Solving under-constrained numerical constraint satisfaction problems with IBEX

Gilles Chabert, Alexandre Goldsztejn, Christophe Jermann

Solving under-constrained numerical constraint satisfaction problems with IBEX

Gilles Chabert[2,4], Alexandre Goldsztejn[1] and Christophe Jermann[3]

[1]CNRS, LS2N, Ecole Centrale de Nantes, France

[2]LS2N, Institut des Mines Télécom Atlantique, France

[3]LS2N, Université de Nantes, France

[4]gilles.chabert@imt-atlantique.fr

Keywords: Numerical constraints, manifolds, parametric interval Newton

A numerical satisfaction problem (NCSP) consists in finding the variables values $x \in \mathbb{R}^n$ in a given box domain $[x] \in \mathbb{IR}^n$ satisfying equality constraints $h(x) = 0$, with $h : \mathbb{R}^n \to \mathbb{R}^{n_e}$ where n_e is the number of equality constraints, and inequality constraints $g(x) \leq 0$, with $h : \mathbb{R}^n \to \mathbb{R}^{n_i}$ where n_i is the number of inequality constraints. The solution set to be computed is denoted $\Sigma := \{x \in [x] : h(x) = 0, g(x) \leq 0\}$. Computing exactly Σ is impossible in general, and numerical constraint solvers usually compute a paving $\mathcal{P} \subseteq \mathbb{IR}^n$, i.e., a finite set of boxes, consisting of inner boxes and unknown boxes, i.e., $\mathcal{P} = \mathcal{I} \cup \mathcal{U}$ and $\mathcal{I} \cap \mathcal{U} = \emptyset$. Inner boxes are proved to contain solutions, but the exact interpretation of these boxes actually depends on the structure of the problem. No information is available for unknown boxes, but the solver looses no solution therefore $\Sigma \subseteq \cup\mathcal{P}$. Constraint solvers handle efficiently NCSPs where there is no equality constraint, in which case an inner boxe contains only solution (see the right graphic of Figure 1), and well constrained systems of equations, i.e., $n_e = n$, where inner boxes contain one unique solution (see the left graphic of Figure 1). This twofold interpretation of inner boxes with respect to the problem structure seems not homogeneous. Intermediate cases with less equations than variables, where the Σ is a manifold of dimension $n - n_e \in \{1, \ldots, n-1\}$, have been the topic of several works but where not included in the general framework of NCSPs. In this case, constraint solvers typically output a large number of unknown boxes, which cover the solution set.

In order to tackle solution sets of arbitrary dimension, we generalize the interpretation of inner boxes as follows: A box $[x]$ is called inner if we can choose $n - n_e$ coordinates x_P with $P \subseteq N := \{1, \ldots, n\}$, called parameters, such that for each $x_P \in [x_P]$ there exists a unique choice of the other n_e coordinates $x_{N\setminus P} \in [x_{N\setminus P}]$, such that x is a solution. This indeed generalizes the classical two fold interpretation of inner boxes since for $n_e = 0$ we obtain $\forall x_N \in [x_N], \exists! x_\emptyset \in [x_\emptyset], x \in \Sigma$, which simplifies to $\forall x \in [x], x \in \Sigma$, while for $n_e = n$ we obtain $\forall x_\emptyset \in [x_\emptyset], \exists! x_N \in [x_N], x \in \Sigma$, which simplifies to $\exists! x \in [x], x \in \Sigma$. When the solution set is of dimension $n - n_e$, it crosses the box in parallel to the subspace x_P (see the middle graphic of Figure 1).

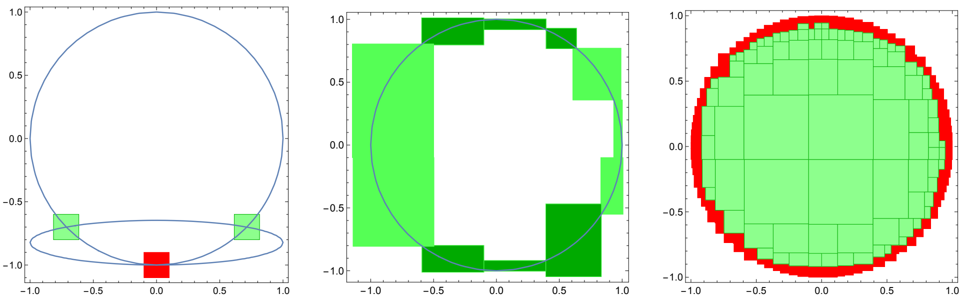

Figure 1: From left to right: Zero, one and full dimensional solution sets. Inner boxes are shown in green and unknown boxes are shown in red. The unknown box of the zero dimensional NCSP contains a singular solution. Dark and light green inner boxes of the one dimensional NCSP correspond to $P = \{1\}$ and $P = \{2\}$ respectively.

Classically, proving that a box $[x]$ is inner for full dimensional NCSPs that contain only inequality constraints $g(x) \leq 0$ is done by checking that the interval evaluation $[g]([x])$ is nonpositive; proving that a box is inner for zero-dimensional NCSPs that contain n equality constraints is done by using the interval Newton operator. For positive but not full dimensional NCSP, we use a parametric interval Newton operator, whose success exactly matches the semantic of inner boxes. Parameters are chosen dynamically by studying the Jacobian of the equality constraints in order to determinate the direction of the solution set (typically by applying a LU-decomposition with pivoting to discover a square sub-matrix with good conditioning). It is critical to allow the interval Newton to perform some inflation of the box by removing the intersection with the previous iterate in the Newton iteration: Indeed, splitting may create solutions on the boundary of variable domains that will never be selected as parameters, and which therefore require an inflation for the parametric interval Newton to succeed. This solving process has been implemented in the solver IBEX, which is available to download at `http://www.ibex-lib.org/`. Several case studies will be presented from geometry, robotics and phase diagrams computation.

Finally, handling carefully the interaction between manifolds defined by equality constraints and inequality constraints, i.e., manifolds with boundary, is nontrivial. While the current theory and implementation allow building an atlas of a manifold without boundary (each inner box giving rise to a chart of the atlas), currently only necessary conditions for a box to be a boundary box have been implemented.