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EQUICONTINUITY AND LI-YORKE PAIRS OF DENDRITE MAPS

GHASSEN ASKRI

ABSTRACT. In this paper, relationships between equicontinuity of a dendrite map f on $\Lambda(f)$, absence of Li-Yorke pairs, collection of minimal sets and regularly recurrent points are investigated.

1. Introduction

Let (X, f) be a dynamical system i.e. X is a compact metric space with metric d and $f : X \rightarrow X$ is a continuous map. Denote by $\mathbb{N}$ the set $\{1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$, $f^n = f^{n-1} \circ f$ where f^0 is the identity map.

1.1. Topological dynamics. Let $x \in X$. The *orbit* of x (under f) is $Orb_f(x) = \{f^n(x); n \in \mathbb{Z}_+\}$ and the ω -*limit* set of x is $\omega_f(x) = \{y \in X : \liminf_{n \rightarrow +\infty} d(f^n(x), y) = 0\}$. The point x is called

- *fixed* if $f(x) = x$ and *N -periodic* (or periodic with period N) if $f^N(x) = x$ and $f^i(x) \neq x$ for $0 < i < N$,
- *regularly recurrent* if for any open set U containing x , $Orb_{f^N}(x) \subset U$ for some $N \in \mathbb{N}$,
- *almost periodic* if for any open set U containing x , there is $N \in \mathbb{N}$ such that for all $i \in \mathbb{Z}_+$, $\{f^i(x), f^{i+1}(x), \dots, f^{i+N}(x)\} \cap U \neq \emptyset$,
- *recurrent* if $x \in \omega_f(x)$.

We denote by $\Lambda(f)$ the union of all ω -limit sets and $Fix(f)$, $P(f)$, $RR(f)$, $AP(f)$ and $R(f)$ the sets of fixed points, periodic points, regularly recurrent points, almost periodic points and recurrent points, respectively. Recall that always we have

$$P(f) \subset RR(f) \subset AP(f) \subset R(f) \subset \Lambda(f).$$

A nonempty subset of X is called *minimal* if it is closed, f -invariant and has no proper subset with these properties. The map f is called *minimal* if X itself is minimal. Notice that M is minimal if and only if any point in it is almost periodic. Let A be a subset of X . Then A is called *N -periodic* (or periodic with period N) if $A, f(A), \dots, f^{N-1}(A)$ are pairwise disjoint and $f^N(A) = A$. The *orbit of A* (under f) is the set $Orb_f(A) = \cup_{n \in \mathbb{Z}_+} f^n(A)$.

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A point $y \in X$ is called an α -limit point of x under f if and only if there is a strictly increasing sequence of positive integers $(k_n)_{n \geq 0}$ and a sequence of points $(y_n)_{n \geq 0}$ in X such that

- $f^{k_n}(y_n) = x$ for all $n \geq 0$, and
- $\lim_{n \rightarrow +\infty} y_n = y$.

The set $\alpha_f(x)$ of all α -limit points of x under f is called the α -limit set.

1.2. General topology. For $A \subset B \subset X$, $\text{Int}(A, B)$ mean the interior of A relatively to B . For any subset A of X , A' is the set of accumulation points of A .

1.3. Li-Yorke pairs and entropy.

- A pair $(x, y) \in X^2$ is called
 - *proximal* if $\liminf_{n \rightarrow +\infty} d(f^n(x), f^n(y)) = 0$,
 - *asymptotic* if $\limsup_{n \rightarrow +\infty} d(f^n(x), f^n(y)) = 0$,
 - *distal* they are not proximal,
 - a *Li-Yorke* pair if it is proximal but not asymptotic.

The map f is called *distal* if $(x, y) \in X^2$ is proximal then $x = y$.

- Let $n \in \mathbb{N}$ and $\varepsilon > 0$. A subset E of X is called (n, ε) -separated if for any distinct points $x, y \in E$, we have $\max_{0 \leq i < n} d(f^i(x), f^i(y)) > \varepsilon$. Denote by $\text{Sep}(n, f, \varepsilon)$ the maximal possible cardinality of an (n, f, ε) -separated set in X . The *topological entropy* of f is defined by

$$h(f) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{\log \text{Sep}(n, f, \varepsilon)}{n}.$$

It was proved by Blanchard et al [4] that any dynamical system with positive topological entropy is Li-Yorke chaotic i.e. it has an uncountable set for which each proper pair is a Li-Yorke pair.

1.4. Equicontinuity. f is called *equicontinuous* (or *stable in the sense of Lyapunov*) at $x \in X$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \geq 0$ provided that $d(x, y) < \delta$. The map f is *equicontinuous* if it is equicontinuous at any point of X or equivalently for any $\varepsilon > 0$ there is $\delta > 0$ such that $d(f^n(u), f^n(v)) < \varepsilon$ for all $n \geq 0$ provided that $d(u, v) < \delta$. Notice that f is equicontinuous at x if and only if f^k is so at $f^i(x)$ for any positive integers k and i .

1.5. Hyperspace. We denote by 2^X the hyperspace of all nonempty closed subsets of X . The Hausdorff metric d_H is defined as follows: Let $A, B \in 2^X$,

$$d_H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}$$

where $d(x, A) = \inf_{y \in A} d(x, y)$. The space 2^X endowed with d_H is a compact metric space.

1.6. Dendrites. A *dendrite* is a locally connected compact connected metric space containing no simple closed curve. Let X be a dendrite. A point $x \in X$ is called *endpoint* (resp. *branch point*) if $X \setminus \{x\}$ is connected (resp. has at least three connected components). We denote by $End(X)$ and $B(X)$ the sets of endpoints and branch points of X , respectively. A *triod* is a dendrite with unique branch point and finitely many endpoints. Any two distinct points a and b in X can be joined by a unique arc, we denote it by $[a, b]$. We denote by $[a, b) = [a, b] \setminus \{b\}$ and $(a, b) = [a, b] \setminus \{a, b\}$. The *convex hull* of a nonempty subset A of X is the connected subset $[A] = \cup_{x \in A} [a, x]$ for any $a \in A$. An arc I of X is called *free* if $I \cap B(X) = \emptyset$. Let Y be a subdendrite of X and $x \in Y$. The *order* of x relatively to Y , denoted by $ord_Y(x)$, is the number of connected components of $Y \setminus \{x\}$. For any $x \in X$, let $r_Y(x)$ be the unique point in Y belonging to the arc joining x and z for any $z \in Y$. The map $r_Y : X \rightarrow Y$ is called the *first point map*.

1.7. Motivation and Main result. Li-Yorke chaotic dynamical systems has been extensively studied in the literature during the last forty years . In such a system various kind of ω -limit sets and recurrent points can occur. It is interesting to study the behavior of points of systems without Li-Yorke pairs. In [9], [5] and [6], Smítal and his co-authors investigated nonchaotic interval maps.

Theorem 1.1. ([6], [5]) *Let $f : I \rightarrow I$ be an interval map. Then the following assertions are equivalent:*

- (1) f has no Li-Yorke pairs,
- (2) $f|_{\Lambda(f)}$ is equicontinuous,
- (3) $\Lambda(f) = RR(f)$.

In this paper we generalise the previous theorem on dendrites. The main result of the paper is the following.

Theorem 1.2. *Let (X, f) be a dynamical system. The relations between the properties:*

- (1) f has no Li-Yorke pairs,
- (2) $f|_{\Lambda(f)}$ is equicontinuous,
- (3) $R(f) = RR(f)$,
- (4) the collection of minimal sets is closed in 2^X ,
- (5) $\Lambda(f) = AP(f)$.

are described by the scheme below where the thick (resp. thin, dashed) arrow means that the implication is true for general dynamical systems (resp. true for dendrites X with $End(X)$ countable and closed, false).

2. Auxiliary results

Definition 2.1. Let $f : X \rightarrow X$ be a dendrite map and let I and J be two arcs of X which are either disjoint or intersect only in their endpoint. If $f^n(I) \cap f^m(J) \supset I \cup J$ for some $n, m \in \mathbb{N}$ then we say that I, J form an arc horseshoe for f .

Lemma 2.2. ([7], Theorem 2) Let f be a continuous map of a dendrite. If there is an integer n such that f^n has an arc horseshoe then $h(f) > 0$.

Lemma 2.3. ([3], Lemma 1) Let (X, f) be a dynamical system, f be onto and $x \in X$. Then $\alpha_f(x)$ is non-empty, closed and f -invariant.

Lemma 2.4. [8] Let X be a compact Hausdorff space and let $f : X \rightarrow X$ be a minimal map. If U is a nonempty open subset of X then there is a positive integer r such that $X = \bigcup_{i=0}^r f^i(U)$.

The following result is a folklore.

Lemma 2.5. Let (X, f) be a dynamical system. If f is equicontinuous then it is distal. In particular, f has no Li-Yorke pairs.

The following Theorem is due to Auslander and Ellis.

Theorem 2.6. Let (X, f) be a dynamical system. Then for any $x \in X$ there is some almost periodic point $y \in \text{Orb}_f(x)$ such that (x, y) is proximal.

Remark. In the previous theorem, y can be chosen in $\omega_f(x)$.

Lemma 2.7. Let (X, f) be a dynamical system without Li-Yorke pairs. Then any ω -limit set is minimal. In other words, $\Lambda(f) = AP(f)$.

Proof. Let $L = \omega_f(x)$ be an ω -limit set of f . By Auslander-Ellis's theorem, there is a minimal set $M \subset L$ and a point $y \in M$ such that (x, y) is proximal. Since f has no Li-Yorke pairs then (x, y) is asymptotic. It follows that $L = M$. $\square$

Lemma 2.8. Let (X, f) be a dynamical system. If $f|_{\Lambda(f)}$ is equicontinuous then the collection of all minimal sets of f is closed in $(2^X, d_H)$.

Proof. Let $(M_n)_{n \geq 1}$ be a sequence of minimal sets converging in $(2^X, d_H)$ to M . Then M is a closed f -invariant set. Suppose that M is not minimal. Then there is a proper minimal subset F of M . Let $z \in M \setminus F$, $\varepsilon = d_H(z, F)$ and $x_0 \in F$. By equicontinuity of $f|_{\Lambda(f)}$, there is $0 < \delta < \frac{\varepsilon}{2}$ such that for

all $x \in \Lambda(f)$ and all $n \in \mathbb{Z}_+$, $d(f^n(x_0), f^n(x)) < \frac{\varepsilon}{2}$ provided that $d(x, x_0) < \delta$. Let $p > 0$ such that $d_H(M, M_p) < \delta$. There is $x \in M_p$ such that $d(x, x_0) < \delta$. As M_p is minimal, there is $k > 0$ such that $d(f^k(x), z) < \frac{\varepsilon}{2}$. Then $d(f^k(x_0), z) \leq d(f^k(x_0), f^k(x)) + d(f^k(x), z) < \varepsilon$. So $d_H(z, F) < \varepsilon$. Absurd. $\square$

Lemma 2.9. *Let (X, f) be a dynamical system such that $f|_{\Lambda(f)}$ is equicontinuous. Then $\Lambda(f) = AP(f)$.*

Proof. It is clear that $AP(f) \subset \Lambda(f)$. Suppose that there is an ω -limit set $L = \omega_f(x)$ and $y \in L$ such that $y \notin AP(f)$. Then there is a minimal set $M \subset L$ and $z \in M$ such that (y, z) is proximal. Since $f|_{\Lambda(f)}$ is equicontinuous then by Lemma 2.5 (y, z) is asymptotic. On the other hand, by Lemma 2.3 $\alpha_{f|_L}(y)$ is a nonempty, closed and f -invariant subset of L .

Claim. $M \subset \alpha_{f|_L}(y)$.

Proof. Suppose that $M \not\subset \alpha_{f|_L}(y)$. Then by minimality of M , $M \cap \alpha_{f|_L}(y) = \emptyset$. Let $\varepsilon_0 = d_H(M, \alpha_{f|_L}(y))$ and let $w \in \alpha_{f|_L}(y)$. Let $\delta > 0$. There is $t \in L$ such that $d(t, w) < \delta$ and $f^n(t) = y$ for some $n \geq 1$. So there is $N \geq 1$ such that $d(f^k(t), f^k(w)) > \frac{\varepsilon_0}{2}$ for all $k \geq N$. Thus $f|_L$ is not equicontinuous. Absurd. This finishes the proof of the Claim.

There is a sequence $(y_n)_{n \geq 1}$ in L converging to z and a nondecreasing sequence of integers $(k_n)_{n \geq 1}$ such that $f^{k_n}(y_n) = y$ for all $n \geq 1$. For any $\alpha > 0$ and any $N \geq 1$, there is $n \geq N$ such that $d(y_n, z) < \alpha$, but $d(f^{k_n}(y_n), f^{k_n}(z)) = d(y, f^{k_n}(z)) > \frac{d(y, M)}{2}$. It follows that $f|_L$ is not equicontinuous at z . Absurd. Thus $\Lambda(f) \subset AP(f)$. Hence $\Lambda(f) = AP(f)$. $\square$

From Lemmas 2.8 and 2.9 we can deduce the following

Corollary 2.10. *Let (X, f) be a dynamical system such that $f|_{\Lambda(f)}$ is equicontinuous. Then $\Lambda(f)$ is closed.*

Lemma 2.11. *Let (X, f) be a dynamical system such that $f|_{\Lambda(f)}$ is equicontinuous. Then $h(f) = 0$.*

Proof. Since $\Lambda(f)$ is closed, then $\overline{R(f)} \subset \Lambda(f)$. Hence by [2] $h(f) = h(f|_{\overline{R(f)}}) = h(f|_{\Lambda(f)})$. As $f|_{\Lambda(f)}$ is equicontinuous, we get $h(f) = 0$. $\square$

3. Splitting minimal sets of maps on dendrites X with

$$\text{Card}(\text{End}(X)) < c$$

Definition 3.1. *Let X be a dendrite. An arc $[x, y]$ in X is called a tracing arc starting from x if $\text{Fix}(f) \cap [x, y] = \emptyset$ and for any $z \in [x, y]$, $(z, f(z)) \cap [x, y] \neq \emptyset$. Moreover, if $f(y) = y$ then $[x, y]$ is called a complete tracing arc.*

Lemma 3.2. ([11], Proposition 2.9) *Let $f : X \rightarrow X$ be a dendrite map. Then, for any given $x \in X \setminus \text{Fix}(f)$, there exists a unique complete f -tracing arc starting from x .*

The proof of the following Lemma is inspired from [1].

Lemma 3.3. *Let $f : X \rightarrow X$ be a zero topological entropy dendrite map with $\text{Card}(\text{End}(X)) < c$ and let M be a nondegenerate (finite or infinite) minimal set. Then there is a connected subset C of X and an integer $k \geq 2$ satisfying the following properties:*

- (i) $C \cap M$ has nonempty interior in M ,
- (ii) $f^k(C) \subset C$,
- (iii) $M \subset \bigcup_{i=0}^{k-1} f^i(C)$,
- (iv) $C, f(C), \dots, f^{k-1}(C)$ are pairwise disjoint.

Proof. Let $Y = [M]$. We distinguish two cases:

Case 1: $Y \cap \text{Fix}(f) \neq \emptyset$. Let $a \in Y$ be a fixed point. Denote by $A = \bigcup_{n \in \mathbb{N}} f^{-n}(a)$ and $A_Y = A \cap Y$. Notice that $\text{ord}_Y(a) < \infty$ (It's clear if M is finite. Suppose that M is infinite and $\text{ord}_Y(a) = \infty$. Then there are infinitely many pairwise disjoint connected subsets $(C_i)_{i \geq 1}$ of $Y \setminus \{a\}$ such that $C_i \cap M \neq \emptyset$ for all $i \geq 1$. So $\lim_{i \rightarrow +\infty} \text{diam}(C_i) = 0$. Since $a \in \overline{C_i}$ for all $i \geq 1$, we get $a \in M$. Absurd).

Claim 1. $[A_Y] \cap M = \emptyset$.

Proof. Suppose the contrarily. Then there is $b \in Y$ and $y \in M$ such that $y \in (a, b)$ and $f^n(b) = a$ for some $n \geq 1$. Let $I = [a, y]$ and $J = [y, b]$. Since M is closed, there is $z \in M$ such that $b \in (y, z)$. So there are integers $p, q > 0$ such that $f^p(I) \cap f^q(J) \supset I \cup J$. Hence I, J is an arc horseshoe. By Lemma 2.2, $h(f) > 0$. Absurd. This finishes the proof of Claim 1.

Proof of (i) We have $M = \bigcup_{e \in \text{End}(X)} [a, e] \cap M$. Since $\text{Card}(\text{End}(X)) < c$, then by the well known theorem of Baire, there is $e_0 \in \text{End}(X)$ such that $\text{Int}([a, e_0] \cap M, M)$ is nonempty. Let $M_0 = M \cap [a, e_0]$. By Claim 1, $[M_0] \cap A_\infty = \emptyset$. So let C be the connected component of $X \setminus A_\infty$ containing M_0 . Hence $\text{Int}(M \cap C, M) \neq \emptyset$.

Proof of (ii) Let $x \in M_0$, there is $k > 0$ such that $f^k(x) \in M_0$. So $f^k(C) \cap C \neq \emptyset$. As $f^k(C) \cap A_\infty = \emptyset$, we get $f^k(C) \subset C$.

We can assume that k is the smallest integer with this property.

Proof of (iii) Since $\text{Int}(M_0, M) \neq \emptyset$, then by [8], we have $\bigcup_{i=0}^N f^i(M_0) = M$ for some $N \geq k$. As $M_0 \subset C$, we get $\bigcup_{i=0}^N f^i(M_0) \subset \bigcup_{i=0}^N f^i(C)$. So using (ii), $M \subset \bigcup_{i=0}^{k-1} f^i(C)$.

Proof of (iv) Suppose on the contrary that $f^i(C) \cap f^j(C) \neq \emptyset$ for some $0 \leq i < j \leq k-1$. Let $s = k - j$. Then $f^{s+i}(C) \cap f^{s+j}(C) \neq \emptyset$. Since $f^{s+j}(C) = f^k(C) \subset C$, $f^{s+i}(C) \cap C \neq \emptyset$. As $0 < s+i < k$, this contradicts the minimality of k .

The integer k is greater than 1 since $\text{ord}_Y(a) > 1$ and $M \subset \bigcup_{i=0}^{k-1} f^i(C)$.

Case 2: $Y \cap \text{Fix}(f) = \emptyset$. Let $x \in M$. Then by Lemma 3.2 there is a complete tracing arc with endpoints x and a where $a \in \text{Fix}(f)$. By Lemma 5.5 of [1], for any $y \in Y$, $[y, a]$ is a complete tracing arc starting from y . Let $b = r_Y(a)$ and $B = \cup_{n \geq 0} f^{-n}(b)$. Notice that $f(b) \notin Y$, $b \notin M$ and $k := \text{ord}_Y(b) < \infty$.

Claim 2. $[B \cap Y] \cap M = \emptyset$.

Proof. Suppose the contrary. Then there are $b' \in Y$ and $x \in M$ such that $x \in (b, b')$ and $f^n(b') = b$ for some $n \in \mathbb{N}$. Let $I = [b, x]$, $J = [x, b']$ and $y \in M$ such that $[x, b'] \subset (b, y)$. Since $b \in [x, f(b)]$, then $f(I) \supset [b, f(x)]$. More generally, we have $f^k(I) \supset [b, f^k(x)]$ for all $k \in \mathbb{N}$. Since $y \in \omega_f(x) = M$, there is $N \geq n$ such that $f^N(I) \supset I \cup J$. On the other hand, $f^n(J) \supset [b, f^n(x)]$. So $f^N(J) \supset I \cup J$. Hence I, J is an arc horseshoe for f . Absurd.

As in Case 1, there is a connected subset C of $X \setminus B$ satisfying properties (i) – (iv).

This ends the proof of the Lemma. $\square$

Proposition 3.4. *Let $f : X \rightarrow X$ be a zero topological entropy dendrite map with $\text{Card}(\text{End}(X)) < c$ and let M be a nondegenerate minimal set. If M is finite then set $\mathcal{I} = \{1, 2, \dots, p\}$. If M is infinite then set $\mathcal{I} = \mathbb{N}$. Then there is a sequence of subdendrites $(D_n)_{n \in \mathcal{I}}$ of X and a sequence of nondecreasing integers $(k_n)_{n \geq 1}$ satisfying the following properties:*

- (P.1) *for any $n \in \mathcal{I}$, D_n is periodic of period k_n and $k_n | k_{n+1}$ whenever $n+1 \in \mathcal{I}$,*
- (P.2) *for any $i, j \in \{0, 1, \dots, k_n - 1\}$ ($i \neq j$), $f^i(D_n)$ and $f^j(D_n)$ are either disjoint or intersect only in one common point,*
- (P.3) *for any $n \geq 1$, $\cup_{i=0}^{q_n-1} f^{ik_n}(D_{n+1}) \subset \text{Orb}_f(D_n)$ where $q_n = \frac{k_{n+1}}{k_n}$,*
- (P.4) *$M \subset \cap_{n \geq 1} \text{Orb}_f(D_n)$,*

Proof. Let C be a connected subset of X satisfying properties (i) – (iv) of Lemma 3.3. For any $0 \leq i \leq k_1 - 1$, we have $f(M_i) = M_{i+1 \bmod k_1}$ where $M_i = f^i(C) \cap M$. Let $D_1 = \cap_{i=0}^{+\infty} f^{k_1 i}(\overline{C})$. Then

- D_1 is a subdendrite of X and $f^{k_1}(D_1) \subset D_1$,
- for any $0 \leq i < j < k_1$, $f^i(D_1)$ and $f^j(D_1)$ are either disjoint or intersect in one common point.
- for any $0 \leq i < k_1$, $M_i \subset f^i(D_1)$. So $M \subset \text{Orb}_f(D_1)$.

Now, $f|_{D_1}^{k_1} : D_1 \rightarrow D_1$ is a dendrite map of zero topological entropy and $\text{Card}(\text{End}(D_1)) < c$. Since M_0 is a minimal set of $f|_{D_1}^{k_1}$, there is a periodic subdendrite $D_2 \subset D_1$ under $f|_{D_1}^{k_1}$ of period $q_1 \geq 2$ such that $M_0 \subset \cup_{j=0}^{q_1-1} f^{k_1 j}(D_2)$, $f^{k_1 i}(D_2)$ and $f^{k_1 j}(D_2)$ ($0 \leq i \neq j < q_1$) are either disjoint or intersect only on one common point. So D_2 is a periodic subdendrite of period $k_2 = k_1 q_1$ such that $M \subset \text{Orb}_f(D_2)$ and $\cup_{i=0}^{q_1-1} f^{ik_1}(D_2) \subset \text{Orb}(D_1)$. We continue this process to construct the sequence $(D_n)_{n \geq 1}$ satisfying properties (P₁) – (P₄). $\square$

Lemma 3.5. *Let X be a dendrite with $\text{End}(X)$ countable and closed and let $(D_n)_{n \in \mathbb{N}}$ be a sequence of subdendrites of X satisfying (P.1), (P.2) and (P.3). Then there is an integer $N \geq 1$ such that $\forall n \geq N$, $f^i(D_n)$ is a free arc for some $0 \leq i < k_n$.*

Proof. Suppose that for any integer n and any $0 \leq i < k_n$, $f^i(D_n) \cap \text{End}(X) \neq \emptyset$. Then for any $n \geq 1$, the subsets $E_{n,i} = f^i(D_n) \cap \text{End}(X)$ ($0 \leq i < k_n$) are pairwise disjoint and closed. Let $\tau = (i_0, i_1, i_2, \dots) \in \prod_{n=0}^{+\infty} \mathbb{Z}/q_n\mathbb{Z}$ where $q_0 = k_0 = 1$. By the property (P₃), the subset $\cap_{n=1}^{+\infty} f^{\sum_{j=0}^{n-1} i_j k_j}(D_n)$ is nonempty, closed and intersect $\text{End}(X)$. Furthermore, $E_\tau \cap E_{\tau'} = \emptyset$ whenever $\tau \neq \tau'$. Pick $x_\tau \in E_\tau \cap \text{End}(X)$. Then the map $\tau \in \prod_{n=0}^{+\infty} \mathbb{Z}/q_n\mathbb{Z} \mapsto x_\tau \in \text{End}(X)$ is injective. This implies that $\text{End}(X)$ is uncountable. Absurd. So there is $N \in \mathbb{N}$ such that $f^i(D_N) \cap \text{End}(X) = \emptyset$ for some $0 \leq i < k_N$. Since $\text{End}(X)$ is closed, $f^i(D_N) \cap B(X)$ is finite. By taking k_n sufficiently large, we have $f^j(D_n) \cap B(X) = \emptyset$ for some $n \geq N$ and $0 \leq j < k_n$. It follows that $f^j(D_n)$ is a free arc. $\square$

4. Li-Yorke pairs and equicontinuity of a dendrite map f on $\Lambda(f)$

Lemma 4.1. *Let X be a dendrite with $\text{End}(X)$ is countable and closed and let $f : X \rightarrow X$ be a continuous map. If $f|_{\Lambda(f)}$ is equicontinuous then f has no Li-Yorke pairs.*

Proof. Suppose on the contrary that (x, y) is a Li-Yorke pair of f . By Lemma 2.9 $\omega_f(x) \cup \omega_f(y) \subset AP(f)$. We can assume that $\omega_f(x)$ is infinite. By [10] $\omega_f(x) \not\subseteq P(f)$. So there is an infinite minimal set $M \subset \omega_f(x)$. By Lemma 2.11 $h(f) = 0$. So by Proposition 3.4 and Lemma 3.5 there is a periodic free arc I of X of period p such that $M \subset \text{Orb}_f(I)$. Set $g = f^p$. Then $M_0 := M \cap I$ is an infinite minimal set of g . There is $n \geq 0$ such that $f^n(x) \in I$. Since $(f^n(x), f^n(y))$ is a Li-Yorke pair of g , there is a nondecreasing sequence of integers $(n_i)_{i \geq 1}$ such that $(g^{n_i}(f^n(x)))_{i \geq 1}$ and $(g^{n_i}(f^n(y)))_{i \geq 1}$ converges to some $z \in I$. If $g^m(f^n(y)) \in I$ for some $m \geq 1$ then $(f^n(x), f^n(y))$ is a Li-Yorke pair for $g|_I$. Otherwise, $z \in \partial(I)$. Hence $g^2(z) = z$. So $(f^n(x), z) \in I^2$ is also a Li-Yorke pair for $g|_I$. In all cases, the interval map $g|_I : I \rightarrow I$ has a Li-Yorke pair. By [6] $g|_I$ is not equicontinuous on $\Lambda(g)$. Then so does for f on $\Lambda(f)$. Absurd. Thus f has no Li-Yorke pairs. $\square$

Remark 4.2. *Lemma 4.1 fails if $\text{End}(X)$ is not closed (See [1], Example 2).*

Lemma 4.3. *Let X be a dendrite with $\text{End}(X)$ countable and closed and let $f : X \rightarrow X$ be a continuous map. Suppose that f has no Li-Yorke pairs. Then $f|_{\Lambda(f)}$ is equicontinuous at any point in $\Lambda(f) \setminus (P(f) \cap \text{End}(X)')$.*

Proof. Let $x \in \Lambda(f) \setminus (P(f) \cap \text{End}(X)')$. Then there is a minimal set M in X containing x . We distinguish two cases.

Case 1: $x \notin P(f)$. Then M is infinite. Let I be an N -periodic free arc in X such that $M \subset \text{Orb}_f(I)$. There is $k \geq 0$ such that I is a neighborhood of $f^k(x)$. Since $f|_I^N$ has no Li-Yorke pairs, then by Theorem 1.1 f^N is equicontinuous at $f^k(x)$. Hence f is equicontinuous at x .

Case 2: $x \in P(f)$. Then $x \notin \text{End}(X)'$. Without loss of generality, we can assume that $f(x) = x$. As $\text{End}(X)$ is closed, there is a neighborhood $V = \cup_{i=1}^r [x, e_i]$ of x such that $V \cap B(X) \subset \{x\}$. Fix $\varepsilon_0 > 0$. There is $0 < \varepsilon < d(x, X \setminus V)$ such that for all $0 \leq i \leq 2r$, $d(f^i(u), f^i(v)) < \varepsilon_0$ whenever $d(u, v) < \varepsilon$. There is $\delta > 0$ such that $d(x, f^i(y)) < \varepsilon$ for all $0 \leq i \leq 2r$ whenever $d(x, y) < \delta$. Let $y \in \Lambda(f) \cap B(x, \delta)$. Then $f^i(y), f^j(y) \in [x, e_s]$ for some $0 \leq i < j \leq r$ and $1 \leq s \leq r$. We can assume that $i = 0$ and $y \in (x, f^j(y))$. If $y \notin P(f)$, as $\omega_f(y)$ has no isolated points, we can assume that $(y, f^j(y)) \subset (w, z)$ for some $w, z \in \omega_f(y) \cap [x, e_s]$. Let Y be the connected component of $X \setminus \{x\}$ containing y .

Claim 1. $\text{Orb}_{f^j}(y) \subset Y$.

Proof. Suppose the contrary. Let $p > 1$ be the smallest integer for which $f^{pj}(y) \notin Y$. Then there is $x' \in (y, f^j(y))$ such that $f^{(p-1)j}(x') = x$. So $[y, x'], [x', f^j(y)]$ is an arc horseshoe, absurd. This finishes the proof of Claim 1.

Claim 2. $[y, f^j(y)] \cap \text{Fix}(f^j) \neq \emptyset$.

Proof. Let $q > 1$ be the smallest integer such that $f^{qj}(y) \in [x, f^j(y))$. Then by Lemma 3.2 there is a point $a \in \text{Fix}(f^j)$ such that $b := r|_{[\text{Orb}_{f^j}(y)]} \in [f^{nj}(y), f^{(n+1)j}(y)]$ for some $n \geq 0$. Let $b' \in (y, f^j(y)) \cap f^{-nj}(b)$. If $b \notin (y, f^j(y))$ then $[y, b'], [b', f^j(y)]$ is an arc horseshoe. Absurd. So $b \in (y, f^j(y))$. As $[y, f^j(y)]$ is a free arc, $f^j(b) = b$. This finishes the proof of Claim 2.

Claim 3. $\text{Orb}_{f^{2j}}(y) \subset [x, b]$.

Proof. Suppose that $f^{2j}(y) \notin [x, y]$. As $y \in R(f^j)$, there is $n > 1$ such that $f^{jn}(y) \in (x, y)$. Suppose that n is the smallest one. So there is $b' \in [f^j(y), f^{2j}(y)]$ such that $f^{(n-2)j}(b') = b$. Thus $[f^j(y), b'], [b', f^{2j}(y)]$ is an arc horseshoe. Absurd. So we get, $f^{2j}(y) \in [x, b]$. The connected subset of $[x, b] \setminus (\cup_{n \geq 0} f^{-nj}(b) \cup \cup_{n \geq 0} f^{-n}(x))$ containing $[y, f^{2j}(y)]$ is f^{2j} -invariant. So $f^{2nj}(y) \in [x, y]$ for all $n \geq 0$. This finishes the proof of Claim 3.

Now, we have $d(x, y) < \delta$. So $d(x, f^{2nj}(y)) \leq d(x, f^j(y)) < \varepsilon$ for all $n \geq 0$. So $d(x, f^{2nj+m}(y)) < \varepsilon_0$ for any $0 \leq m \leq 2r$ and $n \geq 0$. Thus f is equicontinuous at x .

□

Lemma 4.4. *There is a dendrite X with $\text{End}(X)$ is countable and closed and a continuous map $f : X \rightarrow X$ without Li-Yorke pairs such that $\Lambda(f) = P(f)$ but the collection of minimal sets is not closed in 2^X . As a consequence, $f|_{\Lambda(f)}$ is not equicontinuous.*

Proof. **Construction of the dendrite X .**

We construct the dendrite X on the plane $\mathbb{R}^2$ as follows.

- Let $I_0 = [e_0, e_{-1}]$ where $e_0 = (1, 0)$ and $e_{-1} = (-1, 0)$,
- For any $n \geq 1$, let $J_n = [b_{n,0}, e_n]$ where $b_{n,0} = (1 - \frac{1}{n}, 0)$ and $e_n = (1 - \frac{1}{n}, \frac{1}{n})$,
- For any $n \geq 1$ and $k \geq 1$, let $I_{n,k} = [b_{n,k}, e_{n,k}]$ where $b_{n,k} = (1 - \frac{1}{n}, \frac{1}{n}(1 - \frac{1}{k+1}))$ and $e_{n,k} = (1 - \frac{1}{n} - \frac{1}{2^{k+n}}, \frac{1}{n}(1 - \frac{1}{k+1}))$.

Then $X = I_0 \cup \bigcup_{n \geq 1} J_n \cup \bigcup_{n,k \geq 1} I_{n,k}$ is a dendrite with $\text{End}(X) = \{e_n; n \geq -1\} \cup \{e_{n,k}; n, k \geq 1\}$ which is countable and closed.

Construction of the map f .

The notation $[a, b] \xrightarrow{f} [c, d]$ means that f send linearly the arc $[a, b]$ to the arc $[c, d]$ such that $f(a) = c$ and $f(b) = d$. We define the map $f : X \rightarrow X$ as follows. For all $n, k \geq 1$,

- $f(e_1) = f(e_0) = e_0$,
- $[b_{n,0}, b_{n+1,0}] \xrightarrow{f} [b_{n+1,0}, b_{n+2,0}]$,
- $[b_{1,k-1}, b_{1,k}] \xrightarrow{f} [b_{k+1,0}, b_{k+2,0}]$,
- $[b_{1,0}, e_{-1}] \xrightarrow{f} [b_{2,0}, e_{2,1}]$ and $[b_{1,k}, e_{1,k}] \xrightarrow{f} [b_{k+2,0}, e_{k+2,1}]$,
- $[b_{2,0}, b_{2,1}] \xrightarrow{f} [b_{3,0}, b_{1,0}]$ and $[b_{2,k}, b_{2,k+1}] \xrightarrow{f} [b_{1,k-1}, b_{1,k}]$,
- $[b_{n+1,0}, b_{n+1,1}] \xrightarrow{f} [b_{n+2,0}, b_{n,0}]$, $[b_{n+1,1}, b_{n+1,2}] \xrightarrow{f} [b_{n,0}, b_{n,2}]$ and $[b_{n+1,k-1}, b_{n+1,k}] \xrightarrow{f} [b_{n,k}, b_{n,k+1}]$,
- $[b_{2,1}, e_{2,1}] \xrightarrow{f} [b_{1,0}, e_{-1}]$ and $[b_{n+1,k}, e_{n+1,k}] \xrightarrow{f} [b_{n,k+1}, e_{n,k+1}]$.

It is easy to check that the map $f : X \rightarrow X$ is continuous, has no Li-Yorke pairs and the collection of minimal sets is not closed in 2^X . We have $\Lambda(f) = P(f) = \{e_0, e_{-1}\} \cup \{e_{n,k}; n, k \geq 1\}$. Indeed, for any $n \geq 1$, $\text{Orb}_f(e_{n,1})$ is a periodic orbit which converges in 2^X to the set $M = \{e_n; n \geq 1\}$. Since $f(e_0) = e_0$, then M is not minimal. (See Figures 1 and 2)

FIGURE 1. The action of f on I_0 and $\{e_n; n \geq -1\}$ FIGURE 2. The action of f on branch points $b_{n,k}$ ($n \geq 1, k \geq 0$)

□

Lemma 4.5. *Let X be a dendrite with $\text{End}(X)$ countable and closed and let $f : X \rightarrow X$ be a continuous map. If f has no Li-Yorke pairs then $R(f) = RR(f)$.*

Proof. It suffices to prove that $R(f) \subset RR(f)$. Let $x \in R(f)$. Then $M = \omega_f(x)$ is a minimal set. If $x \in P(f)$ then $x \in RR(f)$. Suppose that $x \notin P(f)$. Then there is a p -periodic free arc I in X such that $M \subset \text{Orb}_f(I)$. Let $0 \leq i < p$ such that $x \in f^i(I)$. There is $x' \in I \cap M$ such that $f^i(x') = x$. Let U be an open set in X containing x . Then $f^{-i}(U)$ is an open subset in X containing x' . By Theorem 1.1 $R(f|_I^p) = RR(f|_I^p)$. As $x' \in \omega_{f|_I^p}(x')$, we get

$x' \in RR(f|_U^p)$. So there is an integer $N > 0$ such that $Orb_{f^{pN}}(x') \subset f^{-i}(U)$. It follows that $Orb_{f^{pN}}(x) \subset U$. Thus $x \in RR(f)$. We conclude that $R(f) = RR(f)$. $\square$

In the following Lemma, we strengthen absence of Li-Yorke pairs to get equicontinuity on $\Lambda(f)$.

Lemma 4.6. *Let X be a dendrite with $End(X)$ countable and closed and let $f : X \rightarrow X$ be a continuous map. Suppose that f has no Li-Yorke pairs. If one of the following assertions holds*

- (1) *the collection of minimal sets is closed in $(2^X, d_H)$,*
- (2) *$f|_{\Lambda(f)}$ is equicontinuous at every point in $End(X)'$,*
- (3) *$f|_{P(f)}$ is equicontinuous,*

then $f|_{\Lambda(f)}$ is equicontinuous.

Proof. (1) Suppose that the collection of minimal sets is closed in $(2^X, d_H)$.

Let $x \in \Lambda(f)$. By Lemma 4.3 we can assume that $x \in P(f)$. Also we may assume that $x \in Fix(f)$. Let $(y_n)_{n \geq 1}$ be a sequence in $\Lambda(f)$ converging to x . For any $n \geq 1$, $M_n = \omega_f(y_n)$ is a minimal set containing y_n . We can assume that (M_n) is convergent to some closed set M . Then $d(y_n, M)$ goes to 0 when $n \rightarrow +\infty$. As the collection of minimal sets is closed in $(2^X, d_H)$, then M is a minimal set. So $M = \{x\}$. Now fix $\varepsilon > 0$. There is $\alpha > 0$ such that if $d(x, y) < \alpha$ then $d(x, \omega_f(y)) < \varepsilon$. This implies that $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \geq 0$. thus f is equicontinuous at x .

- (2) and (3) follows from Lemma 4.3

$\square$

Proof of the main Theorem The implications $(2) \Rightarrow (1)$, $(2) \Rightarrow (4)$ and $(1) \Rightarrow (3)$ are given in Lemmas 4.1, 2.8 and 4.5. The implication $(1) \Rightarrow (5)$ is proved in Lemma 2.7. Finally, $(1) \not\Rightarrow (4)$ is given by Lemma 4.4.

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GHASSEN ASKRI, BIZERTE PREPARATORY ENGINEERING INSTITUTE, 7021, JARZOUNA,
TUNISIA

E-mail address: askri.ghassen@gmail.com and elaskri.ghassen@ipeib.u-carthage.tn