HUSIMI, WIGNER, TÖPLITZ, QUANTUM STATISTICS AND ANTICANONICAL TRANSFORMATIONS

Thierry Paul

HUSIMI, WIGNER, TÖPLITZ, QUANTUM STATISTICS AND ANTICANONICAL TRANSFORMATIONS

THIERRY PAUL

ABSTRACT. We study the behaviour of Husimi, Wigner and Töplitz symbols of quantum density matrices when quantum statistics are tested on them, that is when on exchange two coordinates in one of the two variables of their integral kernel. We show that to each of these actions is associated a canonical transform on the cotangent bundle of the underlying classical phase space. Equivalently can one associate a complex canonical transform on the complexification of the phase-space. In the off-diagonal Töplitz representation introduced in [P], the action considered is associated to a complex aanticanonical relation.

1. INTRODUCTION

Quantum statistics is a fundamental hypothesis in quantum mechanics. In insures in particular the stability of matter. At the contrary of many other aspects of non-relativistic quantum mechanics which have a natural "'classical" counterpart, it seems difficult to associate to statistics properties of quantum object a classical corresponding symmetry. Changing the sign after permutation of coordinates of different particle doesn't appeal any classical simple action. Moreover most of the quantities which "passes" at the limit of vanishing Planck constant are quadratic and therefore looks at insensible to the change of sign. Finally, typical fermionic expressions such as exchange term in the Hartree-Fock theory vanishes numerically at the limit $\hbar \to 0$.

In this little note, we will implement this "exchange" action on three (in fact four) different symbols associated to quantum density matrices: the Husimi function (average of the density matrix on coherent states, therefore a probability density), Wigner functions (tat is the Weyl suitably renormalized by a power of the Planck constant in order to be of integral 1 (but non positive) and the Töplitz symbol appearing in the so-called positive quantization procedure.

In these three symbolic situation, the result is that associated to the exchange action appears as the action of a complex or equivalently on a doubled space canonical transformation:

(1) for the Husimi symbol (after a weighting by a Gaussian weight), a direct action on the variables corresponding to a complex canonical transformation: the transform $\bar{z}_i \leftrightarrow \bar{z}_j$ z_i, z_j remaining unchanged. the complex canonical transform is of the form $\begin{pmatrix} 0 & i \\ i & o \end{pmatrix}$.
(2) idem for the Töplitz symbol, with a different Gaussian weight
(3) for thw Wigner symbol (renormalized Weyl symbol), the above-mentioned complex transform is seen as a canonical transformtion on the cotangent bundle of the phase space. This transformation is the composition of permutation of variables and a "Fourier rotation" $q_i \to p_i,\ p_i \to -q_i$ and the exchange acts on the Wigner function by the metaplectic (in a doubled dimension space) representation, namely exchange of coordinates plus Fourier transform. In particular it doesn't act by a metapletic type representation of the complex linear symplectic group.

   To get such a feature, one has to go the off-diagonal Töplitz caluclus introduced in [P] and is this time associated to a an anticanonical transformation, that is a transformation which maps the sympletic form to its opposite.
(4) the off-diagonal symbol is mapped by the action of the metaplectic representation of the anticanonical linear

transformation $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. See Sections 7 and mostly 8 for details.

The conclusion to which all this (sometimes only formal) computations lead is the fact that, at a "classical" level, quantum statistics involve transformation which don't preserve the usual symplectic cotangent bundle of the configuration space: either one has to pass in a non trivial way to the cotangent bundle of the cotangent bundle itself, either one has to non preserve the sympletic structure, and allow anticanonical transformations.

The underlying classical picture of bosons and fermions either lives on the cotangent space of the classical phase space, or involves antisymplectic symmetries.

2. Quantum statistics

On the setting of indistinguishable quantum particles, a state is a density matrix, i.e. a positive trace one operator on $\mathcal{H}^{\otimes N}$, invariant by permutations of the factors in the tensorial product. we have denoted $\mathcal{H} = L^2(\mathbf{R}^d)$.

Definition 2.1. *Let ρ be a density matrix given by an integral kernel $\rho(X;Y),\ X=(x_1,\dots,x_n),\ Y=(y_1,\dots,y_n)$. We define, for $i,j=1,\dots,N$, the mappings*
$U_{i\leftrightarrow j}:\ \rho(X;Y)\to U_{i\to j}\rho(X;Y)=\rho(X;Y)|_{y_i\leftrightarrow y_j}$
and
$V_{i\leftrightarrow j}:\ \rho(X;Y)\to V_{i\to j}\rho(X;Y)=\rho(X;Y)|_{x_i\leftrightarrow x_j}$.

In terms of density matrices, quantum statistics will be seen as looking at density matrices which are eigenvectors of eigenvalue 1 or -1 of the two mappings $U_{i\leftrightarrow j}, V_{i\leftrightarrow j}$.

The indistinguishability property of the quantum system reads as

$$U_{i\leftrightarrow j}V_{i\leftrightarrow j}=V_{i\leftrightarrow j}U_{i\leftrightarrow j},\quad \forall i,j=1,\dots,N. \tag{1}$$

3. Husimi

Let us recall that the Husimi function of a density matrix ρ is defined as

$$\widetilde{W}[\rho](Z,\bar{Z})=\frac{1}{(2\pi\hbar)^{dN}}\langle\varphi_Z|\rho|\varphi_Z\rangle, \tag{2}$$

where, for $Z=q+ip\in\mathbf{Z}^{dN}$ and $x\in\mathbf{R}^{dN}$,

$$\varphi_Z(x)=\frac{1}{(\pi\hbar)^{\frac{dN}{4}}}e^{-\frac{(x-q)^2}{2\hbar}}e^{i\frac{p.x}{\hbar}}. \tag{3}$$

The most elementary properties of the Husimi transform are

$$\widetilde{W}[\rho]\geq 0 \text{ and } \int_{\mathbf{Z}^{dN}}\widetilde{W}[\rho](Z)dZ=\operatorname{trace}\rho=1. \tag{4}$$

Our first link between quantum statistics and the classical underlying space is the contents of the following result.

Lemma 3.1. *Let us consider the Husimi function of* ρ, $\widetilde{W}[\rho](Z,\bar{Z})$ *expressed on the complex variables* $Z=(z_1,\dots,z_n)$, $z_l=\ q_l+ip_l,\ \bar{z}_l=q_l-ip_l$.

Then

$$\widetilde{W}[U_{i\leftrightarrow j}\rho](Z,\bar{Z})=e^{\frac{(\bar{z}_i-\bar{z}_j)(z_i-z_j)}{2\hbar}}\widetilde{W}[\rho](Z,\bar{Z})|_{z_i\leftrightarrow z_j}$$

$$\widetilde{W}[V_{i\leftrightarrow j}\rho](Z,\bar{Z})=e^{\frac{|z_i-z_j|^2}{2\hbar}}\widetilde{W}[\rho](Z,\bar{Z})|_{\bar{z}_i\leftrightarrow\bar{z}_j}$$

Note that, as expected,

$$\widetilde{W}[V_{i\leftrightarrow j}U_{i\leftrightarrow j}\rho](Z,\bar{Z})=\widetilde{W}[\rho](z,\bar{z})|_{z_i\leftrightarrow z_j,\ \bar{z}_i\leftrightarrow\bar{z}_j}$$

Note also that

$$(5)\qquad \begin{pmatrix} z_i \\ z_j \\ \bar{z}_i \\ \bar{z}_j \end{pmatrix} \to \begin{pmatrix} z_j \\ z_i \\ \bar{z}_i \\ \bar{z}_j \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} q_+ \\ q_- \\ p_+ \\ p_- \end{pmatrix} \to \begin{pmatrix} -ip_+ \\ iq_+ \\ q_+ \\ p_+ \end{pmatrix}$$

so the complex metaplectic transform associatedis

$$(6)\qquad S_H^c = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \det S_H^c = 1.$$

Corollary 3.2. *A density matrix ρ is bosonic if and only if, for all $i, j = 1, \dots, n$,*

$$\begin{aligned} \widetilde{W}[\rho](Z, \bar{Z})| &= e^{\frac{(\bar{z}_i - \bar{z}_j)(z_i - z_j)}{2\hbar}} \widetilde{W}[\rho](Z, \bar{Z})|_{z_i \leftrightarrow z_j} \\ &= e^{\frac{(\bar{z}_i - \bar{z}_j)(z_i - z_j)}{2\hbar}} \widetilde{W}[\rho](Z, \bar{Z})|_{\bar{z}_i \leftrightarrow \bar{z}_j}. \end{aligned}$$

Corollary 3.3. *Let $n = 2$. A density matrix ρ is bosonic if and only if*

$$\widetilde{W}[\rho](Z, \bar{Z})| \ = \ e^{\frac{(\bar{z}_1 - \bar{z}_2)(z_1 - z_2)}{4\hbar}} H(z_1 - z_2, \bar{z}_1 - \bar{z}_2, z_1 + z_2, \bar{z}_1 + \bar{z}_2)$$

with H even (separately) in the two first variables.

4. Wigner

The Wigner function of a density matrix is nothing but its Weyl symbol, divided by $(2\pi\hbar)^{dN}$. More precisely the Wigner function of ρ is defined as

$$(7)\qquad W[\rho](X, \Xi) = \int_{\mathbf{R}^{2dN}} \rho(X + \hbar\frac{\delta}{2}, X - \hbar\frac{\delta}{2}) e^{i\frac{X.\Xi}{\hbar}} d\delta$$

At the contrary of the Husimi function, $W[\rho]$ is not positive, but its main elementary properties are

$$(8)\qquad \int_{\mathbf{R}^{2dN}} W[\rho](X, \Xi) dX d\xi = \operatorname{trace} \rho \ = \ 1$$

and

$$\frac{1}{(2\pi\hbar)^{dN}} \int_{\mathbf{R}^{2dN}} W[\rho](X, \Xi 0 W[\rho'](X, \Xi) dX d\Xi \ = \ \operatorname{trace}(\rho\rho').$$

Let us now define the semiclassical symplectic Fourier transform as

$$f(\widehat{q,p}^{\hbar}) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbf{R}^d\times\mathbf{R}^d} f(x,\xi)e^{i\frac{q\xi-px}{\hbar}}dxd\xi.$$

Note that, at the difference of the usual Fourier transform:

$$f(\widehat{\widehat{x,\xi}^{\hbar}}^{\hbar}) = f(x,\xi)$$

Let $a_\mp = \frac{a_i\mp a_j}{\sqrt{2}}$ for $a = q,p,y,\xi$. And let omit the dependence in the variable $q_1,\dots,q_{i-1},q_{i+1},\dots,q_{j-1},q_{j+1},\dots,q_N$ and the same for p.

We denote

$$W^{\frac{\pi}{2}}[\rho](x_+,\xi_+;x_-,\xi_-) = W[\rho](x_i,x_j;\xi_i,\xi_j).$$

Lemma 4.1.

$$W^{\frac{\pi}{2}}[U_{i\leftrightarrow j}\rho](q_+,p_+;p_-,q_-) = W^{\frac{\pi}{2}}[\rho](q_+,p_+;\widehat{q_-,p_-}^{\hbar})$$

$$W^{\frac{\pi}{2}}[V_{i\leftrightarrow j}\rho](q_+,p_+;p_-,q_-) = W^{\frac{\pi}{2}}[\rho](q_+,p_+;\widehat{-q_-,-p_-}^{\hbar})$$

Note that

$$W[V_{i\leftrightarrow j}U_{i\leftrightarrow j}\rho](q_1,p_1,\dots,q_i,p_i,\dots,q_j,p_j,\dots,q_n,p_n) =$$
$$W[\rho](q_1,p_1,\dots,q_{i-1},p_{i-1},q_j,p_j,\dots,q_{j-1},p_{j-1},q_i,p_i,\dots,q_n,p_n)$$

Proof. It is enough to isolate the ij block.

$U_{i\leftrightarrow j}\rho(x_i,x_j;y_1,y_j) = \rho(x_i,x_j : y_j,y_i)$. So

$$\begin{aligned}
&(2\pi\hbar)^{2d}W[U_{i\leftrightarrow j}\rho](q_i,q_j;p_i,p_j)\\
=& \int d\delta_i d\delta_j \rho(q_i+\delta_i,q_j+\delta_j;q_j-\delta_j,q_i-\delta_i)e^{-2ip\cdot\delta/\hbar}\\
=& \int d\delta d\eta W[\rho]\\
&((q_i+q_j+\delta_i-\delta_j)/2,(q_j+q_i+\delta_j-\delta_i)/2;\eta)e^{\frac{i}{\hbar}(q_i-q_j+\delta_i+\delta_j)\eta_i+\eta_j(q_j-q_i+\delta_i+\delta_j)}\\
&e^{-2ip\cdot\delta/\hbar}\ (=e^{-2ip_1(\delta/\hbar}) \\
=& \int d\delta d\eta W[\rho](q_i+q_j+\delta,q_i+q_j-\delta,\eta)\delta(\eta_i+\eta_j-(p_i+p_j))e^{-2i(p_i-p_j)\delta/\hbar}\\
=& \int dyd\eta\delta(\eta_i+\eta_j-(p_i+p_j))\delta(y_i+y_j-(q_i+q_j))\\
&e^{i((q_i-q_j)(\eta_i-\eta_j)-(p_i-p_j)(y_i-y_j))/\hbar}W[\rho](y;\eta)
\end{aligned}$$

Let us perform the change of variable $a_\mp = \frac{a_i \mp a_j}{\sqrt{2}}$ for $a = q, p, y, \eta$. This correspond to the metaplectic mapping:

$$R(\frac{\pi}{2}) = \begin{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} & 0 & 0 & 0 \\ 0 & \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} & 0 & 0 \\ 0 & 0 & \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix} \text{ on } \begin{pmatrix} q_i \\ q_j \\ \xi_i \\ \xi_j \\ p_i \\ p_j \\ x_i \\ x_j \end{pmatrix}$$

on $T^*(T^*\mathbf{R}^d, dq \wedge d\xi + dp \wedge dx)$.

Note that both

$$dq \wedge dp = dq_+ \wedge dp_+ + dq_- \wedge dp_- = d\tilde{q} \wedge d\tilde{p}$$

and

$$dq \wedge d\xi + dp \wedge dx = d\tilde{q} \wedge d\tilde{\xi} + d\tilde{p} \wedge d\tilde{x}$$

where $\tilde{a} = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$.

We denote $W^{\frac{\pi}{2}}[\rho](y_+, \eta_+; \eta_-, y_-)$ and $W^{\frac{\pi}{2}}[U_{i\leftrightarrow j}\rho](q_+, p_+; p_-, q_-)$. We get

$$W^{\frac{\pi}{2}}[U_{i\leftrightarrow j}\rho](q_+, p_+; p_-, q_-) = W^{\frac{\pi}{2}}[\rho](q_+, p_+; \widehat{q_-, p_-}^{\hbar})$$

□

Let us call now W^- the Wigner function (done with the symplectic Fourier transform) on the two variables q_-, p_-[1].

One has

$$W^-\left[W^{\frac{\pi}{2}}[U_{i\leftrightarrow j}\rho]\right](q_+, p_+|p_-, q_-; x_-, \xi_-) =$$

$$W^-\left[W^{\frac{\pi}{2}}[\rho]\right](q_+, p_+| -\xi_-, -x_-; q_-, p_-)$$

[1] namely, $W^-\left[W^{\frac{\pi}{2}}[\rho]\right](q_+, p_+|p_-, q_-; x_-, \xi_-) =$

$\int \overline{W^{\frac{\pi}{2}}[\rho]](q_+, p_+, p_- + 2\delta\hbar, q_- + 2\delta'\hbar)} W^{\frac{\pi}{2}}[\rho]](q_+, p_+, p_- - 2\delta\hbar, q_- - 2\delta'\hbar) e^{i(x_-\delta - \xi_-\delta')} d\delta d\delta'$.

That is, the action of $U_{i\leftrightarrow j}$ on ρ is seen on $W^-\big[W^{\frac{\pi}{2}}[\rho]\big]$ by the pointwise action of the following matrix:

$$S=\begin{pmatrix}S_+ & 0\\ 0 & S_-\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} & 0\\ 0 & \begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\end{pmatrix}\text{ on }\begin{pmatrix}q_+\\ \xi_+\\ p_+\\ x_+\\ p_-\\ q_-\\ x_-\\ \xi_-\end{pmatrix}$$

and this matrix is symplectic.

Defining now $z_\pm=p_\pm+ix_\pm,\ \theta_\pm=q_\pm+i\xi_\pm$ we find that S becomes on these new variables, $S^c=(S^c_+,S^c_-)=(I,i\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))$ And so the complex metapletic transform associated is

$$S^c_W=\begin{pmatrix}0&i\\i&o\end{pmatrix},\ \det S^C_W=1.$$

5. Töplitz

Let ρ be a Töplitz operator of symbol. $\underset{\sim}{W}[\rho]$. This means that ρ can ve written as

$$\rho=\frac{1}{(2\pi\hbar)^{dN}}\int_{\mathbf{C}^{dN}}\underset{\sim}{W}[\rho](Z,\bar{Z})|\varphi_Z\rangle\langle\varphi_Z|dZ \tag{9}$$

(here the integral as to be understood in the weak sense on $\mathcal{H}$. Elementary properties of $\underset{\sim}{W}[\rho]$ are

$$\underset{\sim}{W}[\rho]\geq 0\Rightarrow\rho>0\text{ and }\int_{\mathbf{C}^{dN}}\underset{\sim}{W}[\rho]dZ=\operatorname{trace}\rho. \tag{10}$$

Moreover,the secondd property of (8) can be "disintegrated" in the following couplig between Husimi and Töplitz settings:

$$\int_{\mathbf{C}^{dN}}\widetilde{W}[\rho](Z,\bar{Z})\underset{\sim}{W}[\rho'](Z,\bar{Z})dZ=\operatorname{trace}(\rho\rho'). \tag{11}$$

Lemma 5.1.

$$\underset{\sim}{W}[U_{i\leftrightarrow j}\rho](z_i,\bar z_i,z_j,\bar z_j)=e^{(|z_i|^2+|z_j|^2)/\hbar}\underset{\sim}{W}[\rho](z_j,\bar z_i,z_i,\bar z_j)$$

$$\underset{\sim}{W}[U_{i\leftrightarrow j}\rho](q_-,p_-;q_+,p_+)=e^{\frac{q_i^2+p_i^2+q_j^2+p_j^2}{\hbar}}\underset{\sim}{W}[\rho](-ip_-,iq_-;q_+,p_+)$$

$$\underset{\sim}{W}[V_{i\leftrightarrow j}\rho](q_1,p_1,\dots,q_i,p_i,\dots,q_j,p_j,\dots,q_n,p_n)=e^{\frac{q_i^2+p_i^2+q_j^2+p_j^2}{\hbar}}$$
$$\times\underset{\sim}{W}[\rho](q_1,p_1,\dots,q_{i-1},p_{i-1},-ip_j,iq_j,\dots,q_{j-1},p_{j-1},-ip_i,iq_i,\dots,q_n,p_n)$$
$$=e^{\frac{q_i^2+p_i^2+q_j^2+p_j^2}{\hbar}}\underset{\sim}{W}[\rho]|_{\substack{z_i\leftrightarrow -z_j\\ \bar z_i\leftrightarrow \bar z_j}},\ z_i=q_i+ip_i.$$

In other words, the exchange action on the Töplitz symbol is the same as the one on the Husimi function, modulo a different gaussian weight.

6. ON Wigner again

Let us denote

$$U^W_{i\leftrightarrow j}W[\rho]=W[U_{i\leftrightarrow j}\rho]$$

Let us moreover denote by $W^2[\rho]$ the Wigner function of the Wigner function of ρ (see footnote 1):

$$W^2[\rho]=W[W[\rho]].$$

Let us denote by $Q_i=(q_i,\xi_i)$ and $P_i=(p_i,x_i)$, $i=1,\dots,N$, the variables in $T^*(T^*\mathbf{R}^d)$). We define:

$$Q_i^t=(\xi_i,q_i),\quad P_i^t=(x_i,p_i).$$

Lemma 6.1.

$$W^2[U_{i\leftrightarrow j}\rho](Q_1,P_1,\dots,Q_i,P_i,\dots,Q_j,P_j,\dots,Q_n,P_n)=$$
$$W^2[\rho](Q_1,P_1,\dots,Q_{i-1},P_{i-1},P_j^t,-Q_j^t,\dots,Q_{j-1},P_{j-1},P_i^t,-Q_i^t,\dots,Q_n,P_n)$$
$$W^2[U_{i\leftrightarrow j}\rho]=W[U^W_{i\leftrightarrow j}W[\rho]]=W^2[\rho]|_{\substack{Q_i\leftrightarrow P_j^t\\ P_i\leftrightarrow -Q_j^t}}.$$

$$W^2[V_{i\leftrightarrow j}\rho](Q_1,P_1,\dots,Q_i,P_i,\dots,Q_j,P_j,\dots,Q_n,P_n)=$$
$$W^2[\rho](Q_1,P_1,\dots,Q_{i-1},P_{i-1},-P_j^t,Q_j^t,\dots,Q_{j-1},P_{j-1},-P_i^t,Q_i^t,\dots,Q_n,P_n)$$

$$W^2[V_{i\leftrightarrow j}\rho] = W[V^W_{i\leftrightarrow j}W[\rho]] = W^2[\rho]|_{\substack{Q_i\leftrightarrow -P^t_j\\ P_i\leftrightarrow Q^t_j}}.$$

So $U^W_{i\leftrightarrow j},\ V_{i\leftrightarrow j}$ are metaplectic operators associated to canonical transforms on $T^*(T^*(\mathbf{R}^{dN}))$.

Lemma 6.2. *Denoting now $z_i = q_i + \xi_i,\ \theta_i = p_i + ix_i$ we have*

$$W[U^W_{i\leftrightarrow j}W[\rho]] = W^2[U_{i\leftrightarrow j}\rho] = W^2[\rho]|_{\substack{z_i\leftrightarrow iz_j\\ \theta_i\leftrightarrow i\theta_j}}$$

$$W^2[V_{i\leftrightarrow j}\rho] = W^2[\rho]|_{\substack{z_i\leftrightarrow -iz_j\\ \theta_i\leftrightarrow -i\theta_j}}$$

So $U^W_{i\leftrightarrow j},\ V_{i\leftrightarrow j}$ are metaplectic operators associated to complex canonical transforms on the complexification of $T^*(\mathbf{R}^{dN})$.

7. Off-diagonal Töplitz representations

In this section, we take $d = 1$ and $N = 2$.

A density matrix ρ has an integral kernel $\rho(x_1, x_2; y_1, y_2)$ and

$$(U\rho)(x_1, x_2; y_1, y_2) = \rho(x_1, x_2; y_2, y_1)$$

$$(V\rho)(x_1, x_2; y_1, y_2) = \rho(x_2, x_1; y_1, y_2).$$

therefore, performing a change of variables

$$x = (x_1 - x_2)/\sqrt{2}, x' = (x_1 + x_2)/\sqrt{2},$$

$$y = (y_1 - y_2)/\sqrt{2}, y' = (y_1 - y_2)/\sqrt{2},$$

one get, with a slight abuse of notation that

$$U\rho(x, y; x', y') = \rho(x, -y : x', y')$$

$$V\rho(x, y; x', y') = \rho(-x, y : x', y')$$

In the rest of this section we will omit the variables x', y'.

Let us consider a (generalized) Töplitz operator

$$H = \int h(z)|\psi^\beta_z\rangle\langle\psi^\beta_z|\frac{dzd\bar{z}}{2\pi\hbar},$$

where, for $\beta > 0$, $z = q + ip$,

$$\psi_z^\beta = \frac{e^{-\frac{\beta(x-q)^2}{2\hbar}} e^{i\frac{px}{\hbar}}}{(\pi\hbar/\beta)^{\frac{1}{4}}}.$$

Let us define H^l by its integral kernel $H^l(x,y) = H(-x,y)$ where $H(x,y)$ is the integral kernel of H. Let H^r be defined the same way by $H^r(x,y) = h(x,-y)$.

Obviously

$$H^{\,l}_{\,r} = \int h(z) |\psi^\beta_{\mp z}\rangle\langle\psi^\beta_{\pm z}| \frac{dz d\bar{z}}{2\pi\hbar}.$$

Therefore, we get the following off-diagonal expressions.

Lemma 7.1.

$$\begin{aligned} VH &= \int h(q,p)|\psi_{-z}\rangle\langle\psi_z|\frac{dzd\bar{z}}{2\pi\hbar} \\ UH &= \int h(q,p)|\psi_{z}\rangle\langle\psi_{-z}|\frac{dzd\bar{z}}{2\pi\hbar} \\ UVH &= \int h(q,p)|\psi_{-z}\rangle\langle\psi_{-z}|\frac{dzd\bar{z}}{2\pi\hbar} \\ U^2 = V^2 &= 1 \end{aligned}$$

These expressions have to be compared to the following ones, derived form Section 5.

Lemma 7.2.

$$\begin{aligned} VH &= \int h(ip,-iq)e^{\frac{q^2+p^2}{2\hbar}}|\psi_{z}\rangle\langle\psi_z|\frac{dzd\bar{z}}{2\pi\hbar} \\ UH &= \int h(-ip,iq)e^{\frac{q^2+p^2}{2\hbar}}|\psi_{z}\rangle\langle\psi_z|\frac{dzd\bar{z}}{2\pi\hbar} \\ UVH &= \int h(-q,-p)|\psi_{z}\rangle\langle\psi_z|\frac{dzd\bar{z}}{2\pi\hbar} \end{aligned}$$

The Töplitz symbol of VH (resp. UH) is $h_V(q,p) = h(ip,-iq)e^{\frac{q^2+p^2}{2\hbar}}$ (resp. $h_U(q,p) = h(-ip,iq)e^{\frac{q^2+p^2}{2\hbar}}$).

Lemma 7.3. *Let $h \geq 0, \int h = 1$.*
Then $H^B := \frac{1}{4}(H + VH + UH + UVH)$ is a bosonic state, and $H^F := \frac{1}{4}(H - VH - UH + UVH)$ is a fernionic one.

Proof. One has $H^B = VH^B = UH^B = UVH^B$, Tr $H^B = 1$, $H^F = -VH^B = -UH^B = UVH^B$, Tr $H^B = 1$, and

$$H^B = \tfrac{1}{4} \int h(q,p) |\psi_z + \psi_{-z}\rangle\langle\psi_z + \psi_{-z}| \frac{dzd\bar{z}}{2\pi\hbar} \geq 0.$$

$$H^F = \tfrac{1}{4} \int h(q,p) |\psi_z - \psi_{-z}\rangle\langle\psi_z - \psi_{-z}| \frac{dzd\bar{z}}{2\pi\hbar} \geq 0.$$

□

Finally, H^B is "semiclassical".

8. Link with the complex metaplectic representation

With the notation of [P] we can make the following observations.

Let us define $S_{\substack{l\\r}} = \begin{pmatrix} 0 & \mp i \\ \pm i & 0 \end{pmatrix}$. Then $S^{-1} = \begin{pmatrix} 0 & \mp i \\ \pm i & 0 \end{pmatrix}$, $\bar{S} = \begin{pmatrix} 0 & \pm i \\ \mp i & 0 \end{pmatrix}$ and $\bar{S}^{-1} = \begin{pmatrix} 0 & \pm i \\ \mp i & 0 \end{pmatrix}$ so that $S^{-1}\bar{S}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Moreover, in the case r, $\beta_{\bar{S}} = \frac{1}{\beta}$ and ${}_\beta T_{\bar{S}}(q,p) = (q', p')$ with

$$\frac{1}{\beta} q' + ip' = \frac{1}{\beta} ip + i(-iq).$$

So ${}_\beta T_{\bar{S}_{\substack{l\\r}}} = \pm \begin{pmatrix} \beta & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}$. Also ${}_\beta T_{S^{-1}_{\substack{l\\r}}} = \pm \begin{pmatrix} -\beta & 0 \\ 0 & -\frac{1}{\beta} \end{pmatrix}$.

We will take $\beta = 1$.

We have $D^1_{\bar{S}} = \frac{1}{(-1)^{1/2}}$, $D^1_{S^{-1}} = \frac{1}{(i)^{1/2}}$ so $D^1_{\bar{S}} D^1_{S^{-1}} = 1$.

$\delta = -2\bar{z}^2$, $\mathbb{Q} = q^2 - p^2$

So

$$H^{S^l_r} = \int h(q,p) e^{\frac{q^2-p^2}{\hbar}} |\psi_{\mp z}\rangle\langle\psi_{\pm z}| \frac{dz d\bar{z}}{2\pi\hbar}. \tag{12}$$

Therefore, with the metapletic representation $U(S)$ defined in [P], Theorem 1, we get the following identities, leading, finally, to a direct metapletic representation for the exchange map, but associated to an anti-canonical relation.

Indeed, using (12), Lemma 7.1 and the definition of $U(S)$ in [P] Theorem 1, we get our final result.

Lemma 8.1. *Let H a Töplitz operator of symbol $h(q,p)$. Then*

$$\begin{aligned} VH &= U\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right)^{-1} H' U\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right) \\ UH &= U\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)^{-1} H' U\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) \end{aligned}$$

where H' is the Töplitz operator of symbol $h'(q,p) = h(q,p)e^{-(q^2-p^2)/\hbar}$.

Note that $\det \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -1$.

References

[G75] A. Grosmann, real and complex canonical transforms, Lecture Notes in Physics, Springer 1975

[FLP12] A. Figalli, M. Ligabo, T. Paul: ' Semiclassical limit for mixed states with singular and rough potentials, Indiana University Mathematics Journal.

[L10] N. Lerner, "Metrics on the phase space and non-selfadjoint pseudo-differential operators", Birkhauser Verlag, Basel, 2010.

[P] T. Paul, *On complex quantum flows*, preprint hal-02004992.

(T.P.) CMLS, Ecole polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau Cedex, France

E-mail address: thierry.paul@polytechnique.edu