) \ = \ \begin{cases} \mu(x)L(x,y)\,, & \text{if } x \neq y \\ 0\,, & \text{if } x = y \end{cases} \tag{13}$$

it enables to measure ∂A through $\underline{\mu}(\partial A)$. As a consequence, we can define the isoperimetric ratios

$$\eta(A) \ := \ \frac{\underline{\mu}(\partial A)}{\mu(A)} \qquad\qquad \eta'(A) \ := \ \frac{\underline{\mu}(\partial A)}{\mu(A\cap V)}$$

By convention $\eta'(A) = +\infty$ if $A \cap V = \varnothing$. The ratio $\eta'(A)$ is the discrete analogue of quantities introduced by Escobar [14] and Jammes [23], since in their terminology, ∂A and $A \cap V$ can be seen respectively as the interior and exterior boundaries, when the set V itself is seen as a boundary of M.

Next consider

$$\rho(A) \ := \ \min_{\substack{B\in\mathcal{A}\\ B\subseteq A}} \eta(B) \qquad\qquad \rho'(A) \ := \ \min_{\substack{B'\in\mathcal{A}\\ B'\subseteq A}} \eta'(B')$$

For any $k \in [\![v]\!]$, introduce the k-th Cheeger–Steklov constant of V by

$$\iota_k \ := \ \min_{(A_1,...,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \rho(A_l)\rho'(A_l)$$

Remark that $\iota_1 = 0$ by taking $A = M$. The next result can be seen as an extension to higher order Cheeger inequalities (in the discrete case) of Théorème 1 of Jammes [23]:

Theorem 6 *Assume that L is reversible and let c be the constant of Theorem 5. We have*

$$\forall\ k \in [\![v]\!], \qquad \sigma_k \ \geqslant \ \frac{c}{k^6}\frac{\iota_k}{\|L\|}$$

where $\|L\|$ is the largest absolute value of the elements of the diagonal of L.

Let us consider

$$h'_k \ := \ \min_{(A_1,...,A_k)\in\mathcal{A}_k(V)} \max_{l\in[\![k]\!]} \eta'(A_l)$$

Proposition 7 *Assume that L is reversible. We have*

$$\forall\ k \in [\![v]\!], \qquad \sigma_k \ \leqslant \ h'_k$$

Remark 8 Let L be a reversible Markov generator but not necessarily irreducible. Let $X \coloneqq (X_t)_{t \geqslant 0}$ be a Markov process generated by L, starting from x under the probability $\mathbb{P}_x$. Assume that the reaching time of V denoted by τ:

$$\tau \quad \coloneqq \quad \inf\{t \geqslant 0 \,:\, X_t \in V\}$$

is almost surely finite. Then all of the results above are valid without irreducibility condition. In particular, $\sigma_k = 0$ if and only if $h'_k = 0$. Indeed one way is obvious due to Proposition 7. For the "only if" part, $\sigma_k = 0$, implies $\iota_k = 0$ by Theorem 6. Therefore there exists $(A_1, ..., A_k) \in \mathcal{A}_k(V)$ such that $\underline{\mu}(\partial A_l) = 0$ for all $l \in [\![k]\!]$. It follows $h'_k = 0$. Note that the number of zeros determines the number of communicating classes. Recall that for the eigenvalues of $L = L^{(1)}$, the result of Lee, Oveis Gharan and Trevisan [27] implies that $\lambda_k = 0$ if and only if the k-th Cheeger constant h_k

$$h_k := \min_{(A_1,\dots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \eta(A_l)$$

is zero. In comparison, we see that the h'_k plays the role of h_k for the Steklov problem .

□

Proof of Proposition 2

It is based on the following simple probabilistic interpretation of S. Let $X \coloneqq (X_t)_{t \geqslant 0}$ be a Markov process generated by L, starting from x under the probability $\mathbb{P}_x$. Denote by τ its reaching time of V:

$$\tau \quad \coloneqq \quad \inf\{t \geqslant 0 \,:\, X_t \in V\}$$

it is a.s. finite, since L is irreducible. A usual application of the martingale problem associated to X shows that for any function $G \in \mathcal{F}(M)$, we have

$$\mathbb{E}_x[G(X_\tau)] \quad = \quad G(x) + \mathbb{E}_x\left[\int_0^\tau L[G](X_s)\, ds\right]$$

In particular, for any $f \in \mathcal{F}(V)$, it appears that its harmonic extension defined in (8) is given by

$$\forall\ x \in M, \qquad F(x) \quad = \quad \mathbb{E}_x[f(X_\tau)] \quad = \quad \nu_x[f]$$

where ν_x is the law of X_τ under $\mathbb{P}_x$. More precisely, we get the existence and uniqueness of the solution of (8), even without assuming that L is irreducible (only the finiteness of τ is needed). We deduce that for any $f \in \mathcal{F}(V)$ and any $x \in V$,

$$S[f](x) \quad = \sum_{y\in M\setminus\{x\}} L(x,y)(F(y)-F(x)) \quad = \sum_{y\in M\setminus\{x\}}\sum_{z\in V} L(x,y)\nu_y(z)(f(z)-f(x))$$

namely, the matrix associated to S is given by

$$\forall\ x,z \in V, \qquad S(x,z) \quad \coloneqq \quad \begin{cases} \sum_{y\in M\setminus\{x\}} L(x,y)\nu_y(z)\,, & \text{if } x \neq z \\ -\sum_{y\in V\setminus\{x\}} S(x,y)\,, & \text{if } x = z \end{cases}$$

On this expression, it is clear that S is a Markov generator, namely that it satisfies $S(x,z) \geqslant 0$ for any $x \neq z \in V$ and $\sum_{z\in V} S(x,z) = 0$ for any $x \in V$. It is also irreducible: for any $x, z \in V$, let $x_0 = x, x_1, x_2, ..., x_l = z$ be a sequence of elements of M such that $L(x_j, x_{j+1}) > 0$ for any $j \in [\![0, l-1]\!]$. Let $(y_j)_{j\in[\![0,k]\!]}$ be the subsequence of $(x_j)_{j\in[\![0,l]\!]}$ consisting of the elements belonging to V. We have $y_0 = x$, $y_k = z$ and from the above description of S, it follows that $S(x_j, x_{j+1}) > 0$ for any $j \in [\![0, k-1]\!]$.

It remains to check that ν, the normalized restriction of μ to V, is invariant for S. For any $f \in \mathcal{F}(V)$, we have, with F constructed as in (8),

$$\begin{aligned} \nu[S[f]] \quad &= \quad \frac{1}{\mu(V)}\sum_{x\in V}\mu(x)S[f](x) \quad = \quad \frac{1}{\mu(V)}\sum_{x\in V}\mu(x)L[F](x) \\ &= \quad \frac{1}{\mu(V)}\sum_{x\in M}\mu(x)L[F](x) \quad = \quad \frac{\mu[L[F]]}{\mu(V)} \quad = \quad 0 \end{aligned}$$

It shows that ν is invariant for S.

■

Remark 9 (probabilist point of view) A Markov process $Y := (Y_t)_{t\geqslant 0}$ associated to the generator S and starting from $x \in V$ can be obtained from a Markov process $X := (X_t)_{t\geqslant 0}$ associated to the generator L and also starting from x, by erasing its passages in $M\backslash V$. More precisely, let $(\tau_n)_{n\in\mathbb{Z}_+}$ be the sequence of jump intertimes of X:

$$\begin{aligned} &\tau_0 &:=& \ 0 \\ \forall\, n\in\mathbb{Z}_+, \qquad &\tau_{n+1} &:=& \ \inf\{t\geqslant 0\ :\ X_{t+\tau_n}\neq X_{\tau_n}\} \end{aligned}$$

Let $(N_n)_{n\in\mathbb{Z}_+}$ be the sequence of integers for which $X_{\tau_1+\tau_2+\cdots+\tau_{N_n}}\in V$ and consider

$$\forall\, n\in\mathbb{Z}_+, \qquad \mathtt{\tau}_n \quad := \quad \sum_{p\in[\![n]\!]} \tau_{N_p}$$

Then we can construct the Markov process Y through the relation

$$\forall\, t\geqslant 0, \qquad Y_t \quad := \quad X_{\tau_1+\tau_2+\cdots+\tau_{N_n}}\,, \qquad \text{if}\ \ t\in[\mathtt{\tau}_n,\mathtt{\tau}_{n+1}[$$

This observation inspired the introduction of the generators $L^{(r)}$, for $r>0$: heuristically the generator of Y is $L^{(\infty)}$, namely X is accelerated with an infinite speed in $M\backslash V$ and only its passages on V remain.

The above probabilistic interpretation also enables to see directly that S is irreducible and that the invariant measure ν of S is just μ conditioned on V. Indeed, for the latter assertion, by the ergodic theorem, we must have a.s.

$$\forall\, y\in V, \qquad \nu(y) \quad = \quad \lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{1}_{\{y\}}(Y_s)\,ds$$

so it follows that for any $y,z\in V$,

$$\frac{\nu(y)}{\nu(z)} \quad = \quad \lim_{t\to+\infty}\frac{\int_0^t \mathbb{1}_{\{y\}}(Y_s)\,ds}{\int_0^t \mathbb{1}_{\{z\}}(Y_s)\,ds} \quad = \quad \lim_{t\to+\infty}\frac{\int_0^t \mathbb{1}_{\{y\}}(X_s)\,ds}{\int_0^t \mathbb{1}_{\{z\}}(X_s)\,ds} \quad = \quad \frac{\mu(y)}{\mu(z)}$$

□

Remark 10 (analytic point of view) Recall that the Dirichlet form associated to L (and μ) is the bilinear form $\mathcal{E}_L$ given by

$$\forall\, F,G\in\mathcal{F}(M), \qquad \mathcal{E}_L(F,G) \quad := \quad -\int F L[G]\,d\mu$$

It is symmetrical, if and only if μ is reversible with respect to L.

The carré du champ associated to L is the bilinear functional Γ_L defined by

$$\forall\, F,G\in\mathcal{F}(M),\ \forall\, x\in M, \qquad \Gamma_L[F,G](x) \quad := \quad L[FG](x)-F(x)L[G](x)-G(x)L[F](x) \tag{14}$$

It is not difficult to compute more explicitly that

$$\forall\, F,G\in\mathcal{F}(M),\ \forall\, x\in M, \qquad \Gamma_L[F,G](x) \quad := \quad \sum_{y\in M} L(x,y)(F(y)-F(x))(G(y)-G(x))$$

In particular, when $F=G$, the r.h.s. looks like a weighted discrete gradient square, explaining the name carré du champ.

From (14), we get that

$$\forall\, F,G\in\mathcal{F}(M), \qquad \int \Gamma_L[F,G]\,d\mu \quad = \quad \mathcal{E}_L(F,G)+\mathcal{E}_L(G,F)$$

and in particular

$$\forall\, F\in\mathcal{F}(M), \qquad \int \Gamma_L[F]\,d\mu \quad = \quad 2\mathcal{E}_L(F,F)$$

where $\Gamma_L[F]$ stands for $\Gamma_L[F,F]$. Furthermore, when μ is reversible with respect to L, we get

$$\forall\ F,G\in\mathcal{F}(M),\qquad \int \Gamma_L[F,G]\,d\mu \;=\; 2\mathcal{E}_L(F,G)$$

These definitions are valid for any finite Markov generator L and we can consider similarly $\mathcal{E}_S$ and Γ_S. For any $f,g\in\mathcal{F}(V)$, let F and G be their harmonic extensions. It is clear that

$$\mathcal{E}_S(f,g) \;=\; \frac{\mathcal{E}_L(F,G)}{\mu(V)} \tag{15}$$

and as a consequence, we have

$$\int \Gamma_S[f,g]\,d\nu \;=\; \frac{1}{\mu(V)}\int \Gamma_L[F,G]\,d\mu$$

which is an important relation in the analytical approach to the usual Steklov (or Dirichlet to Neumann) operators.

It follows immediately from (15) that ν is reversible for S when μ is assumed to be reversible for L.

□

Since for any $r>0$, the generator $L^{(r)}$ is irreducible, it admits a unique invariant probability $\mu^{(r)}$.

Lemma 11 *The probability measure $\mu^{(r)}$ is given by*

$$\forall\ x\in M,\qquad \mu^{(r)}(x) \;=\; \begin{cases} \frac{\mu(x)}{Z_r}, & \text{if } x\in V\\ \frac{\mu(x)}{rZ_r}, & \text{if } x\in M\backslash V\end{cases}$$

where $Z_r\coloneqq \mu(V)+(1-\mu(V))/r$ is the normalisation constant.

Furthermore, if μ is reversible for L, then $\mu^{(r)}$ is reversible for $L^{(r)}$.

Proof

These are consequences of more general facts: assume that $H\in\mathcal{F}(M)$ is positive: $H>0$. Consider the operator HL acting on $\mathcal{F}(M)$ via

$$\forall\ F\in\mathcal{F}(M),\ \forall\ x\in M,\qquad HL[F](x) \;\coloneqq\; H(x)L[F](x)$$

It is an irreducible Markov generator. Let $(1/H)\cdot\mu$ be the positive measure admitting $1/H$ for density with respect to μ. We have

$$\forall\ F\in\mathcal{F}(M),\qquad ((1/H)\cdot\mu)[HL[F]] \;=\; \mu[L[F]] \;=\; 0$$

Thus the invariant probability measure of HL is proportional to $(1/H)\cdot\mu$.

Considering $H\coloneqq \mathbb{1}_V+r\mathbb{1}_{M\backslash V}$ (where $\mathbb{1}_V$ is the indicator function of V) leads to the first announced result.

For the second result, note that in general, when μ is reversible for L, for any $F,G\in\mathcal{F}(M)$,

$$((1/H)\cdot\mu)[F(HL)[G]] \;=\; \mu[FL[G]] \;=\; \mu[GL[F]] \;=\; ((1/H)\cdot\mu)[G(HL)[F]]$$

■

Proof of Proposition 3

In the reversible case, $-L$ is diagonalisable with real eigenvalues. In view of Lemma 11, for any $r>0$, the same is true for $-L^{(r)}$, denote by $0=\lambda_1^{(r)}<\lambda_2^{(r)}\leqslant\lambda_3^{(r)}\leqslant\cdots\leqslant\lambda_m^{(r)}$ its eigenvalues. Let $\mathbb{1}=\Phi_1^{(r)},\Phi_2^{(r)},\Phi_3^{(r)},\ldots,\Phi_m^{(r)}$ be corresponding eigenvectors. They are not unique (especially in the case of multiplicities larger than 1), but we can and do choose them so that they are orthogonal with respect to $\mu^{(r)}$:

$$\forall\ r\in(0,+\infty),\ \forall\ k\neq l\in[\![m]\!],\qquad \mu^{(r)}[\Phi_l^{(r)}\Phi_k^{(r)}] \;=\; 0$$

Normalize them with respect to the supremum norm $\|\cdot\|_\infty$ instead of the $\mathbb{L}^2(\mu^{(r)})$ norm:

$$\forall\, r \in (0,+\infty),\, \forall\, l \in [\![m]\!], \qquad \left\|\Phi_l^{(r)}\right\|_\infty \;=\; 1$$

Consider $l \in [\![m]\!]$ such that

$$\left\{\begin{array}{rcl} \liminf_{r\to+\infty} \lambda_l^{(r)} & < & +\infty \\ \liminf_{r\to+\infty} \lambda_{l+1}^{(r)} & = & +\infty \end{array}\right. \tag{16}$$

By compactness, we can find an increasing sequence of positive numbers $(r_n)_{n\in\mathbb{N}}$ and for any $k \in [\![l]\!]$, a non-negative number $\lambda_k^{(\infty)} \in [0,+\infty)$ and a positive function $\Phi_k^{(\infty)} \in \mathcal{F}(M)$ with $\left\|\Phi_k^{(\infty)}\right\|_\infty = 1$ such that

$$\lim_{n\to\infty} r_n \;=\; +\infty \qquad\qquad \lim_{n\to\infty} \lambda_k^{(r_n)} \;=\; \lambda_k^{(\infty)} \qquad\qquad \lim_{n\to\infty} \Phi_k^{(r_n)} \;=\; \Phi_k^{(\infty)}$$

Passing to the limit in the relations

$$\forall\, x \in V, \qquad L[\Phi_k^{(r_n)}](x) \;=\; L^{(r_n)}[\Phi_k^{(r_n)}](x) \;=\; -\lambda_k^{(r_n)}\Phi_k^{(r_n)}(x)$$

we get

$$\forall\, x \in V, \qquad L[\Phi_k^{(\infty)}](x) \;=\; -\lambda_k^{(\infty)}\Phi_k^{(\infty)}(x)$$

For $x \in M\backslash V$, we have instead

$$r_n L[\Phi_k^{(r_n)}](x) \;=\; -\lambda_k^{(r_n)}\Phi_k^{(r_n)}(x)$$

Since the r.h.s. converges to $-\lambda_k^{(\infty)}\Phi_k^{(\infty)}(x)$ for large $n \in \mathbb{N}$, we deduce that

$$\forall\, x \in M\backslash V, \qquad L[\Phi_k^{(\infty)}](x) \;=\; \lim_{n\to\infty} L[\Phi_k^{(r_n)}](x) \;=\; 0$$

Thus denoting φ_k the restriction of $\Phi_k^{(\infty)}$ to V, it appears that $\Phi_k^{(\infty)}$ is the harmonic extension of φ_k. Note that $\varphi_k \neq 0$, otherwise we would conclude that $\Phi_k^{(\infty)} = 0$, in contradiction with $\left\|\Phi_k^{(\infty)}\right\|_\infty = 1$. Thus $\lambda_k^{(\infty)}$ is an eigenvalue of $-S$. Furthermore, passing to the limit in the relations

$$\forall\, j \neq k \in [\![l]\!], \qquad \mu^{(r_n)}[\Phi_j^{(r_n)}\Phi_k^{(r_n)}] \;=\; 0$$

we see that

$$\forall\, j \neq k \in [\![l]\!], \qquad \nu[\varphi_j\varphi_k] \;=\; 0$$

It follows that the $\lambda_k^{(\infty)}$, for $k \in [\![l]\!]$, correspond to different eigenvalues of $-S$ (with multiplicities). Namely, there exists an increasing mapping $N : [\![l]\!] \to [\![v]\!]$ (recall that $v \coloneqq \mathrm{card}(V)$) such that

$$\forall\, k \in [\![l]\!], \qquad \lambda_k^{(\infty)} \;=\; \sigma_{N(k)}$$

and in particular, $v \geqslant l$. Conversely, consider $\psi_1, \psi_2, ..., \psi_v$ a basis of $\mathcal{F}(V)$ consisting of eigenvectors of $-S$ associated respectively to the eigenvalues $\sigma_1, \sigma_2, ..., \sigma_v$. Since ν is reversible for S, we can and do choose these functions to be orthogonal in $\mathbb{L}^2(\nu)$. Let $\Psi_1, \Psi_2, ..., \Psi_v$ be the harmonic extensions of $\psi_1, \psi_2, ..., \psi_v$. We furthermore impose that $\|\Psi_k\|_\infty = 1$ for all $k \in [\![v]\!]$. Consider the vector space $W \subset \mathcal{F}(M)$ generated by these functions

$$W \;\coloneqq\; \mathrm{Vect}(\Psi_k \,:\, k \in [\![v]\!])$$

Due to the variational principle, we have for any $r > 0$,

$$\lambda_v^{(r)} \;\leqslant\; \sup_{F\in W\backslash\{0\}} \frac{-\mu^{(r)}[FL^{(r)}[F]]}{\mu^{(r)}[F^2]}$$

Since the functions from W are harmonic on $M\backslash V$, we have for any $r > 0$, with the notation of Lemma 11,

$$\forall\, F \in W, \qquad -\mu^{(r)}[FL[F]] \;=\; -\frac{\mu(V)}{Z_r}\nu[FL[F]] \;=\; -\frac{\mu(V)}{Z_r}\nu[fS[f]] \;\leqslant\; \frac{\mu(V)}{Z_r}\sigma_v\nu[f^2]$$

where f is the restriction of F to V. We also have

$$\mu^{(r)}[F^2] \quad = \quad \frac{\mu[\mathbb{1}_V f^2] + \mu[\mathbb{1}_{M\backslash V} F^2]/r}{Z_r} \quad \geqslant \quad \frac{\mu(V)}{Z_r}\nu[f^2]$$

We deduce from these two bounds that

$$\lambda_v^{(r)} \quad \leqslant \quad \sigma_v$$

and

$$\limsup_{r\to+\infty} \lambda_v^{(r)} \quad < \quad +\infty \tag{17}$$

i.e. $l \geqslant v$ and finally $l = v$.

It follows that

$$\forall\ k \in [\![v]\!], \qquad \lim_{n\to\infty} \lambda_k^{(r_n)} \quad = \quad \sigma_k \tag{18}$$

Taking into account (17), for any increasing subsequence $(R_n)_{n\in\mathbb{N}}$ of positive numbers diverging to $+\infty$, we can extract another subsequence $(r_n)_{n\in\mathbb{N}}$ such that (18) is true, we conclude by compactness that

$$\forall\ k \in [\![v]\!], \qquad \lim_{r\to+\infty} \lambda_k^{(r)} \quad = \quad \sigma_k$$

The last assertion of Proposition 3 is a consequence of $l = v$ and of the definition of l in (16).

■

Before coming to the proof of Theorem 5, let us check that for any $A \in \mathcal{A}(V)$, S_A is a subMarkovian generator. The argument is similar to that of the proof of Proposition 2 and is based on the probabilistic representation of the solution F of (10):

$$\forall\ x \in M, \qquad F(x) \quad = \quad \mathbb{E}_x[f(X_{\tau_{A\cap V}})\mathbb{1}_{\tau_{A\cap V}<\tau_{M\backslash A}}] \tag{19}$$

where $(X_t)_{t\geqslant 0}$ is a Markov process generated by L and starting from x, and for any $B \subset M$, τ_B is the hitting time of B:

$$\tau_B \quad \coloneqq \quad \inf\{t \geqslant 0 \,:\, X_t \in B\}$$

As a consequence, the first eigenvalue $\sigma_1(A)$ of $-S_A$ is non-negative. It vanishes, if and only if there is no path (whose transitions are permitted by L) going out of A without passing through $A \cap V$.

Assume that μ is reversible with respect to L. By the variational formulation of eigenvalues and using the notation of Remark 9, we have for $A \in \mathcal{A}$,

$$\sigma_1(A) \quad = \quad \inf\left\{\frac{\mathcal{E}_{S_A}(f,f)}{\nu_{A\cap V}[f^2]} \,:\, f \in \mathcal{F}(A\cap V)\backslash\{0\}\right\} \tag{20}$$

where $\nu_{A\cap V}$ is the normalized restriction of μ to $A \cap V$, which is reversible with respect to S_A. As in (15), in the above formula, $\mathcal{E}_{S_A}(f,f)$ can be replaced by $\mathcal{E}_L(F,F)/\mu(A\cap V)$, where F is associated to f via (10).

We can now come to the

Proof of Theorem 5

The upper bound of σ_k is a direct consequence of the variational characterization of σ_k

$$\sigma_k \quad = \quad \min_{H\in\mathcal{F}_k(V)} \max_{f\in H\backslash\{0\}} \frac{\mathcal{E}_S(f,f)}{\nu[f^2]}$$

where $\mathcal{F}_k(V)$ is the set of all k-dimensional subspace of $\mathcal{F}(V)$, by taking H as the space spanned by the first eigenfunctions of S_{A_l}, $l \in [\![k]\!]$.

The proof of the lower bound is based on the higher order Dirichlet-Cheeger inequalities for finite irreducible and reversible Markov generators. So assume that μ is reversible with respect to L and let $0 = \lambda_1(L) < \lambda_2(L) \leqslant \lambda_3(L) \leqslant \cdots \leqslant \lambda_m(L)$ be the eigenvalues of $-L$. Associate to any $A \in \mathcal{A}$ its first Dirichlet eigenvalue

$$\lambda_1(A) \quad \coloneqq \quad \inf\left\{\frac{\mathcal{E}_L(F,F)}{\mu[F^2]} \,:\, F \in \mathcal{F}(M) \text{ with } F \text{ vanishing on } M\backslash A\right\}$$

This is the same definition as (20) if we had taken $V = M$. Next define for any $k \in [\![m]\!]$,

$$\Lambda_k(L) \quad \coloneqq \quad \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \lambda_1(A_l)$$

The higher order Dirichlet-Cheeger inequalities of Lee, Gharan and Trevisan [27] (see also [31] for its Markovian reformulation) assert that there exists a universal constant $c > 0$ such that

$$\forall\ k \in [\![m]\!], \qquad \lambda_k(L) \quad \geqslant \quad \frac{c}{k^6}\Lambda_k(L)$$

In particular, we can apply them to $L^{(r)}$ for $r > 0$:

$$\forall\ k \in [\![m]\!], \qquad \lambda_k^{(r)} \;=\; \lambda_k(L^{(r)}) \;\geqslant\; \frac{c}{k^6}\Lambda_k(L^{(r)}) \eqqcolon \Lambda_k^{(r)} \tag{21}$$

From Proposition 3, we know the behavior for large $r > 0$ of the l.h.s., for $k \in [\![v]\!]$, so it remains to investigate the r.h.s.

Fix $A \in \mathcal{A}$ and consider for $r > 0$,

$$\lambda_1^{(r)}(A) \quad \coloneqq \quad \inf\left\{\frac{\mathcal{E}_{L^{(r)}}(F,F)}{\mu^{(r)}[F^2]}\ :\ F \in \mathcal{F}(M) \text{ with } F \text{ vanishing on } M\backslash A\right\}$$

It is the smallest eigenvalue of $-L_A^{(r)}$, where $L_A^{(r)}$ is the subMarkovian generator acting on $\mathcal{F}(A)$ whose matrix is the $(A \times A)$-restriction of the matrix corresponding to $L^{(r)}$. The proof of Proposition 3 can easily be adapted to this situation to show that as r goes to $+\infty$, the first $\mathrm{card}(A \cap V)$ eigenvalues of $-L_A^{(r)}$ converge to the eigenvalues of $-S_A$. In particular we get

$$\lim_{r\to+\infty} \lambda_1^{(r)}(A) \quad = \quad \sigma_1(A)$$

Since $\mathcal{A}_k$ is a finite set, it follows that

$$\forall\ k \in [\![v]\!], \qquad \lim_{r\to+\infty} \Lambda_k^{(r)} \quad = \quad \kappa_k$$

where the r.h.s. is defined in (11). The wanted result is thus obtained by passing to the limit in (21) as r goes to $+\infty$.

■

Proof of Theorem 6

To relate the κ_k, for $k \in [\![v]\!]$, to isoperimetric quantities, we will adapt a computation of Jammes [23] to the finite setting. Fix $A \in \mathcal{A}$ and let us come back to (20). More precisely, consider $f \in \mathcal{F}(A \cap V)$ a minimizer of the infimum in the r.h.s. of (20) and F the associated solution of (10). From the Perron-Frobenius' theorem, we know that we can and do choose f to be non-negative and from (19), we also have $F \geqslant 0$. We are looking for a lower bound on the ratio

$$\frac{\mathcal{E}_L(F,F)}{\mu[f^2\mathbb{1}_{A\cap V}]} \quad = \quad \frac{\sum_{x\neq y\in M}\mu(x)L(x,y)(F(y)-F(x))^2}{2\sum_{x\in A\cap V}\mu(x)f^2(x)}$$

So multiply the numerator and the denominator by $\sum_{x'\neq y'\in M}\mu(x')L(x',y')(F(y')+F(x'))^2$. In the numerator we get

$$\begin{aligned}
&\sum_{x'\neq y'\in M}\mu(x')L(x',y')(F(y')+F(x'))^2 \sum_{x\neq y\in M}\mu(x)L(x,y)(F(y)-F(x))^2\\
&\qquad\geqslant \left(\sum_{x\neq y\in M}\mu(x)L(x,y)(F(y)+F(x))|F(y)-F(x)|\right)^2 \qquad (22)\\
&\qquad= \left(\sum_{x\neq y\in M}\mu(x)L(x,y)|F^2(y)-F^2(x)|\right)^2
\end{aligned}$$

where for the first bound we used the Cauchy-Schwarz inequality with respect to the measure $\underline{\mu}$ outside the diagonal of $M \times M$. Concerning the denominator, we begin by noting that

$$\sum_{x'\neq y'\in M}\mu(x')L(x',y')(F(y')+F(x'))^2 \quad \leqslant \quad 2\sum_{x'\neq y'\in M}\mu(x')L(x',y')(F^2(y')+F^2(x'))$$

$$\begin{aligned} &= 4 \sum_{x'\neq y'\in M} \mu(x')L(x',y')F^2(x') \\ &= 4 \sum_{x'\in M} \mu(x')\left|L(x',x')\right| F^2(x') \\ &\leqslant 4\,\|L\| \sum_{x'\in M} \mu(x')F^2(x') \end{aligned} \tag{23}$$

where we used the reversibility of μ with respect to L for the first equality. For any $G \in \mathcal{F}(M)$, denote $|dG|$ the function on $M \times M$ given by

$$\forall\ (x,y)\in M, \qquad |dG|(x,y) \ \coloneqq\ |G(y)-G(x)|$$

Putting together the above computations, we have obtained

$$\sigma_1(A) \ \geqslant\ \frac{1}{8\,\|L\|}\frac{\underline{\mu}[|dF^2|]}{\mu[F^2]}\frac{\underline{\mu}[|dF^2|]}{\mu[f^2\mathbb{1}_{A\cap V}]}$$

To deal with the ratios of the r.h.s., recall the co-area formula (see for instance Formula (3.3.2) page 381 of the lecture notes of Saloff-Coste [33]): for any non-negative $G \in \mathcal{F}(M)$ vanishing somewhere, we have

$$\underline{\mu}[|dG|] \ =\ \int_0^\tau \underline{\mu}[\partial D_t]\,dt$$

where

$$\begin{aligned} \forall\ t\geqslant 0, \qquad D_t &\coloneqq \{x\in M\ :\ G(x)\geqslant t\} \\ \tau &\coloneqq \inf\{t\geqslant 0\ :\ D_t=\varnothing\} \ =\ \inf\{t>0\ :\ \underline{\mu}(\partial D_t)=0\} \end{aligned} \tag{24}$$

We also have

$$\mu[G] \ =\ \int_0^\tau \mu[D_t]\,dt$$

Applying these formulas with $G \coloneqq F^2$ (which vanishes somewhere since $A \neq M$), we deduce that

$$\frac{\underline{\mu}[|dF^2|}{\mu[F^2]} \ \geqslant\ \inf\left\{\frac{\underline{\mu}(\partial D_t)}{\mu[D_t]}\ :\ t\geqslant 0\right\} \ \geqslant\ \min\{\eta(B)\ :\ B\in\mathcal{A},\, B\subset A\}$$

since we have $D_t \subset A$ for all $t \geqslant 0$. Furthermore we have

$$\mu[f^2\mathbb{1}_{A\cap V}] \ =\ \mu[F^2\mathbb{1}_{A\cap V}] \ =\ \int_0^{+\infty}\mu[D_t\cap A\cap V]\,dt \ =\ \int_0^{+\infty}\mu[D_t\cap V]\,dt$$

so we deduce similarly that

$$\frac{\underline{\mu}[|dF^2|]}{\mu[f^2\mathbb{1}_{A\cap V}]} \ \geqslant\ \min\left\{\eta'(B)\ :\ B\in\mathcal{A},\, B\subset A\right\}$$

Finally we have shown that

$$\forall\ A\in\mathcal{A}, \qquad \sigma_1(A) \ \geqslant\ \frac{\rho(A)\rho'(A)}{8\,\|L\|}$$

It follows that

$$\forall\ k\in[\![v]\!], \qquad \kappa_k \ \geqslant\ \frac{\iota_k}{8\,\|L\|} \tag{25}$$

and Theorem 6 is now an immediate consequence of Theorem 5.

■

Proof of Proposition 7

Consider the variational characterisation of σ_k:

$$\sigma_k \ =\ \min_{H\in\mathcal{F}_k(V)}\ \max_{f\in H\setminus\{0\}} \frac{\mathcal{E}_S(f,f)}{\nu[f^2]} = \min_{H\in\mathcal{F}_k(V)}\ \max_{f\in H\setminus\{0\}} \frac{\mathcal{E}_L(F_f,F_f)}{\mu[f^2\mathbb{1}_V]}$$

where $\mathcal{F}_k(\cdot)$ is the set of all k-dimensional subspace of $\mathcal{F}(\cdot)$, and F_f is solution to (10), the harmonic extension of f to $M\backslash V$. We can rewrite the variational characterisation in the following equivalent way.

$$\sigma_k \quad = \quad \min_{\substack{H\in\mathcal{F}_k(M)\\ H|_V\in\mathcal{F}_k(V)}} \max_{F\in H\backslash\{0\}} \frac{\mathcal{E}_L(F,F)}{\mu[F^2\mathbb{1}_V]}$$

Indeed for every $f\in\mathcal{F}(V)$, and all $F\in\mathcal{F}(M)$ with $F|_V = f$ we have

$$\mathcal{E}_L(F_f,F_f) \leqslant \mathcal{E}_L(F,F)$$

This is due to the harmonic property of F_f, for more details see (36). Let $(A_1,...,A_k)\in\mathcal{A}_k(V)$ and consider $H := \mathrm{Vect}(\mathbb{1}_{A_l} : l\in[\![k]\!])\in\mathcal{F}_k(M)$. It is also clear that $H|_V\in\mathcal{F}_k(V)$.

$$\begin{aligned}\frac{\mathcal{E}_L(\mathbb{1}_{A_l},\mathbb{1}_{A_l})}{\mu[\mathbb{1}_{A_l\cap V}]} &= \frac{\sum_{x\neq y\in M}\mu(x)L(x,y)(\mathbb{1}_{A_l}(y)-\mathbb{1}_{A_l}(x))^2}{2\mu(A_l\cap V)}\\ &= \frac{\sum_{x\in A_l,\, y\in A_l^c}\mu(x)L(x,y)+\mu(y)L(y,x)}{2\mu(A_l\cap V)} \quad = \quad \eta'(A_l)\end{aligned}$$

It implies

$$\sigma_k \quad \leqslant \quad \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k(V)} \max_{l\in[\![k]\!]} \eta'(A_l) = h'_k$$

and completes the proof.

■

We conclude this section by the proof of Proposition A in the introduction.

Proposition 12 *There is a universal positive constant c' such that*

$$\forall\ k\in[\![v]\!], \qquad \sigma_{2k} \quad \geqslant \quad \frac{c'}{\log^2(k+1)}\frac{\iota_k}{\|L\|}$$

Proof

By [27, Theorem 4.6] and [31, Section 2], we have

$$\forall\ k\in[\![v]\!], \qquad \lambda_k^{(r)} \quad \geqslant \quad \frac{c}{\log^2(k+1)}\Lambda_k^{(r)}$$

where c is a universal positive constant. Passing to the limit and using (25) we get

$$\forall\ k\in[\![v]\!], \qquad \sigma_k = \lim_{r\to\infty}\lambda_k^{(r)} \geqslant \frac{c}{\log^2(k+1)}\kappa_k \geqslant \frac{c}{8\log^2(k+1)}\frac{\iota_k}{\|L\|}$$

and the statement follows.

■

3 The measurable state space framework

Let $(M,\mathcal{M},\mu)$ be a probability measure space, endowed with a Markov kernel P leaving μ invariant (i.e. $\mu[P[F]] = \mu[F]$, for any bounded measurable function F). The Markov kernel P defines a map $P:\mathbb{L}^2(\mu)\to\mathbb{L}^2(\mu)$ by $P[F](x) := \int_M P(x,dy)F(y)$. It has the following properties

$$P[\mathbb{1}] = \mathbb{1}, \quad \text{and} \quad \forall\ F\geqslant 0 \Rightarrow P[F]\geqslant 0$$

We assume that P is **weakly mixing**, in the following sense. Let $Z := (Z(n))_{n\in\mathbb{Z}_+}$ be a Markov chain whose transition kernel is P. As usual, we indicate that Z is starting from $x\in M$, i.e. $Z(0) = x$, by putting x in index of the underlying probability $\mathbb{P}_x$ and expectation $\mathbb{E}_x$ (more generally, this index will stand for the initial law of $Z(0)$). Denote by $\mathcal{A}$ the set of $A\in\mathcal{M}$ such that $0<\mu(A)\leqslant 1$. For any $A\in\mathcal{A}$, define the **hitting time of** A by Z via

$$\tau_A \quad := \quad \inf\{n\in\mathbb{Z}_+ : Z(n)\in A\} \tag{26}$$

The weak mixing assumption asks for τ_A to be $\mathbb{P}_x$-a.s. finite, for any $x \in M$ and any $A \in \mathcal{A}$ (but what follows can be adapted to the situation where τ_A is a.s. finite, μ-a.s. in $x \in M$ and for any $A \in \mathcal{A}$).

Fix some $V \in \mathcal{A}$, we introduce corresponding **Steklov Markov kernel** K and **Steklov generator** S in the following way: let $\mathcal{B}(V)$ be the set of bounded measurable mappings defined on V. To any $f \in \mathcal{B}(V)$, we associate the mapping $F_f \in \mathcal{B}(M)$ given by

$$\forall\ x \in M, \qquad F_f(x) \ := \ \mathbb{E}_x[f(Z(\tau_V))] \tag{27}$$

and we define

$$\forall\ x \in V, \qquad \left\{ \begin{array}{rcl} K[f](x) & := & P[F_f](x) \\ S[f](x) & := & K[f](x) - f(x) \end{array} \right. \tag{28}$$

Note that K is a **Markov transition operator**, in the sense that it preserves the non-negativity of functions, as well as $\mathbb{1}_V$ (the mapping always taking the value 1 on V). It is immediate to check that the function F_f defined in (27) is given by

$$F_f \ = \ \sum_{n\in\mathbb{Z}_+} (\mathbb{1}_{M\setminus V}P)^n \mathbb{1}_V[f]$$

where the indicator functions are seen as multiplication operators. It follows that the transition kernel of K is $\sum_{n\in\mathbb{Z}_+}(P\mathbb{1}_{M\setminus V})^n P\mathbb{1}_V$. The function F_f is called the **harmonic extension** of f to M, because we have

$$\forall\ x \in M\setminus V, \qquad (P-I)[F_f](x) \ = \ 0 \tag{29}$$

where I stands for the identity operator (it will always be so in the sequel, even when the underlying space will not be the same). Indeed, we have on $M\setminus V$,

$$\begin{aligned} P[F_f] \ &= \ \mathbb{1}_{M\setminus V}P[F_f] \ = \ \mathbb{1}_{M\setminus V}P \sum_{n\in\mathbb{Z}_+} (\mathbb{1}_{M\setminus V}P)^n \mathbb{1}_V[f] \ = \ \sum_{n\in\mathbb{N}} (\mathbb{1}_{M\setminus V}P)^n \mathbb{1}_V[f] \\ &= \ \sum_{n\in\mathbb{Z}_+} (\mathbb{1}_{M\setminus V}P)^n \mathbb{1}_V[f] - \mathbb{1}_V[f] \ = \ \sum_{n\in\mathbb{Z}_+} (\mathbb{1}_{M\setminus V}P)^n \mathbb{1}_V[f] \ = \ F_f \end{aligned}$$

where we used that $\mathbb{1}_V = 0$ on $M\setminus V$ in the last but one equality.

Let ν be the normalisation into a probability measure of the restriction of μ to V.

Lemma 13 *The probability measure ν is invariant for K.*

Proof

Indeed, we compute that for any $f \in \mathcal{B}(V)$,

$$\nu[K[f]] \ = \ \frac{1}{\mu(V)}\mu[\mathbb{1}_V K[f]] \ = \ \frac{1}{\mu(V)}\left(\mu[K[f]] - \mu[\mathbb{1}_{M\setminus V}K[f]]\right)$$

By invariance of μ with respect to P, we have

$$\begin{aligned} \mu[\mathbb{1}_{M\setminus V}K[f]] \ &= \ \mu[P[\mathbb{1}_{M\setminus V}K[f]]] \ = \ \mu\left[P\mathbb{1}_{M\setminus V}\left(\sum_{n\in\mathbb{Z}_+}(P\mathbb{1}_{M\setminus V})^n P[\mathbb{1}_V f]\right)\right] \\ &= \ \mu\left[\sum_{n\in\mathbb{N}}(P\mathbb{1}_{M\setminus V})^n P[\mathbb{1}_V f]\right] \ = \ \mu[K[f]] - \mu[P[\mathbb{1}_V f]] \ = \ \mu[K[f]] - \mu[\mathbb{1}_V f] \end{aligned}$$

In conjunction with the previous identity, we get

$$\nu[K[f]] \ = \ \frac{1}{\mu(V)}\mu[\mathbb{1}_V f] \ = \ \nu[f]$$

as wanted.

■

From now on, we will only be concerned with the more specific **reversible situation** where P is symmetric in $\mathbb{L}^2(\mu)$ (or equivalently $\mu(dx)P(x,dy) = \mu(dy)P(y,dx)$). It follows that P can be extended into a bounded self-adjoint operator on $\mathbb{L}^2(\mu)$. Then ν is also reversible with respect to K: for any $f, g \in \mathcal{B}(V)$, we have

$$\begin{aligned}\nu[fK[g]] &= \frac{1}{\mu(V)}\mu[\mathbb{1}_V fK[g]] = \frac{1}{\mu(V)}\mu\left[\mathbb{1}_V f\left(\sum_{n\in\mathbb{Z}_+}(P\mathbb{1}_{M\setminus V})^n P[\mathbb{1}_V g]\right)\right]\\ &= \frac{1}{\mu(V)}\mu\left[\mathbb{1}_V g\left(\sum_{n\in\mathbb{Z}_+}P(\mathbb{1}_{M\setminus V}P)^n[\mathbb{1}_V f]\right)\right] = \frac{1}{\mu(V)}\mu\left[\mathbb{1}_V g\left(\sum_{n\in\mathbb{Z}_+}(P\mathbb{1}_{M\setminus V})^n P[\mathbb{1}_V f]\right)\right]\\ &= \nu[gK[f]]\end{aligned}$$

As a consequence, K can also be extended into a bounded self-adjoint operator on $\mathbb{L}^2(\nu)$. It leads us to introduce the following quantities for $k \in \mathbb{N}$,

$$\sigma_k \quad := \quad \inf_{H\in\mathcal{H}_k(V)}\ \sup_{f\in H\setminus\{0\}} \frac{\nu[f(I-K)[f]]}{\nu[f^2]} \tag{30}$$

where $\mathcal{H}_k(V)$ is the set of subspaces of dimension k of $\mathbb{L}^2(\nu)$. In the above definition and subsequently, the convention $\inf \varnothing := +\infty$ is enforced. When K has no essential spectrum, the finite elements of $(\sigma_k)_{k\in\mathbb{N}}$ are eigenvalues of $I - K = -S$. due to their variational characterization. We want to estimate them via higher order Cheeger inequalities. To go in this direction, let us consider

$$\mathcal{A}(V) \quad := \quad \{A \in \mathcal{A} \,:\, A \cap V \in \mathcal{A}\}$$

and for $A \in \mathcal{A}(V)$, the **Dirichlet–Steklov Markov kernel** K_A defined on $\mathcal{B}(A \cap V)$ as follows. For any $f \in \mathcal{B}(A \cap V)$, consider

$$\forall\ x \in M, \qquad F_{A,f}(x) \quad := \quad \mathbb{E}_x[f(Z(\tau_{A\cap V}))\mathbb{1}_{\{\tau_{A\cap V}\leqslant\tau_{M\setminus A}\}}]$$

where $\tau_{A\cap V}$ is the hitting time of $A \cap V$ by Z according to (26). The operator K_A is then given by

$$\forall\ x \in A \cap V, \qquad K_A[f](x) \quad := \quad P[F_{A,f}](x)$$

Let ν_A be the normalisation into a probability measure of the restriction of μ (or ν) to $A \cap V$. It can be easily checked as above that K_A is Markovian and symmetric in $\mathbb{L}^2(\nu_A)$, so that K_A can be extended into bounded self-adjoint operator on $\mathbb{L}^2(\nu_A)$. As in (30), we could introduce the quantities $(\sigma_k(A))_{k\in\mathbb{N}}$, but only its first element will be important for us:

$$\sigma_1(A) \quad := \quad \inf_{f\in\mathbb{L}^2(\nu_A)\setminus\{0\}} \frac{\nu_A[f(I-K_A)[f]]}{\nu_A[f^2]} \tag{31}$$

More precisely, for any $k \in \mathbb{N}$, let $\mathcal{A}_k(V)$ be the set of k-tuples $(A_1, A_2, ..., A_k)$ of disjoint elements from $\mathcal{A}(V)$. We introduce the Dirichlet–Steklov connectivity spectrum $(\kappa_k)_{k\in\mathbb{N}}$ of K via

$$\forall\ k \in \mathbb{N}, \qquad \kappa_k \quad := \quad \inf_{(A_1,\ldots,A_k)\in\mathcal{A}_k(V)}\ \max_{l\in[\![k]\!]} \sigma_1(A_l)$$

Definition (31) can be considered for any $A \in \mathcal{A}$, but with the usual convention, we get $\sigma_1(A) = +\infty$ when $A \notin \mathcal{A}(V)$, because $\mathbb{L}^2(\nu_A) = \{0\}$ in this case (and we are left with the trivial $K_A = 0$). Nevertheless, it enables to write

$$\forall\ k \in \mathbb{N}, \qquad \kappa_k \quad = \quad \inf_{(A_1,\ldots,A_k)\in\mathcal{A}_k}\ \max_{l\in[\![k]\!]} \sigma_1(A_l) \tag{32}$$

where $\mathcal{A}_k$ be the set of k-tuples $(A_1, A_2, ..., A_k)$ of disjoint elements from $\mathcal{A}$.

The goal of this section is to show that the extension of Theorem 5 holds in this setting:

Theorem 14 *There exists a universal constant $c > 0$ such that*

$$\forall\ k \in \mathbb{N}, \qquad \frac{c}{k^6}\kappa_k \leqslant \sigma_k \leqslant \kappa_k$$

As in the finite setting, the above result leads to higher order Cheeger inequalities presented below. Nevertheless Theorem 14 is more robust than the latter inequalities (34) and (35), as it will appear in its proof. In a future work, we hope to take advantage of Theorem 14 to give an alternative proof, as well as extensions, of Theorem C of the introduction.

We need the natural extensions of the definitions given in the finite case to our present mesurable state space setting. The boundary of any $A \in \mathcal{A}$ is given by

$$\partial A \quad := \quad \{(x, y) \,:\, x \in A,\, y \in M \backslash A\}$$

It is a measurable subset of $M \times M$ endowed with its product σ-field $\mathcal{M} \otimes \mathcal{M}$. Consider the measure $\underline{\mu}$ on $M \times M$ defined by

$$\underline{\mu}(dx, dy) \quad = \quad \mu(dx) P(x, dy) \tag{33}$$

Here there is a slight difference with the finite case, as we do not impose that the diagonal $D := \{(x, x) \,:\, x \in M\}$ is negligible with respect to $\underline{\mu}$: we cannot do so, because we are not sure D belongs to $\mathcal{M} \otimes \mathcal{M}$. It is not important, since we will only integrate with respect to $\underline{\mu}$ functions which vanish on the diagonal. In particular $\underline{\mu}$ enables to measure ∂A through $\underline{\mu}(\partial A)$. As a consequence, we can define for $A \in \mathcal{A}$ the isoperimetric ratios

$$\eta(A) \;:=\; \frac{\underline{\mu}(\partial A)}{\mu(A)} \qquad\qquad \eta'(A) \;:=\; \frac{\underline{\mu}(\partial A)}{\mu(A \cap V)}$$

(by convention, $\eta'(A) = +\infty$ if $A \notin \mathcal{A}(V)$). Again, the ratio $\eta'(A)$ is the measurable analogue of quantities introduced by Escobar [14] and Jammes [23], since in their terminology, ∂A and $A \cap V$ can be seen respectively as the interior and exterior boundaries, when the set V itself is seen as a boundary of M.

Next consider

$$\rho(A) \;:=\; \inf_{\substack{B \in \mathcal{A} \\ B \subseteq A}} \eta(B) \qquad\qquad \rho'(A) \;:=\; \inf_{\substack{B' \in \mathcal{A} \\ B' \subseteq A}} \eta'(B')$$

For any $k \in \mathbb{N}$, introduce the k-th Cheeger–Steklov constant of V by

$$\iota_k \quad := \quad \inf_{(A_1, \ldots, A_k) \in \mathcal{A}_k} \max_{l \in [\![k]\!]} \rho(A_l) \rho'(A_l)$$

The next result can be seen as an extension to higher order Cheeger inequalities of Théorème 1 of Jammes [23], as in Theorem 6:

Theorem 15 *Let c be the constant of Theorem 14. We have*

$$\forall\, k \in \mathbb{N}, \qquad \sigma_k \;\geqslant\; \frac{c}{k^6} \iota_k \tag{34}$$

Proof

The deduction of Theorem 15 from Theorem 14 is very similar to that of Theorem 6 from Theorem 5. For any function $f \in \mathbb{L}^2(\nu_A) \backslash \{0\}$, due to Remark 10 for the measurable situation and Lemma 17 below, we have

$$\begin{aligned}
\frac{\nu_A[f(I - K_A)[f]]}{\nu_A[f^2]} \quad &= \quad \frac{\mu[F_{A,f}(I - P)[F_{A,f}]]}{\mu[\mathbb{1}_{V \cap A} f^2]} \\
&= \quad \frac{\int_{M \times M} \mu(dx) P(x, dy) \mathbb{1}_{F_{A,f}(y) \neq F_{A,f}(x)} (F_{A,f}(y) - F_{A,f}(x))^2}{2\mu[\mathbb{1}_{V \cap A} f^2]}
\end{aligned}$$

We multiply the numerator and the denominator by $\int_{M \times M} \mu(dx) P(x, dy) \mathbb{1}_{F_{A,f}(y) \neq F_{A,f}(x)} (F_{A,f}(y) + F_{A,f}(x))^2$ and follow the same calculation as in the proof of Theorem 6. The key point is that the statement of the co-area formula is the same in the finite and measurable situations, replacing sums by integrals. To illustrate the kind of slight modifications to be taken into account (also that $\|L\|$ of Theorem 5 can be replaced by 1 here), let us present the equivalent of the computation (23)

$$\begin{aligned}
&\int_{M \times M} \mu(dx) P(x, dy) \mathbb{1}_{F_{A,f}(y) \neq F_{A,f}(x)} (F_{A,f}(y) + F_{A,f}(x))^2 \\
&\leqslant \; 2 \int_{M \times M} \mu(dx) P(x, dy) \mathbb{1}_{F_{A,f}(y) \neq F_{A,f}(x)} (F_{A,f}^2(y) + F_{A,f}^2(x)) \\
&= \; 4 \int_{M \times M} \mu(dx) P(x, dy) \mathbb{1}_{F_{A,f}(y) \neq F_{A,f}(x)} F_{A,f}^2(x) \\
&\leqslant \; 4 \int_M \mu(dx) F_{A,f}^2(x)
\end{aligned}$$

The measurable indicator $\mathbb{1}_{F_{A,f}(y')\neq F_{A,f}(x)}$ is inherited from the Cauchy-Schwarz' inequality in (22) and must be kept to avoid the possible drawback that $D \notin \mathcal{M}\otimes\mathcal{M}$.

In the same spirit, in (24) τ should be defined as the r.h.s. Then we apply the above calculation to a family of functions $f_n \in \mathbb{L}^2(\nu_A)$ such that $\frac{\nu_A[f_n(I-K_A)[f_n]]}{\nu_A[f_n^2]} \to \sigma_1(A)$ as n tends to ∞.

■

As in the previous section we consider

$$h'_k \quad := \quad \inf_{(A_1,\ldots,A_k)\in\mathcal{A}_k(V)} \max_{l\in[\![k]\!]} \eta'(A_l) \tag{35}$$

and by the same proof, Proposition 7 is valid in the measurable situation, i.e.

$$\forall\ k \in \mathbb{N}, \qquad \sigma_k \leqslant h'_k$$

The proof of Theorem 14 follows the same pattern as in the finite case: it will be deduced from the higher order Cheeger inequalities from [31], once the above quantities will be shown to be limits of spectra associated to speed-up Markov processes. More precisely, for $r > 0$, consider the jump Markov generator $L^{(r)}$ on M given by the kernel

$$L^{(r)}(x, dy) \quad := \quad \begin{cases} r(P(x, dy) - \delta_x(dy))\,, & \text{if } x \in M\backslash V \\ P(x, dy) - \delta_x(dy)\,, & \text{if } x \in V \end{cases}$$

Define the probability measure $\mu^{(r)}$ on $(M, \mathcal{M})$ by

$$\mu^{(r)}(dx) \quad = \quad \left(\frac{\mathbb{1}_V(x)}{Z_r} + \frac{\mathbb{1}_{M\backslash V}(x)}{rZ_r}\right)\mu(dx)$$

where $Z_r := \mu(V) + (1 - \mu(V))/r$ is the normalisation constant.

The proof of Lemma 11 is still valid and leads to

Lemma 16 *The operator $L^{(r)}$ is self-adjoint in $\mathbb{L}^2(\mu^{(r)})$.*

Similarly to (30) and (31), consider

$$\lambda_k^{(r)} \quad := \quad \inf_{H\in\mathcal{H}_k} \sup_{F\in H\backslash\{0\}} \frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]}$$

where $\mathcal{H}_k$ is the set of subspaces of dimension k of $\mathbb{L}^2(\mu) = \mathbb{L}^2(\mu^{(r)})$, and for any $A \in \mathcal{A}$,

$$\lambda_1^{(r)}(A) \quad := \quad \inf_{F\in\mathbb{L}^2(A,\mu)\backslash\{0\}} \frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]}$$

where $\mathbb{L}^2(A, \mu)$ is the space of $F \in \mathbb{L}^2(\mu)$ which vanish on $M\backslash A$. The larger $\lambda_1^{(r)}(A)$ is, the easier it is for a (continuous time) Markov process associated to the generator $L^{(r)}$ to exit A: the quantity $\lambda_1^{(r)}(A)$ corresponds to the first Dirichlet eigenvalue of A and measures the asymptotical rate of exit from A.

The numerators in the above r.h.s. are only slightly dependent on $r \geqslant 1$ and related to the similar quantities relative to K:

Lemma 17 *We have for any $r > 0$ and $F \in \mathbb{L}^2(\mu)$,*

$$\mu^{(r)}[F(-L^{(r)})[F]] \quad = \quad \frac{1}{2Z_r}\int \mu(dx)P(x, dy)(F(y) - F(x))^2 \quad = \quad \frac{1}{Z_r}\mu[F(I - P)[F]]$$

Furthermore, for any $f \in \mathbb{L}^2(\nu)$,

$$\nu[f(I - K)[f]] \quad = \quad \frac{1}{\mu(V)} \inf\{\mu[F(I - P)[F]] \,:\, F_{|V} = f\} \quad = \quad \frac{1}{\mu(V)}\mu[F_f(I - P)[F_f]]$$

where $F_{|V}$ stands for the restriction of F to V.

Proof
By definition, for any $r>0$ and $F\in\mathbb{L}^2(\mu)$, we have

$$\begin{aligned}
\mu^{(r)}[f(-L^{(r)})[F]] &= -\int_{M\times M}\mu^{(r)}(dx)L^{(r)}(x,dy)F(x)F(y)\\
&= -\int_{V\times M}\mu^{(r)}(dx)L^{(r)}(x,dy)F(x)F(y)-\int_{(M\backslash V)\times M}\mu^{(r)}(dx)L^{(r)}(x,dy)F(x)F(y)\\
&= \frac{1}{Z_r}\int_{V\times M}\mu(dx)(\delta_x(dy)-P(x,dy))F(x)F(y)\\
&\quad +\frac{1}{Z_r}\int_{(M\backslash V)\times M}\mu(dx)(\delta_x(dy)-P(x,dy))F(x)F(y)\\
&= \frac{1}{Z_r}\int_{M\times M}\mu(dx)P(x,dy)(F(x)-F(y))F(x)\\
&= \frac{1}{2Z_r}\int_{M\times M}\mu(dx)P(x,dy)(F(y)-F(x))^2
\end{aligned}$$

where we used the reversibility (under the form $\mu(dx)P(x,dy)=\mu(dy)P(y,dx)$) in the last equality. Note that the last but one r.h.s. is just $\mu[F(I-P)[F]]/Z_r$.

Similarly, we compute that for any $f\in\mathbb{L}^2(\nu)$,

$$\begin{aligned}
\nu[f(I-K)[f]] &= \int_{V\times V}\nu(dx)K(x,dy)(f(x)-f(y))f(x) \;=\; \int_V \nu(dx)K[f(x)-f](x)f(x)\\
&= \int_V \nu(dx)P[f(x)-F_f](x)f(x) \;=\; \int_{V\times M}\nu(dx)P(x,dy)(f(x)-F_f(y))f(x)\\
&= \int_{V\times M}\nu(dx)P(x,dy)(F_f(x)-F_f(y))F_f(x)\\
&= \int_{M\times M}\nu(dx)P(x,dy)(F_f(x)-F_f(y))F_f(x) \;=\; \frac{1}{\mu(V)}\mu[F_f(I-P)[F_f]]
\end{aligned}$$

where in the last but one equality, we used that F_f is harmonic on $M\backslash V$ according to (29). It remains to see that

$$\inf\{\mu[F(I-P)[F]]\,:\,F_{|V}=f\} \;=\; \mu[F_f(I-P)[F_f]] \tag{36}$$

namely that among all $F\in\mathbb{L}^2(\mu)$ coinciding with f on V, the quantity $\mu[F(I-P)[F]]$ is minimum when $F=F_f$. This is a well-known fact, due to the harmonic property of F_f, let us recall the argument. Write any such function F under the form F_f+G where $G\in\mathbb{L}^2(\mu)$ vanishes on V. We have

$$\begin{aligned}
\mu[F(I-P)[F]] &= \mu[F_f(I-P)[F_f]]+\mu[F_f(I-P)[G]]+\mu[G(I-P)[F_f]]+\mu[G(I-P)[G]]\\
&= \mu[F_f(I-P)[F_f]]+2\mu[G(I-P)[F_f]]+\mu[G(I-P)[G]]\\
&= \mu[F_f(I-P)[F_f]]+\mu[G(I-P)[G]]
\end{aligned}$$

where we used reversibility, $G_{|V}=0$ and (29). The announced minimisation comes from the non-negativity of

$$\mu[G(I-P)[G]] \;=\; \int_{M\times M}\nu(dx)P(x,dy)(G(x)-G(y))^2$$

■

Our first approximation results are:

Theorem 18 *Assume that $\lambda:=\lambda_1^{(1)}(M\backslash V)>0$ (this quantity will be subsequently called the **Dirichlet gap** of $M\backslash V$), namely that it is quite easy for the Markov chains $(Z)_{x\in M}$ to enter into V. Then for any $k\in\mathbb{N}$, we have*

$$\lim_{r\to+\infty}\lambda_k^{(r)} \;=\; \sigma_k$$

and for any $A\in\mathcal{A}$,

$$\lim_{r\to+\infty}\lambda_1^{(r)}(A) \;=\; \sigma_1(A) \tag{37}$$

More precisely, the latter convergence is uniform, in the following sense: let $\mathfrak{d}$ be a distance on the compact set $[0,+\infty]$ compatible with its usual topology. We have

$$\lim_{r\to+\infty}\sup_{A\in\mathcal{A}}\mathfrak{d}(\lambda_1^{(r)}(A),\sigma_1(A)) \quad = \quad 0$$

More generally, the proof of (37) will show that $\lim_{r\to+\infty}\lambda_k^{(r)}(A)=\sigma_k(A)$, for any $k\in\mathbb{N}$, but it will not be useful for our purposes.

Proof

The proof is mainly concerned with the first convergence, since the second one will follow by recycling the obtained quantitative bounds.

We begin by checking that for any $k\in\mathbb{N}$, we have

$$\limsup_{r\to+\infty}\lambda_k^{(r)} \quad \leqslant \quad \sigma_k \tag{38}$$

This result does not require that $\lambda_1^{(1)}(M\backslash V)>0$. Note that any $H\in\mathcal{H}_k(V)$ can be seen as an element of $\mathcal{H}_k$, through the one-to-one mapping

$$\mathbb{L}^2(\nu)\ni f \quad \mapsto \quad F_f\in\mathbb{L}^2(\mu)$$

so that we have

$$\lambda_k^{(r)} \quad \leqslant \quad \inf_{H\in\mathcal{H}_k(V)}\max_{f\in H}\frac{\mu^{(r)}[F_f(-L^{(r)})[F_f]]}{\mu^{(r)}[F_f^2]}$$

According to Lemma (17), for any $f\in\mathbb{L}^2(\nu)$,

$$\mu^{(r)}[F_f(-L^{(r)})[F_f]] \quad = \quad \frac{1}{Z_r}\mu[F_f(I-P)[F_f]] \quad = \quad \frac{\mu(V)}{Z_r}\nu[f(I-K)[f]]$$

Furthermore, we compute that

$$\mu^{(r)}[F_f^2] \quad = \quad \frac{1}{Z_r}\left(\mu[\mathbb{1}_Vf^2]+\mu[\mathbb{1}_{M\backslash V}F_f^2]/r\right) \quad \geqslant \quad \frac{1}{Z_r}\mu[\mathbb{1}_Vf^2] \quad = \quad \frac{\mu(V)}{Z_r}\nu[f^2]$$

Thus we get that

$$\lambda_k^{(r)} \quad \leqslant \quad \inf_{H\in\mathcal{H}_k(V)}\max_{f\in H}\frac{\nu[f(I-K)[f]]}{\nu[f^2]} \quad = \quad \sigma_k$$

from which (38) follows at once.

Conversely, to any subspace $H\subset\mathbb{L}^2(\mu)$ associate $\widetilde{H}$ the subspace of $\mathbb{L}^2(\nu)$ generated by the functions $F_{|V}$ for $F\in H$. For $k\in\mathbb{N}$, let $\mathcal{H}_k^*$ stand for the set of $H\in\mathcal{H}_k$ which are such that $\widetilde{H}\in\mathcal{H}_k(V)$, namely such that $\widetilde{H}$ has dimension k. We begin by remarking that for $k\in\mathbb{N}$ such that $k\leqslant\dim(\mathbb{L}^2(\nu))$ ($\leqslant+\infty$) and for any $r>0$,

$$\lambda_k^{(r)} \quad = \quad \inf_{H\in\mathcal{H}_k^*}\max_{F\in H\backslash\{0\}}\frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]} \tag{39}$$

Indeed, fix some $H\in\mathcal{H}_k$ and choose $F_1,F_2,...,F_k$ a basis of H. Consider for $l\in[\![k]\!]$, f_l the restriction of F_l to V. If $(f_l)_{l\in[\![k]\!]}$ is not an independent family of $\mathbb{L}^2(V)$, then we can find another family $(\widehat{f}_l)_{l\in[\![k]\!]}$ of $\mathbb{L}^2(V)$ such that for any $\epsilon\in(0,1]$, the family $(f_l+\epsilon\widehat{f}_l)_{l\in[\![k]\!]}$ is independent. For $\epsilon\in(0,1]$, consider H_ϵ the space generated by $(F_l+\epsilon\widehat{F}_l)_{l\in[\![k]\!]}$, where the $\widehat{F}_l$, $l\in[\![k]\!]$, are the functions coinciding with $\widehat{f}_l$ on V and e.g. vanishing outside. Since $\widetilde{H}_\epsilon$ belongs to $\mathcal{H}_k(V)$, we have that $H_\epsilon\in\mathcal{H}_k^*$. Furthermore, it is clear that

$$\lim_{\epsilon\to0_+}\max_{F\in H_\epsilon\backslash\{0\}}\frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]} \quad = \quad \max_{F\in H\backslash\{0\}}\frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]}$$

showing (39).

Recall that we have by definition

$$\lambda \;\coloneqq\; \inf_{F\in\mathbb{L}^2(M\backslash V,\mu)\backslash\{0\}}\frac{\mu[F(P-I)[F]]}{\mu[F^2]} \quad = \quad \inf_{\substack{F\in\mathbb{L}^2(\mu)\\ \mathbb{1}_{M\backslash V}F\neq0}}\frac{\mu[\mathbb{1}_{M\backslash V}F(I-P)[\mathbb{1}_{M\backslash V}F]]}{\mu[\mathbb{1}_{M\backslash V}F^2]}$$

It follows that for any $F \in \mathbb{L}^2(\mu)$,

$$\begin{aligned}\mu[\mathbb{1}_{M\setminus V}F^2] &\leqslant \frac{1}{\lambda}\mu[\mathbb{1}_{M\setminus V}F(I-P)[\mathbb{1}_{M\setminus V}F]] \leqslant \frac{1}{\lambda}\mu[(F-\mathbb{1}_V F)(I-P)[F-\mathbb{1}_V F]]\\ &\leqslant \frac{2}{\lambda}\left(\mu[F(I-P)[F]]+\mu[\mathbb{1}_V F(I-P)[\mathbb{1}_V F]]\right) \leqslant \frac{2}{\lambda}\left(\mu[F(I-P)[F]]+2\mu[\mathbb{1}_V F^2]\right)\end{aligned}$$

where we used that the mapping $\mathbb{L}^2(\mu) \ni F \mapsto \mu[F(I-P)[F]]$ is a (non-negative) quadratic form (called the **Dirichlet form** associated to the Markov generator $P-I$, see Remark 10) and that the spectrum of the operator $I-P$ is included into $[0,2]$. We deduce that for any $r>0$,

$$\begin{aligned}\mu^{(r)}[F^2] &= \frac{1}{Z_r}\left(\mu[\mathbb{1}_V F^2]+\frac{1}{r}\mu[\mathbb{1}_{M\setminus V}F^2]\right)\\ &\leqslant \frac{1}{Z_r}\left(\left(1+\frac{4}{r\lambda}\right)\mu[\mathbb{1}_V F^2]+\frac{2}{r\lambda}\mu[F(I-P)[F]]\right)\end{aligned}$$

It follows that

$$\frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]} \geqslant \frac{\mu[F(I-P)[F]]}{\left(1+\frac{4}{\lambda r}\right)\mu[\mathbb{1}_V F^2]+\frac{2}{r\lambda}\mu[F(I-P)[F]]} = \phi_r\left(\frac{\mu[F(I-P)[F]]}{\mu[\mathbb{1}_V F^2]}\right)$$

where

$$\phi_r : [0,+\infty] \ni u \mapsto \frac{u}{1+\frac{4}{\lambda r}+\frac{2u}{\lambda r}}$$

Note that the latter mapping is increasing, so taking into account Lemma 17, we have, with $f := F_{|V}$,

$$\phi_r\left(\frac{\mu[F(I-P)[F]]}{\mu[\mathbb{1}_V F^2]}\right) \geqslant \phi_r\left(\frac{\mu[F_f(I-P)[F_f]]}{\mu(V)\nu[f^2]}\right) = \phi_r\left(\frac{\nu[f(I-K)[f]]}{\nu[f^2]}\right)$$

We deduce from the above computations that for $H \in \mathcal{H}_k^*$,

$$\begin{aligned}\max_{F\in H\setminus\{0\}}\frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]} &\geqslant \max_{f\in\widetilde{H}\setminus\{0\}}\phi_r\left(\frac{\nu[f(I-K)[f]]}{\nu[f^2]}\right)\\ &= \phi_r\left(\max_{f\in\widetilde{H}\setminus\{0\}}\frac{\nu[f(I-K)[f]]}{\nu[f^2]}\right) \geqslant \phi_r(\sigma_k)\end{aligned}$$

since $\widetilde{H} \in \mathcal{H}_k(V)$.

When $k \leqslant \dim(\mathbb{L}^2(\nu))$, it follows from (39) that

$$\lambda_k^{(r)} \geqslant \phi_r(\sigma_k)$$

and it remains to let r go to $+\infty$ to get

$$\liminf_{r\to+\infty}\lambda_k^{(r)} \geqslant \lim_{r\to+\infty}\phi_r(\sigma_k) = \sigma_k \tag{40}$$

When $k > \dim(\mathbb{L}^2(\nu))$, for any $H \in \mathcal{H}_k$, we can find $F^* \in H\setminus\{0\}$ such that $F^*_{|V}=0$ and so

$$\max_{F\in H\setminus\{0\}}\frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]} \geqslant \frac{\mu^{(r)}[F^*(-L^{(r)})[F^*]]}{\mu^{(r)}[F^{*2}]} \geqslant \phi_r(+\infty) = \lambda r$$

It follows that $\lambda_k^{(r)} \geqslant \lambda r/2$ and letting r go to $+\infty$ we get

$$\liminf_{r\to+\infty}\lambda_k^{(r)} = +\infty = \sigma_k$$

Thus (40) is always true and in conjunction with (38), we obtain the first announced convergence.

For the second convergence, note that for $A \in \mathcal{A}$, the definition of $\sigma_1(A)$ is similar to that of σ_1 where V is replaced by $V \cup (M\setminus A)$, except we only consider functions that vanish on $M\setminus A$. It leads us to consider

$$\lambda_A := \lambda_1^{(1)}(A\setminus V)$$

and for $r > 0$, the mapping $\phi_{A,r}$ given by

$$\phi_{A,r} \;:\; [0,+\infty] \ni u \quad\mapsto\quad \frac{u}{1+\frac{4}{\lambda_A r}+\frac{2u}{\lambda_A r}}$$

The above computations show that for any $r > 0$,

$$\sigma_1(A) \;\geqslant\; \lambda_1^{(r)}(A) \;\geqslant\; \phi_{A,r}(\sigma_1(A))$$

Note that the mapping $\mathcal{A} \ni B \mapsto \lambda_1^{(1)}(B)$ is non-increasing with respect to the inclusion of sets (because $\lambda_1^{(1)}(B)$ corresponds to an infimum over the space of functions $\mathbb{L}^2(B,\mu)\backslash\{0\}$, which is non-decreasing with respect to B), so we deduce

$$\begin{array}{rrcl} & \lambda_A & \geqslant & \lambda \\ \forall\; r>0, & \phi_{A,r} & \geqslant & \phi_r \end{array}$$

It follows that to get the wanted uniform convergence, it is sufficient to show that

$$\lim_{r\to+\infty}\sup_{u\in[0,+\infty]} \mathfrak{d}(u,\phi_r(u)) \;=\; 0$$

which is an elementary computation, since it can be reduced to

$$\lim_{r\to+\infty}\max\left(\sup_{u\in[0,1]}|u-\phi(u)|,\ \sup_{u\in[1,+\infty]}\left|\frac{1}{u}-\frac{1}{\phi_r(u)}\right|\right) \;=\; 0$$

■

Remark 19 The assumption of positive Dirichlet gap in Theorem 18 is really needed. Indeed, remark that when $\lambda_1^{(1)}(M\backslash V) = 0$, then for any $r > 0$, we have $\lambda_1^{(r)}(M\backslash V) = 0$. Due to Lemma 17, this is an immediate consequence of

$$\forall\; F\in\mathbb{L}^2(\mu), \qquad \frac{1}{\max(1,r)}\frac{\mu[F(I-P)[F]]}{\mu[F^2]} \;\leqslant\; \frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]} \;\leqslant\; \frac{1}{\min(1,r)}\frac{\mu[F(I-P)[F]]}{\mu[F^2]}$$

Furthermore, the fact that $\lambda_1^{(r)}(M\backslash V) = 0$ implies that $\lambda_2^{(r)} = 0$: consider a sequence of functions $(F_n)_{n\in\mathbb{N}}$ from $\mathbb{L}^2(M\backslash V,\mu)\backslash\{0\}$ such that

$$\lim_{n\to\infty}\frac{\mu^{(r)}[F_n(-L^{(r)})[F_n]]}{\mu^{(r)}[F_n^2]} \;=\; 0$$

and consider for $n\in\mathbb{N}$, $H_n := \mathrm{Vect}(\mathbb{1},F_n)\in\mathcal{H}_2$. We easily get that

$$\lim_{n\to\infty}\max_{F\in H_n\backslash\{0\}}\frac{\mu^{(r)}[F(-L^{(r)})[F]]}{\mu^{(r)}[F^2]} \;=\; 0$$

i.e. $\lambda_2^{(r)} = 0$. In particular, we have

$$\lim_{r\to+\infty}\lambda_2^{(r)} \;=\; 0$$

But it may happen that $\sigma_2 > 0$. Consider for instance an ergodic birth and death transition kernel P on $\mathbb{Z}_+$: we take $M = \mathbb{Z}_+$ endowed with a probability measure μ charging all the points. The reversible transition kernel P is defined via a Metropolis procedure:

$$\forall\; x,y\in\mathbb{Z}_+, \qquad P(x,y) \;:=\; \begin{cases} \frac{1}{2}\left(\frac{\mu(x)}{\mu(y)}\wedge 1\right), & \text{if } |y-x|=1 \\ 0, & \text{if } |y-x|\geqslant 2 \\ 1-\sum_{z\in\mathbb{Z}_+\backslash\{x\}}P(x,z), & \text{if } x=y \end{cases}$$

where $p\wedge q := \min\{p,q\}$. The definition of P via the above Metropolis procedure implies that it is irreducible with respect to μ (see for example [4, Section 3.1]). Recall that by definition, P is ergodic if and only if

$$\forall\; F\in\mathbb{L}^2(\mu), \qquad P[F]=F \quad\Rightarrow\quad F\in\mathrm{Vect}(\mathbb{1})$$

Thus, irreducibility implies ergodicity in the above example. As a result, P is also weakly mixing. Assume that the queues of μ are sufficiently heavy, in the sense that

$$\lim_{x\to\infty} \frac{\mu(x)}{\mu([x,\infty))} \quad = \quad 0$$

An application of discrete Hardy's inequalities (see [29], they are given for finite birth and death processes, but are also valid in the denumerable setting) implies that $\lambda_1^{(1)}(\mathbb{Z}_+\backslash\{0,1\}) = 0$. Nevertheless considering for instance $V = \{0,1\}$ we get that $\sigma_2 > 0$, as a consequence of $K(0,1) = P(0,1) > 0$ and $K(1,0) = P(1,0) > 0$. More generally it can be proven that $\sigma_2 > 0$ for any finite subset of $\mathbb{Z}_+$ non-empty and not reduced to a singleton.

Note that under the weak mixing assumption (or under the ergodicity assumption), $\lambda_2^{(1)} = 0$ means that 0 is the lower bound of the essential spectrum, so that $\lambda_k^{(1)} = 0$ for all $1 \leqslant k < \dim(\mathbb{L}^2(\mu)) + 1$ and similarly, $\lambda_k^{(r)} = 0$ for any $r > 0$ and $1 \leqslant k < \dim(\mathbb{L}^2(\mu)) + 1$.

□

To prove Theorem 14 without the assumption of a positive Dirichlet gap on $M\backslash V$, we will accelerate the Markov process associated to the generator $P - I$ more strongly on the slow points of $M\backslash V$ (near ∞ in the above remark). More precisely, we look for a measurable function $\varphi : M \to [1,+\infty)$, taking the value 1 on V, such that by defining for $r > 0$, the jump Markov generator $L^{(r)}$ by

$$L^{(r)}(x,dy) \quad := \quad \begin{cases} r\varphi(x)(P(x,dy) - \delta_x(dy)), & \text{if } x \in M\backslash V \\ \varphi(x)(P(x,dy) - \delta_x(dy)), & \text{if } x \in V \end{cases} \tag{41}$$

we have that $L^{(1)}$ admits a positive Dirichlet gap on $M\backslash V$. Then, with the corresponding spectra, Theorem 18 will hold. Note that the notions of harmonic functions on $M\backslash V$ with respect to $P - I$ and $L^{(r)}$, for all $r > 0$, coincide and the corresponding Steklov Markov kernels and generators are the same.

Let $X := (X(t))_{t\geqslant 0}$ be a jump Markov process of generator $P - I$ (see Chapter 4 in [16] for the definition). Fix some $\chi \in (0,1)$ and consider the function φ defined by

$$\forall\, x \in M, \qquad \varphi(x) \quad := \quad \frac{1}{\mathbb{E}_x[\chi^\tau]}$$

where $\tau := \inf\{t \geqslant 0 \,:\, X_t \in V\}$. Note that when $x \in M\backslash V$ is a point from which it is difficult to hit V, namely such that τ has a propensity to be large, then $\varphi(x)$ is quite large also: the jump Markov process $X^{(1)} := (X^{(1)}(t))_{t\geqslant 0}$ associated to $L^{(1)}$ is strongly accelerated at x in comparison with X, as wanted. From now on, the notation $L^{(r)}$, for $r > 0$, will only refer to the operators given in (41). Here is the consequence of the acceleration procedure:

Lemma 20 *We have, with $\tau^{(1)} := \inf\{t \geqslant 0 \,:\, X_t^{(1)} \in V\}$,*

$$\forall\, x \in M, \qquad \mathbb{E}_x[\tau^{(1)}] \quad \leqslant \quad \frac{1}{\ln(1/\chi)}$$

Proof

Let us recall the time change transformations (cf. for instance Chapter 6 from the book of Ethier and Kurtz [16]), which enable to construct $X^{(1)}$ from X when both processes start from a fixed $x \in M$. Due to [16, Theorem 1.4], if we define $(\theta_t)_{t\geqslant 0}$ via

$$\forall\, t \geqslant 0, \qquad \int_0^{\theta_t} \frac{1}{\varphi(X_s)}\, ds \quad = \quad t$$

then we can take

$$\forall\, t \geqslant 0, \qquad X^{(1)}(t) \quad := \quad X(\theta_t)$$

In particular, we get

$$\tau^{(1)} \quad = \quad \int_0^\tau \frac{1}{\varphi(X_s)}\, ds$$

It follows that

$$
\begin{aligned}
\mathbb{E}_x[\tau^{(1)}] &= \mathbb{E}_x\left[\int_0^\tau \frac{1}{\varphi(X_s)}\,ds\right] = \int_0^{+\infty}\mathbb{E}_x\left[\mathbb{1}_{s\leqslant\tau}\frac{1}{\varphi(X_s)}\right]ds = \int_0^{+\infty}\mathbb{E}_x\left[\mathbb{1}_{s\leqslant\tau}\mathbb{E}_{X_s}[\chi^\tau]\right]ds \\
&= \int_0^{+\infty}\mathbb{E}_x\left[\mathbb{1}_{s\leqslant\tau}\chi^{-s}\mathbb{E}_x[\chi^\tau|(X_u)_{u\in[0,s]}]\right]ds = \int_0^{+\infty}\mathbb{E}_x\left[\mathbb{1}_{s\leqslant\tau}\chi^{-s}\chi^\tau\right]ds
\end{aligned}
$$

where we use the measurability of the event $\{s \leqslant \tau\}$ with respect to the σ-field generated by $(X_u)_{u\in[0,s]}$, the fact that on $\{s\leqslant\tau\}$, we have $\tau = s+\tau\circ\theta_s$, where θ_s is the shift of the trajectories by an amount s of time, and the Markov property, stating that for any measurable functional F on the trajectories, we have a.s. $\mathbb{E}_x[F\circ\theta_s|(X_u)_{u\in[0,s]}] = \mathbb{E}_{X_s}[F]$. In this formula, $\mathbb{E}_{X_s}$ is the expectation with respect to a diffusion X starting from X_s at time 0. Since all the integral elements are non-negative, we can use again Fubini's formula to get that the last integral is equal to

$$
\mathbb{E}_x\left[\int_0^{+\infty}\mathbb{1}_{s\leqslant\tau}\chi^{-s}\chi^\tau\,ds\right] = \mathbb{E}_x\left[\int_0^\tau \chi^{\tau-s}\,ds\right] = \mathbb{E}_x\left[\int_0^\tau \chi^{s}\,ds\right] = \mathbb{E}_x\left[\frac{\chi^\tau-1}{\ln(\chi)}\right] \leqslant \frac{1}{\ln(1/\chi)}
$$

as announced.

■

From the previous uniform boundedness of the expectations of $\tau^{(1)}$, we deduce uniform exponential bounds on its queues:

Lemma 21 *We have, with $\alpha \coloneqq \ln(2)\ln(1/\chi)/2$,*

$$
\forall\, x\in M,\ \forall\, s\geqslant 0,\qquad \mathbb{P}_x[\tau^{(1)}\geqslant s] \leqslant 2\exp(-\alpha s)
$$

Proof

For any $n\in\mathbb{Z}_+$, we have

$$
\forall\, x\in M,\qquad \mathbb{P}_x[\tau^{(1)}\geqslant an] \leqslant 2^{-n}
$$

where

$$
a \coloneqq \frac{2}{\ln(1/\chi)}
$$

This is shown by iteration on $n\in\mathbb{Z}_+$. It is clear for $n=0$ and if it is true for some $n\in\mathbb{Z}_+$, then by the Markov property and Lemma 20: for any $x\in M$,

$$
\begin{aligned}
\mathbb{P}_x[\tau^{(1)}\geqslant a(n+1)] &= \mathbb{E}_x[\mathbb{1}_{\tau^{(1)}\geqslant a}\mathbb{P}_{X^{(1)}(a)}[\tau^{(1)}\geqslant an]] \leqslant 2^{-n}\mathbb{P}_x[\tau^{(1)}\geqslant a] \\
&\leqslant 2^{-n}\frac{\mathbb{E}_x[\tau^{(1)}]}{a} \leqslant 2^{-n}\frac{1}{a\ln(1/\chi)} = 2^{-(n+1)}
\end{aligned}
$$

where in the third line we use the Markov inequality.

For any $s\in\mathbb{R}_+$, write $n\coloneqq\lfloor s/a\rfloor\in\mathbb{Z}_+$, so that for any $\forall\, x\in M$,

$$
\mathbb{P}_x[\tau^{(1)}\geqslant s] \leqslant \mathbb{P}_x[\tau^{(1)}\geqslant na] \leqslant 2^{-n} = 2^{-\lfloor s/a\rfloor} \leqslant 2(2^{-s/a}) = 2\exp(-\alpha s)
$$

as announced.

■

To simplify the notation, we now take $\chi = \exp(-2/\ln(2))$, so that $\alpha = 1$. Uniform exponential bounds on the queues of exit times from a domain are well-known to imply that the associated Dirichlet gap is positive. Here is a simple proof of this fact:

Lemma 22 *We have*

$$
\lambda_1^{(1)}(M\backslash V) \geqslant \frac{1}{2}
$$

where the l.h.s. is relative to the accelerated generator $L^{(1)}$.

Proof

As in Lemma 11, we see that the measure $\frac{1}{\varphi(x)}\mu(dx)$ is reversible for $L^{(1)}$. Its total weight is

$$Z^{(1)} \;\coloneqq\; \int \mathbb{E}_x[\chi^{\tau^{(1)}}]\,\mu(dx) \;\in\; (0,1)$$

which leads us to define $\mu^{(1)}(dx) \coloneqq \frac{1}{Z^{(1)}\varphi(x)}\mu(dx)$, the invariant probability for $L^{(1)}$.

Our goal is to show that

$$\lambda_1^{(1)}(M\backslash V) \;\coloneqq\; \inf_{F\in\mathbb{L}^2(M\backslash V,\mu^{(1)})\backslash\{0\}} \frac{\mu^{(1)}[F(-L^{(1)})[F]]}{\mu^{(1)}[F^2]} \;\geqslant\; \frac{1}{2} \tag{42}$$

So consider F a bounded and measurable function on M, vanishing on V. By the martingale problems associated to $X^{(1)}$, there exists a $\mathbb{L}^2$ martingale $(M_t)_{t\geqslant 0}$ such that

$$\forall\, t\geqslant 0, \qquad F^2(X^{(1)}(t)) \;=\; F^2(X^{(1)}(0)) + \int_0^t L^{(1)}[F^2](X^{(1)}(s))\,ds + M_t$$

Replace in this relation t by $t\wedge\tau^{(1)}$ and take the expectation to get

$$\mathbb{E}[F^2(X^{(1)}(t\wedge\tau^{(1)}))] \;=\; \mathbb{E}[F^2(X^{(1)}(0))] + \mathbb{E}\left[\int_0^{t\wedge\tau^{(1)}} L^{(1)}[F^2](X^{(1)}(s))\,ds\right]$$

where we use the martingale property $\mathbb{E}(M_t) = \mathbb{E}(M_0) = 0$. Via dominated convergence, we can let t go to infinity to obtain

$$\mathbb{E}[F^2(X^{(1)}(\tau^{(1)}))] \;=\; \mathbb{E}[F^2(X^{(1)}(0))] + \mathbb{E}\left[\int_0^{\tau^{(1)}} L^{(1)}[F^2](X^{(1)}(s))\,ds\right]$$

Note that since $X^{(1)}(\tau^{(1)})\in V$ the l.h.s. vanishes, we deduce

$$\mathbb{E}[F^2(X^{(1)}(0))] \;=\; -\mathbb{E}\left[\int_0^{\tau^{(1)}} L^{(1)}[F^2](X^{(1)}(s))\,ds\right]$$

We have not yet specified the initial distribution of $X^{(1)}(0)$, but take it now to be $\mu^{(1)}$, so the l.h.s. is

$$\mathbb{E}_{\mu^{(1)}}[F^2(X^{(1)}(0))] \;=\; \int \mu^{(1)}(dx)F^2(x) = \mu^{(1)}[F^2]$$

Concerning the r.h.s., recall that the **carré du champs** $\Gamma^{(1)}$ associated to $L^{(1)}$ and defined on any bounded and measurable function G on M by

$$\Gamma^{(1)}[G] \;\coloneqq\; L^{(1)}[G^2] - 2GL^{(1)}[G]$$

is a non-negative function (cf. for instance the book of Bakry, Gentil and Ledoux [3]). It follows that

$$\begin{aligned}
-\mathbb{E}_{\mu^{(1)}}\left[\int_0^{\tau^{(1)}} L^{(1)}[F^2](X^{(1)}(s))\,ds\right] &\leqslant -2\mathbb{E}_{\mu^{(1)}}\left[\int_0^{\tau^{(1)}} F(X^{(1)}(s))L^{(1)}[F](X^{(1)}(s))\,ds\right]\\
&\leqslant 2\mathbb{E}_{\mu^{(1)}}\left[\int_0^{\tau^{(1)}} |F(X^{(1)}(s))L^{(1)}[F](X^{(1)}(s))|\,ds\right]\\
&= \int_0^{+\infty} \mathbb{E}_{\mu^{(1)}}\left[\mathbb{1}_{s\leqslant\tau^{(1)}}|F(X^{(1)}(s))L^{(1)}[F](X^{(1)}(s))|\right]ds
\end{aligned}$$

For any $s\geqslant 0$, taking into account Lemma 21, we have

$$\begin{aligned}
\mathbb{E}_{\mu^{(1)}}\left[\mathbb{1}_{s\leqslant\tau^{(1)}}|F(X^{(1)}(s))L^{(1)}[F](X^{(1)}(s))|\right] &= \mathbb{E}_{\mu^{(1)}}\left[\mathbb{P}_{X^{(1)}(s)}[s\leqslant\tau^{(1)}]|F(X^{(1)}(s))L^{(1)}[F](X^{(1)}(s))|\right]\\
&\leqslant 2\exp(-s)\mathbb{E}_{\mu^{(1)}}\left[|F(X^{(1)}(s))L^{(1)}[F](X^{(1)}(s))|\right]\\
&= 2\exp(-s)\mu^{(1)}[|FL^{(1)}[F]|]
\end{aligned}$$

where we used the invariance of $\mu^{(1)}$ (meaning that for any $s \geqslant 0$, the law of $X^{(1)}(s)$ is equal to $\mu^{(1)}$ when the initial law is $\mu^{(1)}$). We have thus proven that

$$\mu^{(1)}[F^2] \leqslant \int_0^{+\infty} 2\exp(-s)\mu^{(1)}[|FL^{(1)}[F]|]\,ds = 2\mu^{(1)}[|FL^{(1)}[F]|] \leqslant 2\sqrt{\mu^{(1)}[F^2]\mu^{(1)}[(L^{(1)}[F])^2]}$$

i.e.

$$\mu^{(1)}[F^2] \leqslant 4\mu^{(1)}[(L^{(1)}[F])^2]$$

The fact that $L^{(1)}$ is a non-positive self-adjoint operator enables to see that this relation extend to any function in the domain of $L^{(1)}$ with Dirichlet condition on V. It follows that the spectrum of $-L^{(1)}$ with Dirichlet condition on V is above $1/2$, which amounts to (42).

■

As already mentioned, the Steklov Markov kernel $K^{(1)}$ associated to $L^{(1)}$ and V is the same as K. Since in general the generator $L^{(1)}$ cannot be written under the form $P^{(1)} - I$, where $P^{(1)}$ would be a Markov kernel on M, the definitions (27) and (28) must be slightly generalized: denote for any $f \in \mathcal{B}(V)$,

$$\begin{aligned} \forall\, x \in M, \qquad F_f^{(1)}(x) &:= \mathbb{E}_x[f(X^{(1)}(\tau^{(1)}))] \\ \forall\, x \in V, \qquad K^{(1)}[f](x) &:= L^{(1)}[F_f^{(1)}](x) + f(x) \end{aligned} \tag{43}$$

where $\tau^{(1)}$ was defined in Lemma 20. The latter expression for $K^{(1)}$ may appear strange at first view; it is due to the fact that it is a Markov kernel operator. If we rather consider the Steklov generator $S^{(1)} := K^{(1)} - I$, we get the more natural formulation: $S^{(1)}[f] = L^{(1)}[F_f]$, for $f \in \mathcal{B}(V)$, as in the section on finite Markov process. Coming back to our previous convention of Steklov Markov kernels, note that for any $x \in V$, we have

$$L^{(1)}[F_f^{(1)}](x) + f(x) = L[F_f^{(1)}](x) + F_f^{(1)}(x) = \int F_f^{(1)}(y)\, P(x, dy) = P[F_f^{(1)}](x)$$

more in adequacy with (28). Note furthermore that the function $F_f^{(1)}$ defined by (43) is the $L^{(1)}$-harmonic extension of f to M: it satisfies

$$\begin{cases} L^{(1)}[F_f] = 0\,, & \text{on } M\backslash V \\ F_f^{(1)} = f\,, & \text{on } V \end{cases}$$

Since $L^{(1)} = \varphi L$, with φ non-vanishing, the condition $L^{(1)}[F_f] = 0$ is the same as $L[F_f] = 0$. It follows that $F_f^{(1)} = F_f$ and finally $K^{(1)}[f] = K[f]$. By completion, this is true on $\mathbb{L}^2(\nu)$, i.e. $K^{(1)} = K$. The equality $F_f^{(1)} = F_f$ is also obvious from the probabilistic point of view, since $X^{(1)}$ is a time change of X (as seen in the proof of Lemma 20), which itself is the Poissonisation of the Markov chain Z with the same initial condition and associated to P: let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ be independent exponential random variables of parameter 1, X can be constructed from Z via

$$\forall\, t \geqslant 0, \qquad X_t = Z_n, \quad \text{where } n \in \mathbb{Z}_+ \text{ is such that } \textstyle\sum_{p=1}^n \mathcal{E}_p \leqslant t < \sum_{p=1}^{n+1} \mathcal{E}_p$$

The previous considerations are also valid for the operators $K_A^{(1)}$, defined in a similar fashion for $A \in \mathcal{A}(V)$ and we get that $K_A^{(1)} = K_A$. We can now apply Theorem 18 with respect to the generator $L^{(1)}$, which by construction admits a Dirichlet gap on $M\backslash V$. The l.h.s. in the two convergences of Theorem 18 correspond to the generators given by (41) and the r.h.s. are given by (30) and (31), according to the above discussion. These convergences are our final approximation results for the quantities $(\sigma_k)_{k\in\mathbb{N}}$ and $(\sigma_1(A))_{A\in\mathcal{A}}$.

We can now come to the

Proof of Theorem 14

The upper bound is an immediate consequence of the definition of σ_k. Indeed for every $(A_1, ..., A_k) \in \mathcal{A}_k$ it is enough to consider the vector space generated by a family $\{f_{l,n} \in L^2(A_l, \mu) : l \in [\![k]\!]\}$ of test functions such that $\frac{\nu_{A_l}[f_{n,l}(I-K_{A_l})[f_{n,l}]]}{\nu_{A_l}[f_{n,l}^2]}$ tends to $\sigma_1(A_l)$ as $n \to \infty$.

For the lower bound, similarly to (32), define for any $r > 0$,

$$\forall\, k \in \mathbb{N}, \qquad \Lambda_k^{(r)} = \inf_{(A_1,\ldots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \lambda_1^{(r)}(A_l)$$

We have seen in [31], extending the similar result Lee, Oveis Gharan and Trevisan [27] gave in a finite setting, that there exists a universal constant $c > 0$ such that

$$\forall\ r > 0,\ \forall\ k \in \mathbb{N}, \qquad \lambda_k^{(r)} \ \geqslant\ \frac{c}{k^6}\Lambda_k^{(r)} \tag{44}$$

Fix some $k \in \mathbb{N}$. The first convergence of Theorem 18 shows that the l.h.s. converges to σ_k as r goes to $+\infty$. Its uniform convergence leads to

$$\lim_{r\to+\infty} \Lambda_k^{(r)} \ =\ \kappa_k$$

so we can pass to the limit in (44) to obtain the announced inequality.

■

We end this section with Proposition A in the introduction.

Proposition 23 *There is a universal positive constant c' such that*

$$\forall\ k \in \mathbb{N}, \qquad \sigma_{2k} \ \geqslant\ \frac{c'}{\log^2(k+1)}\iota_k$$

Proof

By [31], the proof of Proposition 12 can be extended here. In particular, we have

$$\forall\ k \in \mathbb{N}, \qquad \lambda_{2k}^{(r)} \ \geqslant\ \frac{c}{\log^2(k+1)}\Lambda_k^{(r)}$$

and

$$\forall\ k \in [\![v]\!], \qquad \Lambda_k \ \geqslant\ \frac{1}{8}\iota_k$$

and the statement follows.

■

4 The Riemannian manifold framework

Let (M, g) be a compact Riemannian manifold of dimension n with smooth boundary. We assume that M is connected. Recall the Steklov problem (1) considered in the introduction:

$$\begin{cases} \Delta f = 0\,, & \text{in } M \\ \frac{\partial f}{\partial \nu} = \sigma f\,, & \text{on } \partial M \end{cases}$$

where ν is the unit outward normal to the boundary. Our goal, as in the previous sections, is to relate its eigenvalues $0 = \sigma_1 < \sigma_2 \leqslant \cdots \leqslant \sigma_k \leqslant \cdots \nearrow \infty$ to some isoperimetric constants. We first show that that (1) can be seen as a limit of a family of Laplace eigenvalue problems. This is already known due to the results of Lamberti and Provenzano [25, 32]. They showed that the Steklov eigenvalue problem (1) can be considered as the limit of the family of Neumann eigenvalue problems

$$\begin{cases} \Delta f + \lambda\rho_\epsilon f = 0\,, & \text{in } M \\ \frac{\partial f}{\partial \nu} = 0\,, & \text{on } \partial M \end{cases} \tag{45}$$

for ϵ small enough (one can choose ϵ for example smaller than the focal distance of ∂M). Here $M_\epsilon := \{x \in M :\ d(x, \partial M) < \epsilon\}$, and

$$\rho_\epsilon(x) = \begin{cases} \epsilon\,, & \text{if } x \in M \backslash M_\epsilon \\ \frac{1}{\epsilon}\,, & \text{if } x \in M_\epsilon \end{cases} \tag{46}$$

We denote the eigenvalues of problem (45) by

$$0 = \lambda_1^\epsilon < \lambda_2^\epsilon \leqslant \cdots \leqslant \lambda_k^\epsilon \leqslant \cdots \nearrow \infty$$

Then we have

Theorem 24 *[25, 32] For every $k \in \mathbb{N}$*

$$\lim_{\epsilon\to 0} \lambda_k^\epsilon \ =\ \sigma_k \tag{47}$$

Remark 25 We remark that Lamberti and Provenzano [25, 32] stated the above convergence for bounded domains in R^n with smooth boundary, and the definition of ρ_ϵ on ∂M is slightly different. However, a verbatim proof also results in the convergence (47) on a compact Riemannian manifold (M, g) with smooth boundary, see [32, Chapter 3] for the details of the proof.

□

One can see the similarity of the above theorem with the statement of Proposition 3 and Theorem 18. It would be very interesting to have an alternative approach to prove Theorem 24 and Theorem 28 below by using the results of the previous section. We hope to obtain a unified approach in a future work.

Let $A \subset M$ be a nonempty open domain in M. Let $\partial_e A := \bar{A} \cap \partial M$ and $\partial_i A := \partial A \cap \operatorname{Int} M$ be smooth manifolds of dimension $n-1$ when they are nonempty sets. We consider the mixed Dirichlet–Steklov eigenvalue problem

$$\begin{cases} \Delta f = 0 & in \;\; A \\ \frac{\partial f}{\partial \nu} = \sigma f & on \;\; \partial_e A \\ f = 0 & on \;\; \partial_i A \end{cases} \tag{48}$$

We also need to consider the following mixed Dirichlet–Neumann eigenvalue problem

$$\begin{cases} \Delta f + \lambda \rho_\epsilon f = 0 & in \;\; A \\ \frac{\partial f}{\partial \nu} = 0 & on \;\; \partial_e A \\ f = 0 & on \;\; \partial_i A \end{cases} \tag{49}$$

where ρ_ϵ is defined in (46).

If $\partial_i A = \varnothing$, then $A = \operatorname{Int} M$ and the first eigenvalue is zero. Otherwise the first eigenvalues of the eigenvalue problem (48) and (49) are not zero and we denote their eigenvalues by

$$0 < \sigma_1(A) \leqslant \sigma_2(A) \leqslant \cdots \leqslant \sigma_k(A) \leqslant \cdots \nearrow \infty$$

and

$$0 < \lambda_1^\epsilon(A) \leqslant \lambda_2^\epsilon(A) \leqslant \cdots \leqslant \lambda_k^\epsilon(A) \leqslant \cdots \nearrow \infty$$

respectively. When $\partial_e A = \varnothing$, our convension is that $\sigma_k(A) = \infty$, for every $k \in \mathbb{N}$. Denote by $\mathcal{A}$ the set of nonempty open domains in M such that $\partial_i A$ and $\partial_e A$ are smooth sub-manifolds of dimension $n-1$ when they are nonempty. Let $\mathcal{A}_k$ be the set of k-tuple $(A_1, ..., A_k)$ of mutually disjoint elements of $\mathcal{A}$. We define

$$\Lambda_k^\epsilon \;\; := \inf_{(A_1,\ldots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \lambda_1^\epsilon(A_l) \tag{50}$$

The higher order Cheeger inequality for eigenvalues $\lambda_k^\epsilon(M)$, $k \in \mathbb{N}$ was proved by Miclo in [31]:

Theorem 26 ([31]) *There exists a universal constant $c > 0$ such that for any compact Riemannian manifold M with smooth boundary, the eigenvalues λ_k^ϵ of Neumann eigenvalue problem (45) satisfy*

$$\frac{c}{k^6}\Lambda_k^\epsilon \leqslant \lambda_k^\epsilon \leqslant \Lambda_k^\epsilon \qquad \forall\; k \in \mathbb{N}$$

Remark 27 The above theorem in [31] is originally stated for the Laplace eigenvalue problem with smooth coefficients on closed manifolds. But the argument remains the same when we consider the Neumann eigenvalue problem (45) on a compact manifold with smooth boundary.

□

Similar to Defintion (50), we define

$$\kappa_k \;\; := \inf_{(A_1,\ldots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \sigma_1(A_l)$$

Theorem 28 *There exists a universal constant c_1 such that for any compact Riemannian manifold M with boundary and for any $k \in \mathbb{N}$, the eigenvalues $\sigma_k(M)$ of problem (1) satisfy*

$$\frac{c_1}{k^6}\kappa_k \leqslant \sigma_k \leqslant \kappa_k$$

As a consequence of Theorem 28 we get the higher order Cheeger–Steklov inequalities, see Theorem 29 below. We first define the Cheeger–Steklov constants in this setting similar to those already discussed in the previous sections. For any open subset A of M with piecewise smooth boundary, let $\mu(A)$ denote its Riemannian measure and $\underline{\mu}(\partial A)$ be the induced $(n-1)$-dimensional Riemannian measure of ∂A. We define for every $A \in \mathcal{A}$ the isoperimetric ratios

$$\eta(A) \ := \ \frac{\underline{\mu}(\partial_i A)}{\mu(A)} \qquad\qquad \eta'(A) \ := \ \frac{\underline{\mu}(\partial_i A)}{\underline{\mu}(\partial_e A)}$$

Note that $\eta'(A) = \infty$ if $\bar{A} \cap \partial M = \varnothing$. Let

$$\rho(A) \ := \ \inf_{\substack{B\in\mathcal{A}\\ B\subset A\\ \bar{B}\cap\partial_i A=\varnothing}} \eta(B) \qquad\qquad \rho'(A) \ := \ \inf_{\substack{B'\in\mathcal{A}\\ B'\subset A\\ \bar{B}'\cap\partial_i A=\varnothing}} \eta'(B') \tag{51}$$

For any $k \in \mathbb{N}$ we define the k-th Cheeger–Steklov constant of M by

$$\iota_k \ := \ \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \rho(A_l)\rho'(A_l).$$

The following theorem extends the results of Escobar [31] and Jammes [23].

Theorem 29 *There exists a universal constant c such that for any compact Riemannian manifold M with smooth boundary and for any $k \in \mathbb{N}$, the eigenvalues $\sigma_k(M)$ of problem* (1) *satisfy*

$$\sigma_k \ \geqslant \ \frac{c}{k^6}\iota_k$$

Remark 30

i) One can check that for every $k \in \mathbb{N}$ one has $\iota_k \leqslant \iota_{k+1}$. This is also true in finite and measurable situation.

ii) Note that $\eta'(B)$ is scale invariant. Hence, as mentioned in [23], the power of $\eta(B)$ has to be one so that ι_k has the same scaling as σ_k.

□

Note that for $k = 2$, Theorem 29 gives a version of Jammes' result [23]. The above theorem is the direct sequence of Theorem 28 and Lemma 31 below.

Lemma 31 *Let $\sigma_1(A)$ be the first eigenvalue of the Dirichlet-Steklov eigenvalue problem* (48)*. Then we have*

$$\sigma_1(A) \ \geqslant \ \frac{1}{4}\rho(A)\rho'(A)$$

Proof

Let f be the eigenfunction associated with $\sigma_1(A)$. We repeat the same argument as Jammes' argument in [23] to estimate $\sigma_1(A)$.

$$\sigma_1(A) \ = \ \frac{\int_A |\nabla f|^2\, d\mu \int_A f^2 d\mu}{\int_{\partial_e A} f^2 d\underline{\mu} \int_A f^2 d\mu} \geqslant \frac{\left(\int_A |f\nabla f| d\mu\right)^2}{\int_{\partial_e A} f^2 d\underline{\mu} \int_A f^2 d\mu} \ \geqslant \ \frac{1}{4}\left(\frac{\int_A |\nabla f^2| d\mu}{\int_{\partial_e A} f^2 d\underline{\mu}}\right)\left(\frac{\int_A |\nabla f^2| d\mu}{\int_A f^2 d\mu}\right)$$

where $d\mu$ and $d\underline{\mu}$ are n-dimensional and $(n-1)$-dimensional Riemannian volume elements respectively. Let $h := f^2$ and $H_t := h^{-1}[t,\infty)$. Note that $H_t \in \mathcal{A}$ almost surely in t. Then by the co-area formula we have

$$\left(\frac{\int_A |\nabla h| d\mu}{\int_{\partial_e A} h\, d\underline{\mu}}\right)\left(\frac{\int_A |\nabla h| d\mu}{\int_A h\, d\mu}\right) = \left(\frac{\int_0^\infty \underline{\mu}(\partial_i H_t) dt}{\int_0^\infty \underline{\mu}(\partial_e H_t)\, dt}\right)\left(\frac{\int_0^\infty \underline{\mu}(\partial_i H_t) dt}{\int_0^\infty \mu(H_t)\, dt}\right) \geqslant \rho(A)\rho'(A)$$

which completes the proof.

■

It remains to prove Theorem 28.

Proof of Theorem 28

Recall that by the variational characterisation of Steklov eigenvalues

$$\sigma_k \quad \leqslant \quad \max_{j \in [\![k]\!]} \frac{\mathcal{E}_\Delta(f_j, f_j)}{\int_{\partial M} f_j^2 d\underline{\mu}}$$

where $\{f_j\}$ is a family of test functions in $H^1(M)$ with mutually disjoint supports and $\mathcal{E}_\Delta(f,f) := \int_M |\nabla f|^2 d\mu$ is the Dirichlet form associated to Δ. Hence, the upper bound of σ_k is a direct consequence of the variational characterisation of Steklov eigenvalues.

We now prove the lower bound. We need the following key lemma.

Lemma 32 *The following inequality holds.*

$$\lim_{\epsilon \to 0} \Lambda_k^\epsilon \quad \geqslant \quad \frac{1}{4}\kappa_k$$

Proof

Let $(A_1, \cdots, A_k) \in \mathcal{A}_k$ and $H_0^1(A_j, \partial_i A_j)$ be the closure of $\{f \in C^\infty(A_j) \ : \ f \equiv 0 \text{ on } \partial_i A_j\}$ in $H^1(A_j)$. We can assume $\partial_e A_j \neq \emptyset$. For any ϵ small enough (will be determined below) and every $f \in H_0^1(A_j, \partial_i A_j)$, $j \in [\![k]\!]$ we give an upper bound for the denominator of

$$\frac{\int_{A_j} |\nabla f|^2 d\mu}{\int_{A_j} \rho_\epsilon f^2 \, d\mu} \quad = \quad \frac{\int_{A_j} |\nabla f|^2 d\mu}{\frac{1}{\epsilon}\int_{A_{j,e}^\epsilon} f^2 \, d\mu + \epsilon \int_{A_j \setminus A_{j,e}^\epsilon} f^2 \, d\mu} \tag{52}$$

where $A_{j,e}^\epsilon := \{x \in A_j : d(x, \partial M) < \epsilon\}$. For every $f \in H_0^1(A_j, \partial_i A_j)$ consider $\mathbb{1}_{A_j} f$ as an element of $H^1(M)$. Then

$$\frac{1}{\epsilon}\int_{A_{j,e}^\epsilon} f^2 \, d\mu \quad = \quad \frac{1}{\epsilon}\int_{M_\epsilon} \mathbb{1}_{A_j} f^2 \, d\mu$$

There exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the map

$$E : \partial M \times (0, \epsilon) \ni (x,t) \quad \mapsto \quad \exp_x(-t\nu(x)) \in M_\epsilon$$

is a diffeomorphism. Note that $|\det DE(x,t)| = 1 + O(t)$. Hence, by choosing ϵ_0 even smaller, we can impose that for all $(x,t) \in \partial M \times (0, \epsilon)$

$$\sup_{s \in (0,t)} \frac{|\det DE(x,t)|}{|\det DE(x,s)|} \leqslant 2, \quad \text{which also implies,} \quad |\det DE(x,t)| \leqslant 2$$

Let $F \in H^1(M)$ and by abuse of notation, denote $F \circ E$ by F. For a.e. $(x,t) \in \partial M \times (0,\epsilon)$ we have

$$|F(x,t)| \quad \leqslant \quad |F(x,0)| + \int_0^t \left|\frac{\partial F}{\partial s}(x,s)\right| ds$$

Thus

$$\begin{aligned}
\frac{1}{\epsilon}\int_{M_\epsilon} F^2 \, d\mu \quad &\leqslant \quad \frac{1}{\epsilon}\int_0^\epsilon \int_{\partial M} F^2(x,t) |\det DE(x,t)| d\underline{\mu} dt \\
&\leqslant \quad \frac{1}{\epsilon}\int_0^\epsilon \int_{\partial M} \left(|F(x,0)| + \int_0^t \left|\frac{\partial F}{\partial s}(x,s)\right| ds\right)^2 |\det DE(x,t)| d\underline{\mu} dt \\
&\leqslant \quad \frac{2}{\epsilon}\int_0^\epsilon \int_{\partial M} F(x,0)^2 |\det DE(x,t)| d\underline{\mu} dt \\
&\quad + \frac{2}{\epsilon}\int_0^\epsilon \int_{\partial M} \left(\int_0^t \left|\frac{\partial F}{\partial s}(x,s)\right| ds\right)^2 |\det DE(x,t)| d\underline{\mu} dt \\
&\leqslant \quad 4\int_{\partial M} F(x,0)^2 d\underline{\mu} + \frac{2}{\epsilon}\int_0^\epsilon \int_{\partial M} t \int_0^t \left|\frac{\partial F}{\partial s}(x,s)\right|^2 |\det DE(x,s)| \frac{|\det DE(x,t)|}{|\det DE(x,s)|} ds \, d\underline{\mu} dt \\
&\leqslant \quad 4\int_{\partial M} F^2 d\underline{\mu} + 2\epsilon \int_{M_\epsilon} |\nabla F|^2 d\mu
\end{aligned}$$

Taking $F = \mathbb{1}_{A_j} f$ in the above inequality we get

$$\frac{1}{\epsilon}\int_{A_{j,e}^\epsilon} f^2 \, d\mu \quad \leqslant \quad 4\int_{\partial_e A_j} f^2 d\underline{\mu} + 2\epsilon \int_{A_j} |\nabla f|^2 d\mu \tag{53}$$

We proceed with bounding the second term $\epsilon \int_{A_j \setminus A^\epsilon_{j,e}} f^2\, d\mu$. Let $\xi : M \to \mathbb{R}_+$ be a Lipschitz function such that $|\nabla \xi| \leqslant \frac{1}{\epsilon}$ and

$$\begin{cases} \xi \equiv 1\,, & \text{in } M\setminus M^\epsilon \\ 0 \leqslant \xi \leqslant 1\,, & \text{in } M^\epsilon \\ \xi \equiv 0\,, & \text{on } \partial M \end{cases}$$

We get

$$\begin{aligned}
\epsilon \int_{A_j\setminus A^\epsilon_{j,e}} f^2\, d\mu &\leqslant \epsilon \int_{A_j} \xi f^2\, d\mu = \epsilon \int_M \xi \mathbb{1}_{A_j} f^2\, d\mu \\
&\leqslant \epsilon P_1 \int_M |\nabla(\xi \mathbb{1}_{A_j} f^2)| d\mu = \epsilon P_1 \int_{A_j} |\nabla(\xi f^2)| d\mu \\
&\leqslant \epsilon P_1 \left(\int_{A_j} |\nabla \xi| f^2 d\mu + 2\int_{A_j} \xi f |\nabla f| d\mu \right) \\
&\leqslant \epsilon P_1 \left(\frac{1}{\epsilon} \int_{A^\epsilon_{j,e}} f^2 d\mu + 2 \left(\int_{A_j} (\xi f)^2 d\mu \right)^{\frac12} \left(\int_{A_j} |\nabla f|^2 d\mu \right)^{\frac12} \right) \\
&\overset{(53)}{\leqslant} \epsilon P_1 \left(4 \int_{\partial_e A_j} f^2 d\underline{\mu} + 2\epsilon \int_{A_j} |\nabla f|^2 d\mu \right) \\
&+ 2\epsilon P_1 \bar{\lambda}_1(M)^{-1/2} \left(\int_{A_j} |\nabla(\xi f)|^2 d\mu \right)^{\frac12} \left(\int_{A_j} |\nabla f|^2 d\mu \right)^{\frac12} \\
&\leqslant 4\epsilon P_1 \int_{\partial_e A_j} f^2 d\underline{\mu} + 2\epsilon^2 P_1 \int_{A_j} |\nabla f|^2 d\mu \\
&+ 2P_1 \bar{\lambda}_1(M)^{-1/2} \left(\sqrt{\epsilon} \left(\frac{1}{\epsilon} \int_{A^\epsilon_{j,e}} f^2 d\mu \right)^{\frac12} \left(\int_{A_j} |\nabla f|^2 d\mu \right)^{\frac12} + \epsilon \int_{A_j} |\nabla f|^2 d\mu \right) \\
&\overset{(53)}{\leqslant} 4\epsilon P_1 \int_{\partial_e A_j} f^2 d\underline{\mu} + 2\epsilon P_1 (\epsilon + \bar{\lambda}_1(M)^{-\frac12}) \int_{A_j} |\nabla f|^2 d\mu \\
&+ 2\sqrt{\epsilon} P_1 \lambda_1(M)^{-1/2} \left(4 \int_{\partial_e A_j} f^2 d\underline{\mu} + 2\epsilon \int_{A_j} |\nabla f|^2 d\mu \right)^{\frac12} \left(\int_{A_j} |\nabla f|^2 d\mu \right)^{\frac12} \\
&\leqslant 4\epsilon P_1 \int_{\partial_e A_j} f^2 d\underline{\mu} + 2\epsilon P_1 \left(\epsilon + (1+\sqrt{2}) \bar{\lambda}_1(M)^{-\frac12} \right) \int_{A_j} |\nabla f|^2 d\mu \\
&+ 4\sqrt{\epsilon} P_1 \bar{\lambda}_1(M)^{-1/2} \left(\int_{\partial_e A_j} f^2 d\underline{\mu} \right)^{\frac12} \left(\int_{A_j} |\nabla f|^2 d\mu \right)^{\frac12}
\end{aligned}$$

where P_1 is the L^1-Poincaré constant and $\bar{\lambda}_1(M)$ is the first Dirichlet eigenvalue of M. In the second and fifth inequalities we used the Poincaré inequality on Sobolev spaces $W^{1,1}_0(M)$ and $W^{1,2}_0(M)$ respectively. Hence, for any $\epsilon \in (0, \epsilon_0)$ we get

$$\begin{aligned}
\frac{\int_{A_j} |\nabla f|^2 d\mu}{\int_{A_j} \rho_\epsilon f^2\, d\mu} &\geqslant \frac{\int_{A_j} |\nabla f|^2 d\mu}{4(1+\epsilon P_1) \int_{\partial_e A_j} f^2\, d\underline{\mu} + C_1(\epsilon) \int_{A_j} |\nabla f|^2 d\mu + C_2(\epsilon) \left(\int_{\partial_e A_j} f^2 d\underline{\mu} \right)^{\frac12} \left(\int_{A_j} |\nabla f|^2 d\mu \right)^{\frac12}} \\
&= \psi_\epsilon \left(\frac{\int_{A_j} |\nabla f|^2 d\mu}{\int_{\partial_e A_j} f^2\, d\underline{\mu}} \right)
\end{aligned}$$

where

$$C_1(\epsilon) := 2\epsilon \left(1 + P_1 \left(\epsilon + (1+\sqrt{2}) \bar{\lambda}_1(M)^{-\frac12} \right) \right), \qquad C_2(\epsilon) := 4\sqrt{\epsilon} P_1 \bar{\lambda}_1(M)^{-1/2}$$

and $\psi_\epsilon : (0,\infty) \to (0,\infty)$ defined as

$$\psi_\epsilon(u) \ := \ \frac{u}{4(1+\epsilon P_1) + C_1(\epsilon) u + C_2(\epsilon)\sqrt{u}}$$

is an increasing function. Remark that ϵ_0 is independent of the set A_j and depends only on (M, g). Let f_j be the eigenfunction associated with $\lambda_1^\epsilon(A_j)$.

$$\begin{aligned}\max_{j\in[\![k]\!]}\lambda_1^\epsilon(A_j) &= \max_{j\in[\![k]\!]}\frac{\int_{A_j}|\nabla f_j|^2 d\mu}{\int_{A_j}\rho_\epsilon f_j^2\, d\mu} \geqslant \max_{j\in[\![k]\!]}\psi_\epsilon\left(\frac{\int_{A_j}|\nabla f_j|^2 d\mu}{\int_{\partial_e A_j} f_j^2\, d\underline{\mu}}\right)\\ &\geqslant \max_{j\in[\![k]\!]}\psi_\epsilon(\sigma_1(A_j)) = \psi_\epsilon(\max_{j\in[\![k]\!]}\sigma_1(A_j)) \geqslant \psi_\epsilon(\inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k}\max_{j\in[\![k]\!]}\sigma_1(A_j))\end{aligned}$$

Therefore,

$$\lim_{\epsilon\to 0}\Lambda_k^\epsilon \geqslant \frac{1}{4}\kappa_k$$

which completes the proof.

■

We continue the proof of the theorem. By Theorem 26, we have

$$\lambda_k^\epsilon \geqslant \frac{c}{k^6}\Lambda_k^\epsilon$$

Passing to the limit and applying Lemma 32 and Theorem 24 we conclude:

$$\sigma_k = \lim_{\epsilon\to 0}\lambda_k^\epsilon \geqslant \frac{c}{k^6}\lim_{\epsilon\to 0}\Lambda_k^\epsilon \geqslant \frac{c}{5k^6}\kappa_k$$

■

Similar to Propositions 12 and 23, we have the following improvement on manifolds.

Proposition 33 *There is a universal positive constant c' such that*

$$\forall\ k\in\mathbb{N},\qquad \sigma_{2k} \geqslant \frac{c'}{\log^2(k+1)}\iota_k$$

Proof

Due to [27, 31], there is a universal positive constant c_1 such that

$$\forall\ k\in\mathbb{N},\qquad \lambda_{2k}^\epsilon \geqslant \frac{c_1}{\log^2(k+1)}\Lambda_k^\epsilon$$

Passing to the limit and using Lemmas 31 and 32 we get

$$\forall\ k\in\mathbb{N},\qquad \sigma_{2k} \geqslant \frac{c_1}{4\log^2(k+1)}\kappa_k \geqslant \frac{c_1}{16\log^2(k+1)}\iota_k$$

■

Remark 34 The methods and results above can be adapted to a more general Steklov eigenvalue problem

$$\begin{cases}\operatorname{div}(\phi\nabla f) = 0\,, & \text{in } M\\ \frac{\partial f}{\partial\nu} = \sigma\gamma f\,, & \text{on } \partial M\end{cases}$$

where γ is a continuous positive function on ∂M and ϕ is a smooth positive function on M. But in this paper we stick to the so-called homogenous Steklov problem when $\phi = 1$ and $\gamma = 1$.

□

Remark 35 We now give a more explicit relationship between the higher order Cheeger constants and the higher order Cheeger–Steklov constants. Let

$$\rho_k(M) := \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k}\max_{l\in[\![k]\!]}\rho(A_l)$$

We show that

$$\rho_k(M) = \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k}\max_{l\in[\![k]\!]}\eta(A_l) =: h_k(M) \tag{54}$$

where $h_k(M)$ denotes the k-th Cheeger constant. Indeed, it is easy to check that $\rho(A) \leqslant \eta(A)$ which implies $\rho_k(M) \leqslant h_k(M)$. Thus it is enough to show that for every $\epsilon > 0$, we have $h_k(M) \leqslant \rho_k(M) + \epsilon$. Note that

$$\forall\ B \subset A, \qquad \rho(B) \geqslant \rho(A)$$

Recall the definition of $\rho(A)$ in (51). For every $\epsilon > 0$, there exists $B \in \mathcal{A}$ subset of A such that $\bar{B} \cap \partial_i A = \varnothing$ and

$$0 \leqslant \eta(B) - \rho(B) \leqslant \eta(B) - \rho(A) < \epsilon \tag{55}$$

Let $\mathcal{A}_k^\epsilon$ be a subset of $\mathcal{A}_k$ such that

$$\forall\ (A_1, \cdots, A_k) \in \mathcal{A}_k^\epsilon, \quad 0 \leqslant \eta(A_l) - \rho(A_l) < \epsilon, \quad \forall\ l \in [\![k]\!]$$

We claim

$$\inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \rho(A_l) = \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k^\epsilon} \max_{l\in[\![k]\!]} \rho(A_l)$$

Indeed, let

$$[(A_1, \cdots, A_k)] \quad := \quad \left\{(\tilde{A}_1, \cdots, \tilde{A}_k) \in \mathcal{A}_k \ : \ \max_{l\in[\![k]\!]} \rho(A_l) = \max_{l\in[\![k]\!]} \rho(\tilde{A}_l)\right\}$$

The definition of $\rho_k(M)$ does not change if we choose a representation in each class $[(A_1, \cdots, A_k)]$ and take infimum only over the family of representations. By (55), it is clear that each class has a representation in $\mathcal{A}_k^\epsilon$. This proves the claim. Therefore

$$\rho_k(M) = \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k^\epsilon} \max_{l\in[\![k]\!]} \rho(A_l) > \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k^\epsilon} \max_{l\in[\![k]\!]} \eta(A_l) - \epsilon \geqslant h_k(M) - \epsilon$$

This proves identity (54). Now for a given $(A_1, \cdots, A_k) \in \mathcal{A}_k$, let $l_{\max} \in [\![k]\!]$ be such that

$$\eta(A_{l_{\max}}) \quad = \quad \max_{l\in[\![k]\!]} \eta(A_l)$$

Then we define

$$\bar{h}'_k(M) \quad := \quad \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \rho'(A_{l_{\max}})$$

It is easy to check that we have the following lower bound for $\iota_k(M)$

$$\iota_k(M) \quad \geqslant \quad h_k(M)\bar{h}'_k(M) \tag{56}$$

Similarly we can define

$$\rho'_k(M) := \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \rho'(A_l)$$

With the same argument as above, the following equality holds.

$$\rho'_k(M) = \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \eta'(A_l) =: h'_k(M)$$

For a given $(A_1, \cdots, A_k) \in \mathcal{A}_k$, let $l'_{\max} \in [\![k]\!]$ be such that

$$\eta'(A_{l'_{\max}}) = \max_{l\in[\![k]\!]} \eta'(A_l)$$

Then define

$$\bar{h}_k(M) := \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \rho(A_{l'_{\max}})$$

and we get

$$\iota_k(M) \geqslant \bar{h}_k(M)h'_k(M)$$

□

Jammes in [23] considered several examples to show that for $k = 2$ the geometric quantities $\eta(B)$ and $\eta'(B)$ appearing in the definition of $\iota_k(M)$ are both necessary in the lower bound of $\sigma_2(M)$. Inspired by his examples, we give examples which show the necessity of quantities such as $\eta(B)$ and $\eta'(B)$ in the lower bound for all $k \in \mathbb{N}$.

Example 1 *Exemple 4 of [23] can be used to show the necessity of quantities such as $\eta(B)$ and $\eta'(B)$ in the definition of ι_k for all $k \geqslant 2$: Consider $M_m = N \times (-L_m, L_m)$, where N is a closed manifold and $L_m = \frac{1}{m}$. The Steklov spectrum of M_m can be calculated explicitly, see [10, Lemma 6.1]. They are*

$$\left\{ 0\,, {L_m}^{-1}, \sqrt{\lambda_k(N)}\tanh(\sqrt{\lambda_k(N)}L_m), \sqrt{\lambda_k(N)}\coth(\sqrt{\lambda_k(N)}L_m) \,:\, k \in \mathbb{N} \right\} \tag{57}$$

where $\lambda_k(N)$ are the Laplace eigenvalues of N. It is clear that for every $k \in \mathbb{N}$, $\sigma_k = O(\frac{1}{m})$ as $m \to \infty$, while $h_2(M_m) \geqslant c$ for some positive constant c independent of m as shown in [23, Exemple 4]. Note that $h_k(M_m)$ is a non-decreasing sequence in k. Hence we have $h_k(M) \geqslant h_2(M_m) \geqslant c$, for every $k \geqslant 2$. This together with (56) *and Theorem 29 show the necessity of a quantity such as $\eta'(B)$ in the definition of $\iota_k(M_m)$ for all $k \in \mathbb{N}$.*

Example 2 *Let $\mathbb{S}^1$ be the unit circle and $\mathbb{S}^1_m$ denote a circle of radius m with their standard metric. Consider the sequence $(M_m := \mathbb{S}^1_m \times (0, m^{3/2}))_{m\in\mathbb{N}}$ with product metric. The set of Steklov eigenvalues $\sigma_k(M_m)$ is given by* (57) *with $L_m := m^{3/2}$. Note that $\lambda_k(\mathbb{S}^1_m) = \frac{1}{m^2}\lambda_k(\mathbb{S}^1)$. Hence, for any fixed $k \in \mathbb{N}$ we have*

$$\sigma_k(M_m) \sim m^{3/2}\lambda_k(\mathbb{S}^1_m) = \frac{1}{\sqrt{m}}\lambda_k(\mathbb{S}^1) \quad \text{as } m \to \infty$$

Therefore

$$\forall\ k \in \mathbb{N}, \qquad \lim_{m\to\infty} \sigma_k(M_m) = 0$$

It is easy to check that for every $k \in \mathbb{N}$, $\lim_{m\to\infty} h_k(M_m) = 0$. Indeed, if we choose $A_l = S^1_m \times (\frac{(l-1)m^{3/2}}{k}, \frac{lm^{3/2}}{k})$, $l \in [\![k]\!]$ then

$$h_k(M_m) \quad \leqslant \quad \max_{l\in[\![k]\!]} \frac{\underline{\mu}(\partial_i A_l)}{\mu(A_l)} = \frac{4\pi m}{2\pi m^{5/2}/k} = \frac{2k}{m^{3/2}} \to 0, \quad m \nearrow \infty$$

We now show that there exists a positive constant C independent of m such that $h'_k(M_m) \geqslant C$. Note that $h'_k(M_m)$ is a non-decreasing sequence in k. Thus, it is enough to show that $h'_2(M_m) \geqslant C$ for some constant $C > 0$ independent of m. Let (A_1, A_2) be a partition of M_m (w.l.o.g. we can assume A_1 is connected). Let assume $\partial_i A_1$ only intersects one of the boundary components of M_m. Fixing the area of A_1, $\max\left\{\frac{\underline{\mu}_m(\partial_i A_1)}{\underline{\mu}_m(\partial_e A_1)}, \frac{\underline{\mu}_m(\partial_i A_2)}{\underline{\mu}_m(\partial_e A_2)}\right\}$ is minimized when $\partial_i A_1 = \mathbb{S}^1_m \times \{x\}$ for some $x \in (0, m)$ (where $\underline{\mu}_m$ is the one-dimensional Riemannian measure of a set in M_m). Thus,

$$1 \leqslant \max\left\{\frac{\underline{\mu}_m(\partial_i A_1)}{\underline{\mu}_m(\partial_e A_1)}, \frac{\underline{\mu}_m(\partial_i A_2)}{\underline{\mu}_m(\partial_e A_2)}\right\}$$

We now assume otherwise, i.e. $\partial_i A_1$ intersects both boundary components of M_m. We have

$$\max\left\{\frac{\underline{\mu}_m(\partial_i A_1)}{\underline{\mu}_m(\partial_e A_1)}, \frac{\underline{\mu}_m(\partial_i A_2)}{\underline{\mu}_m(\partial_e A_2)}\right\} \quad \geqslant \quad \frac{2m^{\frac{3}{2}}}{2\pi m} = \frac{\sqrt{m}}{\pi}$$

We conclude that for $m > \pi^2$,

$$h'_k(M_m) \geqslant h'_2(M_m) \geqslant 1$$

This example shows the necessity of a quantity such as $\eta(B)$ in the definition of $\iota_k(M_m)$ for all $k \in \mathbb{N}$. For $k = 2$, a similar example has been studied in [23].

Example 3 (Cheeger dumbbell) *Girouard and Polterovich in [20] studied a family of Cheeger dumbbells M_ϵ and showed that $\lim_{\epsilon\to 0} \sigma_k(M_\epsilon) = 0$ for every $k \in \mathbb{N}$. In their example, M_ϵ is a domain in $\mathbb{R}^2$ consisting of the union of two Euclidean unit disks $\mathbb{D}_1 \cup \mathbb{D}_2$ connected with a thin rectangular neck L_ϵ of length ϵ and width ϵ^3. It is easy to check that $h_2(M_\epsilon) \to 0$ as $\epsilon \to 0$. We show that for $k \geqslant 3$, $h_k(M_\epsilon) \geqslant c > 0$, where c is a constant independent of ϵ. Since $h_k(M_\epsilon) \geqslant h_3(M_\epsilon)$, it is enough to show that $h_3(M_\epsilon) > c$. By contrary, we assume that $h_3(M_\epsilon) \to 0$ as $\epsilon \to 0$. Hence there is a family of $(A^\epsilon_1, A^\epsilon_2, A^\epsilon_3)$ such that*

$$\max\left\{\frac{\underline{\mu}(\partial_i A^\epsilon_1)}{\mu(A^\epsilon_1)}, \frac{\underline{\mu}(\partial_i A^\epsilon_2)}{\mu(A^\epsilon_2)}, \frac{\underline{\mu}(\partial_i A^\epsilon_3)}{\mu(A^\epsilon_3)}\right\} \to 0, \quad \epsilon \to 0$$

Hence we have $\partial_i A^\epsilon_l \subset L_\epsilon$, for all $l \in [\![3]\!]$. Therefore, there exists $l \in [\![3]\!]$ such that $A^\epsilon_l \subset L_\epsilon$. (Notice that the argument uses the fact that M_ϵ is a subset of $\mathbb{R}^2$.) Taking $\epsilon = \frac{1}{m}$, $m \in \mathbb{N}$, and then using the similar argument as in [23, Exemple 4], we conclude that for any ϵ small enough

$$\frac{\underline{\mu}(\partial_i A^\epsilon_l)}{\mu(A^\epsilon_l)} \geqslant c > 0$$

where c is independent of ϵ. It is a contradiction.
This example as in Example 1 shows the necessity of $\eta'(B)$ in $\iota_k(M_\epsilon)$. However, in Example 1 the volume of the family of manifolds tends to zero, while in this example the area and the boundary length of M_ϵ are uniformly controlled.

References

[1] Noga Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. Theory of computing (Singer Island, Fla., 1984).

[2] Noga Alon and Vitali D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B*, 38(1):73–88, 1985.

[3] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.

[4] Michel Benaïm and Nicole El Karoui. *Promenade aléatoire. Chaînes de Markov et simulations, martingales et stratégies.* Palaiseau: Les Éditions de l'École Polytechnique, 2004.

[5] Liliana Borcea. Electrical impedance tomography. *Inverse Problems*, 18(6):R99–R136, 2002.

[6] Robert Brooks. The bottom of the spectrum of a Riemannian covering. *J. Reine Angew. Math.*, 357:101–114, 1985.

[7] Peter Buser. On Cheeger's inequality $\lambda_1 \geqslant h^2/4$. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 29–77. Amer. Math. Soc., Providence, R.I., 1980.

[8] Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199, 1970.

[9] Shiu-Yuen Cheng and Kevin Oden. Isoperimetric inequalities and the gap between the first and second eigenvalues of an Euclidean domain. *J. Geom. Anal.*, 7(2):217–239, 1997.

[10] Bruno Colbois, Ahmad El Soufi, and Alexandre Girouard. Isoperimetric control of the Steklov spectrum. *J. Funct. Anal.*, 261(5):1384–1399, 2011.

[11] Yves Colin de Verdière. Construction de laplaciens dont une partie finie du spectre est donnée. *Ann. Sci. École Norm. Sup. (4)*, 20(4):599–615, 1987.

[12] Amir Daneshgar, Ramin Javadi, and Laurent Miclo. On nodal domains and higher-order Cheeger inequalities of finite reversible Markov processes. *Stochastic Process. Appl.*, 122(4):1748–1776, 2012.

[13] Jozef Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[14] José F. Escobar. The geometry of the first non-zero Stekloff eigenvalue. *J. Funct. Anal.*, 150(2):544–556, 1997.

[15] José F. Escobar. An isoperimetric inequality and the first Steklov eigenvalue. *J. Funct. Anal.*, 165(1):101–116, 1999.

[16] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.

[17] Ailana Fraser and Richard Schoen. The first Steklov eigenvalue, conformal geometry, and minimal surfaces. *Adv. Math.*, 226(5):4011–4030, 2011.

[18] Ailana Fraser and Richard Schoen. Sharp eigenvalue bounds and minimal surfaces in the ball. *Invent. Math.*, 203(3):823–890, 2016.

[19] Kei Funano. Eigenvalues of laplacian and multi-way isoperimetric constants on weighted riemannian manifolds, 2013, arXiv:1307.3919.

[20] Alexander Girouard and Iosif Polterovich. On the Hersch-Payne-Schiffer estimates for the eigenvalues of the Steklov problem. *Funktsional. Anal. i Prilozhen.*, 44(2):33–47, 2010.

[21] Alexandre Girouard and Iosif Polterovich. Spectral geometry of the Steklov problem (survey article). *J. Spectr. Theory*, 7(2):321–359, 2017.

[22] Asma Hassannezhad. Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem. *J. Funct. Anal.*, 261(12):3419–3436, 2011.

[23] Pierre Jammes. Une inégalité de Cheeger pour le spectre de Steklov. *Ann. Inst. Fourier (Grenoble)*, 65(3):1381–1385, 2015.

[24] Nikolay Kuznetsov, Tadeusz Kulczycki, Mateusz Kwaśnicki, Alexander Nazarov, Sergey Poborchi, Iosif Polterovich, and Bartłomiej Siudeja. The legacy of Vladimir Andreevich Steklov. *Notices Amer. Math. Soc.*, 61(1):9–22, 2014.

[25] Pier Domenico Lamberti and Luigi Provenzano. Viewing the Steklov eigenvalues of the Laplace operator as critical Neumann eigenvalues. In *Current trends in analysis and its applications*, Trends Math., pages 171–178. Birkhäuser/Springer, Cham, 2015.

[26] Gregory F. Lawler and Alan D. Sokal. Bounds on the L^2 spectrum for Markov chains and Markov processes: a generalization of Cheeger's inequality. *Trans. Amer. Math. Soc.*, 309(2):557–580, 1988.

[27] James R. Lee, Shayan Oveis Gharan, and Luca Trevisan. Multi-way spectral partitioning and higher-order Cheeger inequalities. In *STOC'12—Proceedings of the 2012 ACM Symposium on Theory of Computing*, pages 1117–1130. ACM, New York, 2012.

[28] Joachim Lohkamp. Discontinuity of geometric expansions. *Comment. Math. Helv.*, 71(2):213–228, 1996.

[29] Laurent Miclo. An example of application of discrete Hardy's inequalities. *Markov Process. Related Fields*, 5(3):319–330, 1999.

[30] Laurent Miclo. On eigenfunctions of Markov processes on trees. *Probab. Theory Related Fields*, 142(3-4):561–594, 2008.

[31] Laurent Miclo. On hyperboundedness and spectrum of Markov operators. *Invent. Math.*, 200(1):311–343, 2015.

[32] Luigi Provenzano. On mass distribution and concentration phenomena for linear elliptic partial differential operators. PhD Thesis, 2016.

[33] Laurent Saloff-Coste. Lectures on finite Markov chains. In *Lectures on probability theory and statistics (Saint-Flour, 1996)*, volume 1665 of *Lecture Notes in Math.*, pages 301–413. Springer, Berlin, 1997.

[34] Vladimir Stekloff. Sur les problèmes fondamentaux de la physique mathématique (suite et fin). *Ann. Sci. École Norm. Sup. (3)*, 19:455–490, 1902.

[35] Gunther Uhlmann. Electrical impedance tomography and Calderón's problem. *Inverse Problems*, 25(12):123011, 39, 2009.

[36] Shing Tung Yau. Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold. *Ann. Sci. École Norm. Sup. (4)*, 8(4):487–507, 1975.

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