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ABSTRACT. In this very short note, we consider the Cucker-Smale dynamical system and we derive rigorously the Vlasov-type equation introduced in [4] in the mean-field limit. The vector field we consider is bounded at infinity in the velocity variables, and Lipschitz continuous in the space variables.

CONTENTS

1. INTRODUCTION

We consider on $\mathbf{R}^{2dN}$ the following Cucker-Smale type vector field [1, 2]

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= G_i(t,X,V),\ i=1,\dots,N \end{aligned} \tag{1}$$

where

$$G_i(t,X,V) = \frac{1}{N}\sum_{j=1}^{N} \phi(v_i - v_j)\psi(x_i - x_j). \tag{2}$$

Here the function ϕ is defined by

$$\phi(v) = \frac{v}{\sqrt{1+|v|^2}}$$

and $\psi : \mathbf{R}^d \to \mathbf{R}$ is supposed to be Lipschitz continuous.

Moreover we used the notation $X = (x_1,\dots,x_N), V = (v_1,\dots,v_N)$.

In fact we are rather interested in the Liouville equation associated to (1) [4], namely

$$\partial_t \rho + v \cdot \nabla_x \rho = \sum_{i=1}^{N} \nabla_{v_i}(G_i \rho) \tag{3}$$

with $\rho \in \mathcal{P}(\mathbf{R}^{2dN})$.

We want to prove that the marginals of $\rho(t)$ tend, as $N \to \infty$, to tensorial powers of the solution of a Vlasov type equation.

Such of Vlasov-type equation associated to (3) has be introduced in [4] and reads

$$\partial_t \rho_1(t,x,v) + v.\nabla_x \rho_1 = \nabla_v (G_{\rho_1}\rho_1), \quad \rho_1|_{t=0} = \rho_1^{in} \in L^1(\mathbf{R}^{2d}, dxdv), \tag{4}$$

with

$$G_{\rho_1}(x,v) = \int_{\mathbf{R}^{2d}} \psi(x-y)\phi(v-w)\rho_1(y,w)dydw. \tag{5}$$

2. The result

Let us start this section by recalling the definition of the second order Wasserstein distance $\mathrm{dist}_{\mathrm{MK},2}$ (see [5, 6]).

Definition 2.1 (quadratic Wasserstein distance). *The Wasserstein distance of order two between two probability measures μ,ν on $\mathbf{R}^m$ with finite second moments is defined as*

$$\mathrm{dist}_{\mathrm{MK},2}(\mu,\nu)^2 = \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbf{R}^m\times\mathbf{R}^m} |x-y|^2\gamma(dx,dy)$$

where $\Gamma(\mu,\nu)$ is the set of probability measures on $\mathbf{R}^m\times\mathbf{R}^m$ whose marginals on the two factors are μ and ν.

We can now state the main result of this little note.

Theorem 2.2. *Let $\rho(t)$ be the solution of (3) with initial condition $\rho^{in} = (\rho_1^{in})^{\otimes N} \in L^1(\mathbf{R}^{2dN})$, $\rho_1^{in}\in L^1(\mathbf{R}^{2d})$, and let $\rho(t)_1 \in L^1(\mathbf{R}^{2d})$ be defined by*

$$\rho(t)_1(x,v) = \int_{\mathbf{R}^{2d(N-1)}} \rho(t)(x,v,x_2,\dots,x_N,v_2,\dots,v_N)dx_2\dots dx_N dv_2\dots dv_N. \tag{6}$$

Of course $(\rho^{in})_1 = \rho_1^{in}$.

Let $\rho_1(t)$ be the solution of (4) with initial condition ρ_1^{in}. Then, for all $N>1, t\in\mathbf{R}$,

$$W_2(\rho(t)_1,\rho_1(t)) \le CN^{-\frac{1}{2}} \tag{7}$$

with

$$C := 4\|\psi\|_\infty\|\psi\|_\infty\frac{e^{\Lambda t}-1}{\Lambda} \qquad \Lambda := 2(1+2(\mathrm{Lip}(\psi)+\mathrm{Lip}(\phi))^2). \tag{8}$$

Remark 2.3. *There exists an equivalent result for higher orders marginals of $\rho(t)$ and other Monge-Kantorovich distances but we prefer in this little note to concentrate on the case of first marginal and quadratic Wasserstein distance. The proof in the more general situation is very close to the one presented here.*

3. Proof

The proof will follow directly the proof of Theorem 3.1 in [3]. the only difference will be the presence of the external force F and the dependence in velocities of G_i.

Let $\pi_N^{in}\in\Pi((\rho^{in})^{\otimes N},(\rho^{in})^{\otimes N})$ satisfy

$$T_\sigma\#\pi_N^{in} = \pi_N^{in}\,, \qquad \text{for each } \sigma\in\mathfrak{S}_N\,, \tag{9}$$

where $\mathfrak{S}_N$ is the group of permutations of N elements and

$$T_\sigma(x_1,v_1,\dots,x_N,v_N,y_1,\xi_1,\dots,y_N,\xi_N)$$
$$= (x_{\sigma(1)},v_{\sigma(1)},\dots,x_{\sigma(N)},v_{\sigma(N)},y_{\sigma(1)},\xi_{\sigma(1)},\dots,y_{\sigma(N)},\xi_{\sigma(N)})\,.$$

We will denote $X=(x_1,\dots,x_N), V=(v_1,\dots,v_N), Y=(y_1,\dots,y_N), \Xi=(\xi_1,\dots,\xi_N)$.

The following Lemma will be one of the keys of the proof of our Theorem.

Lemma 3.1. *Let $\pi_N(t)$ be the solution of*

$$\partial_t \pi_N + V.\nabla_X \pi_N + \Xi.\nabla_Y \pi_N = \sum_{i=1}^N \big(\nabla_{\xi_i}.(G_i(Y,\Xi)\pi_N) + \nabla_{v_i}.(G_{\rho_1(t)}(x_i,v_i))\pi_N)\big) \tag{10}$$

with $\pi_N(0) = \pi_N^{in}$.

Then, for all $t \in \mathbf{R}$, $\pi_N(t)$ is a coupling between $\rho(t)$ and $\rho_1(t)^{\otimes N}$. Moreover $\pi_N(t)$ is invariant by permutations T_σ.

Proof. By taking the two marginals of the two sides of the equality, one get that they satisfy the two Liouville and Vlasov equations. the result is then obtained by uniqueness of the solutions of both equations. □

Lemma 3.2. *Let*

$$D_N(t) := \int \frac{1}{N}\sum_{j=1}^N (|x_j - y_j|^2 + |\xi_j - \eta_j|^2)((X-Y)^2 + (V-\Xi)^2)\pi_N(t)\,.$$

Then

$$\frac{dD_N}{dt} \le \Lambda D_N + \frac{1}{N}\sum_{j=1}^N \int \left|G_{\rho_1(t)}(x_i,v_i) - G_i(X,V)\right|^2 d(\rho_1(t))^{\otimes N}\,,$$

with

$$\Lambda := 2(1 + 2\,\mathrm{Lip}(\nabla V)^2)\,.$$

Proof. We first notice that

$$\frac{dD_N}{dt} = 2\int \left((\Xi - V).(X-Y) + \sum_{i=1}^N (v_i - \xi_i).(G_i(Y,\Xi) - G_{\rho_1(t)}(x_i,v_i))\right) d\pi_N$$

Using $2uv \le u^2 + v^2$ we get

$$\begin{aligned}\frac{dD_N}{dt} &\le \frac{1}{N}\int \left((X-Y)^2 + 2(V-\Xi)^2) + \frac{1}{N}\sum_{i=1}^N |G_i(Y,\Xi) - G_{\rho_1(t)}(x_i,v_i)|^2\right) d\pi_N \\ &\le 2D_N(t) + \frac{1}{N}\int \left(\frac{1}{N}\sum_{i=1}^N |G_i(Y,\Xi) - G_{\rho_1(t)}(x_i,v_i)|^2\right) d\pi_N.\end{aligned}$$

Let us introduce in the square inside the last integral the nul term

$$\frac{1}{N}\sum_{j=1}^N \frac{v_i - v_j}{\sqrt{1+|v_i-v_j|^2}}\phi(x_i - x_j) - \frac{1}{N}\sum_{j=1}^N \frac{v_i - v_j}{\sqrt{1+|v_i-v_j|^2}}\phi(x_i - x_j)$$

Then

$$\left|\frac{v_i - v_j}{\sqrt{1+|v_i-v_j|^2}}\phi(x_i - x_j) - \frac{\xi_i - \xi_j}{\sqrt{1+|\xi_i-\xi_j|^2}}\phi(y_i - y_j)\right|^2$$

can be approximated by

$$\begin{aligned}&((v_i - v_j - (\xi_i - \xi_j))^2 + (x_i - x_j - (y_i - y_j))^2)\,\mathrm{Lip}(\phi)^2 \\ &\le 4\,\mathrm{Lip}(\phi)^2((x_i - y_i)^2 + (v_i - \xi_i)^2 + (x_j - y_j)^2 + (v_j - \xi_j)^2)\end{aligned}$$

This part gives the $4\,\mathrm{Lip}(\phi)^2$ part in Λ, and since the remaining part to integrate contains only the $9X, V)$ variable, the integral against π_N can be replace by the one against $\rho_1(t)^{\otimes N}$. □

The following result is verbatim Lemma 3.3. in [3] with the special value $p = 2$ and d replaced by $2d$.

Lemma 3.3. *Let F be a bounded vector field on $\mathbf{R}^{2d}$, and ρ be a probability density on $\mathbf{R}^{2d}$. For each $j = 1, \dots, N$, one has*

$$\int \left| F \star \rho(x_j, v_j) - \frac{1}{N} \sum_{k=1}^{N} F(x_j - x_k, v_i - v_j) \right|^2 \prod_{m=1}^{N} \rho(x_m, v_m) dx_m dv_m \leq \frac{4}{N} (2\|F\|_{L^\infty})^2 .$$

Lemma 3.3 with $F(x, v) = \phi(v)\psi(x)$ together with Lemma 3.2 gives immediatly that

$$\frac{dD_N}{dt} \leq \Lambda D_N + 2^4 \frac{2}{N} \|\nabla V\|^2_{L^\infty}$$

and, by Gronwall's inequality,

$$D_N(t) \leq D_N(0) e^{\Lambda t} + 2^5 \|\phi \otimes \psi\|^2_{L^\infty} \frac{1}{N} \frac{e^{\Lambda t} - 1}{\Lambda} . \tag{11}$$

Lemma 3.4. $(\pi_N(t))_1$ *is a coupling between* $\rho_1(t)$ *and* $(\rho(t))_1$.

Since $\pi_N(t)$ is symmetric by permutations, one has easily taht

$$D_N(t) = \int (|x_1 - y_1|^2 + |v_1 - \xi_1|^2)(\pi_N(t))_1 \tag{12}$$

and Lemma 3.4 immediately implies that

$$D_N(t) \geq W_2(\rho_1(t), (\rho(t))_1)^2 , \tag{13}$$

and by Lemma 3.3

$$W_2(\rho_1(t), (\rho(t))_1)^2 \leq D_N(0) e^{\Lambda t} + 2^5 \|\phi\|^2_{L^\infty} \|\psi\|^2_{L^\infty} \frac{1}{N} \frac{e^{\Lambda t} - 1}{\Lambda} . \tag{14}$$

Remember that (14) is valid for $D(0)$ as defined in Lemma 3.2 for *any* π_N coupling $(\rho_1^{in})^{\otimes N}$ with itself. Since W_2 is a distance, one has

$$W_2((\rho_1^{in})^{\otimes N}, (\rho_1^{in})^{\otimes N})^2 = \int ((X - Y)^2 + (V - \Xi)^2) \pi_N^{op} = 0$$

for some optimal coupling π_N^{op}.

Choosing now this coupling for the definition of $D_N(0)$ in Lemma 3.2 we get that

$$D_N(0) = \frac{1}{N} W_2((\rho_1^{in})^{\otimes N}, (\rho_1^{in})^{\otimes N})^2 = 0$$

so that (14) gives the result. Theorem 2.2 is proven.

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