The algebra of Boolean matrices, correspondence functors, and simplicity

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THE ALGEBRA OF BOOLEAN MATRICES, CORRESPONDENCE FUNCTORS, AND SIMPLICITY

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ABSTRACT. We determine the dimension of every simple module for the algebra of the monoid of all relations on a finite set (i.e. Boolean matrices). This is in fact the same question as the determination of the dimension of every evaluation of a simple correspondence functor. The method uses the theory of such functors developed in [BT2, BT3], as well as some new ingredients in the theory of finite lattices.

1. Introduction

Let k be a field, let $\mathcal{R}_X$ be the monoid of all relations on a finite set X (also known as Boolean matrices), and let $k\mathcal{R}_X$ be the k-algebra of the monoid. This is an algebra of dimension 2^{n^2}, where $n = |X|$, hence growing very fast in terms of n. It was considered many years ago in [CP, Ki, KR, PW, Sc1, Sc2] and more recently in [BE, Br, Di], but the dimensions of the irreducible representations of $k\mathcal{R}_X$ remained unknown in general.

We solve here the open problem of describing all simple $k\mathcal{R}_X$-modules and finding their dimension. This requires to embed the category of $k\mathcal{R}_X$-modules into the larger category of correspondence functors, namely functors from the category of finite sets and correspondences to the category k-Mod. We use methods of the representation theory of categories, as well as some new ingredients in the theory of finite lattices. The proof is based on very delicate arguments about a system of linear equations which was introduced in [BT3]. We also deduce a formula for the dimension of the Jacobson radical of $k\mathcal{R}_X$ (in characteristic zero). The formulas behave exponentially with respect to n.

It is not too hard to show (and known to specialists) that the simple modules for $k\mathcal{R}_X$ are classified by isomorphism classes of triples (E, R, V), where E is finite set with $|E| \leq |X|$, R is a partial order relation on E, and V is a simple module for the group algebra $k\,\mathrm{Aut}(E, R)$. When $E = X$, we get precisely the simple modules for an algebra $k\mathcal{P}_X$ which is a quotient of $k\mathcal{R}_X$ and which we call the algebra of permuted orders on X, because it has a k-basis $\mathcal{P}_X$ consisting of all $R\Delta_\sigma$ where R is a partial order on X and Δ_σ is the graph of a permutation σ of X. The simple $k\mathcal{P}_X$-modules and their dimension are known and can be described explicitly. The more difficult cases occur when $|E| < |X|$. All cases are uniformly treated in the present paper.

The largest part of our work is concerned with correspondence functors. It is only in Section 8 that we actually deal with the simple modules for the algebra $k\mathcal{R}_X$

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and prove the main results mentioned above, which follow easily from the previous sections of a functorial nature. The connection between functors and simple $k\mathcal{R}_X$-modules is provided by the known fact that the evaluation of a simple correspondence functor at a finite set X is either zero or a simple module for the algebra $k\mathcal{R}_X$. Conversely every simple $k\mathcal{R}_X$-module occurs as the evaluation at X of a simple functor. This provides a way to handle simple modules for the algebra $k\mathcal{R}_X$ by studying simple correspondence functors. It is this embedding in the larger category of correspondence functors which allows us to prove our results. A first step, which is not very hard and explained in [BT2], is the description of the parametrization of simple correspondence functors $S_{E,R,V}$ by isomorphism classes of triples (E,R,V), where (E,R) is a finite poset (i.e. R is a partial order relation on a finite set E) and V is a simple module for the group algebra $k\,\mathrm{Aut}(E,R)$.

The main ingredients for this work are our papers [BT2, BT3] about correspondence functors. Some fundamental modules and functors play a crucial role in our approach. Here k is allowed to be an arbitrary commutative ring. For any finite poset (E,R), we described in [BT1] a fundamental module $M_{E,R} := k\mathcal{P}_E f_R$ for $k\mathcal{P}_E$, where $k\mathcal{P}_E$ is the algebra of permuted orders mentioned above and f_R is a suitable idempotent of $k\mathcal{P}_E$ depending on the order relation R (see Section 3). From this, we constructed and studied in [BT2, BT3] a fundamental functor $\mathbb{S}_{E,R}$, which is the key for understanding simple correspondence functors because the simple functor $S_{E,R,V}$ appears as a suitable quotient of the fundamental functor $\mathbb{S}_{E,R}$ (see Section 4).

Another main ingredient is the link between correspondence functors and the theory of finite lattices, see [BT3]. Associated to any finite lattice T, there is a correspondence functor F_T and a surjective morphism

$$\Theta : F_T \longrightarrow \mathbb{S}_{E,R^{op}}$$

where (E,R) denotes the full subposet of join-irreducible elements of T and R^{op} denotes the opposite relation. The main problem is to describe the kernel of Θ and this gives rise to a complicated system of linear equations which was introduced in [BT3]. One of the main contributions of the present paper is to solve this system. From this solution, a k-basis can be found for each evaluation $\mathbb{S}_{E,R^{op}}(X)$ of a fundamental functor. Generators are found in Section 5 and they are proved to be k-linearly independent in Section 6. The cardinality of this basis is given by a well-known combinatorial formula, behaving exponentially as a function of X.

Turning to simple functors (assuming again that k is a field), we need to pass to a quotient of $\mathbb{S}_{E,R}$ in order to obtain the simple functor $S_{E,R,V}$. This requires to show that each evaluation $\mathbb{S}_{E,R}(X)$ has a free right $k\,\mathrm{Aut}(E,R)$-module structure and that the simple functor $S_{E,R,V}$ is isomorphic to a tensor product $\mathbb{S}_{E,R}\otimes_{k\,\mathrm{Aut}(E,R)}V$. This nontrivial part of the argument requires the whole of Section 7 and culminates with an explicit formula for the dimension of each evaluation $S_{E,R,V}(X)$ of a simple correspondence functor. The last step is to go back to the algebra $k\mathcal{R}_X$ and deduce the dimension of every simple $k\mathcal{R}_X$-module as well as a description of the action of relations on it. As mentioned above, this last step is explained in Section 8. Finally, a few examples are described in Section 9.

There is a classical approach to the classification of simple modules for the algebra of a finite semigroup, going back to the work of Munn and Ponizovsky, using Green's theory of $\mathcal{J}$-classes (see the textbook [CP], the more recent article [GMS] for a modern point of view, or the textbook [St2] for a very recent account). For the algebra $k\mathcal{R}_X$ we are interested in, we do not use this point of view for several reasons. First, our approach to the parametrization of simple modules for $k\mathcal{R}_X$ is not classical and does not use at all $\mathcal{J}$-classes in the monoid $\mathcal{R}_X$. It is based instead on the classification of simple correspondence functors, which in turn depends only, for each E, on the quotient algebra $k\mathcal{P}_E$ and its simple modules. These are easy to describe explicitly and depend in an important way on the fundamental module $M_{E,R}$ associated to a poset (E,R). Secondly, taking advantage of the link with the theory of correspondence functors and using the functor F_T associated to a finite lattice T, our main task is the study of the above morphism Θ (which is itself also based on the fundamental module $M_{E,R}$). Finally, we do not only consider relations, namely subsets of $X \times X$, but also correspondences, namely subsets of $Y \times X$, for $Y \neq X$, to the extent that some of them play a crucial role in the large system of linear equations which is finally solved.

Although we do not use the classical way of handling simple modules for the algebra of a finite monoid, it is not surprising that our functorial approach has connections with the classical one. Whenever such connections can be made clear, we mention them, in a series of remarks. This will help the interested reader to establish the link between some of the functorial concepts we are using and the monoid-theoretic classical approach. However, we emphasize that the question of translating all the functorial proofs of our results in monoid-theoretic terms remains wide open. If such a translation is possible, it will probably require much more work.

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2. Preliminaries on lattices

In this section, we define, in any finite lattice, two operations r^∞ and σ^∞, as well as a subset $\widehat{G}$ of special elements, each lying at the bottom of a totally ordered subset with strong properties. We then prove some results which will play a crucial role in the description of the evaluation of fundamental functors and simple functors.

Let us first fix some notation. By an *order* R on a finite set E, we mean a partial order relation on E. In other words, (E,R) is a finite poset. We write $\leq_R$ for the order relation, so that $(a,b) \in R$ if and only if $a \leq_R b$. Moreover $a <_R b$ means that $a \leq_R b$ and $a \neq b$. The opposite relation R^{op} of R is defined by the property that $(a,b) \in R^{op}$ if and only if $(b,a) \in R$.

If T is a finite lattice, we write $\leq_T$, or sometimes simply $\leq$, for the order relation, $\vee$ for the join (least upper bound), $\wedge$ for the meet (greatest lower bound), $\hat{0}$ for the least element and $\hat{1}$ for the greatest element.

2.1. Notation and definitions.

(a) If (E,R) is a finite poset and $a,b \in E$ with $a \leq_R b$, we define intervals

$$
\begin{aligned}
&[a,b]_E := \{x \in E \mid a \leq_R x \leq_R b\}\,, &&]a,b[_E := \{x \in E \mid a <_R x <_R b\}\,,\\
&[a,b[_E := \{x \in E \mid a \leq_R x <_R b\}\,, &&]a,b]_E := \{x \in E \mid a <_R x \leq_R b\}\,,\\
&[a,\cdot[_E := \{x \in E \mid a \leq_R x\}\,, &&]\cdot,b]_E := \{x \in E \mid x \leq_R b\}\,.
\end{aligned}
$$

When the context is clear, we write $[a,b]$ instead of $[a,b]_E$.

(b) If T is a finite lattice, an element $e \in T$ is called join-irreducible*, or simply* irreducible*, if, whenever $e = \bigvee_{a \in A} a$ for some subset A of T, then $e \in A$. In case $A = \emptyset$, the join is $\hat{0}$ and it follows that $\hat{0}$ is not irreducible. If $e \neq \hat{0}$ is irreducible and $e = s \vee t$ with $s,t \in T$, then either $e = s$ or $e = t$. In other words, if $e \neq \hat{0}$, then e is irreducible if and only if $[\hat{0},e[$ has a unique maximal element.*

(c) *If (E,R) is a subposet of a finite lattice T, we say that (E,R) is a* full subposet *of T if for all $e,f \in E$ we have :*

$$e \leq_R f \Longleftrightarrow e \leq_T f \ .$$

In particular $\mathrm{Irr}(T)$ *denotes the full subposet of irreducible elements of T.*

(d) *If (E,R) is a finite poset, $I_\downarrow(E,R)$ denotes the set of* lower R-ideals *of E, that is, the subsets A of E such that, whenever $a \in A$ and $x \leq a$, then $x \in A$. Clearly $I_\downarrow(E,R)$, ordered by inclusion of subsets, is a lattice, the join operation being union of subsets, and the meet operation being intersection. Similarly, $I^\uparrow(E,R)$ denotes the set of upper R-ideals of E, which is also a lattice. Obviously $I^\uparrow(E,R) = I_\downarrow(E,R^{op})$.*

Note that if (E,R) is the poset of irreducible elements in a finite lattice T, then T is *generated* by E in the sense that any element $x \in T$ is a join of elements of E. To see this, define the height of $t \in T$ to be the maximal length of a chain in $[\hat{0},t]_T$. If x is not irreducible and $x \neq \hat{0}$, then $x = t_1 \vee t_2$ with t_1 and t_2 of smaller height than x. By induction on the height, both t_1 and t_2 are joins of elements of E. Therefore $x = t_1 \vee t_2$ is also a join of elements of E.

Recall that a lattice T is *distributive* if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a,b,c \in T$ (or equivalently $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a,b,c \in T$).

2.2. Lemma. *Let (E,R) be a finite poset.*

(a) *The irreducible elements in the lattice $I_\downarrow(E,R)$ are the lower ideals $]\cdot,e]_E$, where $e \in E$. Thus the poset (E,R) is isomorphic to the poset of all irreducible elements in $I_\downarrow(E,R)$ by mapping $e \in E$ to the ideal $]\cdot,e]_E$.*

(b) *$I_\downarrow(E,R)$ is a distributive lattice.*

(c) *For any finite lattice T with $\mathrm{Irr}(T) = (E,R)$, there is a join-preserving surjective map $f : I_\downarrow(E,R) \longrightarrow T$ which sends any lower ideal $A \in I_\downarrow(E,R)$ to the join $\bigvee_{e\in A} e$ in T.*

(d) *The map $f : I_\downarrow(E,R) \longrightarrow T$ above is bijective if and only if T is a distributive lattice. In that case, f is an isomorphism of lattices.*

Proof : This is not difficult and well-known. For details, see Theorem 3.4.1 and Proposition 3.4.2 in [Sta], and also Theorem 6.2 in [Ro]. □

Whenever we use the lattice $I_\downarrow(E,R)$, we shall (abusively) identify E with its image via the map

$$E \longrightarrow I_\downarrow(E,R)\,, \qquad e \mapsto\,]\cdot,e]_E \ .$$

Thus we view (E,R) as a full subposet of $I_\downarrow(E,R)$.

2.3. Notation. *Let T be a finite lattice and $(E,R) = \mathrm{Irr}(T)$. If $t \in T$, then $r(t)$ denotes the join of all elements strictly smaller than t :*

$$r(t) = \bigvee_{a\in[\hat{0},t[} a \ .$$

It follows that $r(t) = t$ if and only if $t \notin E$. More precisely, if $t \notin E$ and $t \neq \hat{0}$, then t can be written as the join of two smaller elements, so $r(t) = t$, while if $e \in E$, then $r(e)$ is the unique maximal element of $[\hat{0},e[$. We put $r^k(t) = r(r^{k-1}(t))$ and $r^\infty(t) = r^n(t)$ if n is such that $r^n(t) = r^{n+1}(t)$.

2.4. Lemma. *Let T be a finite lattice, let $(E,R)=\mathrm{Irr}(T)$, and let $t\in T$.*

(a) *The map $r:T\to T$ is order-preserving.*
(b) *$r^\infty(t)\notin E$.*
(c) *$r^\infty(t)=t$ if and only if $t\in T-E$.*
(d) *If $e\in E$, $r^\infty(e)$ is the unique greatest element of $T-E$ smaller than e.*

Proof : The proof is a straightforward consequence of the definitions. □

2.5. Lemma. *Let T be a finite lattice, let $(E,R)=\mathrm{Irr}(T)$, and let $e\in E$. Let n be the smallest integer such that $r^n(e)=r^\infty(e)$.*

(a) *$[r^\infty(e),e]$ is totally ordered and $[r^\infty(e),e]=\{r^n(e),\ldots,r^1(e),e\}$.*
(b) *$]r^\infty(e),e]$ is contained in E.*
(c) *$r^\infty(r^i(e))=r^\infty(e)$ for all $0\le i\le n-1$.*
(d) *$[\hat{0},e]=\ [\hat{0},r^\infty(e)]\sqcup\,]r^\infty(e),e]$.*

Proof : Since $e\in E$, $r(e)$ is the unique maximal element of $[\hat{0},e[$. Inductively, $r^i(e)\in E$ for each $0\le i\le n-1$ and $r^{i+1}(e)$ is the unique maximal element of $[\hat{0},r^i(e)[$. It follows that $[r^\infty(e),e]$ is totally ordered and consists of the elements $r^n(e),\ldots,r^1(e),e$. This proves (a), (b) and (c).

Now let $f\in[\hat{0},e]$. Then $f\vee r^\infty(e)\in[r^\infty(e),e]$. If $f\vee r^\infty(e)=r^\infty(e)$, then $f\in\ [\hat{0},r^\infty(e)]$. Otherwise, $f\vee r^\infty(e)\in\,]r^\infty(e),e]$, hence $f\vee r^\infty(e)\in E$ by (b), that is, $f\vee r^\infty(e)$ is irreducible. It follows that $f\vee r^\infty(e)=f$ or $f\vee r^\infty(e)=r^\infty(e)$. But the second case is impossible because $f\vee r^\infty(e)>r^\infty(e)$. Therefore $f\vee r^\infty(e)=f$, that is, $f\in\,]r^\infty(e),e]$. □

2.6. Notation and definitions. *Let T be a finite lattice and $(E,R)=\mathrm{Irr}(T)$.*

(a) *Define ΛE to be the subset of T consisting of all meets of elements of E, that is, elements of the form $\bigwedge\limits_{i\in I} e_i$ where I is a finite set of indices and $e_i\in E$ for every $i\in I$. Note that we include the possibility that I be the empty set, in which case one gets the unique greatest element $\hat{1}$.*
(b) *If $t\in T$, define $\sigma(t)$ to be the meet of all the irreducible elements of T which are strictly larger than t. Inductively, $\sigma^k(t)=\sigma(\sigma^{k-1}(t))$ and $\sigma^\infty(t)=\sigma^n(t)$ where n is such that $\sigma^n(t)=\sigma^{n+1}(t)$.*

Notice that the definition of σ is in some sense 'dual' to the definition of r, because $r(t)$ is the join of all the irreducible elements which are strictly smaller than t. (However, the 'true' dual of r is different : it is the operation r in the opposite lattice T^{op}, thus involving meet-irreducible elements.) It is clear that the map $\sigma:T\to T$ is order-preserving.

2.7. Lemma. *Let $(E,R)=\mathrm{Irr}(T)$ and let $t\in T$.*

(a) *If $t\in\Lambda E-E$, then $\sigma(t)=t$.*
(b) *If $t\notin\Lambda E$, then $\sigma(t)>t$.*

Proof : This follows immediately from the definitions. □

2.8. Remark. For completeness, we can also describe the effect of σ on en element $e \in E$, with 3 cases :

(1) If $]e, \hat{1}] \cap E$ has at least two minimal elements, then either $\sigma(e) = e$ or $\sigma(e) > e$ but $\sigma(e)$ is not irreducible.
(2) If $]e, \hat{1}] \cap E$ has a unique minimal element e^+, then $\sigma(e) = e^+$.
(3) If $]e, \hat{1}] \cap E$ is empty (that is, e is maximal in E), then $\sigma(e) = \hat{1}$.

Note that the equality $\sigma(e) = e$ also occurs in the third case for $e = \hat{1}$, provided $\hat{1}$ is irreducible.

2.9. Lemma. *Let $t \in T$ such that $r^\infty \sigma^\infty(t) = t$. Then :*

(a) $t \notin E$.
(b) If $t \in \Lambda E$, then $t \in \Lambda E - E$ and $t = \sigma(t) = r(t)$.
(c) If $t \notin \Lambda E$ and $e = \sigma^\infty(t)$, then $e \in E$, the interval $[t, e]$ is totally ordered, and

$$[t, e] = \{r^n(e), r^{n-1}(e), \dots, r^1(e), e\} = \{t, \sigma^1(t), \dots, \sigma^n(t)\} ,$$

where $n \geq 1$ is the smallest integer such that $\sigma^n(t) = \sigma^\infty(t)$.

Proof : (a) We have $t = r(t)$ because t is in the image of r^∞.

(b) Since $t \in \Lambda E - E$, the equality $t = \sigma(t)$ follows from Lemma 2.7.

(c) We have $e \geq \sigma(t) > t$ by Lemma 2.7 because $t \notin \Lambda E$. Since $t = r^\infty(e) < e$, we obtain $r(e) < e$, hence $e \in E$. By Lemma 2.5, $[t, e]$ is totally ordered, and

$$[t, e] = \{r^n(e), r^{n-1}(e), \dots, r^1(e), e\} .$$

Since $\sigma^\infty(t) = e$, we must have $\sigma(t) > t$ and also $\sigma(r^i(e)) > r^i(e)$ for $1 \leq i \leq n$ because $t \leq r^i(e) < e$ and σ is order-preserving. But $]t, e] \subseteq E$ by Lemma 2.5 again, so the definition of σ implies that $\sigma(r^i(e)) = r^{i-1}(e)$. It follows that $\sigma^i(t) = r^{n-i}(e)$ and therefore

$$[t, e] = \{t, \sigma^1(t), \dots, \sigma^n(t)\} ,$$

completing the proof. □

Lemma 2.9 is of great significance in this paper and justifies to introduce an important notation.

2.10. Notation. *Define*

$$G^\sharp = \{t \in T \mid r^\infty \sigma^\infty(t) = t\} ,$$
$$\widehat{G} = \{t \in G^\sharp \mid t \notin \Lambda E\} ,$$
$$G = E \sqcup G^\sharp = E \sqcup (\Lambda E - E) \sqcup \widehat{G} .$$

It is not hard to show that $\widehat{G}$ can be characterized by the properties that $t \notin \Lambda E$ and there exists $e \in E$ with $t = r^\infty(e)$ and $\sigma(e) = e$. If e is chosen minimal with these properties, then we recover the element e of Lemma 2.9. We note that $\widehat{G}$ (corresponding to case (c) in Lemma 2.9) is usually a rather small subset of T, often even empty (as for instance in Example 9.8).

2.11. Example. If $T = \{0, 1, \dots, m\}$ is totally ordered, $E = \mathrm{Irr}(T) = \{1, \dots, m\}$. Then $G^\sharp = \widehat{G} = \{0\}$ and $G = T$.

The following two propositions will be crucial for our results on evaluations of fundamental functors in Sections 5 and 6. We continue with the assumption that T is a finite lattice and $(E, R) = \mathrm{Irr}(T)$.

2.12. Proposition. *Let $a \in T$ with $a \notin G$, and let $b = r^\infty\sigma^\infty(a)$. There exists an integer $r \geq 0$ such that*

$$a < \sigma(a) < \ldots < \sigma^r(a) < b \leq \sigma^{r+1}(a) ,$$

and $\sigma^j(a) \in E$ for $j \in \{1, \ldots, r\}$. Moreover $b \in G^\sharp$.

Proof : We know that $a = \sigma^0(a) \notin E$ because $a \notin G$, hence $a = r(a)$. Also $a \leq b$ because $a = r^\infty(a) \leq r^\infty\sigma^\infty(a) = b$. Therefore $a < b$ because otherwise $a = b$ would be in $G^\sharp$. There are two cases (which will correspond to b satisfying case (b) or (c) of Lemma 2.9).

Suppose first that there exists an integer $r \geq 0$ such that $c = \sigma^{r+1}(a) \notin E$. In this case, we choose r minimal with this property, so that $\sigma^j(a) \in E$ for $1 \leq j \leq r$. We have $c = \sigma\big(\sigma^r(a)\big) \in \Lambda E - E$, hence $c = \sigma(c) = \sigma^\infty(c) = \sigma^\infty(a)$. Moreover $b = r^\infty(c) = c$, because $c \notin E$. Therefore

$$a < \sigma(a) < \ldots < \sigma^r(a) < b = \sigma^{r+1}(a)$$

and in fact $b \in \Lambda E - E$ because $b = c$.

In the second case, we suppose that $\sigma^r(a) \in E$ for all $r \in \mathbb{Z}_{>0}$. Let m be the smallest positive integer such that $\sigma^\infty(a) = \sigma^m(a)$. Define $e_i = \sigma^i(a)$ for all $0 \leq i \leq m$. Note that $e_1, \ldots, e_{m-1}$ all belong to E because they belong to ΛE (since they are in the image of the operator σ) and moreover $\sigma(e_i) > e_i$ (excluding the possibility $e_i \in \Lambda E - E$). Also $e_m = \sigma^\infty(a) \in E$ by assumption.

We have $a = r^\infty(a) \leq r^\infty(\sigma^\infty(a)) < \sigma^\infty(a)$, because $\sigma^\infty(a) \in E$. Therefore, there is an integer $r \leq m-1$ such that $b \leq e_{r+1}$ but $b \not\leq e_r$. Note that $b \notin E$ because its definition implies that $b = r(b)$. The inequality $b < e_{r+1}$ is strict because $b \notin E$ while $e_{r+1} \in E$. (The case $r = 0$ occurs when $a \leq b < e_1$.)

In particular $b \leq r^\infty(e_{r+1}) \leq r^\infty(\sigma^\infty(a)) = b$, hence $b = r^\infty(e_{r+1})$. Suppose that the element $e_r \vee b$ is irreducible. Then either $e_r \vee b = b$ or $e_r \vee b = e_r$. The first case is impossible because b is not irreducible. The second case is impossible because it would imply $b \leq e_r$, contrary to the definition of r. Therefore $e_r \vee b \notin E$. Since $e_r \vee b \leq e_{r+1}$, we obtain $e_r \vee b \leq r^\infty(e_{r+1}) = b$ by definition of $r^\infty(e_{r+1})$. It follows that $e_r < b < e_{r+1}$, as required.

The relations $\sigma^r(a) < b < \sigma^{r+1}(a)$ imply that $\sigma^\infty(b) = \sigma^\infty(a)$. Therefore we get $r^\infty\sigma^\infty(b) = b$, proving that $b \in G^\sharp$. In fact $b \in \widehat{G}$, otherwise $b \in \Lambda E - E$, hence $\sigma(b) = b$. But this is impossible because $\sigma^r(a) < b$ implies $b < \sigma^{r+1}(a) \leq \sigma(b)$. ∎

2.13. Notation. *Let $\zeta : G \longrightarrow I^\uparrow(E, R)$ be the map defined by*

$$\zeta(t) = \begin{cases} [t, \hat{1}] \cap E & \text{if } t \in E\,, \\ ]\sigma^\infty(t), \hat{1}] \cap E & \text{if } t \notin E\,. \end{cases}$$

For any $B \in I^\uparrow(E, R)$, define $\wedge B = \bigwedge_{e \in B} e$. By definition of $\sigma^\infty(t)$, we obtain

$$\wedge\zeta(t) = \begin{cases} t & \text{if } t \in E\,, \\ \sigma^\infty(t) & \text{if } t \notin E\,. \end{cases}$$

2.14. Proposition. *Let $t \in G$ and $t' \in T$ such that $t' \leq \wedge\zeta(t)$.*

(a) $\sigma^\infty(t') \leq \sigma^\infty(t)$ and $r^\infty(t') \leq r^\infty(t)$.

(b) $t' \leq t$, except possibly if $t \in \widehat{G}$.

Proof : We have $G = (\Lambda E - E) \sqcup E \sqcup \widehat{G}$ and we consider the three cases successively.

If $t \in \Lambda E - E$, then $\wedge\zeta(t) = \sigma^\infty(t) = t$, hence $t' \leq t$ and consequently $\sigma^\infty(t') \leq \sigma^\infty(t)$ and $r^\infty(t') \leq r^\infty(t)$.

If $t \in E$, then $\wedge\zeta(t) = t$, hence $t' \leq t$ and consequently $\sigma^\infty(t') \leq \sigma^\infty(t)$ and $r^\infty(t') \leq r^\infty(t)$.

Finally, if $t \in \widehat{G}$, then $\wedge\zeta(t) = \sigma^\infty(t)$, thus $\sigma^\infty(t') \leq \sigma^\infty(t)$. Moreover, using part (d) of Lemma 2.4 and the fact that $r^\infty(\sigma(t)) = t$ fo $1 \leq i \leq n$ (Lemma 2.9), we obtain

$$r^\infty(t') \leq r^\infty(\wedge\zeta(t)) = r^\infty(\sigma^\infty(t)) = t = r^\infty(t) \ .$$

This proves (a), and also (b) because the relation $t' \not\leq t$ can appear only in the case $t \in \widehat{G}$. □

3. Fundamental modules

For any finite poset (E, R), we describe in this section a specific module $M_{E,R}$ for the algebra $k\mathcal{R}_E$ of the monoid of relations on E. We call it the *fundamental module* because it plays a crucial role throughout our work.

If X and Y are finite sets, a correspondence from X to Y is a subset of $Y \times X$ (using a reverse notation which will later be convenient for left actions). If $S \subseteq Z \times Y$ and $R \subseteq Y \times X$, the composition of correspondences SR is a correspondence from X to Z defined by

$$SR = \{(z, x) \in Z \times X \mid \exists\, y \in Y \text{ such that } (z, y) \in S \text{ and } (y, x) \in R\} \ .$$

When $X = Y$, a correspondence from X to X is called a (binary) relation on X, also called a Boolean matrix. We let $\mathcal{C}(Y, X)$ be the set of all correspondences from X to Y. In particular, whenever E is a finite set, $\mathcal{R}_E := \mathcal{C}(E, E)$ is the monoid of all relations on E and $k\mathcal{R}_E$ is the k-algebra of this monoid, where k is a commutative base ring. If σ is permutation of the set E, we write $\Delta_\sigma = \{(\sigma(x), x) \mid x \in E\}$ and we also write $\Delta_E := \Delta_{\mathrm{id}}$.

Define

$$\mathcal{P}_E := \{\ R\Delta_\sigma \ \mid \ R \text{ is an order relation on } E \text{ and } \sigma \text{ is a permutation of } E\ \} \ ,$$

and let $k\mathcal{P}_E$ be its k-linear span, with k-basis $\mathcal{P}_E$. Any two orders R and S can be multiplied according to the rule that $R \cdot S$ is equal to the transitive closure of $R \cup S$ if this closure is an order, and zero otherwise. Then any two basis elements $R\Delta_\sigma, S\Delta_\tau \in \mathcal{P}_E$ can also be multiplied via the rule

$$(R\Delta_\sigma)(S\Delta_\tau) = (R \cdot {}^\sigma S)\Delta_{\sigma\tau} \ ,$$

where $\cdot$ denotes the product of orders defined above and where ${}^\sigma S = \Delta_\sigma S \Delta_{\sigma^{-1}}$, that is, ${}^\sigma S = \big\{\big(\sigma(e), \sigma(f)\big) \mid (e, f) \in S\big\}$. Thus $k\mathcal{P}_E$ is a k-algebra, which we called the algebra of permuted orders in [BT1]. Moreover, there is a surjective k-algebra map

$$\pi : k\mathcal{R}_E \longrightarrow k\mathcal{P}_E$$

defined as follows. First $\pi(\Delta_\sigma) = \Delta_\sigma$ and if R is a reflexive relation then $\pi(R)$ is the transitive closure $\overline{R}$ if this closure is an order, and zero otherwise. Finally if a relation S does not contain a permutation (i.e. S cannot be written $R\Delta_\sigma$ with R reflexive), then $\pi(R) = 0$.

The full structure of the k-algebra $k\mathcal{P}_E$ can be described. We let Σ_E be the group of all permutations of E and for any order R on E we let $\mathrm{Aut}(E, R)$ be the automorphism group of the poset (E, R), that is, the stabilizer of R in Σ_E.

3.1. Theorem. *The k-algebra $k\mathcal{P}_E$ is isomorphic to a direct product of matrix algebras*

$$k\mathcal{P}_E \cong \prod_R M_{n_R}(k\,\mathrm{Aut}(E,R))\ ,$$

where the product runs over representatives R of conjugacy classes of orders on E and where $n_R = |\Sigma_E : \mathrm{Aut}(E,R)|$.

This is Theorem 7.5 in [BT1] but it is in fact a very special case of a general result about inverse monoids (see Remark 3.4 below). In the R-th term of the product above, we let f_R be the matrix having a single nonzero entry 1 in the first position of the diagonal (and zeroes in all the other terms of the direct product). Then, for any coset representative σ in $\Sigma_E/\,\mathrm{Aut}(E,R)$, the matrix $\Delta_\sigma f_R \Delta_\sigma^{-1} = f_{{}^\sigma R}$ has a single nonzero entry 1 in the σ-th position of the diagonal. Note that the isomorphism of Theorem 3.1 maps $R \in \mathcal{P}_E$ to the sum $\displaystyle\sum_{S\in\mathcal{P}_E,\,R\subseteq S} f_S$.

Then we define

$$M_{E,R} := k\mathcal{P}_E f_R$$

viewed as a left $k\mathcal{P}_E$-module and we call it the *fundamental module* associated to the finite poset (E,R). All we need to know about $M_{E,R}$ is its structure, described in the next result, which is Proposition 8.5 of [BT1].

3.2. Proposition. *Let (E,R) be a finite poset.*

(a) The fundamental module $M_{E,R}$ is a left module for the algebra $k\mathcal{P}_E$, hence also a left module for the algebra of relations $k\mathcal{R}_E$.
(b) $M_{E,R}$ is a free k-module with a k-basis consisting of the elements $\Delta_\sigma f_R$, where σ runs through the group Σ_E of all permutations of E.
(c) $M_{E,R}$ is a $(k\mathcal{P}_E, k\,\mathrm{Aut}(E,R))$-bimodule and the right action of $k\,\mathrm{Aut}(E,R)$ is free. Explicitly, the right action of $\tau \in \mathrm{Aut}(E,R)$ maps the basis element $\Delta_\sigma f_R$ to the basis element $\Delta_{\sigma\tau} f_R$.

Using Theorem 3.1, one can also view $M_{E,R}$ as the set of column vectors of size $|\Sigma_E : \mathrm{Aut}(E,R)|$ with entries in $k\,\mathrm{Aut}(E,R)$, with its obvious $(k\mathcal{P}_E, k\,\mathrm{Aut}(E,R))$-bimodule structure. In particular, if $\tau \in \mathrm{Aut}(E,R)$, then $\Delta_\tau f_R = f_R \Delta_\tau$ is the matrix having a single nonzero entry τ in the first position of the diagonal.

3.3. Remark. The action of the algebra of relations $k\mathcal{R}_E$ on the module $M_{E,R}$ is given by an explicit formula. For any relation $Q \in \mathcal{R}_E$,

$$Q\cdot \Delta_\sigma f_R = \begin{cases} \Delta_{\tau\sigma} f_R & \text{if } \exists \tau \in \Sigma_E \text{ such that } \Delta_E \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R, \\ 0 & \text{otherwise}\ . \end{cases}$$

(Note that τ is unique in the first case.) This is proved in Proposition 8.5 of [BT1] but is not explicitly used in the present paper. It is used implicitly because the proof of Theorem 4.10 below is based on this formula (see Lemma 6.1 in [BT3]).

3.4. Remark (inverse monoids). The k-basis $\mathcal{P}_E$ is almost a monoid and Theorem 3.1 is a very special case of a general result about inverse monoids. Explicitly, let z be a zero element added to $\mathcal{P}_E$ and set $\widetilde{\mathcal{P}}_E = \mathcal{P}_E \sqcup \{z\}$. Then $\widetilde{\mathcal{P}}_E$ is a monoid (by replacing a product equal to 0 by a product equal to z). The algebra of this monoid satisfies $k\widetilde{\mathcal{P}}_E \cong k\mathcal{P}_E \times k$ and $k\mathcal{P}_E$ is the contracted monoid algebra of $\widetilde{\mathcal{P}}_E$ (see Remark 5.3 in [St2]). Moreover $\widetilde{\mathcal{P}}_E$ is an inverse monoid, the inverse of $R\Delta_\sigma$ being $\Delta_{\sigma^{-1}} R = R^\sigma \Delta_{\sigma^{-1}}$, where $R^\sigma = \Delta_{\sigma^{-1}} R \Delta_\sigma$.

By Theorem 4.3 in [St1] or Corollary 9.4 in [St2], the algebra of an inverse monoid is isomorphic to a product of matrix algebras over the group algebras of its maximal subgroups. Thus, if we discard the term k generated by z, we see that

Theorem 3.1 is just a specific instance of this general result. In order to get this special case, one needs to notice that the idempotents of the inverse monoid $\widetilde{\mathcal{P}}_E$ are, apart from z, all orders R and that the maximal subgroup of $\widetilde{\mathcal{P}}_E$ at R is isomorphic to $\mathrm{Aut}(E,R)$. Moreover, two idempotents are in the same $\mathcal{J}$-class if and only if they are conjugate under Σ_E and the number of idempotents in the $\mathcal{J}$-class of R is the number of conjugates of R, that is, $n_R = |\Sigma_E : \mathrm{Aut}(E,R)|$.

Finally, it should be noticed that the fundamental module $M_{E,R}$ is isomorphic to the left Schützenberger module relative to R (see Section 5.2 in [St2]). More precisely, if L_R denotes the $\mathcal{L}$-class of R in the monoid $\widetilde{\mathcal{P}}_E$, then right multiplication by f_R defines a surjective $k\mathcal{P}_E$-linear map $k\widetilde{\mathcal{P}}_E R \to k\mathcal{P}_E f_R$ (notice that $Rf_R = f_R$). The map has kernel $k(\widetilde{\mathcal{P}}_E R - L_R)$ and induces an isomorphism of $(k\mathcal{P}_E, k\,\mathrm{Aut}(E,R))$-bimodules between the Schützenberger module kL_R and the fundamental module $M_{E,R} = k\mathcal{P}_E f_R$.

3.5. Corollary. *Assume that k is a field. The simple $k\mathcal{P}_E$-modules have the form*

$$T_{R,V} = M_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} V$$

where R runs over a set of representatives of conjugacy classes of orders on E and V runs over a set of representatives of isomorphism classes of simple modules for the group algebra $k\,\mathrm{Aut}(E,R)$.

Proof : This follows from the Morita equivalence between $k\,\mathrm{Aut}(E,R)$ and the matrix algebra $M_{n_R}(k\,\mathrm{Aut}(E,R))$, which is given by the tensor product

$$V \mapsto M_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} V \ ,$$

and then applying Theorem 3.1. This is made explicit in [BT1]. □

4. Correspondence functors

In this section, we recall the basic facts we need about correspondence functors. We refer to [BT2] for details. We denote by $\mathcal{C}$ the category of finite sets and correspondences. Its objects are the finite sets and the set $\mathcal{C}(Y,X)$ of morphisms from X to Y is the set of all correspondences from X to Y, namely all subsets of $Y \times X$. The composition of correspondences is described at the beginning of Section 3.

For any commutative ring k, we let $k\mathcal{C}$ be the k-linearization of the category $\mathcal{C}$. The objects are the same, the set of morphisms $k\mathcal{C}(Y,X)$ is the free k-module with basis $\mathcal{C}(Y,X)$, and composition is extended by k-bilinearity from composition in $\mathcal{C}$. For any permutation σ of X, we write $\Delta_\sigma = \{(\sigma(x),x) \mid x \in X\}$. In particular, $\Delta_X := \Delta_{\mathrm{id}}$ is the identity morphism of the object X.

A correspondence functor is a k-linear functor from $k\mathcal{C}$ to the category k-Mod of left k-modules, for some fixed commutative ring k. We let $\mathcal{F}_k$ be the category of all correspondence functors. As already observed in [BT2], there is a set-theoretic observation to be made. In order to have sets of natural transformations, we need to restrict to a small skeleton of $\mathcal{C}$, for instance the full subcategory whose objects are the sets $\{1,2,\ldots,n\}$ for $n \geq 0$. For simplicity of the exposition, we avoid to recall this technical point, which is used throughout.

If F is a correspondence functor and $\psi \in k\mathcal{C}(Y,X)$, we view the k-module homomorphism $F(\psi) : F(X) \to F(Y)$ as a left action of ψ. More precisely, if $\alpha \in F(X)$, we define a left action $\psi \cdot \alpha := F(\psi)(\alpha) \in F(Y)$. In particular, the evaluation of a correspondence functor at a finite set X is a left $k\mathcal{R}_X$-module where $\mathcal{R}_X = \mathcal{C}(X,X)$

is the monoid of all relations on X. Our strategy will be to work with correspondence functors rather than $k\mathcal{R}_X$-modules.

4.1. Remark (finite category). The category $\mathcal{C}$ has infinitely many objects. If one is interested in fixing an object X, then one can consider the full subcategory $\mathcal{C}_{\leq X}$ whose objects are all subsets of X. This has finitely many objects and morphisms and the corresponding category algebra A (the free k-module on all morphisms in $\mathcal{C}_{\leq X}$, as defined for instance in [We]) is a k-algebra which is Morita equivalent to $k\mathcal{R}_X$. This is because if e is the identity morphism of the object X, then $AeA = A$ and $eAe \cong k\mathcal{R}_X$ (see Theorem 4.13 in [St2] for details). Therefore, the representation theory of the category $\mathcal{C}_{\leq X}$ is equivalent to the representation theory of the monoid $\mathcal{R}_X$.

However, we do not see any strong reason for the restrictive choice of fixing an object, as opposed to considering all objects simultaneously. Thus we will always use correspondence functors defined on the whole of $\mathcal{C}$ and we often need to do so. Also, working with the whole category $\mathcal{C}$ allowed us in [BT2] to consider finiteness conditions as well as asymptotic behavior of correspondence functors. This is another important motivation for avoiding to fix an object and it is used again in Corollary 6.7.

4.2. Remark (bi-surjective relations). In Example 4.4 of his recent work [Ste], Stein gives another connection with correspondences, but again fixing a finite set X. He proves that the algebra $k\mathcal{R}_X$ is isomorphic to the category algebra of the category whose objects are the subsets of X and morphisms are bi-surjective correspondences (hence a subcategory of $\mathcal{C}$).

We use in particular the following construction of correspondence functors. For any fixed left $k\mathcal{R}_E$-module W, the correspondence functor $L_{E,W}$ is defined by

$$L_{E,W}(X) := k\mathcal{C}(X,E) \otimes_{k\mathcal{C}(E,E)} W \ ,$$

with an obvious left action of correspondences in $\mathcal{C}(Y,X)$ by composition. Note that $W \mapsto L_{E,W}$ is left adjoint to the evaluation $F \mapsto F(E)$. There is a subfunctor $J_{E,W}$ of $L_{E,W}$ defined as follows (see Lemma 2.5 in [BT2]) :

$$J_{E,W}(X) = \Big\{ \sum_i \varphi_i \otimes w_i \in L_{E,W}(X) \mid \forall \rho \in k\mathcal{C}(E,X)\,, \sum_i (\rho\varphi_i)\cdot w_i = 0 \Big\} \ .$$

Let us mention an important functorial property of the quotient functor $L_{E,W}/J_{E,W}$.

4.3. Lemma. *Let E be a finite set and let W be a left $k\mathcal{R}_E$-module.*

- *(a) $J_{E,W}(E) = \{0\}$ and $L_{E,W}(E)/J_{E,W}(E) \cong L_{E,W}(E) \cong W$.*
- *(b) Let $\alpha \in L_{E,W}(X)/J_{E,W}(X)$ where X is some finite set. Then $\rho \cdot \alpha = 0$ for every $\rho \in k\mathcal{C}(E,X)$ if and only if $\alpha = 0$.*

Proof : (a) It is clear that $L_{E,W}(E) = k\mathcal{C}(E,E) \otimes_{k\mathcal{C}(E,E)} W \cong W$. Corresponding to $w \in W$, let $\mathrm{id} \otimes w \in L_{E,W}(E)$. If $\mathrm{id} \otimes w \in J_{E,W}(E)$, we choose $\rho = \mathrm{id} \in \mathcal{C}(E,E)$ and we obtain $w = (\rho \circ \mathrm{id}) \cdot w = 0$, by the definition of $J_{E,W}(X)$. This shows that $J_{E,W}(E) = \{0\}$.

(b) Let $\rho \in k\mathcal{C}(E,X)$. It follows from (a) that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J_{E,W}(X) \arrow[r] \arrow[d] & L_{E,W}(X) \arrow[r, "\pi"] \arrow[d, "\rho"] & L_{E,W}(X)/J_{E,W}(X) \arrow[r] \arrow[d, "\rho"] & 0 \\
 & 0 \arrow[r] & L_{E,W}(E) \arrow[r, "\cong"] & W \arrow[r] & 0
\end{tikzcd}$$

Let $\sum_i \varphi_i \otimes w_i \in L_{E,W}(X)$ such that $\pi(\sum_i \varphi_i \otimes w_i) = \alpha$. From the assumption that $\rho \cdot \alpha = 0$ for every $\rho \in k\mathcal{C}(E,X)$, we obtain

$$0 = \rho \cdot \big(\sum_i \varphi_i \otimes w_i\big) = \sum_i \rho\varphi_i \otimes w_i \in L_{E,W}(E) \ .$$

Viewing this in W via the isomorphism $L_{E,W}(E) \cong W$, we get $\sum_i (\rho\varphi_i) \cdot w_i = 0$ for every $\rho \in k\mathcal{C}(E,X)$. In other words, $\sum_i \varphi_i \otimes w_i \in J_{E,W}(X)$ and it follows that $\alpha = 0$. □

We now recall the construction of fundamental functors and simple functors, which are special cases of the construction above (see [BT2, BT3]). Using the fundamental $k\mathcal{R}_E$-module $M_{E,R}$ associated with a poset (E,R), we obtain the fundamental functor

$$\mathbb{S}_{E,R} := L_{E,M_{E,R}}/J_{E,M_{E,R}} \ .$$

Its structure will be described in Theorem 4.12 below, which will be our main tool, but we first mention the following properties (see Proposition 2.6 in [BT3]).

4.4. Proposition. *Let (E,R) be a finite poset and X a finite set.*

(a) $\mathbb{S}_{E,R}(X) = \{0\}$ *if* $|X| < |E|$.
(b) $\mathbb{S}_{E,R}(E) \cong M_{E,R}$.

If now V is a left $k\,\mathrm{Aut}(E,R)$-module, we define the $k\mathcal{R}_E$-module

$$T_{R,V} := M_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} V$$

using the right $k\,\mathrm{Aut}(E,R)$-module structure on $M_{E,R}$ described in Proposition 3.2. We then obtain an associated correspondence functor

$$S_{E,R,V} := L_{E,T_{R,V}}/J_{E,T_{R,V}} \ .$$

When k is a field and V is a simple $k\,\mathrm{Aut}(E,R)$-module, we have seen in Corollary 3.5 that $T_{R,V}$ is a simple $k\mathcal{P}_E$-module, hence also a simple $k\mathcal{R}_E$-module via the surjective homomorphism $\pi : k\mathcal{R}_E \to k\mathcal{P}_E$. But we obtain more (see Theorem 4.7 in [BT2]).

4.5. Theorem. *Assume that k is a field.*

(a) If the $k\,\mathrm{Aut}(E,R)$-module V is simple, then the functor $S_{E,R,V}$ is simple and $S_{E,R,V}(E) \cong T_{R,V}$ is a simple $k\mathcal{R}_E$-module.
(b) The map $(E,R,V) \mapsto S_{E,R,V}$ provides a parametrization of all simple correspondence functors by isomorphism classes of triples (E,R,V), where (E,R) is a finite poset and V is a simple $k\,\mathrm{Aut}(E,R)$-module.
(c) If S is a simple correspondence functor, the triple $(E.R,V)$ such that $S \cong S_{E,R,V}$ is obtained as follows. First E is a set of minimal cardinality such that $S(E) \neq \{0\}$. Then (R,V) is the pair corresponding to the simple $k\mathcal{P}_E$-module $S(E) \cong T_{R,V}$.

4.6. Remark. In part (c), it should be noticed that the simple $k\mathcal{R}_E$-module $S(E)$ is actually a $k\mathcal{P}_E$-module (with the kernel of $\pi : k\mathcal{R}_E \to k\mathcal{P}_E$ acting by zero). This follows from Theorems 4.2 and 4.7 in [BT2]. Therefore, $S(E)$ has the form $T_{R,V}$ for some pair (R,V), by Corollary 3.5.

4.7. Remark (apex and minimality). Theorem 4.5 is the corner stone in our approach to the classification of simple $k\mathcal{R}_X$-modules, which will be given in Theorem 8.1. In the classical Munn–Ponizovsky theory using $\mathcal{J}$-classes, an important concept is the apex of a simple $k\mathcal{R}_X$-module M, which is the minimal regular $\mathcal{J}$-class that does not annihilate M (see [GMS]). Here there is also a notion of minimality, but of a different, and very elementary, nature. Every simple $k\mathcal{R}_X$-module will be realized as the evaluation at X of some simple correspondence functor $S_{E,R,V}$ and E is then a minimal set on which this simple functor does not vanish (by part (c) of Theorem 4.5).

An important step in our strategy is to realize every simple correspondence functor as a quotient of a fundamental functor. More generally, the following property holds (see Lemma 2.7 in [BT3]).

4.8. Proposition. *Let (E,R) be a finite poset, let V be a left $k\,\mathrm{Aut}(E,R)$-module generated by a single element v, and let X be a finite set. There is a surjective morphism of correspondence functors*

$$\Phi : \mathbb{S}_{E,R} \longrightarrow S_{E,R,V}$$

such that, on evaluation at the finite set E, we obtain the surjective homomorphism of $k\mathcal{R}_E$-modules

$$\Phi_E : M_{E,R} \longrightarrow T_{R,V} = M_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} V \ , \qquad a \mapsto a \otimes v \ .$$

In order to obtain information about simple functors $S_{E,R,V}$, we shall always work first with the fundamental functor $\mathbb{S}_{E,R}$, which is a precursor of $S_{E,R,V}$ since we recover $S_{E,R,V}$ by means of the surjective morphism $\Phi : \mathbb{S}_{E,R} \to S_{E,R,V}$. This explains why the fundamental functors play a crucial role throughout our work. We shall see in Section 7 that there is an explicit way to recover $S_{E,R,V}$ from $\mathbb{S}_{E,R}$. It is also worth mentioning that both $\mathbb{S}_{E,R}$ and $S_{E,R,V}$ are defined over an arbitrary commutative ring k.

Now we explain the connection between correspondence functors and lattices. Let T be a finite lattice. We defined in [BT3] a correspondence functor F_T as follows. If X is a finite set, then $F_T(X) = kT^X$, the free k-module with basis the set T^X of all functions from X to T. If $R \subseteq Y \times X$ is a correspondence and if $\varphi \in T^X$, then we associate the function $R\cdot\varphi = F_T(R)(\varphi) \in T^Y$, also simply written $R\varphi$, defined by

$$(R\varphi)(y) := \bigvee_{(y,x)\in R} \varphi(x) \ ,$$

with the usual rule that a join over the empty set is equal to $\hat{0}$. The map

$$F_T(R) : F_T(X) \longrightarrow F_T(Y)$$

is the unique k-linear extension of this construction.

4.9. Remark (row spaces and lattices). There is a classical connection between relations and lattices (see [Ki]). Any relation $S \in \mathcal{R}_X$ can be viewed as a Boolean matrix and its rows generate the *row space* of S, which is a lattice under union (i.e. Boolean addition) and the induced meet. If T_S denotes the row space lattice of S and if a relation $U \in \mathcal{R}_X$ belongs to the left ideal $\mathcal{R}_X S$, then the row space of U is contained in T_S. Therefore, for any $x \in X$, the x-th row of U can be viewed as the value at x of a function $\varphi : X \to T_S$. It follows that $\mathcal{R}_X S$ can be identified with T_S^X, hence $k\mathcal{R}_X S \cong F_{T_S}(X)$. It is easy to see that this is an isomorphism of left $k\mathcal{R}_X$-modules, establishing a link between the principal left ideals of $k\mathcal{R}_X$ and the evaluation at X of some of the functors F_T.

However, we wish to emphasize that the functors F_T bring a change of perspective. First, instead of fixing X and varying T_S when S varies in $\mathcal{R}_X$, we fix T (which does not depend anymore on any choice of S) and we allow X to vary in order to get a functor. Thus F_T has more structure, namely a functor structure when X varies, as opposed to principal left ideals in the fixed algebra $k\mathcal{R}_X$. Note also that working with a fixed T is important throughout our approach, in particular in Sections 5 and 6. Secondly, the fact that T_S depends on $S \in \mathcal{R}_X$ implies in particular that T_S is contained in the lattice of subsets of X, so $|T_S|$ is bounded by $2^{|X|}$. Hence $|X|$ must be large enough while this restriction disappears with the change of perspective. We can now consider an evaluation $F_T(Y)$ for a finite set Y which can be arbitrarily small (up to the empty set). Thus we have not only more structure, but we have also extended the realm of possible evaluations.

The functors F_T play an important role in our approach because they are connected to fundamental functors by a morphism described in the following theorem (see Theorem 6.5 in [BT3]), which is the starting point for the proofs of our main results. If E denotes the set of irreducible elements in T, it is elementary to check that the functions $\varphi \in T^X$ such that $E \not\subseteq \varphi(X)$ generate a subfunctor H_T of F_T.

4.10. Theorem. *Let T be a finite lattice, let $(E,R) = \mathrm{Irr}(T)$, and let $\iota : E \to T$ denote the inclusion map.*

(a) There exists a unique surjective morphism of correspondence functors

$$\Theta_T : F_T \longrightarrow \mathbb{S}_{E,R^{op}}$$

such that $\Theta_{T,E}(\iota) = f_{R^{op}}$ (an element in $\mathbb{S}_{E,R^{op}}(E) = M_{E,R^{op}} = \mathcal{P}_E f_{R^{op}}$).

(b) The subfunctor H_T is contained in the subfunctor $\mathrm{Ker}(\Theta_T)$. Explicitly, if X is a finite set and if $f \in T^X$ satisfies the condition $E \not\subseteq f(X)$, then $\Theta_{T,X}(f) = 0$.

(c) The functor F_T is generated by $\iota \in F_T(E)$, while the functor $\mathbb{S}_{E,R^{op}}$ is generated by $f_{R^{op}} \in \mathbb{S}_{E,R^{op}}(E)$.

The precise definition of Θ_T will be recalled in Definition 7.3. In order to have control of the fundamental functor $\mathbb{S}_{E,R^{op}}$, we need to understand the kernel of Θ_T. To this end, we need to consider some correspondences which were introduced in [BT3] and which play again an important role in the present paper.

4.11. Notation. *Let T be a finite lattice and $(E,R) = \mathrm{Irr}(T)$. For any finite set X and any map $\varphi : X \to T$, we associate the correspondence*

$$\Gamma_\varphi := \{(x,e) \in X \times E \mid e \leq_T \varphi(x)\} \subseteq X \times E\ .$$

In the special case where $T = I_\downarrow(E,R)$, we obtain

$$\Gamma_\varphi = \{(x,e) \in X \times E \mid e \in \varphi(x)\}\ .$$

For the description of the kernel of $\Theta_T : F_T \to \mathbb{S}_{E,R^{op}}$, the following result was obtained as Theorem 7.1 in [BT3]. The result actually gives a first explicit description of every fundamental functor and it is one of our main tools in this paper.

4.12. Theorem. *Let T be a finite lattice, let $(E,R) = \mathrm{Irr}(T)$, and let X be a finite set. The kernel of the map*

$$\Theta_{T,X} : F_T(X) \longrightarrow \mathbb{S}_{E,R^{op}}(X)$$

is equal to the set of linear combinations $\sum_{\varphi:X\to T} \lambda_\varphi \varphi$*, where* $\lambda_\varphi \in k$*, such that for any map* $\psi : X \to I^\uparrow(E,R)$

$$\sum_{\substack{\varphi \\ \Gamma_\psi^{op}\Gamma_\varphi = R^{op}}} \lambda_\varphi = 0 \, .$$

Here $\Gamma_\varphi = \{(x,e) \in X \times E \mid e \leq_T \varphi(x)\}$ *and* $\Gamma_\psi^{op} = \big\{(e,x) \in E \times X \mid e \in \psi(x)\big\}$*, as in Notation 4.11.*

In order to use the condition $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$ appearing in Theorem 4.12, we shall also need equivalent formulations. We first fix notation. If $\psi : X \to I^\uparrow(E,R)$ is a map, define the function $\wedge\psi : X \to T$ by

$$\forall x \in X, \;\; \wedge\psi(x) = \bigwedge_{e\in\psi(x)} e \, ,$$

where $\bigwedge$ is the meet in the lattice T. If φ and φ' are two functions $X \to T$, we write $\varphi \leq \varphi'$ if $\varphi(x) \leq \varphi'(x)$ for all $x \in X$. The following result is Theorem 7.3 in [BT3].

4.13. Theorem. *Let* T *be a finite lattice, let* $(E,R) = \mathrm{Irr}(T)$*, let* $\iota : E \to T$ *denote the inclusion map, and let* X *be a finite set. Let* $\varphi : X \to T$ *and* $\psi : X \to I^\uparrow(E,R)$ *be maps with associated correspondence* Γ_φ *and* Γ_ψ^{op}*, as in Theorem 4.12 above. The following conditions are equivalent.*

(a) $\Gamma_\psi^{op}\varphi = \iota$.
(b) $\Gamma_\psi^{op}\Gamma_\varphi\iota = \iota$.
(c) $\Delta_E \subseteq \Gamma_\psi^{op}\Gamma_\varphi \subseteq R^{op}$.
(d) $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$.
(e) $\varphi \leq \wedge\psi$ *and* $\forall e \in E,\ \exists x \in X$ *such that* $\varphi(x) = e$ *and* $\psi(x) = [e,\cdot[_E$.
(f) $\forall t \in T,\ \psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E$ *and* $\forall e \in E,\ \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E$.

As noticed in the proof of this theorem in [BT3], we abuse notation in (f). For any subset Y of X, $\psi(Y)$ is a subset of $I^\uparrow(E,R)$, hence a set of subsets of E, but we write $\psi(Y)$ for the union of these subsets of E, that is, $\psi(Y) = \bigcup_{x\in Y} \psi(x)$.

Conditions (e) and (f) will play a crucial role in our results on fundamental functors and simple functors (Sections 5 and 6).

4.14. Remark (Schützenberger modules). There is another connection between the functorial constructions of the present section and the classical representation theory of the monoid $\mathcal{R}_X$ (see [GMS] or [St2]). Using the notation of Remark 4.9, suppose $U \in \mathcal{R}_X S$ but U and S are not in the same $\mathcal{L}$-class of the monoid $\mathcal{R}_X$, that is, $\mathcal{R}_X U \neq \mathcal{R}_X S$. Then the row space T_U must be strictly contained in the row space T_S and therefore T_U cannot contain the set E of irreducible elements of T_S since the irreducible elements generate T_S. In other words, the function $\varphi : X \to T_S$ corresponding to U (as in Remark 4.9) satisfies $E \not\subseteq \varphi(X)$, hence belongs to $H_{T_S}(X)$, where H_{T_S} denotes the subfunctor of F_{T_S} defined before Theorem 4.10. Thus U is in the same $\mathcal{L}$-class as S if and only if $E \subseteq \varphi(X)$, that is, $\varphi \notin H_{T_S}(X)$. Using the isomorphism $k\mathcal{R}_X S \cong F_{T_S}(X)$ of Remark 4.9, we deduce that the quotient of $k\mathcal{R}_X S$ giving the left Schützenberger module relative to S (see Section 5.2 in [St2]) is isomorphic to $F_{T_S}(X)/H_{T_S}(X)$.

In a more functorial way, given a fixed lattice T, the functor F_T/H_T links together various Schützenberger modules $F_T(X)/H_T(X)$ when X is allowed to vary (but subject to the restriction $|T| \leq 2^{|X|}$, as mentioned in Remark 4.9). It should also be mentioned that the quotient F_T/H_T is only used explicitly at the end of the proof of Theorem 6.1 and only in the special case when T is a totally ordered lattice. However, it is used implicitly in Sections 5 and 6 where we are dealing with functions $\varphi : X \to T$ satisfying the condition $E \subseteq \varphi(X)$.

4.15. Remark (construction of simple functors). In the Munn-Ponizovsky theory, the simple modules for $k\mathcal{R}_X$ are constructed in the following way (see [GMS] or [St2]). One starts with a poset (E,R) (with $E \subseteq X$) and the Schützenberger module kL_R relative to R (quotient of $k\mathcal{R}_X R$, as in Remark 4.14). Then kL_R has a $(k\mathcal{R}_X, k\,\mathrm{Aut}(E,R))$-bimodule structure, noticing that the maximal subgroup of $\mathcal{R}_X$ at the idempotent R is isomorphic to $\mathrm{Aut}(E,R)$. Induction from left $k\,\mathrm{Aut}(E,R)$-modules to left $k\mathcal{R}_X$-modules is given by the exact functor $kL_R \otimes_{k\,\mathrm{Aut}(E,R)} -$. For any simple $k\,\mathrm{Aut}(E,R)$-module V, its image $kL_R \otimes_{k\,\mathrm{Aut}(E,R)} V$ has a unique maximal submodule and the quotient

$$(kL_R \otimes_{k\,\mathrm{Aut}(E,R)} V) \big/ \mathrm{Rad}(kL_R \otimes_{k\,\mathrm{Aut}(E,R)} V)$$

is the required simple $k\mathcal{R}_X$-module corresponding to (E,R,V).

In view of Remark 4.14, an analogous result for constructing simple correspondence functors should involve a functor F_T/H_T. This is indeed the case and the simple functor $S_{E,R^{op},V}$ can be realized as the unique simple quotient of $(F_T/H_T) \otimes_{k\,\mathrm{Aut}(E,R)} V$ via a surjective morphism

$$\pi : (F_T/H_T) \otimes_{k\,\mathrm{Aut}(E,R)} V \longrightarrow S_{E,R^{op},V}$$

where T is a lattice with $\mathrm{Irr}(T) = (E,R)$. But there is a main new aspect which is crucial for our approach. Namely, we consider the intermediate functor $\mathbb{S}_{E,R^{op}}$ and we factorize π as a composite

$$(F_T/H_T) \otimes_{k\,\mathrm{Aut}(E,R)} V \xrightarrow{\overline{\Theta}_T \otimes \mathrm{id}_V} \mathbb{S}_{E,R^{op}} \otimes_{k\,\mathrm{Aut}(E,R)} V \xrightarrow{\Psi} S_{E,R^{op},V} \ ,$$

where $\overline{\Theta}_T : F_T/H_T \to \mathbb{S}_{E,R^{op}}$ is induced by the morphism $\Theta_T : F_T \to \mathbb{S}_{E,R^{op}}$ of Theorem 4.10. It is this new aspect which allows us to prove our main results. In Sections 5 and 6, we control the kernel of Θ_T and find the dimension of each evaluation $\mathbb{S}_{E,R^{op}}(X)$. Then we prove in Section 7 the nontrivial fact that Ψ is actually an isomorphism, allowing us to find the dimension of each evaluation $S_{E,R^{op},V}(X)$.

5. Generators for the evaluations of fundamental functors

As usual, E denotes a fixed finite set and R an order relation on E. Our purpose is to prove that, for an arbitrary commutative ring k and for any finite set X, the evaluation $\mathbb{S}_{E,R}(X)$ of the fundamental correspondence functor $\mathbb{S}_{E,R}$ is a free k-module, by finding an explicit k-basis. In this section, we first deal with k-linear generators.

Let T be any lattice such that $(E,R) = \mathrm{Irr}(T)$. Note that $I_\downarrow(E,R)$ is the largest such lattice and that any other is a quotient of $I_\downarrow(E,R)$ (Lemma 2.2). By Theorem 4.10, the fundamental functor $\mathbb{S}_{E,R^{op}}$ is isomorphic to a quotient of F_T via a morphism

$$\Theta_T : F_T \longrightarrow \mathbb{S}_{E,R^{op}} \ .$$

For this reason, we work with $\mathbb{S}_{E,R^{op}}$ rather than $\mathbb{S}_{E,R}$.

5.1. Notation. *Let $G = G(T)$ be the subset defined in Notation 2.10 and let X be a finite set. We define $\mathcal{B}_X$ to be the set of all maps $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$.*

Our main purpose is to prove that the set $\Theta_{T,X}(\mathcal{B}_X)$ is a k-basis of $\mathbb{S}_{E,R^{op}}(X)$. We first prove in this section that $\Theta_{T,X}(\mathcal{B}_X)$ generates k-linearly $\mathbb{S}_{E,R^{op}}(X)$ and then we shall show in Section 6 that $\Theta_{T,X}(\mathcal{B}_X)$ is k-linearly independent.

Recall that $G = E \sqcup G^\sharp$ where $G^\sharp = \{a \in T \mid a = r^\infty\sigma^\infty(a)\}$, with r^∞ defined in Notation 2.3 and σ^∞ in Definition 2.6. We denote by G^c the complement of G in T, namely

$$G^c = \{a \in T \mid a \notin E,\ a < r^\infty\sigma^\infty(a)\}\ .$$

Recall from Proposition 2.12 that if $a \in G^c$ and $b = r^\infty\sigma^\infty(a)$, there exists an integer $r \geq 0$ such that

$$a < \sigma(a) < \ldots < \sigma^r(a) < b \leq \sigma^{r+1}(a)\ ,$$

and $\sigma^j(a) \in E$ for $j \in \{1, \ldots, r\}$. Moreover, $b \in G^\sharp$. This produces a way to pass from any element a outside G to a uniquely defined element b in G. This justifies the following terminology :

5.2. Definition. *For $a \in G^c$, the sequence $a < \sigma(a) < \ldots < \sigma^r(a) < b$ defined above will be called the* reduction sequence *associated to a.*

5.3. Notation. *Let $n \geq 1$ and let $(a_0, a_1, \ldots, a_n)$ be a sequence of distinct elements of T. We denote by $[a_0, \ldots, a_n] : T \to T$ the map defined by*

$$\forall t \in T,\ \ [a_0, \ldots, a_n](t) = \begin{cases} a_{j+1} & \text{if } t = a_j,\ j \in \{0, \ldots, n-1\} \\ t & \text{otherwise.} \end{cases}$$

If $a \in G^c$, let $(a_0, a_1, \ldots, a_r, a_{r+1})$ be the reduction sequence associated to a, with $a_0 = a$ and $a_{r+1} = b = r^\infty\sigma^\infty(a)$. We then denote by u_a the element of $k(T^T) = F_T(T)$ defined by

$$u_a = [a_0, a_1] - [a_0, a_1, a_2] + \ldots + (-1)^r[a_0, a_1, \ldots, a_{r+1}]\ .$$

We can now describe a family of useful elements in $\operatorname{Ker}\Theta_T$.

5.4. Theorem. *Let T be a finite lattice, let $(E, R) = \operatorname{Irr}(T)$, let $G^c = \{a \in T \mid a \notin E,\ a < r^\infty\sigma^\infty(a)\}$, and let X be a finite set. Then, for any $a \in G^c$ and for any function $\varphi : X \to T$,*

$$\varphi - u_a \circ \varphi \in \operatorname{Ker}\Theta_{T,X}\ ,$$

where $u_a \circ \varphi$ is defined by bilinearity from the composition of maps $T^T \times T^X \to T^X$.

Proof : The kernel of the map $\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$ was described in Theorem 4.12. Let $\sum\limits_{\varphi:X\to T} \lambda_\varphi\varphi \in F_T(X)$, where $\lambda_\varphi \in k$. Then $\sum\limits_{\varphi:X\to T} \lambda_\varphi\varphi$ belongs to $\operatorname{Ker}\Theta_{T,X}$ if and only if the coefficients λ_φ satisfy a system of linear equations indexed by maps $\psi : X \to I^\uparrow(E, R)$. The equation (E_ψ) indexed by such a map ψ is the following :

$$(E_\psi) : \qquad \sum_{\varphi \underset{E,R}{\vdash} \psi} \lambda_\varphi = 0\ ,$$

where $\varphi \underset{E,R}{\vdash} \psi$ means that $\varphi : X \to T$ and $\psi : X \to I^\uparrow(E, R)$ satisfy the equivalent conditions of Theorem 4.13. We shall use condition (f) of Theorem 4.13, namely

$$\varphi \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in T,\ \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E\ , \\ \forall e \in E,\ \ \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E\ . \end{cases}$$

Recall that we abuse notation here. For any subset Y of X, in particular for $Y = \varphi^{-1}(t)$, we write $\psi(Y) = \bigcup_{x\in Y} \psi(x)$, a union of subsets of E instead of a set of subsets of E. This abusive notation is convenient and is used throughout the proof.

Let $a \in G^c$, and let $(a, e_1, e_2, \dots, e_r, b)$ be the associated reduction sequence. Recall that $e_1, \dots, e_r \in E$ but $b \notin E$. If $r \geq 1$, note that $[a, \cdot[_T \cap E = [e_1, \cdot[_E$ because $a < \sigma(a) = e_1 \in E$. Define, for each $i \in \{1, \dots, r\}$,

$$\varphi_i = [a, e_1, \dots, e_i] \circ \varphi\,, \qquad \text{and also } \ \varphi_{r+1} = [a, e_1, \dots, e_r, b] \circ \varphi\,.$$

In particular, for any $i \in \{1, \dots, r+1\}$,

$$\text{if } \varphi(x) \in T - \{a, e_1, \dots, e_r\}\,, \ \text{ then } \varphi_i(x) = \varphi(x)\,.$$

The other values of the maps φ_i are given in the following table :

(5.5)

$x \in$	$\varphi^{-1}(a)$	$\varphi^{-1}(e_1)$	$\varphi^{-1}(e_2)$	$\dots$	$\varphi^{-1}(e_{r-1})$	$\varphi^{-1}(e_r)$
$\varphi(x)$	a	e_1	e_2	$\dots$	e_{r-1}	e_r
$\varphi_1(x)$	e_1	e_1	e_2	$\dots$	e_{r-1}	e_r
$\varphi_2(x)$	e_1	e_2	e_2	$\dots$	e_{r-1}	e_r
$\varphi_3(x)$	e_1	e_2	e_3	$\dots$	e_{r-1}	e_r
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\varphi_r(x)$	e_1	e_2	e_3	$\dots$	e_r	e_r
$\varphi_{r+1}(x)$	e_1	e_2	e_3	$\dots$	e_r	b

We want to prove that the element

$$\varphi - u_a \circ \varphi = \varphi - \varphi_1 + \varphi_2 - \dots + (-1)^{r-1}\varphi_{r+1}$$

belongs to $\operatorname{Ker}\Theta_{T,X}$. We must prove that it satisfies the equation (E_ψ) for every ψ, so we must find which of the functions $\varphi, \varphi_1, \varphi_2, \dots, \varphi_{r+1}$ are linked with ψ under the relation $\underset{E,R}{\vdash}$. We are going to prove that only two consecutive functions can be linked with a given ψ, from which it follows that the corresponding equation (E_ψ) is satisfied because it reduces to either $1 - 1 = 0$, or $-1 + 1 = 0$. Of course, if none of $\varphi, \varphi_1, \varphi_2, \dots, \varphi_{r+1}$ is linked with ψ, then the corresponding equation (E_ψ) is just $0 = 0$. It follows from this that $\varphi - u_a \circ \varphi$ satisfies all equations (E_ψ), hence belongs to $\operatorname{Ker}\Theta_{T,X}$, as required. We note for completeness that it may happen that some of the functions $\varphi, \varphi_1, \varphi_2, \dots, \varphi_{r+1}$ are equal (this occurs if an inverse image is empty in some column of the table), but this does not play any role in the argument.

Assume first that $r \geq 1$. Write $U := T - \{a, e_1, \dots, e_r\}$ and $V := E - \{e_1, \dots, e_r\}$. The linking with a fixed ψ is controlled by the following conditions :

$$\varphi \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \ \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \ \ \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) \subseteq [a, \cdot[_T \cap E = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i, \cdot[_E \ \ \forall i \in \{1, \dots, r\}\,. \end{cases}$$

The subsets $\varphi_j^{-1}(e_i)$ are determined by Table 5.5 and can be written in terms of φ. In particular, $\varphi_1^{-1}(e_1) = \varphi^{-1}(a) \sqcup \varphi^{-1}(e_1)$, so we obtain

$$\varphi_1 \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \ \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \ \ \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a) \sqcup \varphi^{-1}(e_1)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i, \cdot[_E \ \ \forall i \in \{2, \dots, r\}\,. \end{cases}$$

Similarly, for $2 \leq j \leq r$, we have $\varphi_j^{-1}(e_1) = \varphi^{-1}(a)$ and $\varphi_j^{-1}(e_{i+1}) = \varphi^{-1}(e_i)$ if $1 \leq i \leq j-2$, and then $\varphi_j^{-1}(e_j) = \varphi^{-1}(e_{j-1}) \sqcup \varphi^{-1}(e_j)$. Therefore we get successively

$$\varphi_2 \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_1) \sqcup \varphi^{-1}(e_2)\big) = [e_2, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i, \cdot[_E \;\; \forall i \in \{3, \ldots, r\}\,. \end{cases}$$

$$\varphi_3 \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_1)\big) = [e_2, \cdot[_E \\ \psi\big(\varphi^{-1}(e_2) \sqcup \varphi^{-1}(e_3)\big) = [e_3, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i, \cdot[_E \;\; \forall i \in \{4, \ldots, r\}\,. \end{cases}$$

$$\cdots$$

$$\varphi_{r-1} \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_{i+1}, \cdot[_E \; \forall i \in \{1, \ldots, r-3\} \\ \psi\big(\varphi^{-1}(e_{r-2}) \sqcup \varphi^{-1}(e_{r-1})\big) = [e_{r-1}, \cdot[_E \\ \psi\big(\varphi^{-1}(e_r)\big) = [e_r, \cdot[_E\,. \end{cases}$$

$$\varphi_r \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_{i+1}, \cdot[_E \;\; \forall i \in \{1, \ldots, r-2\} \\ \psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big) = [e_r, \cdot[_E\,. \end{cases}$$

$$\varphi_{r+1} \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_{i+1}, \cdot[_E \;\; \forall i \in \{1, \ldots, r-1\} \\ \psi\big(\varphi^{-1}(e_r)\big) \subseteq [b, \cdot[_T \cap E\,. \end{cases}$$

Suppose that $\varphi \underset{E,R}{\vdash} \psi$. This clearly implies $\varphi_1 \underset{E,R}{\vdash} \psi$. Also $\varphi_i \underset{E,R}{\nvdash} \psi$ for $i \geq 2$, because $\varphi \underset{E,R}{\vdash} \psi$ implies $\psi\big(\varphi^{-1}(e_1)\big) = [e_1, \cdot[_E$, but $\varphi_i \underset{E,R}{\vdash} \psi$ implies $\psi\big(\varphi^{-1}(e_1)\big) \subseteq [e_2, \cdot[_E$ when $i \geq 2$. Therefore only φ and φ_1 are involved in this case.

Suppose now that $\varphi_1 \underset{E,R}{\vdash} \psi$ but $\varphi \underset{E,R}{\nvdash} \psi$. Then $\psi\big(\varphi^{-1}(e_1)\big) \subseteq]e_1, \cdot[_T \cap E = [e_2, \cdot[_E$ (because $e_2 = \sigma(e_1)$) and $\psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E$, hence in particular $\varphi_2 \underset{E,R}{\vdash} \psi$, since

$\varphi_1 \underset{E,R}{\vdash} \psi$ also implies $\psi\big(\varphi^{-1}(e_i)\big) = [e_i, \cdot[_E$ for $i \in \{2, \ldots, n\}$. On the other hand, since $\varphi_1 \underset{E,R}{\vdash} \psi$ implies $\psi\big(\varphi^{-1}(e_2)\big) = [e_2, \cdot[_E$, we cannot have $\psi\big(\varphi^{-1}(e_2)\big) \subseteq [e_3, \cdot[_E$ and so $\varphi_i \underset{E,R}{\nvdash} \psi$, for $i \geq 3$. Therefore only φ_1 and φ_2 are involved in this case.

Suppose by induction that $\varphi_i \underset{E,R}{\vdash} \psi$ but $\varphi_{i-1} \underset{E,R}{\nvdash} \psi$, for some $i \in \{1, \ldots, r-1\}$. Then the same argument shows that $\varphi_{i+1} \underset{E,R}{\vdash} \psi$ and that only φ_i and φ_{i+1} are involved in this case.

Suppose now that $\varphi_r \underset{E,R}{\vdash} \psi$ but $\varphi_{r-1} \underset{E,R}{\nvdash} \psi$. Then $\psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big) = [e_r, \cdot[_E$ but $\psi\big(\varphi^{-1}(e_r)\big) \neq [e_r, \cdot[_E$. Hence $\psi\big(\varphi^{-1}(e_r)\big) \subseteq]e_r, \cdot[_E \subseteq [b, \cdot[_T \cap E$, since $\sigma(e_r) \geq b$. Moreover $e_r \in \psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big)$ and $e_r \notin \psi\big(\varphi^{-1}(e_r)\big)$. It follows that $e_r \in \psi\big(\varphi^{-1}(e_{r-1})\big)$, hence $\psi\big(\varphi^{-1}(e_{r-1})\big) = [e_r, \cdot[_E$. Therefore $\varphi_{r+1} \underset{E,R}{\vdash} \psi$ and only φ_r and φ_{r+1} are involved in this case.

Finally, if $\varphi_{r+1} \underset{E,R}{\vdash} \psi$, then $\psi\big(\varphi^{-1}(e_{r-1})\big) = [e_r, \cdot[_E$ and $\psi\big(\varphi^{-1}(e_r)\big) \subseteq [b, \cdot[_T \cap E \subseteq [e_r, \cdot[_E$. Thus $\psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big) = [e_r, \cdot[_E$. Therefore $\varphi_r \underset{E,R}{\vdash} \psi$ and we are again in the case when only φ_r and φ_{r+1} are involved.

The special case $r = 0$ has to be treated separately. There are only 2 terms φ and φ_1 in the alternating sum. If $\varphi \underset{E,R}{\vdash} \psi$, then $\psi\big(\varphi^{-1}(a)\big) \subseteq [a, \cdot[_T \cap E$, hence $\psi\big(\varphi^{-1}(a)\big) \subseteq [b, \cdot[_T \cap E$ because $b \leq \sigma(a)$ when $r = 0$. Therefore

$$\psi\big(\varphi_1^{-1}(b)\big) = \psi\big(\varphi^{-1}(a) \sqcup \varphi^{-1}(b)\big) \subseteq [b, \cdot[_T \cap E$$

and so $\varphi_1 \underset{E,R}{\vdash} \psi$. Conversely, it is straightforward to see that $\varphi_1 \underset{E,R}{\vdash} \psi$ implies $\varphi \underset{E,R}{\vdash} \psi$.

We have proved that only two consecutive functions can be linked with a given ψ, as was to be shown. ∎

We have now paved the way for finding generators of $\mathbb{S}_{E,R^{op}}(X)$.

5.6. Theorem. *Let T be a finite lattice, let $(E, R) = \mathrm{Irr}(T)$, and let*

$$G^c = \{a \in T \mid a \notin E,\ a < r^\infty \sigma^\infty(a)\}\ .$$

For $a \in G^c$, let u_a be the element of $k(T^T)$ introduced in Notation 5.3, and let u_T denote the composition of all the elements u_a, for $a \in G^c$, in some order (they actually commute, see Theorem 5.8 below).

(a) Let X be a finite set. Then for any $\varphi : X \to T$, the element $u_T \circ \varphi$ is a k-linear combination of functions $f : X \to T$ such that $f(X) \subseteq G$.

(b) Let $\mathcal{B}_X$ be the set of all maps $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$. Then the set $\Theta_{T,X}(\mathcal{B}_X)$ generates $\mathbb{S}_{E,R^{op}}(X)$ as a k-module.

Proof : (a) We see in Table 5.5 that the functions $\varphi_1, \varphi_2, \ldots, \varphi_{r+1}$ do not take the value a. It follows that for any $a \in G^c$ and any $\varphi : X \to T$, the element $u_a \circ \varphi = \varphi_1 - \varphi_2 + \ldots + (-1)^r \varphi_{r+1}$ is a k-linear combination of functions φ_i such that $\varphi_i(X) \cap G^c \subseteq \big(\varphi(X) \cap G^c\big) - \{a\}$. We now remove successively all such elements a by applying successively all u_a for $a \in G^c$. It follows that $u_T \circ \varphi$ is a k-linear combination of functions $f : X \to T$ such that $f(X) \cap G^c = \emptyset$, that is, $f(X) \subseteq G$.

(b) Since $\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$ is surjective, $\mathbb{S}_{E,R^{op}}(X)$ is generated as a k-module by the images $\Theta_{T,X}(\varphi)$ of all maps $\varphi : X \to T$. For any $a \in G^c$, $u_a \circ \varphi$ has the same image as φ under $\Theta_{T,X}$, by Theorem 5.4. Therefore $u_T \circ \varphi$ has the same image as φ under $\Theta_{T,X}$. Moreover, $u_T \circ \varphi$ is a k-linear combination of functions $f : X \to T$ such that $f(X) \subseteq G$, by (a). Finally, if $E \nsubseteq f(X)$, then

$f \in \operatorname{Ker}\Theta_{T,X}$ by Theorem 4.10, so we can remove any such function in the linear combination $u_T \circ \varphi$ without changing the image $\Theta_{T,X}(u_T \circ \varphi)$. So we are left with linear combinations of maps $f : X \to T$ such that $E \subseteq f(X) \subseteq G$. ∎

We now mention that much more can be said about the elements u_a appearing in Theorem 5.6.

5.7. Definition. *Let T be a finite lattice. Recall that G^c denotes the complement of G in T. We define an oriented graph structure $\mathcal{G}(T)$ on T in the following way : for $x, y \in T$, there is an edge $x \rightarrowtail y$ from x to y in $\mathcal{G}(T)$ if there exists $a \in G^c$ such that (x, y) is a pair of consecutive elements in the reduction sequence associated to a.*

5.8. Theorem. *Keep the notation of Theorem 5.6 and let $\mathcal{G}(T)$ be the graph structure on T introduced in Definition 5.7.*

- *(a) The graph $\mathcal{G}(T)$ has no (oriented or unoriented) cycles, and each vertex has at most one outgoing edge. Hence $\mathcal{G}(T)$ is a forest.*
- *(b) For $a \in G^c$, the element u_a is an idempotent of $k(T^T)$.*
- *(c) $u_a \circ u_b = u_b \circ u_a$ for any $a, b \in G^c$.*
- *(d) The element u_T is an idempotent of $k(T^T)$.*

There is actually a closed formula for u_T and this is useful for the explicit description of the action of correspondences on the evaluation of simple functors (see Theorem 8.3). Otherwise, Theorem 5.8 has apparently no direct implication on the structure of correspondence functors, so we omit the proof.

6. Linear independence of the generators

In Section 5, we found a set $\Theta_{T,X}(\mathcal{B}_X)$ of generators for the evaluation $\mathbb{S}_{E,R^{op}}(X)$ of a fundamental functor $\mathbb{S}_{E,R^{op}}$. We now move to linear independence.

6.1. Theorem. *Let T be a finite lattice, let $(E, R) = \operatorname{Irr}(T)$, let X be a finite set, and let $\mathcal{B}_X$ be the set of all maps $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$, where $G = G(T)$ is the subset defined in Notation 2.10. The elements $\Theta_{T,X}(\varphi)$, for $\varphi \in \mathcal{B}_X$, are k-linearly independent in $\mathbb{S}_{E,R^{op}}(X)$.*

Proof : We consider again the kernel of the map

$$\Theta_{T,X} : F_T(X) \longrightarrow \mathbb{S}_{E,R^{op}}(X) ,$$

which was described in Theorem 4.12 by a system of linear equations. This can be reformulated by introducing the k-linear map

$$\begin{array}{rcl} \eta_{E,R,X} : F_T(X) & \longrightarrow & F_{I^\uparrow(E,R)}(X) \\ \varphi & \longmapsto & \displaystyle\sum_{\substack{\psi : X \to I^\uparrow(E,R) \\ \varphi \underset{E,R}{\vdash} \psi}} \psi \end{array}$$

where the notation $\varphi \underset{E,R}{\vdash} \psi$ means, as before, that $\varphi : X \to T$ and $\psi : X \to I^\uparrow(E, R)$ satisfy the equivalent conditions of Theorem 4.13. Theorem 4.12 asserts that

$$\operatorname{Ker}(\Theta_{T,X}) = \operatorname{Ker}(\eta_{E,R,X}) .$$

For handling the condition $\varphi \underset{E,R}{\vdash} \psi$, we shall use part (e) of Theorem 4.13, namely

(6.2)

$$\varphi \underset{E,R}{\vdash} \psi \iff \begin{cases} \varphi \leq \wedge\psi \ , \\ \forall e \in E, \ \exists x \in X \ \text{ such that } \ \varphi(x) = e \ \text{ and } \ \psi(x) = [e, \cdot[_E \ . \end{cases}$$

Let $N = N_{E,R,X}$ be the matrix of $\eta_{E,R,X}$ with respect to the standard basis of $F_T(X)$, consisting of maps $\varphi : X \to T$, and the standard basis of $F_{I^\uparrow(E,R)}(X)$, consisting of maps $\psi : X \to I^\uparrow(E,R)$. Explicitly,

$$\textbf{(6.3)} \qquad N_{\psi,\varphi} = \begin{cases} 1 & \text{if } \varphi \underset{E,R}{\vdash} \psi \ , \\ 0 & \text{otherwise.} \end{cases}$$

Note that N is a square matrix in the special case when $T = I_\downarrow(E,R)$, because complementation yields a bijection between $I_\downarrow(E,R)$ and $I^\uparrow(E,R)$. However, if T is a proper quotient of $I_\downarrow(E,R)$, then N has less columns.

In order to prove that the elements $\Theta_{T,X}(\varphi)$, for $\varphi \in \mathcal{B}_X$, are k-linearly independent, we shall prove that the elements $\eta_{E,R,X}(\varphi)$, for $\varphi \in \mathcal{B}_X$, are k-linearly independent. In other words, we have to show that the columns of N indexed by $\varphi \in \mathcal{B}_X$ are k-linearly independent. Now we consider only the rows indexed by elements of the form $\psi = \zeta \circ \varphi'$, where $\varphi' \in \mathcal{B}_X$ and $\zeta : G \to I^\uparrow(E,R)$ is the map defined in (2.13). We then define the square matrix M, indexed by $\mathcal{B}_X \times \mathcal{B}_X$, by

$$\forall \varphi, \varphi' \in \mathcal{B}_X, \ \ M_{\varphi',\varphi} = N_{\zeta\circ\varphi',\varphi} \ .$$

We are going to prove that M is invertible and this will prove the required linear independence.

The invertibility of M implies in particular that the map ζ must be injective, otherwise two rows of M would be equal. Therefore M turns out to be a submatrix of N, but this cannot be seen directly from its definition (unless an independent proof of the injectivity of ζ is provided).

The characterization of the condition $\varphi \underset{E,R}{\vdash} \psi$ given in (6.2) implies that

$$M_{\varphi',\varphi} = \begin{cases} 1 & \text{if } \varphi \leq \wedge\zeta\varphi' \text{ and } \forall e \in E, \exists x \in X, \ \varphi(x) = e = \varphi'(x) \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

because the equality $\zeta\varphi'(x) = [e, \cdot[_E$ is equivalent to $\varphi'(x) = e$, by definition of ζ (see Notation 2.13).

By Proposition 2.14, if $t, t' \in G$ are such that $t \leq \wedge\zeta(t')$, then $r^\infty(t) \leq r^\infty(t')$ and $\sigma^\infty(t) \leq \sigma^\infty(t')$. Let $\preceq$ be the preorder on G defined by these conditions, i.e. for all $t, t' \in G$,

$$t \preceq t' \iff r^\infty(t) \leq r^\infty(t') \text{ and } \sigma^\infty(t) \leq \sigma^\infty(t') \ .$$

We extend this preorder to $\mathcal{B}_X$ by setting, for all $\varphi', \varphi \in \mathcal{B}_X$,

$$\varphi \preceq \varphi' \iff \forall x \in X, \ \varphi(x) \preceq \varphi'(x) \ ,$$

which makes sense because $\varphi(x), \varphi'(x) \in G$ by definition of $\mathcal{B}_X$. We denote by $\preceq\succeq$ the equivalence relation defined by this preorder.

Clearly the condition $M_{\varphi',\varphi} \neq 0$ implies $\varphi \leq \wedge\zeta\varphi'$, hence $\varphi \preceq \varphi'$ by Proposition 2.14 quoted above. In other words the matrix M is block triangular, the blocks being indexed by the equivalence classes of the preorder $\preceq$ on $\mathcal{B}_X$. Showing that M is invertible is equivalent to showing that all its diagonal blocks are invertible. In other words, we must prove that, for each equivalence class C of $\mathcal{B}_X$ for the relation $\preceq\succeq$, the matrix $M_C = (M_{\varphi',\varphi})_{\varphi',\varphi\in C}$ is invertible. Let C be such a fixed equivalence class.

Recall from Definition 2.10 that $G = \Lambda E \sqcup \widehat{G}$. If $t \in \widehat{G}$, then by Lemma 2.9 $e := \sigma^\infty(t)$ belongs to E and $[t,e] = \{r^k(e), r^{k-1}(e), \ldots, r^1(e), e\}$, where $t = r^k(e) = r^\infty(e)$. By Lemma 2.5, all elements of $[\,t, \sigma^\infty(t)\,]_T$ belong to E except t itself. Moreover, if $x \in [\,t, \sigma^\infty(t)\,]_T$ then $r^\infty(x) = t$ by Lemma 2.9. It follows that the sets

$$G_t = [\,t, \sigma^\infty(t)\,]_T \ , \quad \text{for } t \in \widehat{G} \ ,$$

are disjoint, and contained in G. Let

$$G_* = G - \bigsqcup_{t \in \widehat{G}} G_t \ ,$$

so that we get a partition

$$G = \bigsqcup_{t \in \{*\} \sqcup \widehat{G}} G_t \ .$$

6.4. Lemma. *Let $\varphi', \varphi \in \mathcal{B}_X$. If $\varphi' \preceq\succeq \varphi$, then for all $t \in \{*\} \sqcup \widehat{G}$,*

$$\varphi'^{-1}(G_t) = \varphi^{-1}(G_t) \ .$$

Proof : Let $t \in \widehat{G}$ and $x \in \varphi^{-1}(G_t)$. Then $\varphi(x) \in [\,t, \sigma^\infty(t)\,]_T$, hence $r^\infty\varphi(x) = t$ and $\sigma^\infty\varphi(x) = \sigma^\infty(t)$, by Lemma 2.9. But the relation $\varphi' \preceq\succeq \varphi$ implies that $r^\infty\varphi'(x) = r^\infty\varphi(x)$ and $\sigma^\infty\varphi'(x) = \sigma^\infty\varphi(x)$. Therefore $r^\infty\varphi'(x) = t$ and $\sigma^\infty\varphi'(x) = \sigma^\infty(t)$, from which it follows that $\varphi'(x) \in [\,t, \sigma^\infty(t)\,]_T$, that is, $x \in \varphi'^{-1}(G_t)$. This shows that $\varphi^{-1}(G_t) \subseteq \varphi'^{-1}(G_t)$. By exchanging the roles of φ and φ', we obtain $\varphi'^{-1}(G_t) = \varphi^{-1}(G_t)$.

Now G_* is the complement of $\bigsqcup\limits_{t \in \widehat{G}} G_t$ in G and the functions φ', φ have their values in G (by definition of $\mathcal{B}_X$). So we must have also $\varphi'^{-1}(G_*) = \varphi^{-1}(G_*)$. □

For every $t \in \{*\} \sqcup \widehat{G}$, we define

$$X_t = \varphi_0^{-1}(G_t)$$

where φ_0 is an arbitrary element of C. It follows from Lemma 6.4 that this definition does not depend on the choice of φ_0. Therefore, the equivalence class C yields a partition

$$X = \bigsqcup_{t \in \{*\} \sqcup \widehat{G}} X_t \ ,$$

and every function $\varphi \in C$ decomposes as the disjoint union of the functions φ_t, where $\varphi_t : X_t \to G_t$ is the restriction of φ to X_t.

For $t \in \widehat{G}$, define

$$E_t = \,]\,t, \sigma^\infty(t)\,]_T \ .$$

By Lemma 2.5, this consists of elements of E, so $E_t = E \cap G_t$. Then we define $E_* = E - \bigsqcup\limits_{t \in \widehat{G}} E_t = E \cap G_*$, so that we get a partition

$$E = \bigsqcup_{t \in \{*\} \sqcup \widehat{G}} E_t \ .$$

For every $t \in \{*\} \sqcup \widehat{G}$ and for every $\varphi \in C$, the fact that φ belongs to $\mathcal{B}_X$ implies that φ_t belongs to the set $\mathcal{B}_{X,t}$ of all maps $\varphi_t : X_t \to G_t$ such that

$$E_t \subseteq \varphi_t(X_t) \subseteq G_t \ .$$

Moreover, if $\varphi', \varphi \in C$, then

$$M_{\varphi',\varphi} = 1 \iff \forall t \in \{*\} \sqcup \widehat{G},\ \left\{ \begin{array}{l} \forall x \in X_t,\ \varphi_t(x) \leq \wedge\zeta\varphi'_t(x) \\ \forall e \in E_t,\ \exists x \in X_t,\ \varphi'(x) = e = \varphi(x)\ . \end{array} \right.$$

It follows that the matrix M_C is the tensor product of the square matrices $M_{C,t}$ for $t \in \{*\} \sqcup \widehat{G}$, where the matrix $M_{C,t}$ is indexed by the functions $\varphi_t : X_t \to G_t$ in $\mathcal{B}_{X,t}$ and satisfies

$$(M_{C,t})_{\varphi'_t,\varphi_t} = 1 \iff \left\{ \begin{array}{l} \forall x \in X_t,\ \varphi_t(x) \leq \wedge\zeta\varphi'_t(x) \\ \forall e \in E_t,\ \exists x \in X_t,\ \varphi'(x) = e = \varphi(x)\ . \end{array} \right.$$

In order to show that M_C is invertible, we shall prove that each matrix $M_{C,t}$ is invertible.

If $\varphi_* \in \mathcal{B}_{X,*}$ and $x \in X_*$, then $\varphi_*(x) \in G_*$, hence $\varphi_*(x) \notin \widehat{G}$, because $\widehat{G}$ consists of all the bottom elements of the intervals G_t where $t \in \widehat{G}$. Therefore, the condition $\varphi_*(x) \leq \wedge\zeta\varphi'_*(x)$ implies $\varphi_*(x) \leq \varphi'_*(x)$ by Proposition 2.14, because $\varphi'_*(x) \notin \widehat{G}$. It follows that the matrix $M_{C,*}$ is unitriangular, hence invertible, as required.

Now we fix $t \in \widehat{G}$, we consider the matrix $M_{C,t}$ and we discuss the special role played by the elements of the set $\widehat{G}$. The interval $G_t = [t, \sigma^\infty(t)]_T$ is isomorphic to the totally ordered lattice $\underline{n} = \{0, 1, \ldots, n\}$, for some $n \geq 1$, and the set of irreducible elements $E_t =]t, \sigma^\infty(t)]_T$ is isomorphic to $[n] = \{1, \ldots, n\}$. Composing the maps $\varphi_t : X_t \to G_t$ with this isomorphism, we obtain maps $X_t \to \underline{n}$.

Changing notation for simplicity, we write X for X_t and φ for φ_t, and we let $\mathcal{B}_X^{\underline{n}}$ be the set of all maps $\varphi : X \to \underline{n}$ corresponding to maps in $\mathcal{B}_{X,t}$, i.e. satisfying the condition $[n] \subseteq \varphi(X) \subseteq \underline{n}$. We note that this condition is the same as the condition that φ belongs to $\mathcal{B}_X$ for the lattice $\underline{n}$, because the set $G(\underline{n})$ is the whole of $\underline{n}$ by Example 2.11. The matrix $M_{C,t}$, which we write $M^{\underline{n}}$ for simplicity, is now indexed by all the maps in $\mathcal{B}_X^{\underline{n}}$ and we have

$$M^{\underline{n}}_{\varphi',\varphi} = 1 \iff \left\{ \begin{array}{l} \forall x \in X,\ \varphi(x) \leq \wedge\zeta\varphi'(x) \\ \forall e \in [n],\ \exists x \in X,\ \varphi'(x) = e = \varphi(x)\ . \end{array} \right.$$

Here we need to clarify the meaning of the notation $\wedge\zeta$, so we recall that for any $g \in G$, we have

$$\wedge\zeta(g) = \left\{ \begin{array}{ll} g & \text{if } g \in E\ , \\ \sigma^\infty(g) & \text{if } g \notin E\ . \end{array} \right.$$

If g belongs to G_t and is mapped to $a \in \underline{n}$ via the isomorphism $G_t \cong \underline{n}$, then $\sigma^\infty(g)$ is mapped to n and we obtain

$$\wedge\zeta(a) = \left\{ \begin{array}{ll} a & \text{if } a \in [n],\ \text{i.e. } a \neq 0\ , \\ n & \text{if } a = 0\ . \end{array} \right.$$

The point here is that we obtain the same result as the one we would have obtained by working with the lattice $\underline{n}$, that is, by working with the corresponding map $\zeta : \underline{n} \to I^\uparrow([n], \mathrm{tot})$, which is easily seen to be a bijection, mapping 0 to $\emptyset$ and $j \geq 1$ to $[j, n]$.

Now we return to the beginning of the proof of Theorem 6.1 in the special case of the lattice $\underline{n}$, with $\mathrm{Irr}(\underline{n}) = ([n], \mathrm{tot})$, where tot denotes the usual total order. We have

$$\mathrm{Ker}(\Theta_{\underline{n},X}) = \mathrm{Ker}(\eta_{[n],\mathrm{tot},X})$$

and the matrix N of $\eta_{[n],\mathrm{tot},X}$ has entries 0 and 1, with

$$N_{\psi,\varphi} = 1 \iff \left\{ \begin{array}{l} \forall x \in X,\ \varphi(x) \leq \wedge\psi(x) \\ \forall e \in [n],\ \exists x \in X,\ \varphi(x) = e \ \text{ and }\ \psi(x) = [e, \cdot[_{[n]}\ , \end{array} \right.$$

where $\varphi : X \to \underline{n}$ and $\psi : X \to I^{\uparrow}([n], \mathrm{tot})$. But since $\zeta : \underline{n} \to I^{\uparrow}([n], \mathrm{tot})$ is a bijection, we can write $\psi = \zeta\varphi'$ and index the rows by the set of all functions $\varphi' : X \to \underline{n}$. We obtain

$$N_{\varphi',\varphi} = 1 \iff \begin{cases} \forall x \in X,\ \varphi(x) \le \wedge\zeta\varphi'(x) \\ \forall e \in [n],\ \exists x \in X,\ \varphi'(x) = e = \varphi(x)\ , \end{cases}$$

If φ or φ' is not in $\mathcal{B}_X^{\underline{n}}$ (that is, the image of either φ or φ' does not contain $[n]$), then the second condition cannot hold and so $N_{\varphi',\varphi} = 0$. Thus we restrict the matrix N to the rows and columns indexed by $\mathcal{B}_X^{\underline{n}}$. This restriction is exactly the same matrix as the matrix $M^{\underline{n}}$ above. Therefore, in order to prove that $M^{\underline{n}}$ is invertible, it suffices to prove that the columns of N indexed by $\mathcal{B}_X^{\underline{n}}$ are k-linearly independent. This in turn is equivalent to the condition that the set $\eta_{[n],\mathrm{tot},X}(\mathcal{B}_X^{\underline{n}})$ is k-linearly independent, or also that the set $\Theta_{\underline{n},X}(\mathcal{B}_X^{\underline{n}})$ is k-linearly independent in $\mathbb{S}_{[n],\mathrm{tot}^{op}}(X)$, because $\mathrm{Ker}(\Theta_{\underline{n},X}) = \mathrm{Ker}(\eta_{[n],\mathrm{tot},X})$. In other words, we have to prove Theorem 6.1 in the case of a total order.

By Theorem 4.10, we know that the surjective morphism

$$\Theta_{\underline{n}} : F_{\underline{n}} \longrightarrow \mathbb{S}_{[n],\mathrm{tot}^{op}}$$

has $H_{\underline{n}}$ in its kernel, where $H_{\underline{n}}$ is the subfunctor of $F_{\underline{n}}$ generated by all the maps $\varphi : X \to \underline{n}$ such that $[n] \not\subseteq \varphi(X)$. Therefore $\Theta_{\underline{n}}$ induces a surjective morphism

$$\overline{\Theta}_{\underline{n}} : F_{\underline{n}}/H_{\underline{n}} \longrightarrow \mathbb{S}_{[n],\mathrm{tot}^{op}}\ .$$

But Theorem 11.8 in [BT3] asserts that $F_{\underline{n}}/H_{\underline{n}}$ is isomorphic to $\mathbb{S}_{[n],\mathrm{tot}}$, hence also to $\mathbb{S}_{[n],\mathrm{tot}^{op}}$ in view of the poset isomorphism $([n], \mathrm{tot}^{op}) \cong ([n], \mathrm{tot})$ via the map $j \mapsto n - j + 1$. Clearly the set $\mathcal{B}_X^{\underline{n}}$ is a k-basis of $F_{\underline{n}}(X)/H_{\underline{n}}(X)$, so that $\mathbb{S}_{[n],\mathrm{tot}^{op}}(X)$ is also a free k-module of rank $|\mathcal{B}_X^{\underline{n}}|$. Evaluation at X yields a surjective homomorphism

$$\overline{\Theta}_{\underline{n},X} : F_{\underline{n}}(X)/H_{\underline{n}}(X) \longrightarrow \mathbb{S}_{[n],\mathrm{tot}^{op}}(X)$$

between two free k-modules of the same rank, hence an isomorphism (by standard algebraic K-theory, see Lemma 6.8 in [BT3]). Since the elements $\Theta_{\underline{n},X}(\varphi)$, for $\varphi \in \mathcal{B}_X^{\underline{n}}$, are the images under $\overline{\Theta}_{\underline{n},X}$ of the k-basis $\mathcal{B}_X^{\underline{n}}$ of $F_{\underline{n}}(X)/H_{\underline{n}}(X)$, they form a k-basis of $\mathbb{S}_{[n],\mathrm{tot}^{op}}(X)$. In particular, they are k-linearly independent.

This completes the proof of Theorem 6.1. □

In order to obtain formulas for the dimension of the evaluation of a fundamental functor, we need a well-known combinatorial lemma, which is Lemma 8.1 in [BT2], but actually goes back to [BBHM].

6.5. Lemma. *Let E be a subset of a finite set G. For any finite set X, the number N of all maps $\varphi : X \to G$ such that $E \subseteq \varphi(X) \subseteq G$ is equal to*

$$N = \sum_{i=0}^{|E|} (-1)^i \binom{|E|}{i} (|G| - i)^{|X|}\ .$$

We can now prove one of our main results about fundamental correspondence functors. This generalizes a formula obtained in [BT3] in the case of a total order.

6.6. Theorem. *Let (E, R) be a finite poset and let T be any lattice such that $(E, R) = \mathrm{Irr}(T)$. Let X be a finite set and let $\mathcal{B}_X$ be the set of all maps $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$, where $G = G(T)$ is the subset defined in Notation 2.10.*

(a) The set $\Theta_{T,X}(\mathcal{B}_X)$ (more precisely, the injective image of $\mathcal{B}_X$ under $\Theta_{T,X}$) is a k-basis of $\mathbb{S}_{E,R^{op}}(X)$.

(b) The k-module $\mathbb{S}_{E,R^{op}}(X)$ is free of rank

$$\mathrm{rk}_k\big(\mathbb{S}_{E,R^{op}}(X)\big) = |\mathcal{B}_X| = \sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}(|G|-i)^{|X|}\ .$$

Proof : (a) follows from Theorem 5.6 and Theorem 6.1.

(b) The formula follows immediately from (a) and Lemma 6.5. ∎

6.7. Corollary. *With the notation above, $|G|$ only depends on (E,R), and not on the choice of T.*

Proof : The formula of Theorem 6.6 implies that

$$\mathrm{rk}_k\big(\mathbb{S}_{E,R^{op}}(X)\big) \sim |G|^{|X|} \qquad \text{as } |X| \to \infty\ .$$

Since $\mathbb{S}_{E,R^{op}}$ only depends on (E,R), it follows that $|G|$ only depends on (E,R). ∎

We shall prove in a future paper a stronger property : the full subposet G of T only depends on (E,R) up to isomorphism.

7. From fundamental functors to simple functors

In this section, we complete the description of simple functors by showing that they can be constructed directly from fundamental functors. This uses in an essential way the fact, proved in Theorem 7.5 below, that each evaluation of a fundamental functor $\mathbb{S}_{E,R}$ is a free $k\,\mathrm{Aut}(E,R)$-module. The proof will depend on our main Theorem 6.6.

In order to use the action of automorphisms, we will need the following lemma.

7.1. Lemma. *Let (E,R) be a finite poset.*

(a) For any finite lattice T such that $\mathrm{Irr}(T)=(E,R)$, restriction induces an injective group homomorphism $\mathrm{Aut}(T)\to\mathrm{Aut}(E,R)$.

(b) There exists a finite lattice T such that $\mathrm{Irr}(T)=(E,R)$ and such that the restriction homomorphism $\mathrm{Aut}(T)\to\mathrm{Aut}(E,R)$ is an isomorphism.

Proof : (a) For any lattice T such that $\mathrm{Irr}(T)=(E,R)$, any lattice automorphism of T induces an automorphism of the poset (E,R). This gives a group homomorphism $\mathrm{Aut}(T)\to\mathrm{Aut}(E,R)$, which is injective since any element t of T is equal to the join of the irreducible elements $e\leq_T t$.

(b) Requiring that $\mathrm{Aut}(T)\cong\mathrm{Aut}(E,R)$ amounts to requiring that any automorphism of (E,R) can be extended to an automorphism of T. This is clearly possible if we choose $T=I_\downarrow(E,R)$. ∎

Now let T be a finite lattice with $\mathrm{Irr}(T) = (E, R)$. Our first goal is to work with right $k\,\mathrm{Aut}(E, R)$-module structures on F_T and $\mathbb{S}_{E,R}$. First, note that the group $\mathrm{Aut}(T)$ acts on the right on $F_T(X)$ as follows :

$$\forall \varphi : X \to T\,,\ \forall \sigma \in \mathrm{Aut}(T)\,, \qquad \varphi \cdot \sigma := \sigma^{-1} \circ \varphi\,.$$

Next, recall that the fundamental functor associated to a finite poset (E, R) is

$$\mathbb{S}_{E,R} := L_{E,M_{E,R}}/J_{E,M_{E,R}}$$

where $M_{E,R}$ is the fundamental $k\mathcal{R}_E$-module. Using the right $k\,\mathrm{Aut}(E, R)$-module structure on $M_{E,R}$, we can define a right $k\,\mathrm{Aut}(E, R)$-module structure on each evaluation

$$L_{E,M_{E,R}}(X) = k\mathcal{C}(X, E) \otimes_{k\mathcal{C}(E,E)} M_{E,R}$$

and we now show that this right module structure can be carried to $\mathbb{S}_{E,R}(X)$.

7.2. Lemma.

(a) $J_{E,M_{E,R}}(X)$ is a right $k\,\mathrm{Aut}(E, R)$-submodule of $L_{E,M_{E,R}}(X)$.
(b) $\mathbb{S}_{E,R}(X)$ has a right $k\,\mathrm{Aut}(E, R)$-module structure.
(c) The left action of any element of $k\mathcal{C}(Y, X)$ is a homomorphism of right $k\,\mathrm{Aut}(E, R)$-modules $\mathbb{S}_{E,R}(X) \to \mathbb{S}_{E,R}(Y)$.

Proof : (a) Since $k\mathcal{P}_E$ is a quotient algebra of $k\mathcal{C}(E, E)$ and the tensor product defining $L_{E,M_{E,R}}(X)$ is over $k\mathcal{C}(E, E)$, any element of $L_{E,M_{E,R}}(X)$ can be written $\varphi \otimes f_R$ for some $\varphi \in k\mathcal{C}(X, E)$. The right action of $\sigma \in \mathrm{Aut}(E, R)$ on $M_{E,R}$ (see Proposition 3.2) induces the right action given by

$$(\varphi \otimes f_R)\Delta_\sigma = \varphi \otimes (\Delta_\sigma f_R) = (\varphi\Delta_\sigma) \otimes f_R\,.$$

If $\varphi \otimes f_R \in J_{E,M_{E,R}}(X)$, then $(\rho\varphi) \cdot f_R = 0$ for all $\rho \in k\mathcal{C}(E, X)$. Then the element $(\varphi\Delta_\sigma) \otimes f_R$ satisfies

$$(\rho\varphi\Delta_\sigma) \cdot f_R = (\rho\varphi) \cdot f_R\Delta_\sigma = 0$$

for all $\rho \in k\mathcal{C}(E, X)$. Therefore $(\varphi\Delta_\sigma) \otimes f_R = (\varphi \otimes f_R)\Delta_\sigma$ belongs to $J_{E,M_{E,R}}(X)$, as was to be shown.

(b) This follows immediately from (a).

(c) This follows from the fact that the left and right actions commute, by associativity of the composition $k\mathcal{C}(Y, X) \times k\mathcal{C}(X, E) \times k\mathcal{C}(E, E) \to k\mathcal{C}(Y, E)$. □

We need now to explain how the morphism $\Theta_T : F_T \to \mathbb{S}_{E,R^{op}}$ of Theorem 4.10 is defined. This appears in Theorem 6.5 of [BT3].

7.3. Definition. *Let $\varphi : X \to T$ be a map, i.e. a generator of $F_T(X)$. Then $\Theta_{T,X}(\varphi)$ is the class in the quotient $\mathbb{S}_{E,R^{op}}(X)$ of the element*

$$\Gamma_\varphi \otimes f_{R^{op}} \in k\mathcal{C}(X, E) \otimes_{k\mathcal{C}(E,E)} M_{E,R^{op}} = L_{E,M_{E,R^{op}}}(X)\,,$$

where Γ_φ is the correspondence defined in Notation 4.11.

7.4. Lemma. *Let (E,R) be a poset. Let T be a lattice such that $\mathrm{Irr}(T) = (E,R)$ and such that the restriction homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(E,R)$ is an isomorphism. For every finite set X, the map*

$$\Theta_{T,X} : F_T(X) \longrightarrow \mathbb{S}_{E,R^{op}}(X)$$

is a homomorphism of right $k\,\mathrm{Aut}(E,R)$-modules.

Proof : Note that we obviously have an equality $\mathrm{Aut}(E,R) = \mathrm{Aut}(E,R^{op})$. Let us first prove that, for any $\varphi : X \to T$ and any $\sigma \in \mathrm{Aut}(T) \cong \mathrm{Aut}(E,R)$,

$$\Gamma_\varphi \Delta_\sigma = \Gamma_{\sigma^{-1}\varphi} \; .$$

An element $(x,e) \in X \times E$ belongs to the left hand side if and only if $(x,\sigma(e)) \in \Gamma_\varphi$, because $(\sigma(e),e) \in \Delta_\sigma$. By the definition of Γ_φ, this is equivalent to the condition $\sigma(e) \leq_T \varphi(x)$, which in turn is equivalent to $e \leq_T \sigma^{-1}\varphi(x)$ because $\sigma \in \mathrm{Aut}(T)$. Thus we obtain that (x,e) satisfies the condition defining $\Gamma_{\sigma^{-1}\varphi}$, that is, (x,e) belongs to the right hand side.

Now we can compute $\Theta_{T,X}(\varphi \cdot \sigma) = \Theta_{T,X}(\sigma^{-1}\varphi)$. By Definition 7.3, this is the class in the quotient $\mathbb{S}_{E,R^{op}}(X)$ of the element

$$\Gamma_{\sigma^{-1}\varphi} \otimes f_{R^{op}} = \Gamma_\varphi \Delta_\sigma \otimes f_{R^{op}} = \Gamma_\varphi \otimes \Delta_\sigma f_{R^{op}} = (\Gamma_\varphi \otimes f_{R^{op}})\Delta_\sigma \; .$$

The latter equality uses the definition of the right action of σ on $f_{R^{op}}$ (Proposition 3.2). Since the class of $\Gamma_\varphi \otimes f_{R^{op}}$ is $\Theta_{T,X}(\varphi)$, this shows that $\Theta_{T,X}(\varphi \cdot \sigma) = \Theta_{T,X}(\varphi) \cdot \sigma$, as required. ∎

Our next result is crucial for the rest of this section, hence for the computation of the dimension of the evaluations of simple functors (Theorem 7.10).

7.5. Theorem. *Let (E,R) be a poset. Let T be a lattice such that $\mathrm{Irr}(T) = (E,R)$ and such that the restriction homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(E,R)$ is an isomorphism. For every finite set X, the evaluation $\mathbb{S}_{E,R^{op}}(X)$ is a free right $k\,\mathrm{Aut}(E,R)$-module.*

Proof : The set of all maps $\varphi : X \to T$ is a k-basis of $F_T(X)$ and is permuted by the right action of $\mathrm{Aut}(E,R)$. If G is as before (see Theorem 6.6), we claim that the subset $\mathcal{B}_X$ of all maps satisfying $E \subseteq \varphi(X) \subseteq G$ is freely permuted by $\mathrm{Aut}(E,R)$. First note that $\mathrm{Aut}(E,R)$ obviously leaves E invariant. It also leaves G invariant because $\mathrm{Aut}(T) \cong \mathrm{Aut}(E,R)$ preserves the characterization of G given in Notation 2.10. Therefore $\mathrm{Aut}(E,R)$ acts on $\mathcal{B}_X$.

If $\sigma \in \mathrm{Aut}(E,R)$ stabilizes some $\varphi \in \mathcal{B}_X$, that is, $\varphi \cdot \sigma = \varphi$, then $\sigma^{-1}\varphi(x) = \varphi(x)$ for all $x \in X$, hence in particular $\sigma^{-1}(e) = e$ for every $e \in E$ because $E \subseteq \varphi(X)$. It follows that σ is the identity automorphism of E. This proves the claim above.

Now $\mathcal{B}_X$ is mapped bijectively onto $\Theta_{T,X}(\mathcal{B}_X)$, which is a k-basis of $\mathbb{S}_{E,R^{op}}(X)$, by Theorem 6.6. By Lemma 7.4, $\Theta_{T,X}$ is a homomorphism of $k\,\mathrm{Aut}(E,R)$-modules. It follows that $\Theta_{T,X}(\mathcal{B}_X)$ is freely permuted by $\mathrm{Aut}(E,R)$ and therefore $\mathbb{S}_{E,R^{op}}(X)$ is a free (right) $k\,\mathrm{Aut}(E,R)$-module. ∎

Now we construct a morphism $\Psi : \mathbb{S}_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} V \to S_{E,R,V}$, which will be proved later to be an isomorphism (Theorem 7.9).

7.6. Proposition. *Let (E,R) be a finite poset, let $A = \mathrm{Aut}(E,R)$, and let V be a left kA-module, generated by a single element v (e.g. a simple module).*

(a) For any finite set X, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J_{E,M_{E,R}}(X) \arrow[r, "j"] \arrow[d] & L_{E,M_{E,R}}(X) \arrow[r, "\pi"] \arrow[d] & \mathbb{S}_{E,R}(X) \arrow[r] \arrow[d, "\mathrm{id}\otimes v"] & 0 \\
 & J_{E,M_{E,R}}(X)\otimes_{kA} V \arrow[r, "j\otimes \mathrm{id}_V"] \arrow[d] & L_{E,M_{E,R}}(X)\otimes_{kA} V \arrow[r, "\pi\otimes \mathrm{id}_V"] \arrow[d, "\mathrm{id}"] & \mathbb{S}_{E,R}(X)\otimes_{kA} V \arrow[r] \arrow[d, "\Psi_X"] & 0 \\
0 \arrow[r] & J_{E,M_{E,R}\otimes V}(X) \arrow[r, "i"] & L_{E,M_{E,R}\otimes V}(X) \arrow[r] & S_{E,R,V}(X) \arrow[r] & 0
\end{tikzcd}$$

(b) On the right hand side, both maps $\mathrm{id}\otimes v$ and Ψ_X are surjective.

Proof : The first row comes from the definition of $\mathbb{S}_{E,R}$ and j denotes the inclusion map while π is the quotient map. The second row is obtained from the first by tensoring with V (tensoring is right exact), using the right kA-module structure obtained in Lemma 7.2. The three vertical maps from the first to the second row are all given by $\alpha \mapsto \alpha\otimes v$ and they are surjective because v generates V, hence $V = kA\cdot v$. The third row comes from the definition of $S_{E,R,V}$ and i denotes the inclusion map. Now we have to describe the vertical maps from the second to the third row. The middle vertical map is the identity because

$$L_{E,M_{E,R}}(X)\otimes_{kA} V = k\mathcal{C}(X,E)\otimes_{k\mathcal{C}(E,E)} M_{E,R}\otimes_{kA} V = L_{E,M_{E,R}\otimes_{kA}V}(X)\ .$$

We claim that $(j\otimes \mathrm{id}_V)\big(J_{E,M_{E,R}}(X)\otimes_{kA} V\big)$ is contained in $J_{E,M_{E,R}\otimes V}(X)$. It will then follow that $j\otimes\mathrm{id}_V$ defines the vertical map on the left. This in turn shows that id induces the vertical map Ψ_X on the right and Ψ_X is surjective.

In order to prove the claim, let $\varphi\otimes f_R \in J_{E,M_{E,R}}(X)$ where $\varphi\in k\mathcal{C}(X,E)$. This means that

$$\forall \rho\in k\mathcal{C}(E,X)\,,\qquad (\rho\varphi)\cdot f_R = 0\ .$$

It follows that $(\rho\varphi)\cdot(f_R\otimes v) = 0$ in $M_{E,R}\otimes_{kA} V$ because $\rho\varphi$ only acts on the first term of the tensor product. This means that the element

$$\varphi\otimes(f_R\otimes v)\in L_{E,M_{E,R}\otimes V}(X)$$

actually belongs to $J_{E,M_{E,R}\otimes V}(X)$. But this element is equal to

$$(\varphi\otimes f_R)\otimes v = (j\otimes \mathrm{id}_V)(\varphi\otimes f_R\otimes v)\ ,$$

proving the claim. □

7.7. Notation. *Consider the diagram of Proposition 7.6. When X is allowed to vary, the morphisms Ψ_X on the right hand side define a surjective morphism of correspondence functors*

$$\Psi : \mathbb{S}_{E,R}\otimes_{kA} V \longrightarrow S_{E,R,V}\ ,$$

providing a direct link between the fundamental functor $\mathbb{S}_{E,R}$ and the simple functor $S_{E,R,V}$ when V is simple.

Similarly, the right hand side composite $\Psi_X\circ(\mathrm{id}\otimes v)$ yields a surjective morphism

$$\Phi := \Psi\circ(\mathrm{id}\otimes v) : \mathbb{S}_{E,R} \longrightarrow S_{E,R,V}$$

which is a morphism of correspondence functors because it is induced by the middle vertical morphism

$$L_{E,M_{E,R}} \longrightarrow L_{E,M_{E,R}\otimes_{kA}V}\ ,$$

which is obviously a morphism of correspondence functors. This defines the morphism Φ appearing in Proposition 4.8.

Our goal is to prove that $\Psi : \mathbb{S}_{E,R} \otimes_{kA} V \to S_{E,R,V}$ is an isomorphism. We prepare the ground with the following lemma, which is analogous to Lemma 4.3. In the proof, we need the full strength of Theorem 7.5, based in turn on Theorem 6.6. Since we consider simple modules, we assume that k is a field.

7.8. Lemma. *Let k be a field, let (E,R) be a finite poset, let $A = \mathrm{Aut}(E,R)$, and let V be a simple kA-module. Let $\alpha \in \mathbb{S}_{E,R}(X) \otimes_{kA} V$ where X is some finite set. Then $\rho \cdot \alpha = 0$ for every $\rho \in k\mathcal{C}(E,X)$ if and only if $\alpha = 0$.*

Proof : Since V is a simple kA-module and A is a finite group, we claim that there exists an injective homomorphism of kA-modules $\lambda : V \to kA$. This follows from the following argument. If $V^\natural$ denotes the dual simple module, there exists a surjective homomorphism $\pi : kA \to V^\natural$, which we dualize to obtain an injective homomorphism $\pi^\natural : V \to (kA)^\natural$. Now the group algebra of a finite group is a symmetric algebra, so $(kA)^\natural \cong kA$, and this defines the injective homomorphism $\lambda : V \to kA$.

If M is a free right kA-module, then

$$\mathrm{id}_M \otimes \lambda : M \otimes_{kA} V \longrightarrow M \otimes_{kA} kA$$

remains injective. This is clear if M is free of rank one and then it follows in general by taking direct sums. Now we compose with the isomorphism $M \otimes_{kA} kA \cong M$ and we take $M = \mathbb{S}_{E,R}(X)$, which is indeed a free right kA-module by Theorem 7.5. We obtain an injective homomorphism

$$\lambda_X : \mathbb{S}_{E,R}(X) \otimes_{kA} V \longrightarrow \mathbb{S}_{E,R}(X)$$

which is easily seen to define a morphism of correspondence functors

$$\lambda : \mathbb{S}_{E,R} \otimes_{kA} V \longrightarrow \mathbb{S}_{E,R}$$

because we use only the right module structure, whereas correspondences act on the left.

For every $\rho \in k\mathcal{C}(E,X)$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{S}_{E,R}(X) \otimes_{kA} V & \xrightarrow{\lambda_X} & \mathbb{S}_{E,R}(X) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ \mathbb{S}_{E,R}(E) \otimes_{kA} V & \xrightarrow{\lambda_E} & \mathbb{S}_{E,R}(E) \end{array}$$

Whenever our given element $\alpha \in \mathbb{S}_{E,R}(X) \otimes_{kA} V$ satisfies $\rho \cdot \alpha = 0$ for every $\rho \in k\mathcal{C}(E,X)$, we also have $\rho \cdot \lambda_X(\alpha) = 0$ for every $\rho \in k\mathcal{C}(E,X)$. But this implies that $\lambda_X(\alpha) = 0$ by Lemma 4.3. Since λ_X is injective, $\alpha = 0$, as required. $\square$

Now we come to our main description of simple correspondence functors.

7.9. Theorem. *Let k be a field, let (E,R) be a finite poset, let $A = \mathrm{Aut}(E,R)$, and let V be a simple kA-module. The morphism $\Psi : \mathbb{S}_{E,R} \otimes_{kA} V \to S_{E,R,V}$ is an isomorphism.*

Proof : For any finite set X and any $\rho \in k\mathcal{C}(E,X)$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{S}_{E,R}(X) \otimes_{kA} V & \xrightarrow{\Psi_X} & S_{E,R,V}(X) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ \mathbb{S}_{E,R}(E) \otimes_{kA} V & \xrightarrow{\Psi_E} & S_{E,R,V}(E) \end{array}$$

Note that Ψ_E is an isomorphism because

$$\mathbb{S}_{E,R}(E) \otimes_{kA} V = M_{E,R} \otimes_{kA} V = S_{E,R,V}(E)$$

and Ψ is induced by the identity morphism $L_{E,M_{E,R}} \otimes_{kA} V \to L_{E,M_{E,R} \otimes_{kA} V}$.

Let $\alpha \in \mathbb{S}_{E,R}(X) \otimes_{kA} V$ such that $\Psi_X(\alpha) = 0$. Then

$$\Psi_E(\rho \cdot \alpha) = \rho \cdot \Psi_X(\alpha) = 0$$

for every $\rho \in k\mathcal{C}(E,X)$. Since Ψ_E is an isomorphism, we obtain $\rho \cdot \alpha = 0$ for every $\rho \in k\mathcal{C}(E,X)$. Therefore $\alpha = 0$ by Lemma 7.8. This proves that Ψ_X is injective and we know that it is surjective by construction. $\square$

We can finally prove one of our main results, namely the determination of the dimension of any evaluation of a simple correspondence functor. Because of the link with lattices (via the morphism Θ_T), it is convenient to state the result for R^{op} rather than R. But this is actually a minor point because $S_{E,R,V}$ is isomorphic to the dual of $S_{E,R^{op},V^\natural}$ where $V^\natural$ is the dual module, by Theorem 9.8 in [BT3].

7.10. Theorem. *Let k be a field. Let (E,R) be a poset and let V be a simple left $k\,\mathrm{Aut}(E,R)$-module. Let T be a lattice such that $\mathrm{Irr}(T) = (E,R)$ and such that the restriction homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(E,R)$ is an isomorphism. Let $G = E \sqcup \{t \in T \mid t = r^\infty\sigma^\infty(t)\} \subseteq T$ (see Definition 2.10).*

For any finite set X, the dimension of $S_{E,R^{op},V}(X)$ is given by

$$\dim_k S_{E,R^{op},V}(X) = \frac{\dim_k V}{|\,\mathrm{Aut}(E,R)|} \sum_{i=0}^{|E|} (-1)^i \binom{|E|}{i} (|G|-i)^{|X|} \;.$$

Proof : By Theorem 7.5, $\mathbb{S}_{E,R^{op}}(X)$ is isomorphic to the direct sum of n_X copies of the free right module $k\,\mathrm{Aut}(E,R)$, for some $n_X \in \mathbb{N}$. In particular

$$\dim_k \mathbb{S}_{E,R^{op}}(X) = n_X\,|\,\mathrm{Aut}(E,R)| \;.$$

By Theorem 7.9, the simple functor $S_{E,R^{op},V}$ is isomorphic to $\mathbb{S}_{E,R^{op}} \otimes_{k\,\mathrm{Aut}(E,R)} V$, using the obvious equality $\mathrm{Aut}(E,R) = \mathrm{Aut}(E,R^{op})$. Thus we obtain

$$S_{E,R^{op},V}(X) \cong \mathbb{S}_{E,R^{op}}(X) \otimes_{k\,\mathrm{Aut}(E,R)} V \cong n_X\big(k\,\mathrm{Aut}(E,R)\big) \otimes_{k\,\mathrm{Aut}(E,R)} V \cong n_X V \;.$$

Hence $\dim_k \mathbb{S}_{E,R^{op},V}(X) = n_X \dim_k V$. Therefore

$$\dim_k \mathbb{S}_{E,R^{op},V}(X) = \frac{\dim_k V}{|\,\mathrm{Aut}(E,R)|} \dim_k \mathbb{S}_{E,R^{op}}(X) \;.$$

The result now follows from Theorem 6.6. $\square$

8. Simple modules for the algebra of relations

Let X be a fixed finite set and consider the monoid $\mathcal{R}_X = \mathcal{C}(X,X)$ of all relations on X, also known as the monoid of all Boolean matrices of size $|X|$. As before, write $k\mathcal{R}_X$ for the algebra of this monoid. Throughout this section, we assume that the base ring k is a field. We give the parametrization of all simple modules for the algebra $k\mathcal{R}_X$ and then solve the open problem of giving a formula for their dimension. We also give an explicit description for the action of relations on every simple $k\mathcal{R}_X$-module.

We mentioned in the introduction that, for the parametrization of all simple modules for the algebra $k\mathcal{R}_X$, we do not use the Munn-Ponizovsky theory using Green's $\mathcal{J}$-classes. We use instead the parametrization of all simple correspondence functors $S_{E,R,V}$ by isomorphism classes of triples (E,R,V) (see Theorem 4.5). The method is now very elementary since it simply uses the fact that every evaluation of a simple functor yields a simple module, or zero, and that every simple module arises in that way. This mechanism goes back to Green (6.2 in [Gr]) but we also give some more recent references.

8.1. Theorem. *Let X be a finite set.*

(a) *The set of isomorphism classes of simple $k\mathcal{R}_X$-modules is parametrized by the set of isomorphism classes of triples (E,R,V), where E is a finite set with $|E| \leq |X|$, R is an order relation on E, and V is a simple $k\,\mathrm{Aut}(E,R)$-module.*

(b) *The simple module parametrized by the triple (E,R,V) is $S_{E,R,V}(X)$, where $S_{E,R,V}$ is the simple correspondence functor corresponding to the triple (E,R,V).*

Proof : We first recall that the evaluation $S(X)$ of a simple correspondence functor S at a finite set X is either zero or a simple $k\mathcal{R}_X$-module. The proof is very easy and appears in Proposition 3.2 of [We], or also in Proposition 2.7 of [BT2].

Conversely, we claim that any simple $k\mathcal{R}_X$-module W occurs as the evaluation of some simple correspondence functor S, that is, $W \cong S(X)$. Explicitly, S is the functor $L_{X,W}/J_{X,W}$ considered in Lemma 4.3. The claim is Lemma 2.5 of [BT2] which quotes the first lemma of [Bo]. It also appears in Proposition 3.2 of [We], where it is attributed to Green (6.2 in [Gr]). This requires to view $k\mathcal{R}_X = k\mathcal{C}(X,X)$ as a category with a single object X, hence a full subcategory of $k\mathcal{C}$. Proposition 3.2 of [We] or Proposition 2.7 of [BT2] also show that the simple correspondence functor S such that $W \cong S(X)$ is unique up to isomorphism. All these facts actually hold for the simple representations of any small category.

By Theorem 4.5, our simple correspondence functor $S = S_{E,R,V}$ is parametrized by a triple (E,R,V), where E is a finite set, R is an order relation on E, and V is a simple $k\,\mathrm{Aut}(E,R)$-module. Whenever $W = S_{E,R,V}(X) \neq \{0\}$, we have $|E| \leq |X|$ because $S_{E,R,V}$ vanishes on sets Y with $|E| > |Y|$ (by Theorem 4.5). In order to obtain the parametrization of the statement, we also need to show that $W = S_{E,R,V}(X)$ is nonzero if $|E| \leq |X|$. This is clear if $|E| = |X|$ because $S_{E,R,V}(E) = T_{R,V}$ is nonzero (see the construction of $S_{E,R,V}$ in Section 4). Knowing that $S_{E,R,V}(E) \neq \{0\}$, Corollary 3.7 in [BT2] asserts precisely that $S_{E,R,V}(X) \neq \{0\}$ if $|E| < |X|$. This provides the required parametrization and completes the proof. □

Note that we used at the end of the proof the non-vanishing of $S_{E,R,V}(X)$ when $|E| < |X|$. This is a special property of correspondence functors (Corollary 3.7 in [BT2]) and it may not hold for representations of other small categories.

In view of Theorem 8.1, a formula for the dimension of any simple $k\mathcal{R}_X$-module is now given by Theorem 7.10. More explicitly, we fix a poset (E,R) and a finite lattice T having (E,R) as the full subset of its join-irreducible elements. We can also choose T such that $\mathrm{Aut}(T) \cong \mathrm{Aut}(E,R)$ by taking for instance $T = I_\downarrow(E,R)$. We consider the simple $k\mathcal{R}_X$-module $S_{E,R^{op},V}(X)$, continuing to use R^{op} as in Theorem 7.10. We define the subset G of T as in Notation 2.10 and we write $G = G_{E,R}$ to emphasize its dependence on (E,R). Its cardinality $|G|$ only depends on (E,R), by Corollary 6.7.

8.2. Theorem. *With the notation above, the dimension of a simple $k\mathcal{R}_X$-module is given by the formula*

$$\dim(S_{E,R^{op},V}(X)) = \frac{\dim_k V}{|\mathrm{Aut}(E,R)|} \sum_{i=0}^{|E|} (-1)^i \binom{|E|}{i} (|G_{E,R}| - i)^{|X|} \ .$$

Proof : This is a restatement of Theorem 7.10. ∎

An explicit description can be given for the action of relations on the simple $k\mathcal{R}_X$-module $S_{E,R^{op},V}(X)$. We define a subset

$$\mathcal{B}_{E,R,X} = \{\varphi \in T^X \mid E \subseteq \varphi(X) \subseteq G_{E,R}\} \subseteq T^X \ .$$

By Theorem 6.6, the surjective morphism $\Theta_T : F_T \to \mathbb{S}_{E,R^{op}}$ induces a k-module decomposition

$$F_T(X) = k\mathcal{B}_{E,R,X} \oplus \mathrm{Ker}(\Theta_{T,X}) \ ,$$

where $k\mathcal{B}_{E,R,X}$ is the k-subspace of $F_T(X)$ with basis $\mathcal{B}_{E,R,X}$. Thus we have a k-module isomorphism

$$\mathbb{S}_{E,R^{op}}(X) \cong k\mathcal{B}_{E,R,X} \ .$$

The family of subspaces $k\mathcal{B}_{E,R,X}$ do not form a subfunctor of F_T, but they can be used to describe the evaluations of the functors $\mathbb{S}_{E,R^{op}}$ and $S_{E,R^{op},V}$.

We explain a procedure for modifying an element $\varphi \in T^X$ modulo $\mathrm{Ker}(\Theta_{T,X})$ in order to project it in $k\mathcal{B}_{E,R,X}$. In Theorem 5.6, we introduced an element $u_T \in k(T^T)$ which has the property that, for any $\varphi \in T^X$, the composition $u_T \circ \varphi$ is a k-linear combination of maps $f \in T^X$ such that $f(X) \subseteq G_{E,R}$. (Actually, u_T is idempotent, by Theorem 5.8.) Moreover,

$$u_T \circ \varphi \equiv \varphi \pmod{\mathrm{Ker}(\Theta_{T,X})} \ .$$

Let $\pi_{T,X}$ be the k-linear idempotent endomorphism of $k(T^X)$ defined by

$$\forall \varphi \in T^X, \ \ \pi_{T,X}(\varphi) = \begin{cases} \varphi & \text{if } E \subseteq \varphi(X) \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

By Theorem 4.10,

$$\pi_{T,X}(\varphi) \equiv \varphi \pmod{\mathrm{Ker}(\Theta_{T,X})} \ .$$

Then, for any map $\varphi \in T^X$, we obtain

$$\pi_{T,X}(u_T \circ \varphi) \in k\mathcal{B}_{E,R,X} \ ,$$

that is, a k-linear combination of maps $f \in T^X$ such that $E \subseteq f(X) \subseteq G_{E,R}$. Moreover,

$$\pi_{T,X}(u_T \circ \varphi) \equiv \varphi \pmod{\mathrm{Ker}(\Theta_{T,X})} \ .$$

Thus if we lift arbitrarily a basis element of $\mathbb{S}_{E,R^{op}}(X)$ to $\varphi \in F_T(X)$, we can modify it modulo $\mathrm{Ker}(\Theta_{T,X})$ to obtain an element of $k\mathcal{B}_{E,R,X}$. Applying this procedure to the action of a relation $U \in \mathcal{C}(X,X)$ on an element $\varphi \in k\mathcal{B}_{E,R,X}$, we obtain

$$U\varphi \equiv \pi_{T,X}(u_T \circ U\varphi) \pmod{\mathrm{Ker}(\Theta_{T,X})} \ ,$$

and $\pi_{T,X}(u_T \circ U\varphi)$ belongs to $k\mathcal{B}_{E,R,X}$.

As in Section 7, we tensor on the right with the $k\,\mathrm{Aut}(E,R)$-module V, using the right action of $\mathrm{Aut}(E,R)$ on $\mathcal{B}_{E,R,X}$ defined by $\varphi\cdot\sigma := \sigma^{-1}\circ\varphi$ for all $\sigma \in \mathrm{Aut}(E,R)$. By Theorem 7.9, we have isomorphisms

$$S_{E,R^{op},V}(X) \cong \mathbb{S}_{E,R^{op}}(X) \otimes_{k\,\mathrm{Aut}(E,R)} V \cong k\mathcal{B}_{E,R,X} \otimes_{k\,\mathrm{Aut}(E,R)} V \ ,$$

the second isomorphism being only k-linear.

This analysis proves the following result, which provides a computational method for describing the action of a relation on a simple $k\mathcal{R}_X$-module.

8.3. Theorem. *Fix the notation above.*

(a) $S_{E,R^{op},V}(X) \cong k\mathcal{B}_{E,R,X} \otimes_{k\,\mathrm{Aut}(E,R)} V$ *as k-vector spaces.*

(b) *Transporting the action of relations via this isomorphism, the action of a relation $U \in \mathcal{C}(X,X)$ on an element*

$$\varphi \otimes v \in k\mathcal{B}_{E,R,X} \otimes_{k\,\mathrm{Aut}(E,R)} V\,, \qquad (\varphi \in \mathcal{B}_{E,R,X}\,,\ v \in V)$$

is given by

$$U \cdot (\varphi \otimes v) = \pi_{T,X}(u_T \circ U\varphi) \otimes v\,.$$

Our last result gives the dimension of the Jacobson radical $J(k\mathcal{R}_X)$ of the k-algebra $k\mathcal{R}_X$. We assume for simplicity that the field k has characteristic zero.

8.4. Theorem. *Assume that k is a field of characteristic zero. Let $J(k\mathcal{R}_X)$ be the Jacobson radical of the k-algebra $k\mathcal{R}_X$ and let $n = |X|$. Then*

$$\dim J(k\mathcal{R}_X) = 2^{n^2} - \sum_{e=0}^{n} \sum_{R} \frac{1}{|\mathrm{Aut}(E,R)|} \Big(\sum_{i=0}^{e} (-1)^i \binom{e}{i} (|G_{E,R}| - i)^n \Big)^2\,,$$

where R runs over a set of representatives of Σ_e-conjugacy classes of order relations on the set $E = \{1,\ldots,e\}$. The integer $|G_{E,R}|$ is the cardinality of the set $G_{E,R}$ defined in Notation 2.10. Note that if $e = 0$, then $E = \emptyset$, $R = \emptyset$ and $|G_{E,R}| = 1$ (by Example 2.11).

Proof : Since k has characteristic zero, the semi-simple algebra $k\mathcal{R}_X/J(k\mathcal{R}_X)$ is separable, that is, it remains semi-simple after scalar extension to an algebraic closure $\overline{k}$ of k. In other words, $\dim J(k\mathcal{R}_X)$ does not change after this scalar extension. Therefore, we can assume that $k = \overline{k}$.

By Theorem 8.1, every simple $k\mathcal{R}_X$-module has the form $S_{E,R,V}(X)$ with $|E| \leq |X|$, where $S_{E,R,V}$ is the simple correspondence functor parametrized by the triple (E,R,V). In order to have a parametrization, we take $E = \{1,\ldots,e\}$ with $0 \leq e \leq n$, we take R in a set of representatives as in the statement, and finally we take V in a set of representatives of isomorphism classes of simple $k\,\mathrm{Aut}(E,R)$-modules.

Since the endomorphism algebra of a simple module is isomorphic to k, by Schur's lemma and the assumption that k is algebraically closed, the dimension of the semi-simple algebra $k\mathcal{R}_X/J(k\mathcal{R}_X)$ is equal to the sum of the squares of the dimensions of all simple modules, by Wedderburn's theorem. From the formula for the dimension of simple modules, we obtain

$$\begin{aligned}
\dim\big(k\mathcal{R}_X/J(k\mathcal{R}_X)\big) &= \sum_{E,R,V} \big(\dim S_{E,R,V}(X)\big)^2 = \sum_{E,R,V} \big(\dim S_{E,R^{op},V}(X)\big)^2 \\
&= \sum_{E,R,V} \Big(\frac{\dim V}{|\mathrm{Aut}(E,R)|}\Big)^2 \Big(\sum_{i=0}^{|E|} (-1)^i \binom{|E|}{i} (|G_{E,R}| - i)^{|X|} \Big)^2 \\
&= \sum_{e=0}^{n} \sum_{R} \Big(\sum_{V} \frac{(\dim V)^2}{|\mathrm{Aut}(E,R)|^2} \Big) \Big(\sum_{i=0}^{e} (-1)^i \binom{e}{i} (|G_{E,R}| - i)^n \Big)^2 \\
&= \sum_{e=0}^{n} \sum_{R} \frac{1}{|\mathrm{Aut}(E,R)|} \Big(\sum_{i=0}^{e} (-1)^i \binom{e}{i} (|G_{E,R}| - i)^n \Big)^2\,,
\end{aligned}$$

because $\displaystyle\sum_{V} (\dim V)^2 = \dim(k\,\mathrm{Aut}(E,R)) = |\mathrm{Aut}(E,R)|$, by semi-simplicity of the group algebra in characteristic zero (Maschke's theorem). Now

$$\dim J(k\mathcal{R}_X) = \dim k\mathcal{R}_X - \dim\big(k\mathcal{R}_X/J(k\mathcal{R}_X)\big) = 2^{n^2} - \dim\big(k\mathcal{R}_X/J(k\mathcal{R}_X)\big)$$

and the result follows. $\square$

If k is an algebraically closed field of prime characteristic p, the formula has to be modified in a straightforward manner, in order to take into account the Jacobson radical of $k\,\mathrm{Aut}(E,R)$. Then it seems likely that the same formula holds over any field of characteristic p (that is, $k\mathcal{R}_X/J(k\mathcal{R}_X)$ is likely to be a separable algebra), but we leave this question open.

9. Examples

We state here without proofs a list of examples. For simplicity, we assume that the base ring k is a field (but many results actually remain true over an arbitrary commutative ring k). We first describe a few small cases for modules over the algebra $k\mathcal{R}_X$, using the notation of Section 8. Then we give the decomposition of the functors F_T associated to some particular lattices T.

9.1. Example. Let $X=\emptyset$. There is a single relation on $\emptyset$, namely $\emptyset$, and $k\mathcal{R}_\emptyset \cong k$. Then $S_{\emptyset,\emptyset,k}(\emptyset) \cong k\mathcal{B}_{\emptyset,\emptyset,\emptyset}\otimes_k k \cong k$ and the unique relation $\emptyset$ acts as the identity on k.

9.2. Example. Let $X=\{1\}$. There are 2 relations on $\{1\}$, namely $\emptyset$ and $\Delta_{\{1\}}$.

For $E=\emptyset$, we get $S_{\emptyset,\emptyset,k}(\{1\}) \cong k\mathcal{B}_{\emptyset,\emptyset,\{1\}}\otimes_k k \cong k$ and both relations act as the identity on k.

For $E=\{1\}$, we obtain $S_{\{1\},\Delta_{\{1\}},k}(\{1\}) \cong k\mathcal{B}_{\{1\},\Delta_{\{1\}},\{1\}}\otimes_k k\cong k$, the relation $\emptyset$ acts by zero, while $\Delta_{\{1\}}$ acts as the identity.

Moroever, $k\mathcal{R}_{\{1\}}\cong k\times k$ is a semi-simple algebra.

9.3. Example. Let $X=\{1,2\}$. There are $2^4=16$ relations on $\{1,2\}$, so $k\mathcal{R}_{\{1,2\}}$ has dimension 16.

For $E=\emptyset$, we get a simple $k\mathcal{R}_{\{1,2\}}$-module $S_{\emptyset,\emptyset,k}(\{1,2\})$ of dimension 1.

For $E=\{1\}$, we get a simple $k\mathcal{R}_{\{1,2\}}$-module $S_{\{1\},\Delta_{\{1\}},k}(\{1,2\})$ of dimension 3.

For $E=\{1,2\}$, there are two relations in $\mathcal{P}_E$ up to conjugacy, namely the equality relation $\Delta_{\{1,2\}}$ and the usual total order tot. Moreover, $\mathrm{Aut}(\{1,2\},\Delta_{\{1,2\}})$ is a group of order 2, with two simple modules k_+ and k_- (assuming that the characteristic of k is not 2). Therefore, we obtain two simple $k\mathcal{R}_{\{1,2\}}$-modules of dimension 1

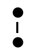
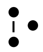
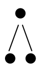
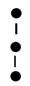
$$\begin{aligned} S_{\{1,2\},\Delta_{\{1,2\}},k_+}(\{1,2\}) &\cong k\mathcal{B}_{\{1,2\},\Delta_{\{1,2\}},\{1,2\}}\otimes_{kC_2} k_+ \;, \\ S_{\{1,2\},\Delta_{\{1,2\}},k_-}(\{1,2\}) &\cong k\mathcal{B}_{\{1,2\},\Delta_{\{1,2\}},\{1,2\}}\otimes_{kC_2} k_- \;, \end{aligned}$$

For the other relation tot, we obtain a simple $k\mathcal{R}_{\{1,2\}}$-module of dimension 2

$$S_{\{1,2\},\mathrm{tot},k}(\{1,2\}) \cong k\mathcal{B}_{\{1,2\},\mathrm{tot},\{1,2\}}\otimes_k k \cong k\mathcal{B}_{\{1,2\},\mathrm{tot},\{1,2\}} \;,$$

Altogether, there are 5 simple $k\mathcal{R}_{\{1,2\}}$-modules and the Jacobson radical has dimension 0. Therefore $k\mathcal{R}_{\{1,2\}}$ is semi-simple (provided the characteristic of k is not 2).

9.4. Example. For $|X|=3$, the algebra $k\mathcal{R}_X$ is not semi-simple. The dimension of the Jacobson radical of $k\mathcal{R}_X$ is equal to 42, using either the computer software [GAP4] or the computer calculations obtained in [Br]. According to Theorem 8.4, this value can be recovered directly as follows :

Size e	Poset(E,R)	$\lvert \mathrm{Aut}(E,R)\rvert$	$\lvert G_{E,R}\rvert$	$\sum\limits_{i=0}^{e}(-1)^i\binom{e}{i}(\lvert G_{E,R}\rvert-i)^3$	total
0	$\emptyset$	1	1	1	1
1	•	1	2	7	49
2	••	2	4	18	162
		1	3	12	144
3	• • •	6	5	6	6
		1	5	6	36
		2	5	6	18
		2	5	6	18
		1	6	6	36

In this case, the algebra $k\mathcal{R}_X$ has dimension $2^{3^2} = 512$. The sum of the last column of this table is equal to 470, so we recover the dimension of the radical $42 = 512 - 470$.

9.5. Example. For $|X| = 4$, the algebra $k\mathcal{R}_X$ has dimension $2^{4^2} = 65{,}536$. The direct computation of the radical of such a big algebra seems out of reach of usual computers. However, using the formula of Theorem 8.4 and the structure of the 16 posets of cardinality 4, one can show by hand that the radical of $k\mathcal{R}_X$ has dimension 32,616.

For larger values of $n = |X|$, a computer calculation using Theorem 8.4 yields the following values for the dimension of $J(k\mathcal{R}_X)$:

$n = 5$	$n = 6$	$n = 7$	$n = 8$
29,446,050	67,860,904,320	562,649,705,679,642	18,446,568,932,288,588,616

We now move to examples of fundamental functors and functors associated to lattices.

9.6. Example. There are many examples of fundamental functors $\mathbb{S}_{E,R}$ for which the set G is the whole of T, for instance when $T = \Lambda E$. In all such cases, we have $F_T/H_T \cong \mathbb{S}_{E,R}$. Moreover, in many such cases, $\mathrm{Aut}(E,R)$ is the trivial group. Take for instance (E,R) to be a disjoint union of trees with branches of different length. In any such case, $F_T/H_T \cong \mathbb{S}_{E,R} \cong S_{E,R,k}$ is simple, provided k is a field.

Our next purpose is to decompose the functor F_T for some small lattices T. In order to use an inductive process, we use inclusions $A \to T$ where A is a distributive sublattice. We could as well use surjective morphisms $T \to A$, as in Section 10 of [BT3], but the following general result shows that it does not matter.

9.7. Lemma. *Let T and A be finite lattices and assume that A is distributive. Let $\sigma : A \to T$ be an injective join-preserving map. Then there is a surjective join-preserving map $\pi : T \to A$ such that $\pi\sigma = \mathrm{id}_A$.*

Proof : We define $\pi(t) = \bigwedge\limits_{\substack{a\in A\\ \sigma(a)\geq t}} a$. Then π is order-preserving and therefore $\pi(t_1)\vee\pi(t_2)\leq\pi(t_1\vee t_2)$ for any $t_1,t_2\in T$. Now we have

$$\pi(t_1)\vee\pi(t_2) = \Big(\bigwedge_{\substack{a_1\in A\\ \sigma(a_1)\geq t_1}} a_1\Big)\vee\Big(\bigwedge_{\substack{a_2\in A\\ \sigma(a_2)\geq t_2}} a_2\Big) = \bigwedge_{\substack{a_1,a_2\in A\\ \sigma(a_1)\geq t_1\\ \sigma(a_2)\geq t_2}} (a_1\vee a_2)$$

by distributivity of A. For any such pair (a_1,a_2), the join $a_1\vee a_2$ belongs to the set $\{a\in A\mid \sigma(a)\geq t_1\vee t_2\}$ and therefore

$$\bigwedge_{\substack{a_1,a_2\in A\\ \sigma(a_1)\geq t_1\\ \sigma(a_2)\geq t_2}} (a_1\vee a_2) \;\geq\; \bigwedge_{\substack{a\in A\\ \sigma(a)\geq t_1\vee t_2}} a \;=\; \pi(t_1\vee t_2)\ .$$

The equality $\pi(t_1)\vee\pi(t_2)=\pi(t_1\vee t_2)$ follows.

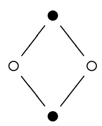
If $\sigma(a_1)\leq\sigma(a_2)$, then $\sigma(a_1\vee a_2)=\sigma(a_1)\vee\sigma(a_2)=\sigma(a_2)$, and therefore $a_1\vee a_2 = a_2$ by injectivity of σ, i.e. $a_1\leq a_2$. It follows from this observation that, for any $b\in A$,

$$\pi\sigma(b) = \bigwedge_{\substack{a\in A\\ \sigma(a)\geq\sigma(b)}} a \;=\; \bigwedge_{\substack{a\in A\\ a\geq b}} a = \; b\ ,$$

hence $\pi\sigma = \mathrm{id}_A$. ∎

The property of Lemma 9.7 is reflected in the fact that the morphism $F_A\to F_T$ induced by σ must split, because the functor F_A is projective by Theorem 4.12 in [BT3] and injective by Theorem 10.6 in [BT2].

9.8. Example. Let $T=\Diamond$ be the lozenge, in other words the lattice of subsets of a set of cardinality 2 :

$\Diamond =$ 

By Theorem 11.12 in [BT3], for any finite lattice T, we can split off from F_T simple functors $\mathbb{S}_n := \mathbb{S}_{\underline{n},\mathrm{tot}} \cong S_{\underline{n},\mathrm{tot},k}$ corresponding to all totally ordered sequences $\hat{0}\leq d_0<d_1<\ldots<d_n=\hat{1}$ in T. In the case of $F_\Diamond$, we obtain

$$F_\Diamond \cong \mathbb{S}_0\oplus 3\mathbb{S}_1\oplus 2\mathbb{S}_2\oplus L$$

for some subfunctor L. We know that $F_\Diamond$ maps surjectively onto the fundamental functor $\mathbb{S}_{\circ\circ}$ associated to the (opposite) poset of irreducible elements of $\Diamond$, that is, a set of cardinality 2 ordered by the equality relation. Moreover, all the factors $\mathbb{S}_n$ lie in the subfunctor $H_\Diamond$, because no totally ordered subset contains the two irreducible elements of $\Diamond$ (figured with an empty circle in the above picture). Therefore L maps surjectively onto $\mathbb{S}_{\circ\circ}$.

We can evaluate $F_\Diamond$ at a set X of cardinality x, and take dimensions over k. By Theorem 6.6, we obtain

$$4^x = 1^x + 3(2^x-1^x) + 2(3^x-2\cdot 2^x+1^x) + \dim_k L(X)\ .$$

It follows that

$$\dim_k L(X) = 4^x - 2\cdot 3^x + 2^x\ .$$

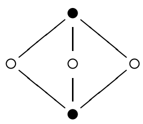
Now we apply Theorem 6.6 to the fundamental functor $\mathbb{S}_{\circ\circ}$. The set G is the whole of T in this case, so $\dim_k\mathbb{S}_{\circ\circ}(X) = 4^x-2\cdot 3^x+2^x$.

Since L maps surjectively onto $\mathbb{S}_{\circ\circ}$ and $\dim_k L(X) = \dim_k \mathbb{S}_{\circ\circ}(X)$ for any finite set X, this surjection is an isomorphism. Hence

$$F_\Diamond \cong \mathbb{S}_0 \oplus 3\mathbb{S}_1 \oplus 2\mathbb{S}_2 \oplus \mathbb{S}_{\circ\circ}\ .$$

Since the lattice $\Diamond$ is distributive, $F_\Diamond$ is projective by Theorem 4.12 in [BT3]. It follows that each summand is a projective object in the category $\mathcal{F}_k$ of correspondence functors.

9.9. Example. Let T be the following lattice :

$T =$ 

As in the previous example, F_T admits a direct summand isomorphic to

$$\mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 3\mathbb{S}_2\ .$$

Moreover, there are three obvious sublattices of T isomorphic to $\Diamond$, which provide three direct summands of F_T isomorphic to $\mathbb{S}_{\circ\circ}$. Thus we have a decomposition

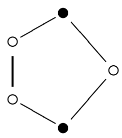
$$F_T \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 3\mathbb{S}_2 \oplus 3\mathbb{S}_{\circ\circ} \oplus M$$

for some subfunctor M of F_T. Using arguments similar to those of the previous example, we get

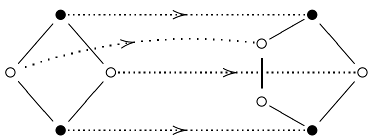
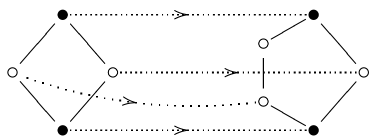
$$F_T \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 3\mathbb{S}_2 \oplus 3\mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\circ\circ\circ}\ .$$

All the summands in this decomposition of F_T, except possibly $\mathbb{S}_{\circ\circ\circ}$, are projective functors. Since the lattice T is not distributive, the functor F_T is not projective (Theorem 4.12 in [BT3]), thus $\mathbb{S}_{\circ\circ\circ}$ is actually not projective either.

9.10. Example. Let D be the following lattice :

$D =$ 

As before, we know that F_T admits a direct summand isomorphic to a direct sum $\mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 4\mathbb{S}_2 \oplus \mathbb{S}_3$. Moreover, there are two inclusions

 and 

of the lattice $\Diamond$ into D, which yield two direct summands of F_D isomorphic to $\mathbb{S}_{\circ\circ}$. So there is a decomposition

$$F_D \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 4\mathbb{S}_2 \oplus \mathbb{S}_3 \oplus 2\mathbb{S}_{\circ\circ} \oplus N$$

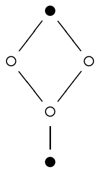
for a suitable subfunctor N of F_D. As in the previous examples, the subfunctor N maps surjectively onto the fundamental functor $\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}$ associated to the (opposite) poset $\substack{\circ\\|\\\circ}\,\circ$ of irreducible elements of D. Computing dimensions, we obtain $N \cong \mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}$, and therefore

$$F_D \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 4\mathbb{S}_2 \oplus \mathbb{S}_3 \oplus 2\mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}\ .$$

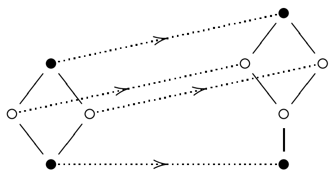
Again D is not distributive, so that F_D is not projective. Thus the functor $\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}$ is not projective either.

Actually, the lattice D and the lattice T of the previous example are the smallest non-distributive lattices and they are used for the well-known characterization of distributive lattices (see Theorem 4.7 in [Ro]).

9.11. Example. Let C be the following lattice :

$C =$ 

Again, we know that F_C admits a direct summand isomorphic to a direct sum $\mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3$. Moreover, the inclusion



of $\Diamond$ in C yields a direct summand of F_C isomorphic to $\mathbb{S}_{\circ\circ}$. So there is a decomposition

$$F_C \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3 \oplus \mathbb{S}_{\circ\circ} \oplus Q$$

for some direct summand Q of F_C.

Now F_C maps surjectively onto the fundamental functor $\mathbb{S}_{\substack{\circ\\ /\backslash\\ \circ\;\circ}}$ associated to the opposite poset of its irreducible elements, and arguments as before yield an isomorphism $Q \cong \mathbb{S}_{\substack{\circ\\ /\backslash\\ \circ\;\circ}}$, hence a decomposition

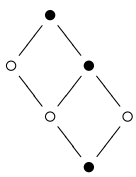
$$F_C \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3 \oplus \mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\substack{\circ\\ /\backslash\\ \circ\;\circ}} \ .$$

Since C is distributive, F_C is projective and we conclude that $\mathbb{S}_{\substack{\circ\\ /\backslash\\ \circ\;\circ}}$ is projective. Taking dual functors corresponds to taking opposite lattices (see Theorem 8.9 and Remark 9.7 in [BT3]), so we get a decomposition

$$F_{C^{op}} \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3 \oplus \mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\substack{\circ\;\circ\\ \backslash/\\ \circ}} \ .$$

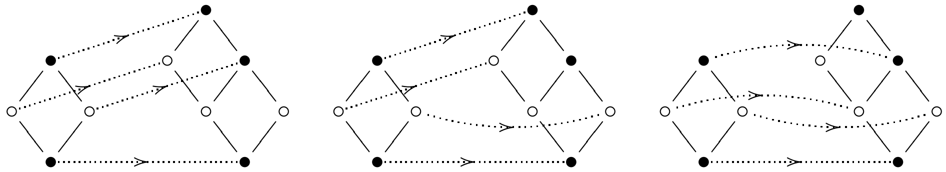
Therefore $\mathbb{S}_{\substack{\circ\;\circ\\ \backslash/\\ \circ}}$ is also projective.

9.12. Example. Let P be the following lattice :

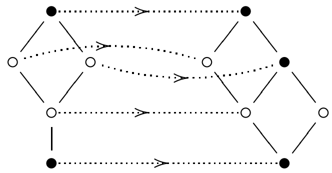
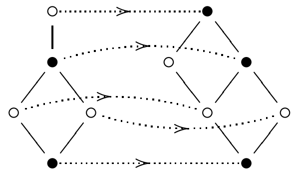
$P =$ 

that is, the direct product of a totally ordered lattice of cardinality 3 and a totally ordered lattice of cardinality 2.

We know that F_P admits a direct summand isomorphic to $\mathbb{S}_0 \oplus 5\mathbb{S}_1 \oplus 7\mathbb{S}_2 \oplus 3\mathbb{S}_3$ and the inclusions



of $\Diamond$ in P yield 3 direct summands of F_P isomorphic to $\mathbb{S}_{\circ\circ}$. Moreover, the inclusions

 and 

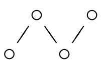
of C and C^{op} in P yield direct summands $\mathbb{S}_{\substack{\circ\;\circ\\ \backslash/\\ \circ}}$ and $\mathbb{S}_{\substack{\circ\\ /\backslash\\ \circ\;\circ}}$ of F_P, hence there is a direct summand U of F_P such that

$$F_P \cong \mathbb{S}_0 \oplus 5\mathbb{S}_1 \oplus 7\mathbb{S}_2 \oplus 3\mathbb{S}_3 \oplus 3\mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\substack{\circ\;\circ\\ \backslash/\\ \circ}} \oplus \mathbb{S}_{\substack{\circ\\ /\backslash\\ \circ\;\circ}} \oplus U \ .$$

Since the lattice P is distributive, the functor F_P is projective, hence U is projective. Now F_P maps surjectively onto the fundamental functor $\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}$, and H_P is contained in the kernel of this surjection. It follows that U maps surjectively onto $\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}$, which is a simple functor, as k is a field and the poset $\substack{\circ\\|\\\circ}\,\circ$ has no nontrivial automorphisms.

A more involved analysis shows that U is indecomposable and is a projective cover of the simple functor $\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}$. Moreover, one can show that the functor U is uniserial, with a filtration

$$0 \underbrace{\quad\subset\quad}_{\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}} W \underbrace{\quad\subset\quad}_{\mathbb{S}_{\substack{\circ\;\circ\\ /\backslash/\\ \circ\;\circ}}} V \underbrace{\quad\subset\quad}_{\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}} U \ ,$$

where $W \cong U/V \cong \mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}$, and V/W is isomorphic to the simple functor $\mathbb{S}_{\substack{\circ\;\circ\\ /\backslash/\\ \circ\;\circ}}$ associated to the poset  of cardinality 4. An easy consequence of this is that

$$\mathrm{Ext}^1_{\mathcal{F}_k}(\mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}, \mathbb{S}_{\substack{\circ\;\circ\\ /\backslash/\\ \circ\;\circ}}) \cong \mathrm{Ext}^1_{\mathcal{F}_k}(\mathbb{S}_{\substack{\circ\;\circ\\ /\backslash/\\ \circ\;\circ}}, \mathbb{S}_{\substack{\circ\\|\\\circ}\,\circ}) \cong k \ .$$

References

[Bo] S. Bouc. Foncteurs d'ensembles munis d'une double action, *J. Algebra* 183 (1996), 664–736.

[BT1] S. Bouc, J. Thévenaz. The algebra of essential relations on a finite set, *J. reine angew. Math.* 712 (2016), 225–250.

[BT2] S. Bouc, J. Thévenaz. Correspondence functors and finiteness conditions, *J. Algebra* 495 (2018), 150–198.

[BT3] S. Bouc, J. Thévenaz. Correspondence functors and lattices, *J. Algebra* 518 (2019), 453–518.

[BBHM] R.L. Brandon, K.H. Butler, D.W. Hardy, G. Markowsky. Cardinalities of $\mathcal{D}$-classes in B_n, *Semigroup Forum* 4 (1972), 341–344.

[BE] M.R. Bremner, M. El Bachraoui. On the semigroup algebra of binary relations, *Comm. Algebra* 38 (2010), 3499–3505.

[Br] M.R. Bremner. Structure of the rational monoid algebra for Boolean matrices of order 3, *Linear Alg. and its Applications* 449 (2014), 381–401.
[CP] A.H. Clifford, G.B. Preston. *The algebraic theory of semigroups*, Vol. I, Mathematical Surveys, No. 7, American Mathematical Society, Providence,1961.
[Di] Y.I. Diasamidze. Complete semigroups of binary relations, *J. Math. Sci. (N. Y.)* 117 (2003), no. 4, 4271–4319.
[GAP4] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.8*, 2015. `(http://www.gap-system.org)`.
[GMS] O. Ganyushkin, V. Mazorchuk, B. Steinberg. On the irreducible representations of a finite semigroup, *Proc. Amer. Math. Soc.*, 137 (2009), 3585–3592.
[Gr] J.A. Green. Polynomial representations of GL_n, Springer Lecture Notes in Mathematics no. 830 (1980).
[Ki] K.H. Kim. *Boolean Matrix Theory and Applications*, Dekker, 1982.
[KR] K.H. Kim, F. Roush. Linear Representations of Semigroups of Boolean Matrices, *Proc. Amer. Math. Soc.* 63 (1977), 203–207.
[PW] R.J. Plemmons, M.T. West. On the semigroup of binary relations, *Pacific J. Math.* 35 (1970), 743–753.
[Ro] S. Roman. *Lattices and ordered sets*, Springer, 2008.
[Sc1] S. Schwarz. On the semigroup of binary relations on a finite set, *Czechoslovak Math. J.* 20(95) (1970), 632–679.
[Sc2] S. Schwarz. The semigroup of fully indecomposable relations and Hall relations, *Czechoslovak Math. J.* 23(98) (1973), 151–163.
[Sta] R.P. Stanley. *Enumerative Combinatorics, Vol. I*, Second edition, Cambridge studies in advanced mathematics 49, Cambridge University Press, 2012.
[Ste] I. Stein. Algebras of Ehresmann semigroups and categories, *Semigroup Forum* 95 (2017), 509–526.
[St1] B. Steinberg. Möbius functions and semigroup representation theory, *J. Combin. Theory (Series A)* 113 (2006), 866–881.
[St2] B. Steinberg. *Representation Theory of Finite Monoids*, Springer, 2016.
[We] P. Webb. An introduction to the representations and cohomology of categories, in: M. Geck, D. Testerman, J. Thévenaz (Eds.), "Group Representation Theory", EPFL Press, Lausanne, 2007, pp. 149–173.

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