Recovering time-dependent singular coefficients of the wave-equation-One Dimensional Case

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1 Introduction

Let $\Omega =]0, b[\subset \mathbb{R}$, $b > 0$, and consider the following initial boundary value problem

$$\left\{ \begin{array}{rcl} \mathcal{L}_\gamma u & = & 0 \quad \text{in} \quad (0,T) \times \Omega = \Omega_T, \\ u|_{x=0} & = & f(t) \quad \text{on} \quad (0,T), \\ u|_{x=b} & = & 0 \quad \text{on} \quad (0,T), \\ u\Big|_{t=0} & = & u_0 \quad \text{on} \quad \Omega, \\ \partial_t u\Big|_{t=0} & = & u_1 \quad \text{on} \quad \Omega, \end{array} \right. \tag{1.1}$$

where $\mathcal{L}_\gamma u = \partial_t^2 u - \nabla_x \cdot (\gamma \nabla_x u)$, $\gamma = \gamma(t,x)$ has the following properties : There exist a positive constant $k \neq 1$ and a smooth function $t \mapsto a(t) \in]0, b[$ such that

$$\gamma(t,x) = \left\{ \begin{array}{ll} 1 \quad \text{if} \quad x < a(t), \\ k^2 \quad \text{if} \quad x \in]a(t), b[= D(t). \end{array} \right. \tag{1.2}$$

We make the following assumption

$$\text{(H1D)} \quad \|\dot{a}(t)\|_\infty < \min(1,k),$$

where $\dot{a} = \frac{da}{dt}$. The inverse problem were are concern with is to obtain some informations on $a(\cdot)$ and k, by choosing carefully the data f and then measuring $\partial_x u(t,x)$ at $x = 0$.

Since the velocity of waves in $\Omega \setminus D(t)$ is one, it is quite natural to consider the following functions. We set

$$\xi(t) = t - a(t), \tag{1.3}$$

$$\mu(t) = t + a(t). \tag{1.4}$$

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For simplicity, and if it is unambiguous, we shall write $\xi(t) = \xi$, $\mu(t) = \mu$.
If needed, we extend $a(t)$ in $\mathbb{R} \setminus [0, T]$ by a smooth extention, and so we extend $D = \{\{t\} \times (a(t), b)),\ t \in [0, T]\}$, $D^C = \{\{t\} \times (0, a(t)),\ t \in [0, T]\}$, $\partial D = \{(t, a(t)),\ t \in [0, T]\}$ too (with the same notation) by replacing $[0, T]$ by $\mathbb{R}$ in their definition, in such a way that

$$\delta := \frac{1}{2}\mathrm{d}(\partial D, \mathbb{R} \times \Omega) > 0, \quad |\dot{a}|_\infty < \min(1, k).$$

We put

$$t_s := \inf\{t \geq s; a(t) = t - s\}, \quad t^*(s) = 2t_s - s, \quad s \in [0, T].$$

Remark 1.1. *Since $|\dot{a}| < 1$ and $a > 0$, it becomes obvious that $\{t \geq s; a(t) = t - s\} = \{t_s\}$, and that $s \mapsto t_s$ and $s \mapsto t^*(\cdot)$ are smooth and increasing.*

In fact, t_0 is the necessary time delay to have the first information on $D(t)$, and t_s is the same, but with initial time at $t = s$. We set

$$\mu_0 := t_0 + a(t_0) = 2t_0.$$

Remark 1.2. *We obviously have $\mu(t_s) = t^*(s)$ and $\xi(t_s) = s$. Hence $\mu = t^* \circ \xi$ and $\xi^{-1}(\cdot) = t_{(\cdot)}$.*

We also define the coefficient of reflexion/transmition by

$$\alpha(t) := \frac{1 - k + (k - \frac{1}{k})\dot{a}(t)}{1 + k + (k - \frac{1}{k})\dot{a}(t)} = \left(\frac{1-k}{1+k}\right)\left(\frac{1 - (1 + \frac{1}{k})\dot{a}(t)}{1 + (1 - \frac{1}{k})\dot{a}(t)}\right), \qquad (1.5)$$

$$\beta(t) := \frac{2}{1 + k + (k - \frac{1}{k})\dot{a}(t)}. \qquad (1.6)$$

Thanks to (H1D), the functions α and β are well-defined in $[0, T]$. We shall deal with data and measurements as functions in the usual Sobolev space $H^s(I)$, where $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ is an non empty open interval. If $s \in (0, 1)$ it can be defined by

$$H^s(I) = \Big\{q \in L^2(I);\ \iint_{I \times I} \frac{|q(x) - q(y)|^2}{|x - y|^{1+2s}} dx\, dy < \infty\Big\}, \quad 0 < s < 1.$$

Our main result is the following

Theorem 1.3. *Assume that $(u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0 - 1}(\Omega)$ for some $r_0 \in (0, \frac{1}{2})$. Fix $f \in L^2(-\infty, T)$ such that*

1. *$f|_{(-\infty,0)} \in H^{r_0}(-\infty, 0)$;*
2. *$f|_{(0,t)} \in H^{r_0(1-t'/T)}((0, t))$ for $0 < t < t' \leq T$;*

3. $f|_{(0,t')} \notin H^{r_0(1-t/T)}((0,t'))$ *for* $0 \le t < t' \le T$.

Then, the following statements hold.
1) There exists a unique solution u *of (1.1) in* $L^2(\Omega_T)$.
2) The quantity $\partial_x u|_{x=0}$ *is defined in* $H^{-1}(0,T)$ *by continuous extension.*
3) The distribution $g = \partial_x u|_{x=0} + f' \in H^{-1}(0,T)$ *has the following form*

$$g = g_A + g_E,$$

where g_A, g_E *satisfy the following properties:*

(i) $g_A(\mu) = 2\alpha(t) f'(\xi), \quad \forall \mu \in [0,T]$.

(ii) $g_A|_{(0,\mu)} \in H^{r_0(1-\tilde{\xi}/T)-1}(0,\mu)$ *for all* $\mu_0 < \mu \le T$ *and all* $\tilde{\xi} > \xi$.

(iii) *If* $\dot{a}(t) \ne \frac{k}{1+k}$ *then* $g_A|_{(0,\mu)} \notin H^{r_0(1-\tilde{\xi}/T)-1}(0,\mu)$, $\forall \tilde{\xi} < \xi$.

(iv) *There exists* $\varepsilon > 0$ *such that*

$$g_E|_{(0,\mu)} \in H^{\varepsilon + r_0(1-\xi/T)-1}(0,\mu), \quad \forall \mu \in [0,T]. \tag{1.7}$$

The main consequence of this is

Corollary 1.4. *Assume that* $\dot{a}(t) \ne \frac{k}{1+k}$ *for all* t*, and* $(u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ *for some* $r_0 > 0$*. Let* $T > 0$*. We claim that:*
1) We can know if $T \le \mu_0$ *or if* $T > \mu_0$.
2) Assume that $T > t^*(0) = \mu_0$*. Set*

$$s^* := {t^*}^{-1}(T), \quad t_{max} := t_{s^*}.$$

Then we can recover the functions $s \mapsto t_s$, $0 \le s \le s^*$, $t \mapsto a(t)$, $t_0 \le t \le t_{max}$. *The constant* k *is the root of a second degree equation with known coefficients. If* $\dot{a} \le 0$ *then this equation has no more than one positive root, and so, we are able to reconstruct* k.

Remark 1.5. *Obviously, from Corollary 1.4 and Remark 1.1, and since* $t_0 = a(t_0) < b$*, we can ensure the condition* $T > \mu_0$ *by choosing* $T \ge 2b$.

In Theorem 1.3, the existence of such a function f is ensured, thanks to the following

Lemma 1.6. *For all* $R > 0$*, there exists a function* $G(t)$, $0 \le t \le 1$*, such that*

1. $G|_{(0,t)} \in H^{(1-t')/R}(0,t)$ *for all* $0 < t < t' \le 1$.

2. $G|_{(0,t')} \notin H^{(1-t)/R}(0,t')$ *for all* $0 < t < t' \le 1$.

Remark 1.7. *In Theorem 1.3, if* $(u_0, u_1) \in H_0^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ *for some* $r_0 \in (\frac{1}{2}, 1]$*, and if* $u_0(0)$ *is known, then we can fix* $f \in L^2(0,T)$ *such that*

1. $f|_{[0,t]} \in H^{r_0(1-t/T)}([0,t])$ *for* $0 < t \leq T$*;*
2. $f|_{[0,t']} \notin H^{r_0(1-t/T)}([0,t'])$ *for* $0 < t < t' \leq T$*,*

and with $f(0) = u_0(0)$*. Then, the same result holds than in Theorem 1.3, but with* $r_0 \in (\frac{1}{2}, 1]$*.*
If $(u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ *for some* $r_0 \in (\frac{1}{2}, 1]$*, but if we don't know the value of* $u_0(0)$*, then the information is not sufficient (with our approach) to construct* f *so that the result of Theorem 1.3 holds* with this value $r_0 \in (\frac{1}{2}, 1]$*, and so, we are obliged to come back to the situation* $(u_0, u_1) \in H^{r_1}(\Omega) \times H^{r_1-1}(\Omega)$*, where* $r_1 < \frac{1}{2}$*.*

The paper is organized as follows. In Section 2, we analyse the direct problem (1.1). In Section 3 we construct an ansatz u_A for (1.1) where f is the function of Theorem 1.3. In Section 4, we first prove Corollary 1.4, then Theorem 1.3. In particular, we analyse the error $u_E = u - u_A$.

2 Study of the direct problem

2.1 Notations

We denote by $(|)$ the usual scalar product in $L^2(\Omega; dx)$, by $(|)_H$ the scalar product in a Hilbert space H, by $<;>_{H^* \times H}$ the duality product between a Hilbert space H and its dual space H^*, by $<;>$ the duality product in $\mathcal{D}'(\Omega_T) \times \mathcal{D}(\Omega_T)$ or in $\mathcal{D}'(0,T) \times \mathcal{D}(0,T)$. We put $\mathcal{H}^1 = L^2(0,T; H_0^1(\Omega))$, $\mathcal{H}^{-1} = L^2(0,T; H^{-1}(\Omega)) = \mathcal{H}^{1*}$, $W = \{v \in \mathcal{H}^{-1};\ \partial_t v \in \mathcal{H}^{-1}\}$ with obvious norms. We denote

$$E^r = H^r(\Omega) \times H^{r-1}(\Omega) \times H^r(0,T),$$

and

$$E_0^r = \begin{cases} \{(u_0, u_1, f) \in E^r;\ u_0(0) = f(0), u_0(b) = 0\}, & \frac{1}{2} < r \leq 1, \\ E^r, & 0 \leq r < \frac{1}{2}. \end{cases}$$

(For $r = \frac{1}{2}$ we could set E_0^r as in the case $r > \frac{1}{2}$, but the relations $u_0(0) = f(0)$ and $u_0(b) = 0$ should be modified).
We denote $\Omega_{t_1,t_2} = (t_1, t_2) \times \Omega$.

For data v_0, v_1, F, let v satisfying in some sense:

$$\begin{cases} \mathcal{L}_\gamma v &= F \quad \text{in} \quad \Omega_T, \\ v(t,x) &= 0\,, \quad x \in \partial\Omega, \quad t \in (0,T), \\ v|_{t=0} &= v_0 \quad \text{on} \quad \Omega, \\ \partial_t v|_{t=0} &= v_1 \quad \text{on} \quad \Omega. \end{cases} \tag{2.1}$$

We formally define the following operators:

$$\begin{aligned} u &= \tilde{P}(u_0, u_1, f), \\ \partial_x u|_{x=0} + f' &= \tilde{Z}(u_0, u_1, f), \\ (u|_{t=s}, \partial_t u|_{t=s}) &= \tilde{X}(s)(u_0, u_1, f), \quad 0 \le s \le T, \\ v &= P(v_0, v_1, F), \\ \partial_x v|_{x=0} &= Z(v_0, v_1, F), \\ (v|_{t=s}, \partial_t v|_{t=s}) &= X(s)(v_0, v_1, F), \quad 0 \le s \le T, \end{aligned}$$

where u, v, are respectively solutions of (1.1), (2.1).

2.2 Main results

In this section and the one above, we state that Problems (1.1), (2.1) have a unique solution for adequate spaces.

Lemma 2.1. *1. The operator P is a continuous linear mapping from $H_0^1(\Omega) \times L^2(\Omega) \times (L^2(\Omega_T) + W)$ into $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.*
2. The operator $X(s)$ is continuous from $H_0^1(\Omega) \times L^2(\Omega) \times (L^2(\Omega_T) + W)$ into $H_0^1(\Omega) \times L^2(\Omega)$, for all $s \in [0,T]$.

Lemma 2.2. *1. The operator P continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ into $L^2(\Omega_T)$.*
2. The operator $X(s)$ continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ into $L^2(\Omega) \times H^{-1}(\Omega)$, for all $s \in [0,T]$.

Lemma 2.3. *1. The operator $\tilde{P}$ is a continuous linear mapping from E_0^1 into $C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, and continuously extends as a continuous operator from E^0 into $L^2(\Omega_T)$.*
2. The operator $\tilde{X}(s)$ is continuous from E_0^1 into $H^1(\Omega) \times L^2(\Omega)$, and continuously extends as a continuous operator from E^0 into $L^2(\Omega) \times H^{-1}(\Omega)$, for all $s \in [0,T]$.

Lemma 2.4. *The operator Z (respect., $\tilde{Z}$) is continuous from $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$ (respect., E_0^1) into $L^2(0,T)$ and continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ (respect., E^0) into $H^{-1}(0,T)$.*

Lemma 2.5. *Let $t_1 \in [0,T]$. Assume that $F \in \mathcal{H}^{-1}$ has a compact support in $\mathcal{O}(t_1)$. Let $v = P(v_0, v_1, F)$. Then there exists a neighborhood $\tilde{K}$ of $K(t_1)$ in $\overline{D^C}$ such that $v|_{\tilde{K}}$ does not depend on F, that, is, if $v_0 = v_1 = 0$, then $v|_{\tilde{K}}$ vanishes, and, in particular, supp $\partial_x v|_{x=0} \subset (\mu(t_1), T]$.*

2.3 Proofs

Let us consider the familly of bilinear forms $b(t)$, $t \in \mathbb{R}$, defined by

$$b(t; u, v) = \int_\Omega \gamma(t, x)\nabla_x u(x)\ \nabla_x v(x)\ dx, \quad \forall u, v \in H^1(\Omega).$$

Lemma 2.1 is a corollary of the following general theorem (proof in appendix), which is an extension of [1, XV section 4] where γ did not depend on the variable t.

Theorem 2.6. *Let $T > 0$ and $\Omega \subset \mathbb{R}^n$, $n \geq 1$, such that $H_0^1(\Omega)$ is compact in $L^2(\Omega)$. Let $\gamma(t, x) > 0$ be such that $\gamma, \gamma^{-1} \in C^0([0, T]; L^\infty(\Omega))$, $\partial_t\gamma \in L^\infty(\Omega_T)$. Let $F \in W \cup L^2(\Omega_T)$ and $v_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$. Then, there exists a unique weak solution v to (2.1), that is, $v \in C([0, T]; H_0^1(\Omega))$, $\partial_t v \in C([0, T]; L^2(\Omega))$, $v|_{t=0} = v_0$, $\partial_t v|_{t=0} = v_1$, and*

$$\frac{d}{dt}(\partial_t v|\phi) + b(t; v(t, \cdot), \phi) = < F(t, \cdot); \phi >,$$

in the sense of $\mathcal{D}'(]0, T[)$, for all $\phi \in H_0^1(\Omega)$. Moreover there exists a constant C such that

$$\|\partial_t v(t, \cdot)\|_{L^2(\Omega)} + \|\partial_x v(t, \cdot)\|_{L^2(\Omega)} \leq C \left(\|F\|_{L^2(\Omega_t)+W} + \|v_0\|_{H_0^1(\Omega)} + \|v_1\|_{L^2(\Omega)} \right), \quad \forall t \in [0, T]. \tag{2.2}$$

Let us show that Lemma 2.2 is a straightforward consequence of Lemma 2.1 with the operator P replaced by its adjoint P^*. Let $(v_0, v_1, F) \in L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$. By the principle of duality, we can write (2.1) as

$$(v|g)_{L^2(\Omega_T)} \quad = \quad < v_1, w(0) >_{H^{-1}\times H_0^1} - (v_0|\partial_t w(0)) + < F, w >_{\mathcal{H}^{-1}\times\mathcal{H}^1},$$

for all $g \in L^2(\Omega_T)$, where we put $w = P^*(0, 0, g)$. Consequently (thanks to Lax-Milgram theorem), Equation (2.1) admits a unique solution $v \in L^2(\Omega_T)$, and this shows Point 1 of Lemma 2.2. Once again, we have

$$\begin{aligned} < \partial_t v|_{t=T}, f_0 >_{H^{-1}\times H_0^1} - (v|_{t=T}|f_1) \quad &= \quad < v_1, w(0) >_{H^{-1}\times H_0^1} - (v_0|\partial_t w(0)) \\ &\quad + < F, w >_{\mathcal{H}^{-1}\times\mathcal{H}^1}, \end{aligned}$$

for all $(f_0, f_1) \in H_0^1(\Omega) \times L^2(\Omega)$, where we put $w = P^*(f_0, f_1, 0)$. This shows that $(v|_{t=T}, \partial_t v|_{t=T}) \in L^2(\Omega) \times H^{-1}(\Omega)$. This proves Point 2 of Lemma 2.2 in the non-restrictive case $s = T$.

Let us prove Lemma 2.3. Let $\Phi(x) \in C^\infty(\mathbb{R})$ with $\Phi(0) = 1$ and with support in $[0, a_m]$, where $a_m \leq a(t)$ for all t. Let us consider $f \in H^1_{loc}(\mathbb{R})$ first. Set

$$u_{in}(t, x) = f(t - x)\Phi(x). \tag{2.3}$$

Problem (1.1) with unknown u is (at least formally) equivalent to the following one: find $v = u - u_{in}$ satisfying (2.1) with

$$v_0(x) = u_0(x) - f(-x)\Phi(x), \quad v_1(x) = u_1(x) - f'(-x)\Phi(x), \tag{2.4}$$
$$F(t,x) = -\mathcal{L}_\gamma u_{in}(t,x) = -\mathcal{L}_1 u_{in}(t,x) = -2f'(t-x)\Phi(x) + f(t-x)\Phi''(x). \tag{2.5}$$

Relation (2.5) shows that $F \in L^2(\Omega_T)$. In fact, we have $F \in W$ also, since

$$\partial_t F(t,x) = -2f''(t-x)\Phi(x) + f'(t-x)\Phi''(x),$$

and, for all $\varphi \in \mathcal{D}(\Omega_T)$,

$$\begin{aligned} < f''(t-x)\Phi(x), \varphi(t,x) > &= < f''(t-x,)\Phi(x)\varphi(t,x) > \\ &= < f'(t-x), \partial_x(\Phi(x)\varphi(t,x)) > \le C\|\varphi\|_{\mathcal{H}^1}, \end{aligned}$$

which shows that $\partial_t F(t,x) \in \mathcal{H}^{-1}$. Similarly, we have

$$< f'(t-x)\Phi(x), \varphi(t,x) > = < f(t-x), \partial_x(\Phi(x)\varphi(t,x)) \le C\|\varphi\|_{L^2(\Omega_T)},$$

which shows that $F \in \mathcal{H}^{-1}$ if $f \in L^2_{loc}(\mathbb{R})$ only. We set

$$R: \begin{array}{ccc} H^1_{loc}(\mathbb{R}) & \to & L^2(\Omega_T) \cap W \\ f & \mapsto & F \text{ defined by } (2.5) \end{array},$$

$$S: \begin{array}{ccc} E^1_0 & \to & H^1_0(\Omega) \times L^2(\Omega) \\ (u_0, u_1, f) & \mapsto & (v_0, v_1) \text{ defined by } (2.4). \end{array}$$

The above analysis shows that R continuously extends as a continuous operator from $L^2_{loc}(\mathbb{R})$ into $\mathcal{H}^{-1}$. Similarly, S continuously extends as a continuous operator from E^0 into $L^2(\Omega) \times H^{-1}(\Omega)$. Consequently, and since a solution to (1.1) can be written $u = v + u_{in}$ with $v = P(S(u_0, u_1, f), R(f))$, Point 1 of Lemma 2.3 is proved. Similarly, we prove Point 2 of Lemma 2.3, since we have $\tilde{X}(s)(u_0, u_1, f) = X(s)(S(u_0, u_1, f), Rf) + (u_{in}|_{t=s}, \partial_t u_{in}|_{t=s})$.

Let us prove Lemma 2.4. Let $(v_0, v_1, F) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$.
As above, for all $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(T) = 0$, there exists a unique solution $q = q_\varphi \in L^2(\Omega_T)$ to

$$\left\{ \begin{array}{rcll} \mathcal{L}_\gamma q &=& 0 & \text{in} \quad \Omega_T, \\ (q(t,0), q(t,b)) &=& (\varphi, 0) & \text{on} \quad (0,T), \\ q|_{t=T} &=& 0 & \text{on} \quad \Omega, \\ \partial_t q|_{t=T} &=& 0 & \text{on} \quad \Omega, \end{array} \right. \tag{2.6}$$

since it is a particular case of Lemma 2.3 with reversal time.
Moreover, we have $q_\varphi \in C([0,T]; H^1_0(\Omega))$, $\partial_t q_\varphi \in C([0,T]; L^2(\Omega))$ with

$$\|q_\varphi|_{t=0}\|_{L^2(\Omega)} + \|\partial_t q_\varphi|_{t=0}\|_{H^{-1}(\Omega)} + \|q_\varphi\|_{L^2(\Omega_T)} \le C\|\varphi\|_{L^2(0,T)}, \tag{2.7}$$
$$\|q_\varphi|_{t=0}\|_{H^1(\Omega)} + \|\partial_t q_\varphi|_{t=0}\|_{L^2(\Omega)} + \|q_\varphi\|_{\mathcal{H}^1} + \|\partial_t q_\varphi\|_{L^2(\Omega_T)} \le C\|\varphi\|_{H^1(0,T)} \tag{2.8}$$

By the duality principle, and thanks to (2.7), we have in the sense of $\mathcal{D}'([0,T))$,

$$\begin{aligned} < \partial_x v|_{x=0}, \varphi > &= - < v_0, \partial_t q_\varphi|_{t=0} > + < v_1, q_\varphi|_{t=0} > + < F, q_\varphi > \quad (2.9) \\ &\leq C \left(\|v_0|_{t=0}\|_{H^1(\Omega)} + \|v_1\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega_T)} \right) \|\varphi\|_{L^2(0,T)}, \quad (2.10) \end{aligned}$$

which shows that $\partial_x v|_{x=0} \in L^2(0,T)$ and that Z is a continuous mapping from $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$ into $L^2(0,T)$.

Now, let $(v_0, v_1, F) \in L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$. Then, Relation (2.9) and Estimate (2.8) imply

$$< \partial_x v|_{x=0}, \varphi > \leq C \left(\|v_0|_{t=0}\|_{L^2(\Omega)} + \|v_1\|_{H^{-1}(\Omega)} + \|F\|_{\mathcal{H}^{-1}} \right) \|\varphi\|_{H^1(0,T)}, \quad (2.11)$$

which shows that $\partial_x v|_{x=0} \in (H_T^1)' \subset H^{-1}(0,T)$, the dual space of $H_T^1 = \{f \in H^1(0,T); f(T) = 0\}$, and that Z continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ into $H^{-1}(0,T)$.
This ends the proof of the property of Z in Lemma 2.4. Since $\partial_x u_{in}|_{x=0} = -f'$, we have $\tilde{Z}(u_0, u_1, f) = Z(S(u_0, u_1, f), Rf)$, and Point 2 of Lemma 2.4 is proved. □

By the well-known Sobolev interpolation theory, we have also proved:

Proposition 2.7. *The operator P (respect., $\tilde{P}$) continuously maps $H^s(\Omega) \times H^{s-1}(\Omega) \times L^2(0,T; H^{s-1}(\Omega))$ (respect., E_0^s) into $L^2(0,T; H^s(\Omega))$, $s \in [0,1](\backslash \frac{1}{2})$. The operator Z (respect., $\tilde{Z}$) continuously maps $H^s(\Omega) \times H^{s-1}(\Omega) \times L^2(0,T; H^{s-1}(\Omega))$ (respect., E_0^s) into $H^{s-1}(0,T)$, $s \in [0,1](\backslash \frac{1}{2})$.*

Proof of Lemma 2.5. Denote $K = K(t_1)$. Notice that $K \cap \overline{D^C} = \{(t_1, a(t_1))\}$. We assume that $v_0 = v_1 = 0$. Since supp$F \cap \overline{\Omega_{t_1}} = \emptyset$, then, thanks to Lemma 2.2 with T replaced by t_1, v vanishes in Ω_{t_1}. Let $K' = \text{int}\, K$ the interior of K. The function $v|_K \in L^2(K')$ satisfies the following equations:

$$\begin{aligned} \partial_t^2 v - \Delta_x v = 0 \quad &\text{in} \quad K', \\ v(t,0) = 0, \quad &t_1 < t < \mu(t_1), \\ v|_{t=t_1} = \partial_t v|_{t=t_1} = 0 \quad &\text{in} \quad (0, a(t_1)). \end{aligned}$$

It is well-known that this implies $v|_{K'} = 0$, and so, supp $\partial_x v|_{x=0} \subset [\mu(t_1), T]$. But since the support of F does not touch ∂K, we similarly have $v|_{K_\varepsilon(t_1)} = 0$, supp $\partial_x v|_{x=0} \subset [\mu(t_1) + \delta, T]$, for some $\varepsilon > 0$ sufficiently small.
However, let us give a more straightforward and simple proof to the fact that supp $\partial_x v|_{x=0} \subset [\mu(t_1) + \delta, T]$. Fix $\delta, \varepsilon > 0$ such that $\mu(t_1) + \delta > \mu(t_1 + \varepsilon)$ and supp $F \cap K_\varepsilon(t_1) = \emptyset$. Let $t_2 \in [t_1, t_1 + \varepsilon]$, $\varphi \in H_0^1(0, \mu(t_2))$ and set $w(t,x) = \varphi(t+x)$ for $t_2 \leq t \leq \mu(t_2)$. Observe that $w = q_\varphi$ of (2.6), but with $(0,T)$ replaced by

$(t_2, \mu(t_2))$. In fact, supp $w \subset K(t_2)$, and so w vanishes in $D \cap \Omega_{t_2,\mu(t_2)}$. We then have, similarly to (2.9),

$$< \partial_x v|_{x=0,t_2<t<\mu(t_2)}, \varphi >= - < v|_{t_2}, \partial_t w|_{t_2} > + < \partial_t v|_{t_2}, w|_{t_2} > + < F, w >= 0$$

since $v|_{t_2} = \partial_t v|_{t_2} = 0$ and supp $F \cap$ supp $w = \emptyset$. Since φ is arbitrarily chosen, this shows that supp $\partial_x v|_{x=0} \cap (t_2, \mu(t_2)) = \emptyset$, for all $t_2 \in [t_1, t_1 + \varepsilon]$. Hence, supp $\partial_x v|_{x=0} \subset [\mu(t_1 + \varepsilon), T]$. $\square$.

3 Ansatz

3.1 Notations

For $t \in [0, T]$ we put

$$K(t) = \{(s, x) \in [t, \mu(t)] \times \overline{\Omega};\ s+x \leq \mu(t)\}, \quad \mathcal{O}(t) = \{(s, x) \in \overline{\Omega_{t,T}}; s+x > \mu(t)\}.$$

(Notice that $K(t) \subset \overline{D^C}$ and $K(t) \cap \overline{D} = \{(t, a(t))\}$).
For $\varepsilon > 0$, $t \in [0, T]$, we put $K_\varepsilon(t) = \cup_{t \leq s \leq t+\varepsilon} K(s)$.

If $q(x)$ is sufficiently smooth in Ω, then $[q]_t := q(a(t) + 0) - q(a(t) - 0)$.
We write $g_1 \overset{s}{\simeq} g_2$ if g_1 or $g_2 \in H^s(0, T)$ and $g_1 - g_2 \in H^{s+\varepsilon}(0, T)$ for some $\varepsilon > 0$.
We put $C^j_+ = \{f \in C^j(\mathbb{R});\ f|_{\mathbb{R}^-} = 0\}$, $j \in \mathbb{N}$, which is dense in $L^2(\mathbb{R}^+) \approx \{f \in L^2(\mathbb{R});\ f|_{(-\infty,0)} = 0\}$. We consider for all $t \in [0, T]$ the formal operator $\mathcal{A}(t) = -\nabla_x(\gamma(t, \cdot)\nabla_x)$ defined from $H^1(\Omega)$ into $H^{-1}(\Omega)$ by duality:

$$< \mathcal{A}(t)u, w >_{H^{-1}(\Omega)\times H^1_0(\Omega)} = (\gamma(t)\nabla_x u|\nabla_x w), \quad \forall u, w \in H^1(\Omega) \times H^1_0(\Omega).$$

Let f be a measurable function, we define the ansatz $u_A = U_A(f)$ for (1.1) as follows. Recall that $\xi(t)$ and $\mu(t)$ are defined by (1.3), (1.4), and we have

$$\xi_0 = t_0 - a(t_0) = 0, \tag{3.1}$$

$$\mu_0 = t_0 + a(t_0) = 2t_0. \tag{3.2}$$

In addition, we put, for $t \in [0, T]$,

$$\nu = t - \frac{a(t)}{k}, \quad \nu_0 = t_0 - \frac{a(t_0)}{k}. \tag{3.3}$$

Thanks to Assumption (H1D), $t \mapsto \nu(t)$ is invertible. Recall also that the coefficient of reflexion/transmition, α and β, are defined by (1.5), (1.6). Note that we have

$$\alpha(t)\frac{d\mu}{d\xi} - \beta(t)\frac{d\nu}{d\xi} = -1, \tag{3.4}$$

$$\alpha(t) + k\beta(t) = 1. \tag{3.5}$$

We also define:

$$f_2(\mu) = \alpha(t)\frac{d\mu}{d\xi}f(\xi), \tag{3.6}$$

$$f_3(\nu) = \beta(t)\frac{d\nu}{d\xi}f(\xi). \tag{3.7}$$

We put

$$u_A(t,x) = \begin{cases} f(t-x)+f_2(t+x)-f_2(t-x)\Phi_\varepsilon(x), & 0\le t\le T, \quad 0<x<a(t), \\ \Phi_\varepsilon(x-b+2\varepsilon)f_3(t-\frac{x}{k}), & 0\le t\le T \quad a(t)<x<b, \end{cases}$$

where we fix $\Phi_\varepsilon \in C^\infty(\mathbb{R})$ so that $\Phi_\varepsilon(r)=1$ if $r<\frac{1}{2}\varepsilon$, $\Phi_1(r)=0$ if $r>\varepsilon$, $0<\varepsilon\le\frac{1}{2}\mathrm{d}(\partial D,\partial\Omega_T)$. It is clear that the linear operator $U_A:\ f\mapsto u_A$ is bounded from $L^2(\mathbb{R})$ into $L^2(\Omega_{\mu_T})$.

3.2 Properties of the Ansatz

Lemma 3.1. *Let $f\in C^2(\mathbb{R})$. Then we have*
1) $u_A\in C^2([0,T];H^1(\Omega))$, $u_A|_D\in C^2(\overline{D})$, $u_A|_{D^C}\in C^2(\overline{D^C})$.
2) There exists a smooth function $\tau(t)$ with support in $[t_0,\mu_0]$ such that

$$[\gamma\partial_x u_A(t)]_t=\tau(t)f(\xi(t)).$$

3) a) u_A vanishes near $x=b$.
b) Let $g_A=\partial_x u_A|_{x=0}+f'$. Then $g_A(\mu)=2\alpha(t)f'(\xi)$ for $0\le\mu\le T$, where t, ξ, μ are related by (1.3), (1.4), (3.3).
4) Put $F_A=\mathcal{L}_\gamma u_A$ in the sense that $F_A(t,\cdot)=\frac{d^2}{dt^2}u_A(t)+\mathcal{A}(t)u_A(t)\in H^{-1}(\Omega)$ for all t, and $F_A\in C([0,T];H^{-1}(\Omega))$. Then, F_A can be written $F_A(t,x)=F_1(t,x)-\tau(t)f(\xi(t))\delta_{a(t)}(x)$, where τ is smooth, and $F_1\in C([0,T];L^2(\Omega))$ is defined for $0\le t\le T$ by

$$F_1(t,x) = \begin{cases} \Phi_2(x)f_2(t-x)+\Phi_3(x)f_2'(t-x) & 0<x<a(t), \\ \Phi_4(x)f_3(t-\frac{x}{k})+\Phi_5(x)f_3'(t-\frac{x}{k}), & a(t)<x<b, \end{cases} \tag{3.8}$$

where the functions Φ_j are smooth and independant of f, with compact support in $[\varepsilon/2,\varepsilon]$ for $j=2,3$, and in $[b-\varepsilon,b-\varepsilon/2]$ for $j=4,5$.

Proof. Point 1. is obvious, since we have, thanks to (3.4),

$$\begin{aligned}[u_A(t,\cdot)]_t &= f_3(\nu(t))-f(\xi(t))-f_2(\mu(t)) \\ &= \left(\beta(t)\frac{d\nu}{d\xi}-1-\alpha(t)\frac{d\mu}{d\xi}\right)f(\xi)=0.\end{aligned}$$

Let us consider Point 2. For $0\le t\le T$ we have

$$\gamma\partial_x u_A(t,a(t)-0) = -f'(\xi)+f_2'(\mu)=(-1+\alpha)f'(\xi)+\frac{d(\alpha\frac{d\mu}{d\xi})}{d\mu}f(\xi),$$

$$\gamma\partial_x u_A(t,a(t)+0) = -kf_3'(\nu)=-k\beta(t)f'(\xi)-k\frac{d(\beta(t)\frac{d\nu}{d\xi})}{d\nu}f(\xi).$$

Thanks to (3.5) we get

$$[\gamma\partial_x u_A(t)]_t = -\tau(t)f(\xi),$$

with

$$\tau(t) = -k\frac{d(\beta(t)\frac{d\nu}{d\xi})}{d\nu} - \frac{d(\alpha\frac{d\mu}{d\xi})}{d\mu}.$$

This ends Point 2.
Let us consider Point 3 b), since 3 a) is obvious. For $0 \leq \mu \leq T$ we have

$$\partial_x u_A(\mu, 0) \quad = \quad -f'(\mu) + 2f_2'(\mu) = -f'(\mu) + 2\alpha(t)f'(\xi).$$

This ends Point 3.
Let us prove Point 4. A short computation yields (3.8). Thanks to Point 2, we obtain $F_A = F_1 + \tau(t)f(\xi)$ in the required sense. This ends the proof of the lemma. □

We define the bounded operators $U_A : C^2(\mathbb{R}) \ni f \mapsto u_A \in C^2([0,T]; H^1(\Omega))$, T_0: $C^2(\mathbb{R}) \ni f \mapsto T_0 f \in C([0,T]; H^{-1}(\Omega))$ such that $T_0 f(t) = \tau(t)f(\xi)\delta_{a(t)}(x)$, and T_1: $C^2(\mathbb{R}) \ni f \mapsto T_1 f = F_1 \in C([0,T]; L^2(\Omega))$, T_A: $C^2(\mathbb{R}) \ni f \mapsto T_A f = F_A \in C([0,T]; H^{-1}(\Omega))$. Notice that $T_0 f(t) \in H^{-s}(\Omega)$ for all $s > \frac{1}{2}$, $t \in [0,T]$. Obviously we have the following propositions and Lemma.

Proposition 3.2. *The operator U_A continuously extends as a bounded operator from $L^2(0,T)$ into $C([0,T]; H^{-1}(\Omega))$.*

Proposition 3.3. *The operator T_0 continuously extends as a bounded operator from $L^2(0,T)$ into $L^2(0,T; H^{-s}(\Omega))$, $\forall s > \frac{1}{2}$.*

Lemma 3.4. *1) The operator T_A is continuous from $C^2(\mathbb{R})$ into $L^2(0,T; H^{-1}(\Omega))$ and, for all $s \in [0, \frac{1}{2})$, it extends as a continuous operator from $H^s(0,T)$ into $L^2(0,T; H^{s-1}(\Omega))$.*
2) The operator G_A : $f \mapsto \partial_x U_A(f)|_{x=0} + f'$ is continuous from $C^2(\mathbb{R})$ into $C^0([0,T])$, and, for all $s \in [0, \frac{1}{2})$, it extends as a continuous operator from $H^s(0,T)$ into $H^{s-1}(0,T)$.
3) Let f such as in Theorem 1.3, then $g_A := G_A f$ satisfies (ii) and (iii) of Theorem 1.3.

Proof of Lemma 3.4. Point 1). Thanks to Lemma 3.3, it is sufficient to prove this with T_A replaced by T_1. Thanks to the interpolation theory, it is sufficient to prove that T_1 is a bounded operator from $L^2(0,T)$ into $L^2(0,T; H^{-1}(\Omega))$ and from $H_0^1(0,T)$ into $L^2(\Omega_T)$, that is obvious. Hence Point 1) holds. Point 2) is obvious for the same reason. Point 3) is obvious, since $\alpha(t) \neq 0$ for all t. □

3.3 Modification of F_1

The regularity of F_1 is not sufficient for us, we replace it by the following one, $F_{\varepsilon,\tilde{\mu}}$, which is equivalent to F_1 in the sense of Lemma 2.5.
Let $\tilde{\mu} \in [0,T]$, put $\tilde{t} = \mu^{-1}(\tilde{\mu})$, $\tilde{\xi} = \xi(\tilde{t})$, $\tilde{\nu} = \nu(\tilde{t})$, and consider a smooth function $\phi(\cdot;\varepsilon,\tilde{\mu})$ defined in $\mathbb{R}^2$ such that $\phi(t,x;\varepsilon,\tilde{\mu}) = 1$ for $(t,x) \in \overline{\Omega_{\tilde{t}}} \cup K_{\varepsilon/2}(\tilde{t})$, $\phi(t,x;\varepsilon,\tilde{\mu}) = 0$ for $t \geq \tilde{t} + \varepsilon$ and $(t,x) \notin K_\varepsilon(\tilde{t})$. For $s \in [0, \frac{1}{2})$, $f \in H^s(\mathbb{R})$ and $F_1 = T_1(f)$ we put

$$F_{\varepsilon,\tilde{\mu}}(t,x) \quad = \quad F_1(t,x)\phi(t,x;\varepsilon,\tilde{\mu}).$$

We have the two following properties.

Lemma 3.5. *For $\varepsilon < \delta$, the support of $F_1 - F_{\varepsilon,\tilde{\mu}}$ is contained in $\mathcal{O}(\tilde{t})$.*

Proof. Since $F_1 - F_{\varepsilon,\tilde{\mu}} = (1 - \phi(\cdot;\varepsilon,\tilde{\mu})F_1$, the support of $F_1 - F_{\varepsilon,\tilde{\mu}}$ is contained in supp $(1-\phi(\cdot;\varepsilon,\tilde{\mu}))\cap$ supp F_1. But supp $(1-\phi(\cdot;\varepsilon,\tilde{\mu})) \subset \overline{\Omega_T} \setminus (\Omega_{\tilde{t}} \cup \mathrm{int}(K_{\varepsilon/2}(\tilde{t})))$. Then the proof is done if we show that $(\tilde{t}, a(\tilde{t})) \notin$ supp $(F_1 - F_{\varepsilon,\tilde{\mu}})$. But, thanks to (3.8), the support of F_1 is localized in $\{x \leq \varepsilon\} \cup \{x \geq b - \varepsilon\}$ that does not touch ∂D. ☐

Lemma 3.6. *Let f as in Theorem 1.3. There exists $c > 0$ and $\varepsilon_0 > 0$, independent of f, such that, for all $\varepsilon \in (0,\varepsilon_0)$, $\tilde{\mu} \in [0,T]$, $F_{\varepsilon,\tilde{\mu}} \in C([0,T]; H^{r_0(1-\tilde{\xi}/T)+c\varepsilon-1}(\Omega))$.*

To prove it, we use the following well-known property.

Proposition 3.7. *Let $g \in H^s(\mathbb{R})$ for some $s \in [-1, 0]$. Let $r \in \mathbb{R}^*$ and $G(t,x) = g(t + rx)$, $(t,x) \in \Omega_T$. Then $G \in C([0,T]; H^s(\Omega))$.*

Let us prove Lemma 3.6. Observe that, by definition of $\phi(\cdot;\varepsilon,\tilde{\mu})$, and thanks to (3.8), the support of $F_{\varepsilon,\tilde{\mu}}|_{\Omega_{\tilde{\mu}}}$ is a subset of the set

$$E(\varepsilon,\tilde{\mu}) = K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D^C}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D} \cap \{b - \varepsilon \leq x \leq b\}).$$

Firstly, let $(t,x) \in K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D^C})$. Then we have $t - x \leq \tilde{t} + \varepsilon$, and so

$$\xi(\mu^{-1}(t-x)) < \xi(\mu^{-1}(\tilde{t}+\varepsilon)) < \xi(\mu^{-1}(\tilde{\mu} - \delta + \varepsilon)),$$

since the functions ξ and $\mu^{-1}$are smooth and non decreasing, and $\delta < a(\tilde{t}) = \tilde{\mu} - \tilde{t}$. So, for ε sufficiently small and some $c > 0$ (values that are independent of t, x), we have

$$\xi(\mu^{-1}(t-x)) < \tilde{\xi} - c\varepsilon, \quad (t,x) \in K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D^C}). \tag{3.9}$$

Secondly, let $(t,x) \in \overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D} \cap \{b - \varepsilon \leq x \leq b\}$. Then $t - \frac{x}{k} \leq \nu(t) - \frac{\delta-\varepsilon}{k}$ and so, for ε sufficiently small and some $c > 0$,

$$\xi(\nu^{-1}(t - \frac{x}{k})) \leq \xi(\nu^{-1}(\nu(t) - \frac{\delta - \varepsilon}{k})) < \tilde{\xi} - C\varepsilon.$$

We thus have

$$\xi(\nu^{-1}(t-\frac{x}{k})) < \tilde{\xi} - C\varepsilon, \quad (t,x) \in \overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D} \cap \{b-\varepsilon \leq x \leq b\}. \tag{3.10}$$

Since F_1 is expressed in terms of $f_2'(t-x)$, $f_2(t-x)$ in D^C, and in terms of $f_3'(t-\frac{x}{k})$, $f_3(t-\frac{x}{k})$ in D, and since the support of $F_{\varepsilon,\tilde{\mu}}$ is contained in $E(\varepsilon,\tilde{\mu})$, then, thanks to (3.9), (3.10), we see that $F_{\varepsilon,\tilde{\mu}}$ can be expressed in terms of $f|_{(-\infty,r)}$ and $f'|_{(-\infty,r)}$, $r = \tilde{\xi} - c\varepsilon$ only. Hence, thanks to Proposition 3.7, the conclusion follows. □

4 Proof of the main results

4.1 Proof of Corollary 1.4

Firstly, notice that $\alpha(t) \neq 0 \Longleftrightarrow \dot{a}(t) \neq \frac{k}{1+k}$.
1) If $T \leq \mu_0$ then $g = 0$ in $(0,T)$, and if $T > \mu_0$ then $g \neq 0$ since $g|_{(\mu_0,T)} \notin H^{r_0(1-s^*/T)-1}(\mu_0,T)$. Hence, the knowledge of g provides $T \leq \mu_0$ or $T > \mu_0$.
2)

- Let $\mu \in [\mu_0, T]$. Thanks to Theorem 1.3, we can construct

  $$\xi = \inf\{r > 0;\ g|_{(0,\mu)} \in H^{r_0(1-r/T)-1}(0,\mu)\},$$

  and so the invertible function $\mu \mapsto \xi$ from $[\mu_0, T]$ into $[0, s^*]$. (This implies that s^* is recovered too). Putting $t = \frac{1}{2}(\mu+\xi)$, we recover t_{s^*} which is t for $\mu = T$, and also the functions $t \mapsto \xi = \xi(t)$, $t \mapsto \mu(t)$, $t \mapsto a(t) = \frac{1}{2}(\mu(t) - \xi(t))$, for $t \in [t_0, t_{s^*}]$. We then construct the functions $t_{(\cdot)} = (\xi(\cdot))^{-1}$, $t^*(\cdot) = 2t_{(\cdot)} - \mathrm{id}$.

- Thanks to the above point and to (i) of Theorem 1.3, the smooth function $\alpha(\cdot)$ can be recover as the unique one such that $\mu \mapsto g(\mu) - \alpha(t)f'(\xi)$ belongs to $H^{\varepsilon + r_0(1-\xi/T)}(0,\mu)$ for some $\varepsilon > 0$ and all $\mu \in (0,T)$. Then, k is root of the following equation:

  $$(\alpha + 1 + \dot{a}(\alpha - 1))k^2 + (\alpha - 1)k + \dot{a}(1-\alpha) = 0. \tag{4.1}$$

  Denote by k_1, k_2 the roots, such that $k_1 \leq k_2$. We show that $k_1 \leq 0$. A short computation shows that

  $$(\alpha + 1 + \dot{a}(\alpha-1)) = \frac{2}{D}\left(\frac{(1-\dot{a})^2}{1+\dot{a}}\right) > 0, \quad D = k(1+\dot{a}) + 1 - \dot{a}/k > 0.$$

We have

$$k_1k_2 = \frac{\dot{a}(1-\alpha)}{\alpha+1+\dot{a}(\alpha-1)} = \dot{a}(k_1+k_2). \tag{4.2}$$

If $\dot{a} \leq 0$ then, the second equality in (4.2) implies that it is impossible to have $0 < k_1 \leq k_2$.

?????????,

□

Remark 4.1. *Theorem 1.3 allows us to recover* $t^*(\cdot) = \mu \circ \xi^{-1}$ *as:*

$$t^*(s) := \sup\{t > s;\ g|_{[s,t]} \in H^{r_0(1-t/T)-1}([0,t])\},$$

and shows that

$$t^*(s) = \sup\{t > s;\ g_A|_{[s,t]} \in H^{r_0(1-t/T)-1}([0,t])\}.$$

4.2 Analysis of the error

Let (u_0, u_1, f), r_0 as in Theorem 1.3. Put $u = \tilde{P}(u_0, u_1, f)$, $g = \tilde{Z}(u_0, u_1, f)$, $u_A = U_A(f)$ and

$$u_E = u - u_A, \quad F_A = T_A f, \quad g_A = \partial_x u_A|_{x=0}, \quad g_E = g - g_A = \partial_x u_E|_{x=0},$$

where u_A is defined in Section 3. Let us prove the estimate (1.7) (see (iv) of Theorem 1.3). For the sake of clarity, we replace μ, t, ξ, respectively by $\tilde{\mu}$, $\tilde{t} = \mu^{-1}(\tilde{\mu})$, $\tilde{\xi} = \xi(\tilde{t})$. Put $u_{E,0} = u_0 - u_A(0)$, $u_{E,1} = u_1 - \partial_t u_A\big|_{t=0}$. In view of Subsection 3, the function u_E satisfies

$$\begin{cases} \mathcal{L}_\gamma u_E &= -F_A \quad \text{in} \quad \Omega_{\tilde{\mu}}, \\ u_E|_{x=0,b} &= 0 \quad \text{on} \quad (0,\tilde{\mu}), \\ u_E\big|_{t=0} &= u_{E,1} \quad \text{on} \quad \Omega, \\ \partial_t u_E\big|_{t=0} &= u_{E,1} \quad \text{on} \quad \Omega. \end{cases} \tag{4.3}$$

So we have $u_E = P(u_{E,0}, u_{E,1}, -F_A)$. Recall that, thanks to Lemma **??**, we have $T_0(f) \in L^2(0,\tilde{\mu}; H^{-s}(\Omega))$, for all $s > \frac{1}{2}$. Thanks to Proposition 2.7, we have

$$Z(0,0,T_0(f))\big|_{(0,\tilde{\mu})} \in H^{-s}(0,\tilde{\mu}), \quad \forall s > \frac{1}{2}. \tag{4.4}$$

Let us prove that $u_{E,0} \in H^{r_0}(\Omega)$, $u_{E,1} \in H^{r_0-1}(\Omega)$. Observe that $u_A(0)(x) = (f(-x) + f_2(x) + f_2(-x)\Phi_\varepsilon(x))\chi_{x<a(0)} + f_3(-x/k)\Phi_\varepsilon(x-b+2\varepsilon))\chi_{x>a(0)}$. For $x < a(0) = t_0$ we have

$$\xi(\mu^{-1}(x)) < \xi(\mu^{-1}(t_0)) < \xi(\mu^{-1}(\mu_0)) = \xi(t_0) = 0,$$

and, similarly, $\xi(\mu^{-1}(-x)) \leq \xi(\mu^{-1}(0)) < 0$. For $x > a(0)$ we have

$$\xi(\nu^{-1}(-x/k)) < \xi(\nu^{-1}(-t_0/k)) < \xi(\nu^{-1}(\nu_0)) = 0.$$

Hence, $u_A(0)$ can be expressed in terms of $f(\xi)$ for $\xi < 0$. Since $f|_{(-\infty,0]} \in H^{r_0}(-\infty, 0)$, then $u_A(0) \in H^{r_0}(\Omega)$. Thanks to the asumption on u_0, we then have $u_{E,0} \in H^{r_0}(\Omega)$. Similarly, we have $u_{E,1} \in H^{r_0-1}(\Omega)$. Thanks to (3.6), the regularity of $f_2|_{(0,\tilde{\mu})}$ is given by those of $f|_{(0,\tilde{\xi})}$, that is, $f_2|_{(0,\tilde{\mu})} \in H^{r_0(1-\xi'/T)}((0, \tilde{\mu}))$, for all $\xi' > \tilde{\xi}$. Thus, thanks to Proposition 2.7, we have

$$Z(u_{E,0}, u_{E,1}, 0)\big|_{(0,\tilde{\mu})} \in H^{r_0-1}(0, \tilde{\mu}). \tag{4.5}$$

Thanks to Lemma 2.5 with t_1 replaced by $\tilde{t}$ and T by $\tilde{\mu}$, and to Lemma 3.5, we have

$$Z(0, 0, -F_1)\big|_{(0,\tilde{\mu})} = Z(0, 0, -F_{\varepsilon,\tilde{\mu}})\big|_{(0,\tilde{\mu})}. \tag{4.6}$$

Thanks to Lemma 3.6, if $\varepsilon > 0$ is sufficiently small, we have

$$F_{\varepsilon,\tilde{\mu}}\big|_{\Omega_{\tilde{\mu}}} \in L^2([0, \tilde{\mu}]; H^{r_0(1-\tilde{\xi}/T)+c\varepsilon-1}(\Omega)),$$

and so, thanks to (4.6) and by applying Proposition 2.7, we obtain

$$Z(0, 0, -F_1)\big|_{(0,\tilde{\mu})} \in H^{r_0(1-\tilde{\xi}/T)+\varepsilon-1}(0, \tilde{\mu}), \tag{4.7}$$

for some $\varepsilon > 0$ (independent of $\tilde{\mu}$).

Thanks to (4.4), (4.5) (4.7), and since $g_E = Z(u_{E,0}, u_{E,1}, 0)+Z(0, 0, T(0)f)+Z(0, 0, -F_1)$, the proof of (1.7) is done.

□

5 Appendix: the function G

Let $I = (0, 1)$ and a dense sequence $\{a_n\}_{n\in\mathbb{N}^*}$ in $\overline{I}$. We set

$$f_n(x) = ((x - a_n)_+)^{1/2-a_n},$$

$$G(x) = \sum_{n\in\mathbb{N}^*} \frac{1}{2^n} f_n(x), \quad x \in I,$$

where $z_+ = \max(0, z)$ for $z \in \mathbb{R}$. The function G is increasing.

For $0 < s < 1$ we set the following Sobolev space:

$$H^s(I) = \left\{ q \in L^2(I);\ \int\int_{I\times I} \frac{|q(x) - q(y)|^2}{|x - y|^{1+2s}}\, dx\, dy < \infty \right\}.$$

Lemma 5.1. *Let $b \in (0, 1]$, $r > -\frac{1}{2}$, $s \in (0, 1)$, $a \in [0, b)$. Set $f(x) = ((x - a)_+)^r$, $I_b = (0, b)$. We have $f \in H^s(I_b)$ if, and only if, $r > s - 1/2$. In such a case, we have*

$$\int\int_{I\times I} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}}\, dx\, dy \le C_s \left(\frac{1}{2r + 1} + \frac{r^2}{2r - 2s + 1} \right) (b - a)^{2r-2s+1}, \tag{5.1}$$

for some $C_s > 0$.

Proof. Firstly, let $b = 1$. We have

$$\begin{aligned}
J &:= \int\int_{I_1\times I_1} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}}\, dx\, dy = 2\int_0^1 dy \left(\int_0^y \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}}\, dx \right) \\
&= 2(K_1 + K_2), \\
K_1 &:= \int_a^1 dy \left(\int_0^a \frac{(y - a)^{2r}}{(y - x)^{1+2s}}\, dx \right), \\
K_2 &:= \int_a^1 dy \left(\int_a^y \frac{((y - a)^r - (x - a)^r)^2}{(y - x)^{1+2s}}\, dx \right).
\end{aligned}$$

We have

$$K_1 = \frac{1}{2s} \int_a^1 (y - a)^{2r} \left[\frac{1}{(y - x)^{2s}} \right]_0^a dy = \frac{1}{2s} \int_a^1 \left((y - a)^{2r-2s} - \frac{(y - a)^{2r}}{y^{2s}} \right) dy.$$

If $a = 0$, then $K_1 = 0$. If $a > 0$, then $K_1 < \infty$ if, and only if, $2r > 2s - 1$. In such a case, we have

$$K_1 \le \frac{1}{2s(2r - 2s + 1)} (1 - a)^{2r-2s+1}. \tag{5.2}$$

Let $2r > 2s-1$. We have

$$K_2 = \int_0^{1-a} dy \left(\int_0^y \frac{(y^r-x^r)^2}{(y-x)^{1+2s}}\,dx\right) = \int_0^{1-a} y^{2r-2s}dy \left(\int_0^1 \frac{(1-t^r)^2}{(1-t)^{1+2s}}\,dt\right)$$
$$= \frac{C(r,s)}{2r-2s+1}(1-a)^{2r-2s+1},$$

where

$$C(r,s) = \int_0^1 \frac{(1-t^r)^2}{(1-t)^{1+2s}}\,dt = \int_0^{1/2} \frac{(1-t^r)^2}{(1-t)^{1+2s}}\,dt + \int_{1/2}^1 \frac{(1-t^r)^2}{(1-t)^{1+2s}}\,dt$$
$$\leq C_s\left(\frac{1}{2r+1}+\frac{r^2}{2r-2s+1}\right). \tag{5.3}$$

Since $C(r,s) > 0$, then $K_2 = +\infty$ if $2r \leq 2s-1$. Hence, the sum K_1+K_2 converges iff $2r > 2s-1$. If $2r > 2s-1$, thanks to (5.2) and (5.3), we obtain (5.1).

Secondly, the case $b \in (0,1)$ is easily proved by setting $a = a'b$, $x = x'b$, $y = y'b$. □

Lemma 5.2. *For $0 < s < 1$ and $b \in (0,1]$, we have $G \in H^s(0,b)$ if $s < 1-b$ and $G \notin H^s(0,b)$ if $s > 1-b$.*

Proof. For $x, y \in I$, we have, thanks to the Schwarz inequality,

$$|G(x)-G(y)|^2 \leq \left(\sum_{n\geq 1}\frac{1}{2^n}\right)\left(\sum_{n\geq 1}\frac{1}{2^n}|f_n(x)-f_n(y)|^2\right) = \sum_{n\geq 1}\frac{1}{2^n}|f_n(x)-f_n(y)|^2. \tag{5.4}$$

Let $I_b = (0,b)$, $A_b = \{n \in \mathbb{N}^*;\ a_n \geq b\}$, $B_b = \mathbb{N}^* \setminus A_r = \{n; a_n < b\}$.
For all $n \in B_b$, thanks to Lemma 5.1, we have $f_n \in H^{1-b}(0,1)$, since $1/2 - a_n > (1-b) - 1/2$. For all $n \in A_b$, we have $f_n \in H^{1-b}(I_b)$, since $f_n|_{I_b} = 0$.
Let $0 < s < 1-b$. By using (5.4), and (5.1), we have

$$J_{b,s} := \int\int_{I_b\times I_b} \frac{|G(x)-G(y)|^2}{|x-y|^{1+2s}}\,dx\,dy \leq \sum_{n\in B_b}\frac{1}{2^n}\int\int_{I_b\times I_b}\frac{|f_n(x)-f_n(y)|^2}{|x-y|^{1+2s}}\,dx\,dy$$
$$\leq C_s \sum_{n\in B_b}\frac{1}{2^n}\left(\frac{1}{1-a_n}+\frac{1}{1-a_n-s}\right)(b-a_n)^{2(1-s-a_n)}$$
$$\leq C_s \sum_{n\in B_b}\frac{1}{2^n}\left(\frac{1}{1-b}+\frac{1}{1-b-s}\right)(b-a_n)^{2(1-s-a_n)} < \infty$$

since $(b-a_n)^{2(1-s-a_n)} \leq 1$ for all $n \in B_b$, $0 < s < 1-b$.

Let $s \in (1-b, 1)$. For all $n \in \mathbb{N}^*$ and $x > y$ we have $G(x) - G(y) \geq f_n(x) - f_n(y)$. Fix $n \in A_{1-s} \cap B_b$, that is, $1-s \leq a_n < b$. Thanks to Lemma 5.1, we have $f_n \notin H^s(I_b)$, and then

$$J_{b,s} \geq \frac{1}{2^n}\int\int_{I_b\times I_b}\frac{|f_n(x)-f_n(y)|^2}{|x-y|^{1+2s}}\,dx\,dy = \infty.$$

This ends the proof. □

6 Proof of Theorem 2.6

Let $F \in L^2(\Omega_T)$, $v_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$. Denote $M^1 := \{v \in C([0,T]; H_0^1(\Omega)),$ $\partial_t v \in C([0,T]; L^2(\Omega))\}$, $M_0^1 = \{v \in M;\ v|_{t=0} = 0,\ \partial_t v|_{t=0} = 0\}$,

6.1 Energy estimate.

Put
$$E(t)(v) = \frac{1}{2}\int_\Omega |\partial_t v|^2 + \frac{1}{2}\int_\Omega \gamma(t,\cdot)|\partial_x v|^2, \quad v \in M^1.$$
We claim that, for all $v \in M^1$ such that $L_\gamma v =: f \in L^2(\Omega_T) + W$, the following (standart) estimate, which implies (2.2), holds.

$$E(t)(v) \le C\left(\|f\|^2_{L^2(0,t;\Omega)} + E(0)(v)\right), \quad \forall t \in [0,T], \tag{6.1}$$

for some constant C.

Proof. It is sufficient to show (6.1) for $t = T$. Assume that $f \in L^2(\Omega_T)$. Put $\rho = \sup_Q \frac{|\dot\gamma|}{\gamma}$ and $\Pi_0 \in C^1([0,T]; (0,+\infty))$ such that $\delta^{-1}\Pi_0 \le -\Pi_0'$ for some $\delta \in (0, \frac{1}{\rho})$. (For example, $\Pi_0 = e^{-\frac{t}{\delta}}$). Put

$$Q(v) = \int_0^T E(t)(v)\, \Pi_0 dt, \quad C_0(f) = \int_Q f^2 \Pi_0.$$

We formally have, thanks te the Schwarz inequality,

$$\begin{aligned}
\delta^{-1}Q(v) &\le -\int_0^T E(t)(v)\, \Pi_0' dt = [-E(t)(v)\, \Pi_0(t)]_0^T - \int_0^T \frac{dE(t)(v)}{dt}\, \Pi_0 dt \\
&\le E(0)(v)\, \Pi_0(0) - E(T)(v)\, \Pi_0(T) - \frac{1}{2}\int_Q \Pi_0 \dot\gamma |\partial_t v|^2 - \int_Q \Pi_0 f \partial_t v \\
&\le E(0)(v)\, \Pi_0(0) - E(T)(v)\, \Pi_0(T) + \rho Q(v) + \sqrt{2C_0(f)}\sqrt{Q(v)},
\end{aligned}$$

Hence, we obtain

$$(\delta^{-1} - \rho)Q(v) + E(T)(v)\, \Pi_0(T) \le E(0)(v)\, \Pi_0(0) + \sqrt{2C_0(f)}\sqrt{Q(v)},$$

and so,
$$Q(v) + E(T)(v)\, \Pi_0(T) \le C(C_0(f) + E(0)(v)\, \Pi_0(0)). \tag{6.2}$$

Then (6.2) follows.

6.2 Uniqueness

Consequently, if $v \in M_0^1$ satisfies (2.1) with $F = 0$, then $E(t)(v) \equiv 0$ for all t, and so $v \equiv 0$. This shows that Problem (2.1) admits at most one solution in M^1.

6.3 Existence

Let $(\lambda_j, e_j)_{1 \le j}$ be the familly of spectral values of the positive operator $-\Delta_x$ in $H_0^1(\Omega)$, i.e such that $(e_i, e_j)_{L^2(\Omega)} = \delta_{ij}$, $-\Delta e_j = \lambda_j e_j$, and $\lambda_j \nearrow +\infty$. The data v_0, v_1, F are then written $v_0 = \sum_{j=1}^{\infty} v_{0,j} e_j$, $v_1 = \sum_{j=1}^{\infty} v_{1,j} e_j$, $F(t,\cdot) = \sum_{j=1}^{\infty} F_j(t) e_j$, with

$$\sum_{j=1}^{\infty} \{\lambda_j |v_{0,j}|^2 + |v_{1,j}|^2 + \int_0^T |F_j(t)|^2 dt\} < \infty.$$

Let $N \in \mathbb{N}^*$, and put $E_N = \text{span}\{e_1, \ldots, e_N\}$, $V_{k,N} = (v_{k,1}, \ldots, v_{k,N})$, $k = 0, 1$, $F_N = \sum_{j=1}^{N} F_j(t) e_j$, $B_N(t) = (b_{i,j}(t))_{1 \le i,j \le N}$ with $b_{i,j}(t) = (\nabla e_i, \nabla e_j)_{L^2(\Omega; \gamma(t,\cdot)dx)}$, and consider the following vectorial differential equation: find $V_N(t) = (v_1(t), \ldots, v_N(t))$ such that

$$\frac{d^2}{dt^2} V_N(t) + V_N(t) B_N(t) = F_N(t), \quad 0 \le t \le T,$$

with the initial condition $V_N(0) = V_{0,N}$, $\frac{d}{dt} V_N(0) = V_{1,N}$. Since $B_N(\cdot)$ is continuous, the theorem of Cauchy-Lipschitz implies existence and uniqueness for $V_N(t)$. Note that $B_N(t)$ is positive since, for all $U = (u_1, \ldots, u_N)$, setting $u(x) = \sum_{j=1}^{N} u_j e_j(x)$, we have

$$U B_N(t)\, {}^t U = \int_\Omega |\nabla_x u|^2 \gamma(t,x) dx \ge C \|\nabla_x u\|_{L^2(\Omega)}^2 = C \sum_{j=1}^{N} \lambda_j |u_j|^2,$$

where C is a constant such that $0 < C \le \gamma$ in Q. Let $v_N(t) = \sum_{j=1}^{N} v_j(t) e_j(x)$. Then, a standart energy estimate for $E_N(t)(v_N) = \frac{1}{2}(\dot{V}_N^2(t) + V_N(t) B_N(t) V_N(t))$, as above, implies that there exists a positive constant C such that

$$\|\dot{v}_N(t)\|_{L^2(\Omega)} + \|\partial_x v_N(t)\|_{L^2(\Omega)} \le C(\|v_0\|_{H^1(\Omega)} + \|v_1\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}), \quad 0 \le t \le T.$$

Passing to the limit $N \to +\infty$, we can conclude by standard arguments that $(v_N)_N$ converges to a function $v \in C([0,T]; H_0^1(\Omega))$ satisfying (2.1).

The proof of Theorem 2.6 in done in the case $F \in L^2(\Omega_T)$. The case $F \in W$ is similar.

References

[1] R. Dautray. and J.L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*,**7**, Masson, Paris Milan Barcelone Mexico (1988).

[2] P. Grisvard, *Singularities in boundary value problems*, RMA**22**, Masson, Springer-Verlag (1992).