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The Six Cylinders Problem: $\mathbb{D}_3$-symmetry Approach

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Abstract

Motivated by a question of W. Kuperberg, we study the 18-dimensional manifold of configurations of 6 non-intersecting infinite cylinders of radius r, all touching the unit ball in $\mathbb{R}^3$. We find a configuration with

$$r = \frac{1}{8}\left(3 + \sqrt{33}\right) \approx 1.093070331 \ .$$

We believe that this value is the maximal possible.

1 Introduction

The question: – how many unit cylinders can touch a unit ball? – was asked by W. Kuperberg, [K]. He presented several arrangements of 6 non-intersecting (open) unit cylinders touching the unit ball; it is difficult to imagine that 7 unit cylinders can do it, though no proof of this statement is known. At first glance one can even think that 6 non-intersecting cylinders of radius $r > 1$ can not touch the unit ball. This, however, is not the case, and an example was presented by M. Firsching in his thesis, [F]. In this example the radius r equals to 1.049659. This example was obtained by a numerical exploration of the corresponding 18-dimensional configuration manifold.

The situation thus looks somewhat similar to the case of 12 unit balls touching the central unit ball. There one can similarly ask whether 13 unit balls can do it (the answer is negative, [SW]), or whether 12 balls of bigger radius $r > 1$ can touch the central unit ball. The answer to the latter question

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is positive: it is known that 12 balls of radius $\left(\sqrt{\frac{5+\sqrt{5}}{2}}-1\right)^{-1} \approx 1.10851$, positioned at the 12 vertices of the icosahedron with edge $2r$, touch the central unit ball. This fact makes plausible the idea that the two very symmetric configurations of 12 unit balls touching the central unit one – the FCC (Face Centered Cubic) and the HCP (Hexagonal Closed Packed) configurations (see Figures 1 and 2 for the explanation) – can be *unlocked* by rolling the 12 balls over the central one to a configuration where none of the 12 balls touch each other. This is indeed correct, see [KKLS] for details.

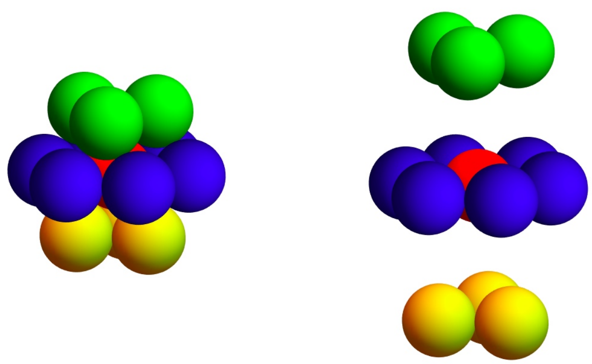

Figure 1: FCC configuration (left) and its layers (right)

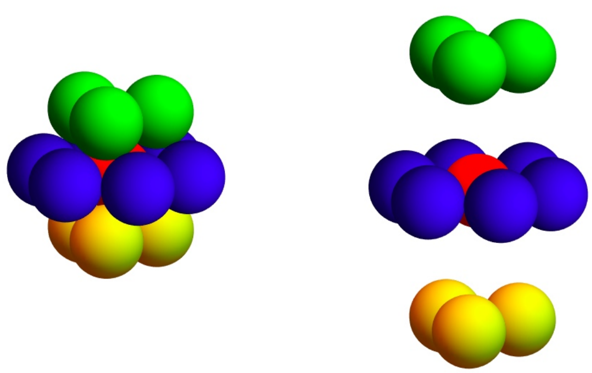

Figure 2: HCP configuration (left) and its layers (right)

The precise meaning of the *unlocking* is the following. Let G be a collection of solid bodies, $G = \{\Lambda_1, ...\Lambda_k\}$, where each Λ_i touches the unit central ball, while some distances between bodies of G are zero. We say that G can be unlocked if there exists a continuous deformation $G(t)$, $t \geq 0$, of G (i.e. $G(0) = G$), such that for any $t > 0$ all the distances between the members in the configuration $G(t)$ are positive, while each Λ_i touches the central ball while moving.

In the present paper we address a similar question – of unlocking the configuration of six unit parallel (right circular) cylinders, touching the central unit ball. We denote this configuration by C_6, see Figure 3.

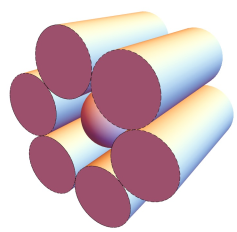

Figure 3: Configuration C_6

The configuration C_6 is not rigid. Indeed, let $H \subset \mathbb{R}^3$ be a half-space, containing three cylinders of C_6, and h be the normal vector to the plane ∂H. Then one can rotate the three cylinders about h, keeping the remaining three intact, see Figure 4.

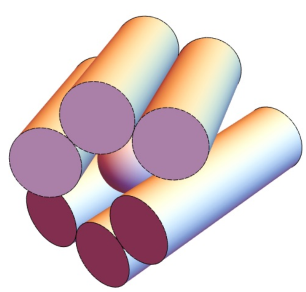

Figure 4: Non-rigidity of C_6

So our configuration C_6 is movable, but this is not yet the unlocking, since some distances stay zero. We will demonstrate that the configuration C_6 is indeed unlockable. Namely, we will present its continuous deformation $C_6(t)$, along which quite a spacing opens between the cylinders, so at some value of t it becomes possible to arrange 6 non-intersecting cylinders of radius

$$r_{\mathfrak{m}} = \frac{1}{8}\left(3 + \sqrt{33}\right) \approx 1.093070331. \tag{1}$$

We believe that our configuration of 6 cylinders with radius $r_{\mathfrak{m}}$ is in fact optimal. In a forthcoming publication [OS] we are going to show that our cylinder arrangement with value $r_{\mathfrak{m}}$, is a local maximum, i.e. any small perturbation of our configuration decreases the corresponding radius.

The search of the maximal radius r is equivalent to finding a point in a certain 18-dimensional manifold M^6, see the definition (4) below, where the minimum of 15 mutual distances attains its maximal value. Guided by our belief that the optimal configuration should possess nice symmetries, we restricted our search to a certain 3-dimensional submanifold $\mathcal{C}^3 = C_6(\varphi, \delta, \varkappa)$ of M^6, see the definition (5) below, comprised by the fixed points of the action of the group $\mathbb{D}_3 \subset SO(3)$ on M^6, i.e. by $\mathbb{D}_3$-symmetric configurations. On $\mathcal{C}^3$, only 4 of 15 distances are different, and only 3 of them are relevant. Our next reduction comes from the observation that the situation when three 'nice' functions g_1, g_2, g_3 on a three-dimensional manifold N coincide on a smooth curve γ is a general position situation, as the dimension counting immediately shows. In such a case the point $x_{\mathfrak{m}} \in N$ where max of the function $\min_i \{g_i(x)\}$ is attained, belongs to γ. It so happens that our case (with g_1, g_2, g_3 being the three relevant distances) falls into it, with $\gamma = C_6(\varphi, \delta(\varphi), \varkappa(\varphi))$, for certain functions $\delta(\varphi), \varkappa(\varphi)$. What is left then is the study of a single function $g_i|_\gamma$ of one variable. We were able to explicitly describe this curve $\gamma \subset M^6$ and to compute the maximal value $r_{\mathfrak{m}}$ of the function r on it. It gives a lower bound for the maximal radius r possible.

We also analyze the generalized situation, with $2n$ cylinders instead of 6. We show that it can be unlocked for $n > 2$ along our curve. For $n = 2$ the configuration is not rigid but it is not unlockable along our curve. However, we conjecture that all possible configurations of four cylinders belong to the curve.

The description of our 18-dimensional manifold M^6 and the choice of coordinates there is in the next section. Section 3 contains the definition of the submanifold $\mathcal{C}^3 \subset M^6$ and the formulation of our main result. The optimization problem on $\mathcal{C}^3$ is solved in Sections 4 and 5, thus proving our main theorem. Section 6 we consider the problem of n equal cylinders touching the unit ball. The last Section 7 contains our conclusions.

We finish the introduction by the brief history of how the present paper was evolving. Our first goal was to convince ourselves that the configuration C_6 is *infinitesimally* unlockable (see Proposition 3). Next, we were trying to analyze the humongous trigonometric formulas for the functions $g_i(x)$, and we used both Wolfram Mathematica [W] and the analog machinery:

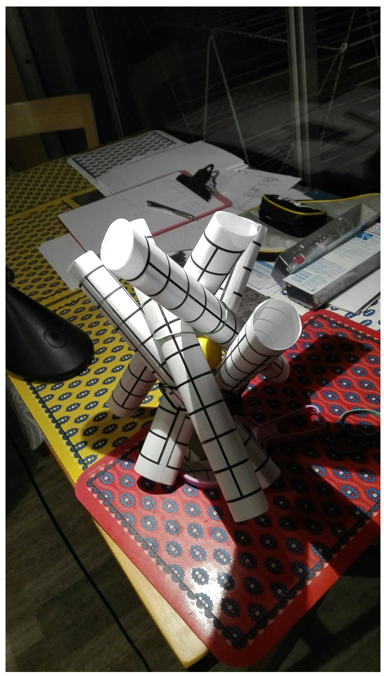

Figure 5: The analog computer. The yellow ball is visible in the center.

to numerically solve the minimax problem. We got an estimate 1.09 for $r_{\mathtt{m}}$. The last phase came with the realization that it is possible to pass from trigonometric expressions to such algebraic ones that our minimax problem becomes *'integrable'*, i.e. can be solved explicitly. In our eyes this is quite a surprising feature of the six cylinder problem, which is beyond our initial expectations. Probably, it points to some hidden symmetry features of the problem.

2 Configuration manifold

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere, centered at the origin. For every $x \in \mathbb{S}^2$ we denote by TL_x the set of all (unoriented) tangent lines to $\mathbb{S}^2$ at x. We denote by M the manifold of tangent lines to $\mathbb{S}^2$. We represent a point in M by a pair (x, τ), where τ is a unit tangent vector to $\mathbb{S}^2$ at x, though such a pair is not unique: the pair $(x, -\tau)$ is the same point in M. We shall use the following coordinates on M. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be the standard coordinate axes in $\mathbb{R}^3$. Let $R_{\mathbf{x}}^{\alpha}, R_{\mathbf{y}}^{\alpha}$ and $R_{\mathbf{z}}^{\alpha}$ be the counterclockwise rotations about these axes by an angle α, viewed from the tips of axes. We call the point $\mathsf{N} = (0, 0, 1)$ the North pole, and $\mathsf{S} = (0, 0, -1)$ – the South pole. By *meridians* we mean geodesics on $\mathbb{S}^2$ joining the North pole to the South pole. The meridian in the plane $\mathbf{xz}$ with positive $\mathbf{x}$ coordinates will be called Greenwich. The angle φ will denote the latitude on $\mathbb{S}^2$, $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the angle $\varkappa \in [0, 2\pi)$ – the longitude, so that Greenwich corresponds to $\varkappa = 0$. Every point $x \in \mathbb{S}^2$ can be written as $x = (\varphi_x, \varkappa_x)$. Finally, for each $x \in \mathbb{S}^2$, we denote by R_x^{α} the rotation by the angle α about the axis joining $(0, 0, 0)$ to x, counterclockwise if viewed from its tip, and by $(x, \uparrow)$ we denote the pair (x, τ_x), $x \neq \mathsf{N}, \mathsf{S}$, where the vector τ_x points to the North. We also abbreviate the notation $(x, R_x^{\alpha} \uparrow)$ to $(x, \uparrow_{\alpha})$.

Let $u = (x', \tau')$, $v = (x'', \tau'')$ be two lines in M. We denote by d_{uv} the distance between u and v; clearly $d_{uv} = 0$ iff $u \cap v \neq \varnothing$. If the lines u, v are not parallel then the square of d_{uv} is given by the formula

$$d_{uv}^2 = \frac{\det^2[\tau', \tau'', x'' - x']}{1 - (\tau', \tau'')^2} ,$$

where $(*, *)$ is the scalar product. For the future use we note that if $d_{uv} = d > 0$, then the cylinders $C_u(r)$ and $C_v(r)$, touching $\mathbb{S}^2$ at x', x'', having directions τ', τ'', and radius r, touch each other iff

$$r = \frac{d}{2 - d}. \tag{2}$$

Indeed, if the cylinders touch each other, we have the proportion:

$$\frac{d}{1} = \frac{2r}{1 + r}. \tag{3}$$

We denote by M^6 the manifold of 6-tuples

$$\mathbf{m} = \{u_1, ..., u_6 : u_i \in M, i = 1, ..., 6\}. \tag{4}$$

Our interest is in the function

$$D\left(\mathbf{m}\right)=\min_{1\leq i<j\leq 6}d_{u_iu_j}.$$

We are especially interested in knowing its maximum, since it defines, via (2), the maximal radius of 6 non-intersecting equal cylinders touching the unit ball.

The generators of the cylinders in C_6 touching the ball define a point in M^6, shown on Figure 6. We denote it by the same symbol C_6. Note that $D\left(C_6\right)=1$.

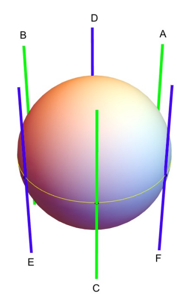

Figure 6: Configuration C_6 of tangent lines

3 Points $\mathbf{m}\in M^6$ with high $D\left(\mathbf{m}\right)$ value

Here we describe the 'good' configurations $\mathbf{m}$ with high values of the function $D\left(\mathbf{m}\right)$. We obtain them by deforming in a certain way the configuration C_6 which in our notation can be written as

$$\begin{aligned}C_6 \quad &\equiv C_6\left(0,0,0\right)=\left\{\left[\left(0,\tfrac{\pi}{6}\right),\uparrow\right],\left[\left(0,\tfrac{\pi}{2}\right),\uparrow\right],\left[\left(0,\tfrac{5\pi}{6}\right),\uparrow\right],\right.\\ &\left.\left[\left(0,\tfrac{7\pi}{6}\right),\uparrow\right],\left[\left(0,\tfrac{3\pi}{2}\right),\uparrow\right],\left[\left(0,\tfrac{11\pi}{6}\right),\uparrow\right]\right\}.\end{aligned}$$

Namely, we will explore the 6-tuples $C_6\left(\varphi,\delta,\varkappa\right)$, of the form

$$\begin{aligned}C_6\left(\varphi,\delta,\varkappa\right)=\Big\{A&=\left[\left(\varphi,\tfrac{\pi}{6}-\varkappa\right),\uparrow_\delta\right],D=\left[\left(-\varphi,\tfrac{\pi}{2}+\varkappa\right),\uparrow_\delta\right],\\ B&=\left[\left(\varphi,\tfrac{5\pi}{6}-\varkappa\right),\uparrow_\delta\right],E=\left[\left(-\varphi,\tfrac{7\pi}{6}+\varkappa\right),\uparrow_\delta\right],\\ C&=\left[\left(\varphi,\tfrac{3\pi}{2}-\varkappa\right),\uparrow_\delta\right],F=\left[\left(-\varphi,\tfrac{11\pi}{6}+\varkappa\right),\uparrow_\delta\right]\Big\}.\end{aligned}\tag{5}$$

In words, the three points $\left[\left(0,\frac{\pi}{6}\right),\uparrow\right]$, $\left[\left(0,\frac{5\pi}{6}\right),\uparrow\right]$ and $\left[\left(0,\frac{3\pi}{2}\right),\uparrow\right]$ go upward by φ, then 'horizontally' by $-\varkappa$, and then the three vectors $\uparrow$ are rotated by δ, while the three remaining points go downward by φ, then 'horizontally' by $\varkappa$, and, finally, the three vectors $\uparrow$ are rotated by δ.

For all $\varphi, \delta, \varkappa$ these configurations possess $\mathbb{D}_3 \equiv \mathbb{Z}_3 \times \mathbb{Z}_2$ symmetry. The group $\mathbb{D}_3$ is generated by the rotations $R_{\mathbf{z}}^{120^\circ}$ and $R_{\mathbf{x}}^{180^\circ}$. We denote by $\mathcal{C}^3 \in M^6$ the 3-dimensional submanifold formed by 6-tuples (5).

We claim that there exists a curve γ in the manifold $\mathcal{C}^3$,

$$\gamma(\varphi) = C_6\big(\varphi, \delta(\varphi), \varkappa(\varphi)\big)\ ,\ \varphi \in \left[0; \frac{\pi}{2}\right]\ , \tag{6}$$

which starts at $C_6(0,0,0)$ for $\varphi = 0$,

$$\gamma(0) = C_6(0,0,0)\ , \tag{7}$$

such that the function $D\big(\gamma(\varphi)\big)$ is unimodal on γ, with maximal value $\sqrt{\frac{12}{11}}$, which corresponds to the value $r_{\mathfrak{m}}$, given in (1), of the radii of the touching cylinders. This is summarized in our main result below. Its proof constitutes a part of Section 5.

Theorem 1 *The configuration $C_6(0,0,0)$ can be unlocked. Moreover,*

i. *There is a continuous curve γ, see (6) and (7), on which the function $D\big(\gamma(\varphi)\big)$ grows for $\varphi \in [0, \varphi_{\mathfrak{m}}]$ and decays for $\varphi > \varphi_{\mathfrak{m}}$, with $\varphi_{\mathfrak{m}} = \arcsin\sqrt{\frac{3}{11}}$. The explicit description of γ is given in (25)-(27).*

ii. *At the point $\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}} = \delta(\varphi_{\mathfrak{m}}), \varkappa_{\mathfrak{m}} = \varkappa(\varphi_{\mathfrak{m}})$ we have*

$$D\Big(C_6(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})\Big) = \sqrt{\frac{12}{11}}\ ,$$

so the radii of the corresponding cylinders are equal to

$$r_{\mathfrak{m}} = \frac{1}{8}\left(3 + \sqrt{33}\right).$$

We stress again that the existence of analytic expression for the curve γ comes beyond expectations, and seems quite surprising.

The record configuration is shown on Figures 7, 8 and 9.

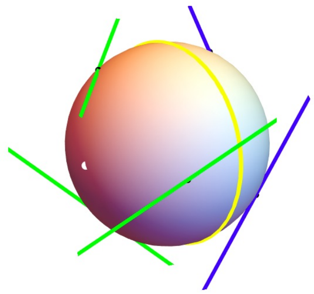

Figure 7: Record configuration, side view, the equator is yellow, the north pole is white

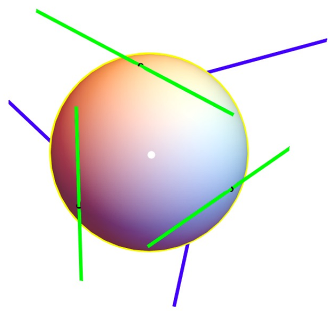

Figure 8: Record configuration again, three upper tangency points shown

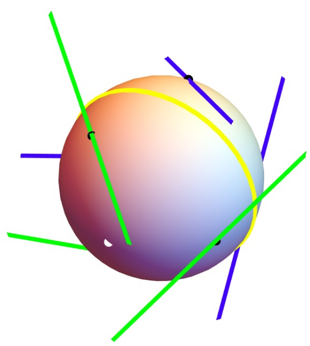

Figure 9: Record configuration once more, two upper and one lower tangency points shown

4 Formulas for $\mathbb{D}_3$-symmetric configurations

Now we present the explicit formulas which we use for exploring the manifold $\mathcal{C}^3$. Because of the $\mathbb{D}_3$-symmetry, $d_{AB} = d_{BC} = d_{CA} = d_{DE} = d_{EF} = d_{FD}$, so we need only one of these. It is given by

$$d_{AB}^2 = \frac{48\sin^2(\delta)\cos^2(\delta)\cos^4(\varphi)}{(6\cos^2(\delta)\cos(2\varphi) + 3\cos(2\delta) + 7)\left(\cos^2(\delta)\sin^2(\varphi) + \sin^2(\delta)\right)} \ ,$$

which, naturally, does not depend on $\varkappa$. Also, $d_{AD} = d_{BE} = d_{CF}$, with

$$d_{AD}^2 = \frac{\mu_{AD}^2}{4(1-\nu_{AD}^2)} \ , \text{ where}$$

$$\begin{aligned} \mu_{AD} &= \sin(2\delta)\Big(2\cos^2(\varphi) - (\cos(2\varphi) - 3)\sin\left(2\varkappa - \tfrac{\pi}{6}\right)\Big) \\ &+ 4\cos(2\delta)\sin(\varphi)\cos\left(2\varkappa - \tfrac{\pi}{6}\right) \ , \\ \nu_{AD} &= \sin\left(2\varkappa - \tfrac{\pi}{6}\right)\Big(\sin^2(\delta) - \cos^2(\delta)\sin^2(\varphi)\Big) \\ &+ \sin(2\delta)\sin(\varphi)\cos\left(2\varkappa - \tfrac{\pi}{6}\right) - \cos^2(\delta)\cos^2(\varphi) \ . \end{aligned}$$

The third triplet of functions is $d_{BD} = d_{CE} = d_{AF}$, with

$$d_{BD}^2 = \frac{\mu_{BD}^2}{4(1-\nu_{BD}^2)} \ , \text{ where}$$

$$\begin{aligned} \mu_{BD} &= \sin(2\delta)\Big(2\cos^2(\varphi) - (\cos(2\varphi) - 3)\sin\left(2\varkappa - \tfrac{5\pi}{6}\right)\Big) \\ &+ 4\cos(2\delta)\sin(\varphi)\cos\left(2\varkappa - \tfrac{5\pi}{6}\right) \ , \\ \nu_{BD} &= \sin\left(2\varkappa - \tfrac{5\pi}{6}\right)\Big(\sin^2(\delta) - \cos^2(\delta)\sin^2(\varphi)\Big) \\ &+ \sin(2\delta)\sin(\varphi)\cos\left(2\varkappa - \tfrac{5\pi}{6}\right) - \cos^2(\delta)\cos^2(\varphi). \end{aligned}$$

The last triplet is $d_{AE} = d_{BF} = d_{CD}$, with

$$d_{AE}^2 = \frac{\mu_{AE}^2}{\nu_{AE}} \ , \text{ where}$$

$$\mu_{AE} = 2\Big(\cos(\delta)\cos(\varkappa) - \sin(\delta)\sin(\varphi)\sin(\varkappa)\Big),$$

$$\nu_{AE} = \cos^2(\delta)\cos^2(\varphi) + (\sin(\delta))\sin(\varkappa) - \cos(\delta)\sin(\varphi)\cos(\varkappa)\Big)^2 .$$

The derivation of the above formulas is straightforward, though tedious. It is difficult to explore these formulas directly. However, there is a suitable choice of variables, such that instead of ratios of trigonometric polynomials involving various sin-s and cos-s of various angles the square of each distance becomes a rational function.

Proposition 2 *Let*

$$S = \sin(\varphi)\ ,\ T = \tan(\delta)\ , \tag{8}$$

and

$$U = \tan(\varkappa - \frac{\pi}{6})\ ,\ \bar{U} = -\tan(\varkappa + \frac{\pi}{6})\ .$$

Then

$$d_{AB}^2 = \frac{12T^2\left(1 - S^2\right)^2}{(4 - 3S^2 + T^2)(S^2 + T^2)}\ , \tag{9}$$

$$d_{AD}^2 = \frac{4\left(TS + U\right)^2}{1 + U^2 + T^2 - S^2 + 2STU}\ , \tag{10}$$

$$d_{BD}^2 = \frac{4\left(-TS + \bar{U}\right)^2}{1 + \bar{U}^2 + T^2 - S^2 - 2ST\bar{U}}\ . \tag{11}$$

Since $(\varkappa + \frac{\pi}{6}) = (\varkappa - \frac{\pi}{6}) + \frac{\pi}{3}$, we have $\left(\text{via } \tan(\beta_1 + \beta_2) = \frac{\tan(\beta_1) + \tan(\beta_2)}{1 - \tan(\beta_1)\tan(\beta_2)}\right)$:

$$\bar{U} = -\frac{U + \sqrt{3}}{1 - \sqrt{3}U} \quad \text{or} \quad -\sqrt{3}U\bar{U} + U + \bar{U} + \sqrt{3} = 0\ . \tag{12}$$

The proof of the proposition is elementary: one needs just to check various trigonometric identities. Yet to find the right choice of variables, allowing further analysis, was the longest part of the present work, involving lengthy and painful computations.

5 Solving $d_{AB}^2 = d_{BD}^2 = d_{AD}^2$.

In this section we will write the functions $d_{AB}^2, d_{BD}^2, d_{AD}^2$, given by relations $(9-11)$, on the curve $d_{AB}^2 = d_{BD}^2 = d_{AD}^2$ as functions of one parameter, and then will maximize them. We will also use, of course, the relation (12).

The angle φ is positive in the region of our interest, so the factors in the denominator of d^2_{AB} do not vanish. The denominator of d^2_{AD} (similarly for d^2_{BD}) can be written in the form $(U+ST)^2+(1+T^2)(1-S^2)$ so it does not vanish as well.

The equality $d^2_{AD}=d^2_{BD}$ gives

$$(T^2+1)(S^2-1)(U+\bar{U})(U-\bar{U}+2ST)=0\ . \tag{13}$$

The factor (T^2+1) is non-zero, the factor (S^2-1) is non-zero at any point in our $(\varphi,\varkappa,\delta)$-space except the initial point $\varphi=0$, the factor $(U+\bar{U})$ is non-zero (see (12)) so we conclude

$$U-\bar{U}+2ST=0\ . \tag{14}$$

Together, eqs. (12) and (14) imply

$$K_1\equiv-\sqrt{3}U^2+2U\left(1-\sqrt{3}ST\right)+2ST+\sqrt{3}=0.$$

The equality $d^2_{AB}=d^2_{AD}$ leads to the equality

$$(T^2+1)\left[(-4S^2+3S^4-T^2)(U+ST)^2-3T^2(S^2-1)^3\right]=0$$

The factor (T^2+1) does not vanish, so we obtain another relation:

$$K_2\equiv(-4S^2+3S^4-T^2)(U+ST)^2-3T^2(S^2-1)^3=0.$$

Therefore

$$(-4S^2+3S^4-T^2)K_1-\sqrt{3}K_2=0,$$

which reads

$$-\sqrt{3}T^2(S^2-1)^3+\left(4S^2-3S^4+T^2\right)\ \left(\sqrt{3}+2ST+\sqrt{3}S^2T^2+2U\right)=0\ .$$

Since the factor $(4S^2-3S^4+T^2)$ does not vanish on our trajectory, we have

$$2U=\frac{\sqrt{3}T^2(S^2-1)^3}{(4S^2-3S^4+T^2)}-\sqrt{3}-2ST-\sqrt{3}S^2T^2\ . \tag{15}$$

Substituting the expression (15) for U into either K_1 or K_2 we find

$$\frac{S^2(4-3S^2+T^2)}{(4S^2-3S^4+T^2)^2}\Psi=0\ ,$$

where

$$\Psi = 4S^2 - 8T^2 - 3S^4 + 29S^2T^2 - 4T^4 - 22S^4T^2 + 14S^2T^4 + 4S^6T^2 - 7S^4T^4 + S^2T^6 \ .$$

Again, the factor $S^2(4-3S^2+T^2)$ does not vanish, so our trajectory is defined by (15) and a component of the curve

$$\Psi = 0 \ .$$

The leading term of Ψ at 0 is $4S^2 - 8T^2$ so there are two components of the curve passing through 0. These two components are related by the reflection of the initial sphere, so we can, without loosing generality take the component for which $T > 0$ for $S > 0$ for small S and T.

We are now looking at the maximal value of the square of the distance $d^2_{AB} = d^2_{AD} = d^2_{BD}$ on our trajectory. The simplest way to do this is to find the maximal value of d^2_{AB} constrained to the curve $\Psi = 0$ since both expressions, d^2_{AB} and Ψ, do not contain the variable U. Moreover, only even powers of S and T appear in d^2_{AB} and Ψ so we set $s = S^2$ and $t = T^2$ and look for the maximal value of the function

$$F = \frac{12t\,(1-s)^2}{(4-3s+t)(s+t)}$$

with the constraint

$$\psi = 0 \ , \ \text{ where } \ \psi = 4s - 8t - 3s^2 + 29st - 4t^2 - 22s^2t + 14st^2 + 4s^3t - 7s^2t^2 + st^3 \ .$$

Let

$$x = \frac{1-s}{t+1} \left(= \cos^2(\varphi)\cos^2(\delta)\right) . \tag{16}$$

In the variables t and x the expression ψ has the following form:

$$\psi = -(1+t)^3 \left(-1 - 2x + tx + 3x^2 + 7tx^2 + 4tx^3\right) \ .$$

The factor $(1+t)$ is non-zero, hence the relation $\psi = 0$ implies

$$t = \frac{1 + 2x - 3x^2}{x\,(1 + 7x + 4x^2)} = \frac{(1+3x)(1-x)}{x\,(1+7x+4x^2)} \tag{17}$$

along our component of the constraint curve. Note that the zeros of the polynomial $1 + 7x + 4x^2$ are negative while the values of the variable x are, by construction, positive, see (16).

The function F in the variables t and x reads

$$F = \frac{12tx^2}{(1-x)(1+3x)} \ .$$

Substituting the expression (16) for t we find that along our curve

$$F = \frac{12x}{1+7x+4x^2} \ . \tag{18}$$

By construction, the variable x decreases from 1 to 0 on our trajectory. It is straightforward to find that the fraction (18) on the interval $(0,1)$ attains its maximum at the point

$$x_{\mathfrak{m}} = \frac{1}{2} \tag{19}$$

with the value

$$F(x_{\mathfrak{m}}) = \frac{12}{11} \ . \tag{20}$$

From (17) we now obtain the value of δ corresponding to this point:

$$t_{\mathfrak{m}} = \tan^2(\delta_{\mathfrak{m}}) = \frac{5}{11} \ , \tag{21}$$

and then, by (16), the value of φ:

$$s_{\mathfrak{m}} = \sin^2(\varphi_{\mathfrak{m}}) = \frac{3}{11} \ . \tag{22}$$

Finally, eq. (15) gives the value of $\varkappa$:

$$U_{\mathfrak{m}} = \tan(\varkappa_{\mathfrak{m}} - \frac{\pi}{6}) = -\frac{1}{11}\sqrt{3}\left(4+\sqrt{5}\right) \ , \tag{23}$$

which means that

$$\tan(\varkappa_{\mathfrak{m}}) = -\frac{1}{\sqrt{15}} \ .$$

At the point $(\varphi_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}, \delta_{\mathfrak{m}})$ the square of the distance in the last triplet is

$$d_{AE}^2 = \frac{540}{143} > \frac{12}{11} \ .$$

The radius of the touching cylinders is given by (2) :

$$r_{\mathfrak{m}} = \frac{\sqrt{\frac{12}{11}}}{2-\sqrt{\frac{12}{11}}} = \frac{1}{8}\left(3+\sqrt{33}\right) \approx 1.093070331 \ . \tag{24}$$

Our trajectory γ is parameterized by the variable x:

$$S = 2\sqrt{\frac{(1-x)x(1+x)}{1+7x+4x^2}} \ , \tag{25}$$

$$T = \sqrt{\frac{(1-x)(1+3x)}{x+7x^2+4x^3}} \ , \tag{26}$$

$$U = \frac{1}{2}\left(-\sqrt{3} - \frac{4(1-x)\sqrt{(1+x)(1+3x)}}{1+7x+4x^2} + \frac{\sqrt{3}(-1+5x)}{1+7x+4x^2}\right) \ .$$

The last equation can be rewritten in the form

$$\tan(\varkappa) = \frac{x-1}{\sqrt{(1+x)(1+3x)}} \ . \tag{27}$$

It is interesting to note that the point where the function F gets back its initial value 1 is also (as $x_{\mathfrak{m}}$) rational: $x = 1/4$.

6 Generalizations

In this section we briefly consider the analogous deformation of $2n$ equal parallel cylinders touching the unit ball, for values of n different from 3. We start by presenting the formulas needed and then prove that for $n > 2$ the configuration is unlockable. The case $n = 2$ is special and we consider it in details.

6.1 Various distances

Let α be the 'angle' between two neighboring vertical cylinders (α is $\frac{\pi}{3}$ for $n = 3$). Our initial configuration, generalizing the configuration of three lines A, B and D, is

$$C_3 \equiv C_3\left(0,0,0\right) = \left\{\left[\left(0, \tfrac{\alpha}{2}\right), \uparrow\right], \left[\left(0, \tfrac{3\alpha}{2}\right), \uparrow\right], \left[\left(0, \tfrac{5\alpha}{2}\right), \uparrow\right]\right\} \ .$$

We will study its deformations

$$C_3\left(\varphi, \delta, \varkappa\right) = \left\{A, B, D\right\} \ \ , \text{ where}$$

$$A = \left[\left(\varphi, \tfrac{\alpha}{2} - \varkappa\right), \uparrow_\delta\right], B = \left[\left(\varphi, \tfrac{5\alpha}{2} - \varkappa\right), \uparrow_\delta\right], D = \left[\left(-\varphi, \tfrac{3\alpha}{2} + \varkappa\right), \uparrow_\delta\right].$$

For the future use we introduce the notation

$$\gamma := \varkappa - \frac{\alpha}{2} \ , \ \bar{\gamma} := \varkappa + \frac{\alpha}{2} \ .$$

In the coordinates (8) we find, after lengthy computations, that

$$d_{AB}^2 = \frac{4\sin(\alpha)^2(1-S^2)^2T^2}{(S^2+T^2)(1-\sin(\alpha)^2S^2+\cos(\alpha)^2T^2)} \ . \tag{28}$$

Next, putting

$$U = \tan(\gamma), \quad \bar{U} = \tan(\bar{\gamma}) \ , \tag{29}$$

we get

$$d_{AD}^2 = \frac{4(ST+U)^2}{1-S^2+T^2+U^2+2STU} \ , \tag{30}$$

while

$$d_{BD}^2 = d_{AD}^2|_{\gamma\to\bar{\gamma},\delta\to-\delta} \ .$$

Again, the initial trigonometric formulas for these distances involve several different trigonometric functions for each angle $\varphi, \varkappa, \delta$. The advantage of the formulas above is that every variable $\varphi, \varkappa, \delta$ enters each distance only via a single trigonometric function, and so these expressions become algebraic, which permits us to write down the final formulas.

6.2 When can the distances d_{AB}, d_{AD}, d_{BD} grow?

Here we look for the range of α in which all the distances d_{AB}, d_{AD}, d_{BD} increase above the value $4\sin(\alpha/2)^2$ (the initial distance between the lines A and D) as we move away from the point $C_3\,(0,0,0)$ in $C_3\,(\varphi,\delta,\varkappa)$. As a result, we will prove the following statement.

Proposition 3 *For any $n = 2k \geq 6$ the configuration of n equal parallel non-intersecting cylinders, touching the unit ball, can be unlocked.*

Proof. Let us consider the curve

$$\varphi = \sum_{j>0} \varphi_j t^j \ , \ \delta = \sum_{j>0} \delta_j t^j \ , \ \varkappa = \sum_{j>0} \varkappa_j t^j \ , \tag{31}$$

and study the expansions of the distances d^2_{**} in t. The coefficient in t^k, $k = 0, 1, ...$, is denoted by $[d^2_{**}]_k$. We have

$$\left[d^2_{AB}\right]_0 = \frac{4\delta_1^2 \sin(\alpha)^2}{\delta_1^2 + \varphi_1^2} \ .$$

This is bigger or equal than $4\sin(\alpha/2)^2$ iff

$$\delta_1^2 \sin(\alpha)^2 \geq (\delta_1^2 + \varphi_1^2) \sin(\alpha/2)^2 \ . \tag{32}$$

Next,

$$\left[d^2_{AD}\right]_0 = \left[d^2_{BD}\right]_0 = 4\sin(\alpha/2)^2$$

and

$$\left[d^2_{AD}\right]_1 = -4\varkappa_1 \sin(\alpha) \ , \ \left[d^2_{BD}\right]_1 = 4\varkappa_1 \sin(\alpha) \ .$$

For both distances to weakly grow, we have to set $\varkappa_1 = 0$. Then

$$\left[d^2_{AD}\right]_2 = -\sin(\alpha)\left(2\delta_1\varphi_1 + 4\varkappa_2 + 2\delta_1\varphi_1\cos(\alpha) + \sin(\alpha)(\delta_1^2 - \varphi_1^2)\right) \ ,$$

$$\left[d^2_{BD}\right]_2 = \sin(\alpha)\left(2\delta_1\varphi_1 + 4\varkappa_2 + 2\delta_1\varphi_1\cos(\alpha) - \sin(\alpha)(\delta_1^2 - \varphi_1^2)\right) \ .$$

Both are positive ($\sin(\alpha) > 0$) iff

$$-2\delta_1\varphi_1(1 + \cos(\alpha)) + (\delta_1^2 - \varphi_1^2)\sin(\alpha) \leq 4\varkappa_2$$

and

$$4\varkappa_2 \leq -2\delta_1\varphi_1(1 + \cos(\alpha)) - (\delta_1^2 - \varphi_1^2)\sin(\alpha) \ .$$

This can be solved for $\varkappa_2$ if

$$-2\delta_1\varphi_1(1+\cos(\alpha)) + (\delta_1^2 - \varphi_1^2)\sin(\alpha) \leq -2\delta_1\varphi_1(1+\cos(\alpha)) - (\delta_1^2 - \varphi_1^2)\sin(\alpha)$$

or

$$\delta_1^2 \leq \varphi_1^2 \ . \tag{33}$$

The inequalities (32) and (33) are compatible iff

$$\cos(\alpha/2)^2 \geq 1/2 \ \text{ or } \ \alpha \leq \pi/2 \ .$$

For $\alpha < \pi/2$ this analysis is sufficient to show that there is a room for the cylinders A, B and D to grow. It is not too difficult to see that the other values of distances between pairs of cylinders in our configuration of

$2n$ cylinders are not relevant. Thus, our Proposition is proven. In particular, this analysis proves also the infinitesimal version of our Theorem 1. ■

For $\alpha = \pi/2$ (the case of four cylinders) we find $\delta_1^2 = \varphi_1^2$ and $4\varkappa_2 = -2\delta_1\varphi_1$, so further analysis is needed. It turns out that there is only one possible motion here: the cylinders A and D stay parallel, the remaining two stay parallel as well, so, up to a global rotation of all four cylinders, one parallel pair is fixed while the other one rotates. We will show this in Subsection 6.3.

We have analyzed also another strategy of unlocking, when the family of possible motions $C_3(\varphi, \delta, \varkappa)$ is replaced by $\bar{C}_3(\varphi, \delta, \varkappa) = \{A, B, D\}$ where

$$A = \left[\left(\varphi, \tfrac{\alpha}{2} - \varkappa\right), \uparrow_\delta\right], B = \left[\left(\varphi, \tfrac{5\alpha}{2} - \varkappa\right), \uparrow_\delta\right], D = \left[\left(-\varphi, \tfrac{3\alpha}{2} + \varkappa\right), \uparrow_{-\delta}\right] .$$

The difference here is the change of δ to $-\delta$ for the cylinder D. This other strategy corresponds to a different embedding of the symmetry group $\mathbb{D}_3$ (in case of $n = 3$, i.e. $\alpha = \frac{\pi}{3}$) in $O(3)$.

The function d_{AB}^2 stays, obviously, the same, while

$$d_{AD}^2 = \frac{4U^2S^2(1+T^2)^2}{(S^2+T^2)(1-S^2+U^2+T^2U^2)}, \quad d_{BD}^2 = d_{AD}^2|_{\gamma\to\bar{\gamma},\delta\to-\delta}.$$

For the curve (31), we have

$$\left[d_{AD}^2\right]_0 = \left[d_{BD}^2\right]_0 = \frac{4\varphi_1^2\sin(\alpha/2)^2}{\delta_1^2+\varphi_1^2} .$$

This is bigger or equal than $4\sin(\alpha/2)^2$ if $\varphi_1^2 \geq \delta_1^2 + \varphi_1^2$ which implies $\delta_1 = 0$. Then (32) implies $\varphi_1 = 0$ so for $0 < \alpha < \pi$ the cylinders cannot be unlocked using this other strategy.

6.3 Four cylinders

The initial position of four cylinders is shown on Figure 10.

Proposition 4 *The configuration of four parallel non-intersecting cylinders of radius $r = 1 + \sqrt{2}$ touching the unit ball, being not rigid, is not unlockable in our regime.*

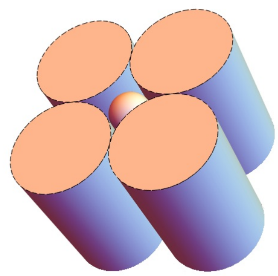

Figure 10: Initial position

Proof. For $n = 4$, the angle $\alpha = \frac{\pi}{2}$, so $\bar{U} = -1/U$, see (29) , and

$$d_{AB}^2 = \frac{4\,(1-S^2)\,T^2}{S^2+T^2} \ ,$$

$$d_{AD}^2 = \frac{4(ST+U)^2}{1-S^2+T^2+2STU+U^2} \ , \ d_{BD}^2 = \frac{4(-1+STU)^2}{1-2STU+U^2-S^2U^2+T^2U^2} \ .$$

Let $Q := T^2 - S^2 - 2S^2T^2$. The system of inequalities

$$d_{AB}^2 \geq 2 \ ,$$

$$d_{AD}^2 \geq 2 \ , \ d_{BD}^2 \geq 2$$

is equivalent to

$$Q \geq 0 \ , \tag{34}$$

$$U^2 + 2STU \geq 1 + Q \ , \ 1 - 2STU \geq U^2(1+Q) \ . \tag{35}$$

The sum of two last inequalities is

$$U^2 + 1 \geq (U^2+1)(1+Q) \ \text{ so } \ 1 + Q \leq 1 \ ,$$

which, together with (34) gives

$$Q = 0 \quad , \quad \text{or} \ \ S^2 = \frac{T^2}{1+2T^2} \ . \tag{36}$$

Now the inequalities (35) become

$$U^2 + 2STU \geq 1 \ , \ 1 - 2STU \geq U^2$$

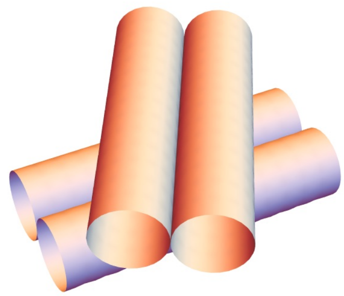

Figure 11: Motion of four cylinders

so

$$U^2 + 2STU = 1 \ . \tag{37}$$

Eqs. (36) and (37) define uniquely the trajectory, depicted in Figure 11. ■

Conjecture. We believe that these are all possible positions of four cylinders of radius $r = 1 + \sqrt{2}$ touching the unit ball.

If that would be the case, then, obviously, one could not put 5 non-intersecting cylinders of radius $r = 1 + \sqrt{2}$ in contact with unit ball, thus answering to the analogue of the initial $n = 6$ question of Kuperberg. But this last statement can be proven independently of the above conjecture.

Proposition 5 *It is not possible to place five non-intersecting cylinders of radius $r = 1 + \sqrt{2}$ in such a way that all of them touch a unit ball.*

Proof. Suppose the opposite. Consider the corresponding configuration of 5 cylinders. Let us inscribe into them 5 balls of the same radius $r = 1 + \sqrt{2}$, each touching the central unit ball. As we will explain in the next paragraphs (see also [KKLS]), any configuration of five non-intersecting balls of radius $r = 1+\sqrt{2}$ touching the central unit ball contains a triple, which, mod $SO(3)$, is formed by a ball on the North pole, a ball on the South pole and a ball on the intersection of Greenwich and the equator. The three non-intersecting cylinders of radius $r = 1+\sqrt{2}$, containing these three balls, have to be parallel (and perpendicular to Greenwich), which leaves a uniquely defined place for just one more cylinder.

In order to understand the configuration manifold of five non-intersecting balls of radius $r = 1 + \sqrt{2}$ touching the central unit ball, let us position one of them at the South. Consider the set T of the three balls closest to this S ball. Their centers lie in the (closed) northern hemisphere.

Consider first the case when no ball from T touches the two others. If at least one of them has its center not on the equatorial plane, then there is no place left for the fifth ball. So all three centers must be on the equator plane, and then the fifth ball is fixed to be the N ball. Our three equatorial balls are then free to use the equatorial plane. (This shows that the dimension of the configuration manifold of our 5 balls is two, mod $SO(3)$.) The N ball, the S ball and any one from the equatorial balls make then the triple sought.

In the remaining case, when one ball from T touches the other two, the triple itself forms a configuration of the type needed. ■

7 Conclusion

In this paper we were attempting to understand better the question of W. Kuperberg about the maximal number of non-intersecting equal (infinite) cylinders of radius $r \geq 1$ touching the unit ball in $\mathbb{R}^3$. The open conjecture is that this number is 6. We were able to clarify a related question of how large the radius r of six cylinders can be in order that the non-intersection condition is satisfied.

We believe that the *record* configuration $C_6(\varphi_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}, \delta_{\mathfrak{m}})$ we found, which has all the relevant distances equal to $\sqrt{\frac{12}{11}}$, gives in fact the best possible value for r, see (24).

It is interesting to note that all the angles describing the configuration $C_6(\varphi_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}, \delta_{\mathfrak{m}})$ are pure geodetic, in the sense of [CRS]: an angle α is pure geodetic if the square of its sine is rational. Formally it is explained as follows: for any rational x the formulas (25), (26) and (27) define pure geodetic angles and our record configuration is attained at $x = 1/2$.

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