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POLYNOMIALITY OF GROTHENDIECK GROUPS FOR FINITE GENERAL LINEAR GROUPS, DELIGNE-LUSZTIG CHARACTERS, AND INJECTIVE UNSTABLE MODULES

HÉLÈNE PÉRENNOU

ABSTRACT. Let $K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ denote the Grothendieck group of finitely generated projective $\mathbb{F}_p GL_n(\mathbb{F}_p)$ -modules. We show that the algebra $\mathbb{C} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$ with multiplication given by induction functors, is a polynomial algebra. We explicit generators and their relation with Deligne-Lusztig characters.

This work is motivated by several conjectures about unstable modules. Let $K(\mathcal{U})$ denote the Grothendieck group of reduced injective unstable modules of finite type over the mod. p Steenrod algebra. We obtain that $\mathbb{C} \otimes K(\mathcal{U})$, with multiplication given by tensor product of unstable modules, is a polynomial algebra. And we identify the Poincaré series associated to elements of $\mathbb{C} \otimes K(\mathcal{U})$.

INTRODUCTION

This paper studies the structure of modular representations of finite general linear groups in natural characteristic. Let $K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ denote the Grothendieck group of finite type projective $\mathbb{F}_p GL_n(\mathbb{F}_p)$ -modules, where \mathbb{F}_p is the prime field with p elements. The graded group $\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ is also a graded ring with multiplication given by induction functors. Using the characterization of modular projective modules by their Brauer characters, we identify the complexification $\mathbb{C} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$ with the set of class functions over $GL_n(\mathbb{F}_p)$ vanishing outside semi-simple classes. A basis of $\mathbb{C} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$ is given by the characteristic functions of semi-simple classes. A semi-simple class is entirely determined by the characteristic polynomial. Thus, monic polynomials over \mathbb{F}_p with a non-zero constant coefficient parametrize the characteristic functions of semi-simple classes. It happens that this indexing behaves well under multiplication. This gives the following,

Theorem. *The algebra $\mathbb{C} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$ is polynomial and a family of generators is given by the characteristic functions associated to irreducible polynomials over \mathbb{F}_p with a non-zero constant coefficient.*

Tensoring Steinberg modules St_n with Deligne-Lusztig [4] characters provides actual $\mathbb{F}_p GL_n(\mathbb{F}_p)$ -projective modules. We use it to give another description of the polynomial generators.

These results on modular characters of general linear groups are motivated by conjectures coming from the study of modules over the Steenrod algebra in algebraic topology. Denote by $K^n(\mathcal{U})$ the free abelian group generated by isomorphism classes of direct summands in the Steenrod algebra module $H^*V_n := H^*(V_n, \mathbb{Z}/p)$, where $V_n = (\mathbb{F}_p)^n$. Viewing a summand of H^*V_n as a summand of H^*V_{n+1} induces a monomorphism from $K^n(\mathcal{U})$ to $K^{n+1}(\mathcal{U})$. Thus $K(\mathcal{U}) = \bigcup_{n \geq 0} K^n(\mathcal{U})$ is a

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filtered free abelian group. It is also a ring with multiplication given by the tensor product of unstable modules. Carlisle and Kuhn state the following conjecture about this structure.

Conjecture. [2, §4] *The ring $K(\mathcal{U})$ is polynomial.*

It is known by [2, Thm. 3.2] that the rings $\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ and $K(\mathcal{U})$ are isomorphic. Theorem then shows that $K_{\mathbb{C}}(\mathcal{U}) := \mathbb{C} \otimes K(\mathcal{U})$ is a polynomial algebra. Indeed the Conjecture holds also over the rationals. We describe a family of polynomial generators (over \mathbb{C}) using Campbell-Selick direct summands [1], as in [9]. This gives a new proof of a conjecture of Schwartz [3, Conj. 3.2], already proved by Hai [7–9], see also [6].

Proposition 1. *The action of Lannes' T functor on $K_{\mathbb{C}}^n(\mathcal{U})$ is diagonalizable with spectrum $\{1, p, \dots, p^n\}$.*

We also identify Poincaré series associated to elements of $K_{\mathbb{C}}(\mathcal{U})$. Let $P_{\mathbb{C}}(\cdot, t)$ denotes the algebra map from $K_{\mathbb{C}}(\mathcal{U})$ to $\mathbb{C}[[t]]$ which associates to the class of a module M , its Poincaré series. Fix θ an embedding of $\overline{\mathbb{F}_p}^{\times}$ in \mathbb{C} .

Proposition 2. *The image of $P_{\mathbb{C}}(\cdot, t)$ is generated as an algebra by the following series*

$$P_{\mathbb{C}}(P_f, t) = \begin{cases} \frac{1}{(w-t)(w^2-t)\dots(w^{2^n-1}-t)}, & \text{for } p = 2, \\ \frac{(w+t)(w^p+t)\dots(w^{p^n-1}+t)}{(w-t^2)(w^p-t^2)\dots(w^{p^n-1}-t^2)}, & \text{for } p > 2, \end{cases}$$

where f runs through the set of irreducible polynomials over \mathbb{F}_p with a non-zero constant term and $w = \theta(\alpha)$ with α a root of f .

We recover, that the kernel of $P_{\mathbb{C}}(\cdot, t)$ is non-trivial for $p = 2$ [3] and our argument answers a more precise question by Hai [9, §5] :

Proposition 3. *For $p = 2$, the restriction of $P_{\mathbb{C}}(\cdot, t)$ to the 1-eigenspace of $K_{\mathbb{C}}(\mathcal{U})$ admits a non-trivial kernel.*

Organisation of the paper. The first section is devoted to the description of the polynomial structure of $\mathbb{C} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$. Section 2 explains the link with Deligne-Lusztig characters. Finally, section 3 gives applications to unstable modules over the Steenrod algebra.

1. POLYNOMIALITY

Let A be a ring, and denote by $K_0(A\text{-mod})$ (resp. $K_0(A\text{-proj})$) the Grothendieck group of A -modules (resp. projective A -modules) of finite type. The graded groups $\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-mod})$ and $\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ are also commutative rings with multiplication given by the induction functors $\text{Ind}_{GL_k(\mathbb{F}_p) \times GL_l(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p)}(-)$ where k, l et n are natural numbers satisfying $k+l = n$. Given a (projective) finitely generated $\mathbb{F}_p GL_n(\mathbb{F}_p)$ -module V , denote by $[V]$ its class in $K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-mod})$ (or in $K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$). In this first section, we show that $\mathbb{C} \otimes \bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ is a polynomial algebra by giving an explicit family of generators.

Remark 4. *We work over \mathbb{F}_p . Since it is a splitting field for $GL_n(\mathbb{F}_p)$, one can replace it by any \mathbb{F}_q , where q is a power of p .*

1.1. Characters of general linear groups. Using the link between a projective module and its Brauer character, for each natural number n , we identify $\mathbb{C} \otimes K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ with the space of class functions over $GL_n(\mathbb{F}_p)$, vanishing outside p -regular elements (*i.e.* elements of order prime to p). Recall that an element of $GL_n(\mathbb{F}_p)$ has order prime to p if, and only if, it is semi-simple (*i.e.* diagonalizable in a finite extension of \mathbb{F}_p). With this point of view, the multiplication in $\mathbb{C} \otimes K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj})$ is given by the formula :

$$\rho_n \cdot \rho_m(g) = \frac{1}{|GL_n(\mathbb{F}_p)| \cdot |GL_m(\mathbb{F}_p)|} \sum_{\substack{h \in GL_{n+m}(\mathbb{F}_p), \\ hgh^{-1} = \begin{pmatrix} g_n & 0 \\ 0 & g_m \end{pmatrix}}} \rho_n(g_n) \rho_m(g_m),$$

where ρ_n and ρ_m are p -regular class functions over $GL_n(\mathbb{F}_p)$ and $GL_m(\mathbb{F}_p)$ respectively.

For g in $GL_n(\mathbb{F}_p)$, denote by χ_g the characteristic polynomial of g .

Definition 1. Let f in $\mathbb{F}_p[x]$ of degree n , $f \neq x$. We denote by π_f the class function over $GL_n(\mathbb{F}_p)$ defined by

$$\pi_f(g) = \delta_{\chi_g, f},$$

where δ is the Kronecker's symbol.

Thus, π_f is the characteristic function of the conjugacy class of semi-simple elements of $GL_n(\mathbb{F}_p)$ for which the characteristic polynomial is f . This choice of index coincides with the multiplication in the following way. For all f_n, f_m in $\mathbb{F}_p[x]$ of degree n and m respectively, and g in $GL_{n+m}(\mathbb{F}_p)$,

$$(1) \quad \pi_{f_n} \cdot \pi_{f_m}(g) = \frac{1}{|GL_n(\mathbb{F}_p)| \cdot |GL_m(\mathbb{F}_p)|} \sum_{\substack{h \in GL_{n+m}(\mathbb{F}_p), \\ hgh^{-1} = \begin{pmatrix} g_n & 0 \\ 0 & g_m \end{pmatrix}}} \pi_{f_n}(g_n) \pi_{f_m}(g_m) = c_{f_n, f_m} \pi_{f_n \cdot f_m}$$

where c_{f_n, f_m} is an integer. Thus, we have the following.

Theorem 5. *The algebra $\mathbb{C} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$ is polynomial with one generator for each irreducible polynomial in $\mathbb{F}_p[x]$ with non-zero constant coefficient. Precisely,*

$$\mathbb{C} \otimes \left(\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) \right) \cong \mathbb{C}[\pi_f, f \text{ irreducible in } \mathbb{F}_p[x], f \neq x].$$

Corollary 6. *The algebra $\mathbb{Q} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$ is polynomial with one generator for each irreducible polynomial in $\mathbb{F}_p[x]$ with non-zero constant coefficient.*

1.2. Multiplicative constants. In order to complete the description of this algebra, it remains to compute the constants c_{f_n, f_m} from (1).

Consider the ring $\bigoplus_{n \geq 0} K_0(\mathbb{C} GL_n(\mathbb{F}_p)\text{-mod})$ where now the multiplication is defined using parabolic induction PInd (also called Harish-Chandra induction). The functor

$$\text{PInd} : \mathbb{C} GL_k(\mathbb{F}_p)\text{-mod} \times \mathbb{C} GL_l(\mathbb{F}_p)\text{-mod} \rightarrow \mathbb{C} GL_n(\mathbb{F}_p)\text{-mod},$$

with $k+l=n$, is the inflation from $GL_k(\mathbb{F}_p) \times GL_l(\mathbb{F}_p)$ to the parabolic subgroup $P_{k,l}$ of $GL_n(\mathbb{F}_p)$, composed with the (ordinary) induction $\text{Ind}_{P_{k,l}}^{GL_n(\mathbb{F}_p)}(-)$ (see [15, §8]). This ring is studied by Springer and Zelevinsky in [14, 15]. They focus on its polynomial structure. We also define the graded ring $(\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-mod}), \text{PInd})$ in a same way.

The main tools to link these rings are the Steinberg modules, denoted by St_n , and Brauer theory.

Theorem 7. [12] [5, 9.6] *The map*

$$\begin{aligned} (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-mod}), \text{PInd}) &\rightarrow (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}), \text{Ind}) \\ [V] &\mapsto [\text{St}_n \otimes V] \end{aligned}$$

is a ring isomorphism.

Corollary 8. *The map*

$$\begin{aligned} (\bigoplus_{n \geq 0} K_0(\mathbb{C} GL_n(\mathbb{F}_p)\text{-mod}), \text{PInd}) &\rightarrow (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}), \text{Ind}) \\ [V] &\mapsto [\text{St}_n \otimes d_n(V)] \end{aligned}$$

where $d_n : \bigoplus_{n \geq 0} K_0(\mathbb{C} GL_n(\mathbb{F}_p)\text{-mod}) \mapsto \bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-mod})$ is the reduction map [13, §15.2], is an epimorphism.

We can now compute the constants from (1).

Proposition 9. *Let f and g be monic polynomials in $\mathbb{F}_p[x]$ with non-zero constant terms.*

- *If f and g are relatively prime, $\pi_f \cdot \pi_g = \pi_{f \cdot g}$, that is*

$$c_{f,g} = 1.$$

- *If f is irreducible of degree d . For all natural numbers n and m ,*

$$c_{f^n, f^m} = p^{dmn} \frac{\psi_{n+m}(p^d)}{\psi_n(p^d)\psi_m(p^d)}$$

$$\text{where } \psi_k(p^d) = \prod_{i=1}^k (p^{id} - 1).$$

Proof. The first point follows from [14, §1.4]. For the second point let n be any integer. The Steinberg character is described as follow. Let f_1, \dots, f_k be irreducible polynomials over \mathbb{F}_p with non-zero constant term and of degree d_1, \dots, d_k respectively. For a semi-simple element g in $GL_n(\mathbb{F}_p)$ with characteristic polynomial $f_1^{n_1} \dots f_k^{n_k}$,

$$(2) \quad \text{St}_n(g) = (-1)^N p^D,$$

where $N = n - \sum_{i=1}^k n_i$ et $D = \sum_{i=1}^k d_i \binom{n_i}{2}$ (see [14, §1.13]).

Thus,

$$\pi_{f^n} = \frac{(-1)^{nd-n}}{p^{d \binom{n}{2}}} \text{St}_{dn} \otimes \pi_{f^n} \quad \text{and} \quad \pi_{f^m} = \frac{(-1)^{md-m}}{p^{d \binom{m}{2}}} \text{St}_{dm} \otimes \pi_{f^m}.$$

So,

$$\pi_{f^n} \cdot \pi_{f^m} = \frac{(-1)^{d(m+n)-m+n}}{p^{d(\binom{n}{2} + \binom{m}{2})}} \text{St}_{d(n+m)} \otimes \text{PInd}_{GL_{dn}(\mathbb{F}_p) \times GL_{dm}(\mathbb{F}_p)}^{GL_{d(n+m)}(\mathbb{F}_p)} (\pi_{f^n} \otimes \pi_{f^m}).$$

And by Theorem 7 and [15, Prop. 10.1],

$$\text{St}_{d(n+m)} \otimes \text{PInd}_{GL_{dn}(\mathbb{F}_p) \times GL_{dm}(\mathbb{F}_p)}^{GL_{d(n+m)}(\mathbb{F}_p)} (\pi_{f^n} \otimes \pi_{f^m}) = \text{St}_{d(n+m)} \otimes \frac{\psi_{n+m}(p^d)}{\psi_n(p^d)\psi_m(p^d)} \pi_{f^{n+m}}.$$

The result follows again by (2). □

Corollary 10. *Let f be an irreducible polynomial of degree d over \mathbb{F}_p with a non-zero constant coefficient,*

$$(\pi_f)^n = p^{d \binom{n}{2}} \frac{\psi_n(p^d)}{\psi_1(p^d)^n} \pi_{f^n}.$$

1.3. Grouplike elements. The ring $\bigoplus_{n \geq 0} K_0(GL_n(\mathbb{F}_p))$ is also a bi-algebra [10, Thm. 6.10] (which is not graded) with co-multiplication Δ defined in degree n by

$$\text{Ind}_{GL_n(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)} : K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) \rightarrow K_0(\mathbb{F}_p(GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p))\text{-proj})$$

composed with the inverse of

$$- \otimes - : K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) \otimes K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) \rightarrow K_0(\mathbb{F}_p(GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p))\text{-proj}).$$

The functions π_f have the following nice property.

Lemma 11. *For f a monic polynomial over \mathbb{F}_p with a non-zero constant coefficient, π_f is grouplike:*

$$\Delta\pi_f = \pi_f \otimes \pi_f.$$

Proof. Let f be as above and of degree n . Fix (a, b) in $GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)$. The induction formula for class functions gives :

$$\begin{aligned} \text{Ind}_{GL_n(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)}(\pi_f)(a, b) &= \frac{1}{|GL_n(\mathbb{F}_p)|} \sum_{\substack{(h,l) \in GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p) \\ (hah^{-1}, lbl^{-1}) \in GL_n(\mathbb{F}_p)}} \pi_f(hah^{-1}) \\ &= \frac{1}{|GL_n(\mathbb{F}_p)|} \sum_{\substack{(h,l) \in GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p) \\ hah^{-1} = lbl^{-1}}} \pi_f(hah^{-1}) \end{aligned}$$

Thus

$$\begin{aligned} \text{Ind}_{GL_n(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)}(\pi_f)(a, b) &= \text{Ind}_{GL_n(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)}(\pi_f)(a, a) \delta_{\chi_a, \chi_b} \\ &= \frac{1}{|GL_n(\mathbb{F}_p)|} \sum_{\substack{(h,l) \in GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p) \\ hah^{-1} = lal^{-1}}} \pi_f(a) \delta_{\chi_a, \chi_b}. \end{aligned}$$

The one to one correspondence between

$$\{(h, l) \in GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p) | hah^{-1} = lal^{-1}\} \text{ and } \text{Stab}_{GL_n(\mathbb{F}_p)}(a) \times \text{Orb}(a),$$

gives

$$\#\{(h, l) \in GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p) | hah^{-1} = lal^{-1}\} = |GL_n(\mathbb{F}_p)|$$

and thus,

$$\text{Ind}_{GL_n(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)}(\pi_f)(a, a) = \pi_f(a) = \delta_{\chi_a, f}.$$

Finally,

$$\text{Ind}_{GL_n(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)}(\pi_f)(a, b) = \delta_{(\chi_a, \chi_b), (f, f)}$$

Thus $\text{Ind}_{GL_n(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p)}(\pi_f)$ is the image of $\pi_f \otimes \pi_f$ by

$$K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) \otimes K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) \xrightarrow{\otimes} K_0(\mathbb{F}_p(GL_n(\mathbb{F}_p) \times GL_n(\mathbb{F}_p))\text{-proj})$$

and $\Delta(\pi_f) = \pi_f \otimes \pi_f$. □

2. DELIGNE-LUSZTIG CHARACTERS

In this section, we use Deligne-Lusztig characters to give another description of the class functions π_f . Let α be a primitive root of degree n over \mathbb{F}_p (that is a cyclic generator of $\mathbb{F}_{p^n}^\times$), let θ be an embedding of $\overline{\mathbb{F}_p}^\times$ in \mathbb{C}^\times and let $w = \theta(\alpha)$. We denote by T_n the cyclic group generated by α , thus $T_n \cong \mathbb{Z}/(p^n - 1)$. The group T_n can be viewed as a subgroup of $GL_n(\mathbb{F}_p)$ by choosing a basis of \mathbb{F}_{p^n} as an \mathbb{F}_p -vector space. Finally, denote for all natural number i in $\{0, \dots, p^n - 2\}$,

$$\begin{aligned} \varphi_i &: T_n \rightarrow \mathbb{C}^\times \\ \alpha^k &\mapsto \omega^{ki} \end{aligned}$$

the irreducible characters of T_n . The induced characters $\text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\varphi_i)$, for i in $\{0, \dots, p^n - 2\}$, are projective and they only depend on the orbit of α^i under the action of the Frobenius map. Moreover, they are closely related to Deligne-Lusztig characters.

Lemma 12. [4, Prop. 7.3] For all i in $\{0, \dots, p^n - 2\}$,

$$\text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\varphi_i) = (-1)^{n-1} R_{T_n}^{\varphi_i} \otimes \text{St}_n,$$

where $R_{T_n}^{\varphi_i}$ is the Deligne-Lusztig character associated to the character φ_i of T_n [4].

Now consider, for all i in $\{0, \dots, p^n - 2\}$, $\hat{\varphi}_i$ the Fourier transform of φ_i . For all i and k in $\{0, \dots, p^n - 2\}$,

$$\hat{\varphi}_i(\alpha^k) = \frac{1}{p^n - 1} \sum_{j=0}^{p^n-2} w^{-kj} \varphi_i(\alpha^j) = \delta_{i,k}.$$

Thus, $\hat{\varphi}_i$ is the indicator function of α^i over T_n . We have the following identity,

Lemma 13. For all k in $\{0, \dots, p^n - 2\}$,

$$\hat{\varphi}_k = \frac{1}{p^n - 1} \sum_{j=0}^{p^n-2} w^{-jk} \varphi_j.$$

Lemma 13 gives a particular case of [4, Prop. 7.5] :

$$(3) \quad \text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\hat{\varphi}_k) = \frac{(-1)^{n-1}}{p^n - 1} \sum_{j=0}^{p^n-2} w^{-jk} R_{T_n}^{\varphi_j} \otimes \text{St}_n.$$

Proposition 14. For all k in $\{0, \dots, p^n - 2\}$,

$$\text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\hat{\varphi}_k) = \frac{\psi_{m_k}(p^{d_k})}{\psi_1(p^n)} \pi_{f_k^{m_k}}$$

where f_k is the irreducible polynomial over \mathbb{F}_p with roots $\{\alpha^k, \alpha^{pk}, \dots, \alpha^{p^{n-1}k}\}$, d_k is the degree of f_k , $m_k = n/d_k$, and $\psi_m(p^d) = \prod_{i=1}^m (p^{id} - 1)$ as in Proposition 9.

Proof. We saw that $\hat{\varphi}_k$ is the indicator function of α^k . Thus $\text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\hat{\varphi}_k)$ is collinear to the characteristic function of the conjugacy class of α^k in $GL_n(\mathbb{F}_p)$. This is exactly the class function $\pi_{f_k^{m_k}}$.

It remains to compute $\text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\hat{\varphi}_k)(\alpha^k)$:

$$\text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\hat{\varphi}_k)(\alpha^k) = \frac{1}{|T_n|} \sum_{\substack{h \in GL_n(\mathbb{F}_p), \\ h\alpha^k h^{-1} \in T_n}} \hat{\varphi}_k(\alpha^k) = \frac{|Z_{GL_n(\mathbb{F}_p)}(\alpha^k)|}{|T_n|}$$

Now $|Z_{GL_n(\mathbb{F}_p)}(\alpha^k)|$ depends on the orbit of the α^k under the Frobenius. Denote by d_k the cardinal of $\{\alpha^k, \alpha^{p^k}, \dots, \alpha^{k p^{n-1}}\}$, the orbit of α^k , and $m_k = n/d_k$. One has,

$$|Z_{GL_n(\mathbb{F}_p)}(\alpha^k)| = |GL_{m_k}(\mathbb{F}_{p^{d_k}})|.$$

Thus,

$$\begin{aligned} \frac{|Z_{GL_n(\mathbb{F}_p)}(\alpha^k)|}{|T_n|} &= \frac{(p^{d_k m_k} - 1)(p^{d_k m_k - p^{d_k}}) \dots (p^{d_k m_k - p^{d_k m_k - d_k}})}{p^{n-1}} \\ &= \frac{\psi_{m_k}(p^{d_k})}{\psi_1(p^n)}. \end{aligned}$$

□

Recall from Corollary 10 that for f_k irreducible monic polynomial with roots set $\{\alpha^k, \alpha^{p^k}, \dots, \alpha^{p^{n-1}k}\}$,

$$\pi_{f_k}^{m_k} = p^{d_k \binom{m_k}{2}} \frac{\psi_{m_k}(p^{d_k})}{\psi_1(p^{d_k})^{m_k}} \pi_{f_k}^{m_k}.$$

Then Proposition 14 implies :

Proposition 15.

$$\pi_{f_k}^{m_k} = (-1)^{n-1} \frac{p^{d_k \binom{m_k}{2}}}{(p^{d_k} - 1)^{m_k}} \sum_{j=0}^{p^n - 2} w^{-jk} R_{T_n}^{\varphi_j} \otimes \text{St}_n,$$

and inverting the Fourier transform gives,

$$R_{T_n}^{\varphi_i} \otimes \text{St}_n = \sum_{k=0}^{p^n - 2} w^{ik} \frac{\psi_1(p^{d_k})^{m_k}}{p^{d_k \binom{m_k}{2}} \psi_1(p^n)} \pi_{f_k}^{m_k}.$$

Thus, we can choose polynomial generators for $\mathbb{C} \otimes (\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}))$ among the representations $(-1)^{n-1} R_{T_n}^{\varphi_j} \otimes \text{St}_n$.

3. APPLICATIONS FOR UNSTABLE MODULES

We use the notations of the introduction. A conjecture of Carlisle et Kuhn [2, §8] states that,

Conjecture 1. [2, §4] *The ring $K(\mathcal{U})$ is polynomial over \mathbb{Z} .*

This conjecture is discussed in [9, §4]. This conjecture translates in terms of modular representations of the general linear groups as follows. For a projective $\mathbb{F}_p GL_n(\mathbb{F}_p)$ -module P , consider the unstable module $\text{Hom}_{GL_n(\mathbb{F}_p)}(P, H^* V_n)$. It is isomorphic to a direct summand of $H^* V_n$. The resulting map

$$(4) \quad \begin{array}{ccc} \bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) & \rightarrow & K(\mathcal{U}) \\ [P] & \mapsto & [\text{Hom}_{GL_n(\mathbb{F}_p)}(P, H^* V_n)], \end{array}$$

is a ring isomorphism by [2, Thm. 3.2]. We extend it to an \mathbb{C} -algebra isomorphism,

$$(5) \quad \mathbb{C} \otimes \left(\bigoplus_{n \geq 0} K_0(\mathbb{F}_p GL_n(\mathbb{F}_p)\text{-proj}) \right) \xrightarrow{\sim} K_{\mathbb{C}}(\mathcal{U}).$$

Theorem 5 proves a weak form of the statement of Conjecture 1.

Theorem 16. *The algebra $K_{\mathbb{C}}(\mathcal{U})$ is polynomial and a family of generators is*

$$\{P_f, f \text{ in } \mathbb{F}_p[x] \text{ irreducible, } f(0) \neq 0\},$$

where P_f is the image of π_f by the isomorphism (5).

Corollary 6 gives the result over the rationals.

Corollary 17. *The algebra $\mathbb{Q} \otimes K(\mathcal{U})$ is polynomial with one generator for each irreducible polynomial over \mathbb{F}_p with a non-zero constant coefficient.*

Recall from Proposition 14 that the class functions π_f are complex linear combination of tensor products of Deligne-Lusztig characters with St_n . The isomorphism (4) provides a one-to-one correspondence between Campbell-Selick [1] direct summands $M_n(j)$ and characters $\text{St}_n \otimes R_{T_n}^{\varphi_i}$ [9, §4]. Proposition 15 now reads :

$$P_f = \frac{1}{p^n - 1} \sum_{j=0}^{|T_n|-1} w^{-j} M_n(j).$$

Note that summands $M_n(j)$ were used by Hai to describe eigenvectors of the Lannes' T functor over the rationals [9, §1].

3.1. Eigenvectors of Lannes' T functor. We now consider the action of the Lannes' T functor [11]. The functor T is defined as the left adjoint to $-\otimes H^* \mathbb{F}_p : \mathcal{U} \rightarrow \mathcal{U}$. This is an exact functor and it commutes with tensor product [11]. In particular, it induces a ring endomorphism of $K(\mathcal{U})$, and we still denote it by T . By [11],

$$T(\text{Hom}_{GL_n(\mathbb{F}_p)}(P, H^* V_n)) \cong \text{Hom}_{GL_n(\mathbb{F}_p)}(P \otimes \mathbb{F}_p[V_n^*], H^* V_n).$$

By the isomorphism (4), we also consider T as an endomorphism of $\bigoplus_{n \geq 0} K_0(GL_n(\mathbb{F}_p)\text{-proj})$. Then,

$$T([P]) = [P \otimes \mathbb{F}_p[V_n^*]].$$

Let $T_{\mathbb{C}}$ denote the complexification of T .

Proposition 18. *Let $f = (x-1)^n g$ in $\mathbb{F}_p[x]$ with $g(0)g(1) \neq 0$. The function π_f is an p^n -eigenvector of $T_{\mathbb{C}}$.*

Proof. Let f be a polynomial of degree n over \mathbb{F}_p with a non-zero constant coefficient. One has, $T(\pi_f) = \pi_f \otimes \mathbb{F}_p[V_n^*]$. Then, let g be in $GL_n(\mathbb{F}_p)$ such that its characteristic polynomial is f . One has

$$\pi_f \otimes \mathbb{F}_p[V_n^*](g) = \begin{cases} \mathbb{F}_p[V_n^*](g), & \text{if } \chi_g = f \\ 0 & \text{else.} \end{cases}$$

If $f \neq x-1$ and $\chi_g = f$, then $g : V_n^* \rightarrow V_n^*$ has no fixed point. Thus $\mathbb{F}_p[V_n^*](g) = 1$ and $T(\pi_f) = \pi_f$. Else, $f = x-1$ and for g satisfying $\chi_g = f$, one has $\mathbb{F}_p[V_1^*](g) = p$. Thus $T(\pi_f) = p\pi_f$. \square

As a corollary, we recover Schwartz' conjecture [3, Conj. 3.2] about the diagonalization of $T_{\mathbb{C}}$. This conjecture was proved in [7] and [8] by different methods.

Corollary 19. *The action of T on $K_{\mathbb{C}}^n(\mathcal{U})$ is diagonalizable and its eigenvalues are $1, p, \dots, p^n$. Furthermore, for $0 \leq i \leq n$, a basis of the p^i -eigenspace is*

$$((H^* \mathbb{F}_p)^{\otimes i} \otimes P_f, \deg(f) = n - i, f(1) \neq 0).$$

Its dimension is $p^{n-i} - p^{n-i-1}$ for $i < n$, and it is one-dimensional for $i = n$.

3.2. Poincaré series. To conclude, we identify the Poincaré series associated to elements of $K_{\mathbb{C}}(\mathcal{U})$. As in the introduction, let $P(\cdot, t)$ denote the map which associates to the class of an unstable module its Poincaré serie,

$$\begin{aligned} K(\mathcal{U}) &\rightarrow \mathbb{Z}[[t]] \\ [M] &\mapsto \sum_{d \geq 0} \dim M^d t^d. \end{aligned}$$

We denote by $P_{\mathbb{C}}(\cdot, t)$ its complexification. Finally, recall that for f a polynomial over \mathbb{F}_p with a non-zero constant coefficient, P_f denotes the element of $K_{\mathbb{C}}(\mathcal{U})$ associated to π_f . The following proposition describes the image of $P_{\mathbb{C}}(\cdot, t)$.

Proposition 20. *Let f be an irreducible polynomial over \mathbb{F}_p with a non-zero constant term,*

$$P_{\mathbb{C}}(P_f, t) = \begin{cases} \frac{1}{(w-t)(w^2-t)\dots(w^{2^{n-1}}-t)}, & p = 2, \\ \frac{(w+t)(w^p+t)\dots(w^{p^{n-1}}+t)}{(w-t^2)(w^p-t^2)\dots(w^{p^{n-1}}-t^2)}, & p > 2. \end{cases}$$

Proof. Since

$$P_f = \frac{1}{p^n - 1} \sum_{j=0}^{|T_n|-1} w^{-j} M_n(j),$$

and

$$M_n(j) = \text{Hom}_{GL_n(\mathbb{F}_p)}(\text{Ind}_{T_n}^{GL_n(\mathbb{F}_p)}(\varphi_j), H^*V_n),$$

the result follows from Molien's formula. \square

The case $p = 2$. In this case, the formula in the Proposition 20 takes a simple form.

Let f be an irreducible polynomial over \mathbb{F}_2 and $\alpha_1, \dots, \alpha_n$ its roots in $\overline{\mathbb{F}_2}^{\times}$. We denote by \tilde{f} the polynomial $(\theta(\alpha_1) - t) \dots (\theta(\alpha_n) - t)$ in $\mathbb{C}[t]$. The image of $P_{\mathbb{C}}(\cdot, t)$ is the subalgebra of $\mathbb{C}(t)$ generated by $\{1/\tilde{f} \mid f \in \mathbb{F}_2[x], f(0) \neq 0\}$.

Let g be the product of irreducible polynomials f_1, \dots, f_N , we denote $P_g = P_{f_1} \otimes \dots \otimes P_{f_N}$ and $\tilde{g} = \tilde{f}_1 \dots \tilde{f}_N$. Thus we have $P_{\mathbb{C}}(P_g, t) = 1/\tilde{g}$.

Proposition 21. *For $p = 2$, the kernel $P_{\mathbb{C}}(\cdot, t)$ is non-trivial.*

Proof. Let N be a natural number and f_1, \dots, f_{k_N} be irreducible polynomial over \mathbb{F}_2 of degree N . By Proposition 20, for all $\alpha_1, \dots, \alpha_{k_N}$, in \mathbb{C} , the serie

$$\sum_{i=1}^{k_N} \frac{\alpha_i}{\prod_{j=1, j \neq i}^{k_N} \tilde{f}_j} = \sum_{i=1}^{k_N} \frac{\alpha_i \tilde{f}_i}{\prod_{j=1}^{k_N} \tilde{f}_j}$$

is in the image. Thus, every equation

$$\alpha_1 \tilde{f}_1 + \dots + \alpha_N \tilde{f}_N = 0$$

in $\mathbb{C}[t]$ gives an element in the kernel. In particular, when the number of irreducible polynomials of degree N is strictly bigger than the dimension of the vector space $\mathbb{C}[t]_{\leq n}$, there is this kind of relation. For example, it happens in degree $N = 6$. \square

In particular, the proof gives a negative answer to the question raised in [9, 4.6] : the restriction of $P_{\mathbb{C}}(\cdot, t)$ on the 1-eigenspace of $K_{\mathbb{C}}(\mathcal{U})$ for T is not injective.

We can also describe the kernel of $P_{\mathbb{C}}(\cdot, t)$. Every element of P of $K_{\mathbb{C}}(\mathcal{U})$ is a linear combination of the P_{f_1}, \dots, P_{f_N} , where f_i is monic of degree d_i with a non-zero constant coefficient :

$$P = \sum_{i=1}^N \alpha_{f_i} P_{f_i}.$$

So P is in the kernel if, and only if,

$$\sum_{i=1}^N \frac{\alpha_{f_i}}{\tilde{f}_i} = 0.$$

That is,

$$\sum_{i=1}^N \frac{\alpha_{f_i} \prod_{j=1, j \neq i}^N \tilde{f}_j}{\prod_{j=1}^N \tilde{f}_i} = 0,$$

or,

$$\sum_{i=1}^N \alpha_{f_i} \prod_{j=1, j \neq i}^N \tilde{f}_j = 0,$$

where the last equation is in the vector space of complex polynomials of degree less or equal to $\prod_{i=1}^N d_i$.

Example 1. Consider polynomial of degree 4 over \mathbb{F}_2 , prime to $x + 1$, we obtain an element of the kernel in degree 12 :

$$P_{FGH} - 5P_{EGH} + 3P_{EFH} + P_{EFG},$$

where

$$\begin{aligned} E &= x^4 + x^3 + x^2 + x + 1 \\ F &= x^4 + x^3 + 1 \\ G &= x^4 + x^2 + 1 = (x^2 + x + 1)^2 \\ H &= x^4 + x + 1. \end{aligned}$$

By definition, this is a 1-eigenvector for the action of T , and this gives an answer to [9, Conj. 5.6].

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