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VAN HAO CAN AND SHUTA NAKAJIMA

ABSTRACT. In this paper, we show that the passage time in the frog model on $\mathbb{Z}^d$ with $d \geq 2$ has a sublinear variance. The proof is based on the method introduced in [8] combining with tessellation arguments to estimate the martingale difference. We also apply this method to get the linearity of the lengths of optimal paths.

1. INTRODUCTION

Frog models are simple but well-known models in the study of the spread of infection. In these models, individuals (also called frogs) move on the integer lattice $\mathbb{Z}^d$, which have one of two states infected (active) and healthy (passive). We assume that at the beginning, there is only one infected frog at the origin, and there are healthy frogs at other sites of $\mathbb{Z}^d$. When a healthy frog encounters with an infected one, it becomes infected forever. While the healthy frogs do not move, the infected ones perform independent simple random walks once they get infected. The object we are interested in is the long time behavior of the infected individuals.

To the best of our knowledge, the first result on frog models is due to Tecls and Wormald [19], where they proved the recurrence of the model (more precisely, they showed that the origin is visited infinitely often a.s.). Since then, there are numerous results on the behavior of the model under various settings of initial configurations, mechanism of walks, or underlying graphs, see [1, 3, 5, 10, 11, 12, 13, 14]. In particular, Popov and some authors study the phase transition of the recurrence vs transience for the model with Bernoulli initial configurations and for the model with drift, see [2, 9, 11, 18]. Another interesting feature in the frog model is that it can be described in the first passage percolation contexts, which is explained below. In fact, Alves, Machado and Popov used this property to prove a shape theorem [1]. Moreover, the large deviation estimates for the passage time are derived in [7, 16] recently.

The frog model can be defined formally as follows. Let $d \geq 2$ and $\{(S_j^x)_{j\in\mathbb{N}}, x \in \mathbb{Z}^d\}$ be independent SRWs such that $S_0^x = x$ for any $x \in \mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$, let

$$t(x,y) = \inf\{j : S_j^x = y\}.$$

The first passage time from x to y is defined by

$$T(x,y) = \inf\Big\{\sum_{i=1}^{k} t(x_{i-1},x_i) : x = x_0,\dots,x_k = y \text{ for some } k\Big\}.$$

The quantity $T(x,y)$ can be seen as the first time when the frog at y becomes infected assuming that the frog at x was the only infected one at the beginning. For the simplicity of notation, we write $T(x)$ instead of $T(0,x)$.

It has been shown in [1] that the passage time is subadditive, i.e. for any $x, y, z \in \mathbb{Z}^d$

$$T(x,z) \leq T(x,y) + T(y,z). \tag{1.1}$$

The authors of [1] also show that the sequence $\{T((k-1)z, kz)\}_{k\geq 1}$ is stationary and ergodic for any $z \in \mathbb{Z}^d$. As a consequence of Kingman's subadditive ergodic theorem (see [15] or [1, Theorem 3.1]), one has

$$\lim_{n\to\infty} \frac{T(nz)}{n} \to \kappa_z \quad a.s., \tag{1.2}$$

with

$$\kappa_z = \inf_{n\geq 1} \frac{\mathbb{E}(T(nz))}{n}.$$

Furthermore, a shape theorem for the set of active frogs has been also proved, see [1, Theorem 1.1]. The convergence (1.2), which can be seen as a law of large numbers, implies that for any $x \in \mathbb{Z}^d$ the passage time $T(x)$ grows linearly in $|x|_1$. A natural question is whether the standard central limit theorem hold for $T(x)$. The first task is to understand the behavior of variance of $T(x)$. In [16], the author proves some large deviation estimates for $T(x)$, see in particular Lemma 2.2 below. As a consequence, one can show that $\mathrm{Var}(T(x)) = \mathcal{O}(|x|_1(1+\log|x|_1)^{2A})$, for some constant A, see Corollary 2.3. However, this result is not enough to answer the question on standard central limit theorem.

Our main result is to show that the passage time has sublinear variance and thus the standard central limit theorem is not true.

Theorem 1.1. *There exists a positive constant $C = C(d)$, such that for any $x \in \mathbb{Z}^d$,*

$$\mathrm{Var}(T(x)) \leq \frac{C|x|_1}{\log|x|_1}.$$

The sublinearity of variance as in Theorem 1.1, which is also called the superconcentration [6], was first discovered in the classical first passage percolation by Benjamini, Kali and Schram [4]. Hence, this result is sometimes called BKS-inequality. Chatterjee found the connection of superconcentration with chaos and multiple valleys in the gaussian polymer and SK model. This relation is expected to hold in general models. Therefore, the superconcentration is not only an interesting property itself but also an important object to study the structure of optimal paths and the energy landscape.

The superconcentration has been proved for several models such as the classical first passage percolation and the gaussian polymer model. In these proofs, one usually has to estimate the martingale difference carefully, which needs the model-dependent arguments. In the frog model, the correlation between passage times is problematic for this kind of estimate. A key observation to pass this difficulty is that the passage times are locally-dependent. Indeed, by large deviation estimates (see Lemma 2.1), $T(x,y) \leq C|x-y|_1$ for some $C > 0$ with very high probability. Thus $T(x,y)$ mainly depends on SRWs $(S^z_\cdot)$ with $|z-x|_1 \leq C|x-y|_1$. Therefore, if the two pairs (x,y) and (u,v) are far enough from each other, the passage times $T(x,y)$ and $T(u,v)$ are weakly dependent. From this observation, using tessellation arguments, we decompose the martingale difference to some groups of the independent passage times. After that, we apply the percolation estimate to get the desired bound. This approach seems to be useful for other problems. Indeed, we also prove the linearity of the length of optimal path by using a similar method.

Given $x, y \in \mathbb{Z}^d$, let us denote by $\mathbb{O}(x,y)$ the set of all optimal paths from x to y. We simply write $\mathbb{O}(x)$ for $\mathbb{O}(0,x)$. For any path $\gamma = (y_i)_{i=1}^{\ell} \subset \mathbb{Z}^d$, we denote $l(\gamma) = \ell$ the number of vertices in this path, and call it the length of γ. We will prove that the length of optimal paths from 0 to x grows linearly in $|x|_1$.

Proposition 1.2. *There exist positive constants ε, c and C, such that for any $x \in \mathbb{Z}^d$*

$$\mathbb{P}\left(c|x|_1 \leq \min_{\gamma\in\mathbb{O}(x)} l(\gamma) \leq \max_{\gamma\in\mathbb{O}(x)} l(\gamma) \leq C|x|_1\right) \geq 1 - e^{-|x|_1^{\varepsilon}}.$$

1.1. Notation.

- If $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, we denote $|x|_1 = |x_1| + \ldots + |x_d|$.
- For any $n \geq 1$, we denote by $B(n) = [-n, n]^d$.
- For any $\ell \geq 1$, we call a sequence of ℓ distinct vertices $\gamma = (y_i)_{i=1}^{\ell}$ in $\mathbb{Z}^d$ a path of length ℓ, we denote $|\gamma|_1 = |y_2 - y_1|_1 + \ldots + |y_\ell - y_{\ell-1}|_1$.
- Given $y = y_i \in \gamma$, we define $\bar{y} = y_{i+1}$ the next point of y in γ with the convention that $\bar{y}_\ell = y_\ell$.

- We write $y \sim \bar{y} \in \gamma$ if $\bar{y}$ is the next point of y in γ.
- For $L \geq 1$, we write

$$\mathcal{P}_L = \{\gamma = (y_i)_{i=1}^{\ell} \subset B(L), |\gamma|_1 \leq L\}.$$

- If f and g are two functions, we write $f = \mathcal{O}(g)$ if there exists a positive constant $C = C(d)$, such that $f(x) \leq Cg(x)$ for any x.
- We use $C > 0$ for a large constant and ε for a small constant. Note that they may change from line to line.

1.2. **Organization of this paper.** The paper is organized as follows. In Section 2, we present some preliminary results including large deviation estimates on the passage time, a lemma to control the tail distribution of maximal weight of paths in site-percolation, the introduction and properties of entropy. In Sections 3 and 4, we prove the main theorem 1.1 and Proposition 1.2.

2. Preliminaries

2.1. **Large deviation estimates on passage time.** We present here some useful estimates on the deviation of passage time.

Lemma 2.1. [1, Lemma 4.2] *There exist a positive integer number C_1 and a positive constant ε_1, such that for any $x, y \in \mathbb{Z}^d$ and $k \geq 0$,*

$$\mathbb{P}\left(T(x,y) \geq C_1|x-y|_1 + k\right) \leq e^{-(C_1|x-y|_1+k)^{\varepsilon_1}}.$$

Notice that in [1], the authors only prove Lemma 2.1 for the case $k = 0$. However, we can easily generalize their arguments to all $k \geq 1$. We safely leave the proof of this lemma to the reader. It follows from Lemma 2.1 that there exists $C > 0$ such that for any $x \in \mathbb{Z}^d$,

$$\mathbb{E}T(x) \leq C|x|_1. \tag{2.1}$$

The following concentration inequality is derived in [16].

Lemma 2.2. [16, Theorem 1.4] *For any $C > 0$, there exist positive constants a, b and A, such that for any $x \in \mathbb{Z}^d$ and $(2 + \log|x|_1)^A \leq t \leq C\sqrt{|x|_1}$,*

$$\mathbb{P}(|T(x) - \mathbb{E}T(x)| \geq t\sqrt{|x|_1}) \leq e^{-bt^a}.$$

As a direct consequence of Lemmas 2.1 and 2.2, we have

Corollary 2.3. *There exists positive constant A, such that*

$$\mathrm{Var}(T(x)) = \mathcal{O}(|x|_1(1 + \log|x|_1)^{2A}).$$

Proof. We take a positive constant C sufficiently large such that Lemma 2.1 and (2.1) hold. By using the fact $\mathbb{E}(X^2) = \int_0^\infty 2t\mathbb{P}(X \geq t)dt$ for any non-negative random variable X, we get

$$\begin{aligned}\mathrm{Var}(T(x)) &= \int_0^\infty 2t\mathbb{P}(|T(x) - \mathbb{E}T(x)| \geq t)dt \\ &= \left(\int_0^{(2+\log|x|_1)^A\sqrt{|x|_1}} + \int_{(2+\log|x|_1)^A\sqrt{|x|_1}}^{2C|x|_1} + \int_{2C|x|_1}^\infty\right) 2t\mathbb{P}(|T(x) - \mathbb{E}T(x)| \geq t)dt. \end{aligned} \tag{2.2}$$

The first term of the right hand side (2.2) can be bounded from above by

$$\int_0^{(2+\log|x|_1)^A\sqrt{|x|_1}} 2tdt \leq (2 + \log|x|_1)^{2A}|x|_1.$$

By Lemma 2.2, the second term is bounded from above by

$$2C|x|_1 \int_0^\infty 2te^{-bt^a}dt = \mathcal{O}(|x|_1).$$

Finally, by (2.1) and Lemma 2.1, the third term is bounded from above by

$$\int_{2C|x|_1}^{\infty} 2t\mathbb{P}(T(x) \geq t/2)dt \leq \int_{2C|x|_1}^{\infty} 2te^{-(t/2)^{\varepsilon_1}} dt = \mathcal{O}(1).$$

Combining these estimates, we get the conclusion. □

Lemma 2.4. *There exists a positive constant ε_2, such that for any $x, y \in \mathbb{Z}^d$, and $M \geq 1$*

$$\mathbb{P}(T(x,y) = t(x,y) = M) \leq e^{-M^{\varepsilon_2}}.$$

Proof. If $|x-y|_1 \leq M^{2/3}$, then the result follows from Lemma 2.1. Assume that $|x-y|_1 \geq M^{2/3}$. Then a well-known estimate for the trajectory of random walk (see [17, Proposition 2.1.2]) shows that for some positive constants c and C,

$$\mathbb{P}\left(\max_{0\leq j\leq k} |S_j^x - x|_1 \geq r\right) \leq Ce^{-cr^2/k}. \tag{2.3}$$

Therefore,

$$\mathbb{P}(t(x,y) = M) \leq \mathbb{P}\left(\max_{0\leq j\leq M} |S_j^x - x|_1 \geq M^{2/3}\right) \leq Ce^{-cM^{1/3}},$$

for some $c, C > 0$. □

2.2. **A result on the maximal weight of paths in site-percolation.** Let $\tilde{\mathcal{P}}_L$ be the set of self-avoiding nearest-neighbor paths in $B(L)$ whose length is bounded by L, i.e.,

$$\tilde{\mathcal{P}}_L = \left\{(y_i)_{i=1}^{\ell} \subset B(L) \cap \mathbb{Z}^d \mid \ell \leq L,\ |y_i - y_{i-1}|_1 = 1 \text{ for } 2 \leq i \leq \ell,\ y_i \neq y_j \text{ if } i \neq j\right\}.$$

Let $\{X_x\}_{x\in\mathbb{Z}^d}$ be a collection of independent and identical distribution random variables such that $\mathbb{P}(X_x = 1) = 1 - \mathbb{P}(X_x = 0) = p$ with a parameter $p \in [0,1]$. For any path γ, we define $X(\gamma) = \sum_{x\in\gamma} X_x$ the weight of γ. The maximal weight of paths in $\tilde{\mathcal{P}}_L$ and $\mathcal{P}_L$ are defined respectively as

$$\tilde{X}_L = \max_{\gamma\in\tilde{\mathcal{P}}_L} X(\gamma),\ X_L = \max_{\gamma\in\mathcal{P}_L} X(\gamma).$$

Note that for any $\gamma \in \mathcal{P}_L$, there exists $\tilde{\gamma} \in \tilde{\mathcal{P}}_L$ such that $\gamma \subset \tilde{\gamma}$. This implies $X_L \leq \tilde{X}_L$.

The tail distribution and expectation of $\tilde{X}_L$ can be controlled as in the following lemma.

Lemma 2.5. [8, Lemma 6.8] *There exist positive constants A_1 and A_2, such that for any $p \in (0,1)$ and $L \geq 1$, the following statements hold.*

(i) *For any $s \geq A_1$,*

$$\mathbb{P}\left(\tilde{X}_L \geq sLp^{1/d}\right) \leq \exp\left(-sLp^{1/d}/2\right).$$

(ii) *We have*

$$\mathbb{E}\left(\tilde{X}_L\right) \leq A_2 Lp^{1/d}.$$

In particular, the above estimates hold if we replace $\tilde{X}_L$ by X_L.

We notice that in [8], the authors prove these results for the edge-percolation, i.e. for the setting where $(X_e)_{e\in\mathcal{E}^d}$ (with $\mathcal{E}^d$ the edge set of $\mathbb{Z}^d$) are the edge-indexed i.i.d. Bernoulli random variables and $\mathcal{Q}_L$ is the set of edge-paths in $B(L)$. However, their proof can be easily adapted to the case of site-percolation as in Lemma 2.5. We also remark that in Lemma 6.8 of [8], the authors only stated Part (ii), but in fact, they have proved (i) and derived (ii) from (i).

2.3. **Entropy.** We first recall the definition of entropy with respect to a probability measure. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $X \in L^1(\Omega, \mu)$ be a non-negative. Then

$$\mathrm{Ent}_\mu(X) = \mathbb{E}_\mu(X \log X) - \mathbb{E}_\mu(X) \log \mathbb{E}_\mu(X).$$

Note that by Jensen's inequality, $\mathrm{Ent}_\mu(X) \geq 0$. The following tensorization property of entropy is proved in [8].

Lemma 2.6. [8, Theorem 2.3] *Let X be a nonnegative L^2 random variable on a product space*

$$\left(\prod_{i=1}^{\infty} \Omega_i, \mathcal{F}, \mu = \prod_{i=1}^{\infty} \mu_i\right),$$

where $\mathcal{F} = \bigvee_{i=1}^{\infty} \mathcal{G}_i$, and each triple $(\Omega_i, \mathcal{G}_i, \mu_i)$ is a probability space. Then

$$\mathrm{Ent}_\mu(X) \leq \sum_{i=1}^{\infty} \mathbb{E}_\mu \mathrm{Ent}_i(X),$$

where $\mathrm{Ent}_i(X)$ is the entropy of $X(\omega) = X((\omega_1, \ldots, \omega_i, \ldots))$ with respect to μ_i, as a function of the i-th coordinate (with all other values fixed).

In the following lemma, we prove a generalization of Bonami inequality for simple random variables.

Lemma 2.7. *Assume that $k \geq 2$. Let $f : \{1, \ldots, k\} \mapsto \mathbb{R}$ be a function and ν be the uniform distribution on $\{1, \ldots, k\}$. Then*

$$\mathrm{Ent}_\nu(f^2) \leq kE((f(U) - f(\tilde{U}))^2),$$

where E is the expectation with respect to two independent random variables $U, \tilde{U}$, which have the same distribution ν.

Proof. Let us denote $a_i = f(i)$. Then

$$\begin{aligned}
\mathrm{Ent}_\nu(f^2) &= E(f^2(U) \log f^2(U)) - E(f^2(U)) \log E(f^2(U)) \\
&= \frac{1}{k} \sum_{i=1}^{k} a_i^2 \log a_i^2 - \frac{1}{k} \sum_{i=1}^{k} a_i^2 \log \left(\frac{\sum_{j=1}^k a_j^2}{k}\right) \\
&= \frac{\sum_{j=1}^k a_j^2}{k} \sum_{i=1}^{k} \frac{a_i^2}{\sum_{j=1}^k a_j^2} \log \left(\frac{k a_i^2}{\sum_{j=1}^k a_j^2}\right) \\
&\leq \frac{\sum_{j=1}^k a_j^2}{k} \log \left(\sum_{i=1}^{k} \frac{k a_i^4}{(\sum_{j=1}^k a_j^2)^2}\right),
\end{aligned}$$

where we have used Jensen's inequality in the last inequatliy. Moreover,

$$\begin{aligned}
\log \left(\sum_{i=1}^{k} \frac{k a_i^4}{(\sum_{j=1}^k a_j^2)^2}\right) &= \log \left(1 + \frac{\sum_{i<j} (a_i^2 - a_j^2)^2}{(\sum_{j=1}^k a_j^2)^2}\right) \\
&\leq \frac{\sum_{i<j} (a_i^2 - a_j^2)^2}{(\sum_{j=1}^k a_j^2)^2},
\end{aligned}$$

since $\log(1 + x) \leq x$ for any $x \geq 0$. Therefore,

$$\mathrm{Ent}_\nu(f^2) \leq \frac{1}{k} \frac{\sum_{i<j} (a_i^2 - a_j^2)^2}{\sum_{j=1}^k a_j^2}.$$

On the other hand,

$$kE((f(U) - f(\tilde{U}))^2) = \frac{1}{k} \sum_{i,j} (a_i - a_j)^2 = \frac{2}{k} \sum_{i<j} (a_i - a_j)^2.$$

Hence,

$$\begin{aligned}\mathrm{Ent}_\nu(f^2) - kE((f(U)-f(\tilde{U}))^2) &\leq \frac{1}{k\sum_{j=1}^k a_j^2}\left[\sum_{i<j}(a_i-a_j)^2\left((a_i+a_j)^2 - 2\sum_{\ell=1}^k a_\ell^2\right)\right] \\ &\leq -\frac{1}{k\sum_{j=1}^k a_j^2}\left[\sum_{i<j}(a_i-a_j)^4\right] \\ &\leq 0,\end{aligned}$$

which proves Lemma 2.7. □

3. Proof of Theorem 1.1

3.1. **Spatial average of the passage time.** We consider a spatial average of $T(x)$ defined by

$$F_m = \frac{1}{\#B(m)}\sum_{z\in B(m)} T(z, z+x),$$

where

$$m = [|x|_1^{1/4}].$$

Proposition 3.1. *For any $\varepsilon > 0$, it holds that*

$$|\mathrm{Var}(T(x)) - \mathrm{Var}(F_m)| = \mathcal{O}(|x|_1^{3/4+\varepsilon}).$$

Proof. For any variables X and Y, by writing $\hat{X} = X - \mathbb{E}(X)$ and $||X||_2 = (\mathbb{E}(X^2))^{1/2}$ and using Cauchy-Schwartz inequality, we get

$$\begin{aligned}|\mathrm{Var}(X) - \mathrm{Var}(Y)| = |E(\hat{X}^2 - \hat{Y}^2)| &\leq ||\hat{X}+\hat{Y}||_2||\hat{X}-\hat{Y}||_2 \\ &\leq (||\hat{X}||_2 + ||\hat{Y}||_2)||\hat{X}-\hat{Y}||_2. \qquad (3.1)\end{aligned}$$

We aim to apply (3.1) for $T(x)$ and F_m. Observe that

$$||\hat{F}_m||_2 \leq \frac{1}{\#B(m)}\sum_{z\in B_m}||\hat{T}(z,z+x)||_2 = ||\hat{T}(0,x)||_2, \qquad (3.2)$$

by translation invariance. By Corollary 2.3,

$$||\hat{T}(0,x)||_2 = \sqrt{\mathrm{Var}(T(x))} = \mathcal{O}(|x|_1^{1/2}(1+\log|x|_1)^A). \qquad (3.3)$$

Using the subadditivity (1.1),

$$\begin{aligned}||\hat{T}(0,x) - \hat{F}_m||_2 &= ||T(x) - F_m||_2 \\ &= \frac{1}{\#B(m)}\left\|\sum_{z\in B(m)}(T(x) - T(z,z+x))\right\|_2 \\ &\leq \frac{1}{\#B(m)}\left\|\sum_{z\in B(m)}(T(z) + T(x,z+x))\right\|_2.\end{aligned}$$

Using Cauchy-Shwartz inequality and the translation invariance, this is further bounded from above by

$$\begin{aligned}&\frac{1}{\#B(m)}\left(\left\|\sum_{z\in B(m)}T(z)\right\|_2 + \left\|\sum_{z\in B(m)}T(x,x+z)\right\|_2\right) \\ =&\frac{2}{\#B(m)}\left\|\sum_{z\in B(m)}T(z)\right\|_2 \\ \leq&2\left\|\max_{z\in B(m)}T(z)\right\|_2.\end{aligned}$$

Using Lemma 2.1 and the union bound, we have

$$\mathbb{P}\left(\max_{z\in B(m)} T(z) \geq C_1|x|_1^{1/4}+k\right) \leq (\#B(m))e^{-(C_1|x|_1^{1/4}+k)^{\varepsilon_1}}.$$

Therefore, by a similar argument as in Corollary 2.3, we have

$$\begin{aligned}\mathbb{E}\left(\max_{z\in B(m)} T(z)^2\right) &\leq C_1^2|x|_1^{1/2} + (\#B(m))\sum_{k\geq 0}(C_1|x|_1^{1/4}+k)^2 e^{-(C_1|x|_1^{1/4}+k)^{-\varepsilon_1}} \\ &= \mathcal{O}(|x|_1^{1/2}). \qquad (3.4)\end{aligned}$$

Combining (3.1)–(3.4), we get the desired result. □

3.2. **Martingale decomposition of F_m and the proof of Theorem 1.1.** Enumerate the vertices of $\mathbb{Z}^d$ as $x_1, x_2, \ldots$. We consider the martingale decomposition of F_m as follows

$$F_m - \mathbb{E}(F_m) = \sum_{k=1}^{\infty}\Delta_k,$$

where

$$\Delta_k = \mathbb{E}(F_m \mid \mathcal{F}_k) - \mathbb{E}(F_m \mid \mathcal{F}_{k-1}),$$

with $\mathcal{F}_k$ the sigma-algebra generated by SRWs $\{(S_j^{x_i})_{j\in\mathbb{N}}, i=1,\ldots,k\}$ and $\mathcal{F}_0$ the trivial sigma-algebra. In [8], using Falik-Samorodnitsky lemma, the authors give an upper bound for variance of F_m in term of $\mathrm{Ent}(\Delta_k^2)$, and $\mathbb{E}(|\Delta_k|)$.

Lemma 3.2. [8, Lemma 3.3] *We have*

$$\sum_{k\geq 1}\mathrm{Ent}(\Delta_k^2) \geq \mathrm{Var}(F_m)\log\left[\frac{\mathrm{Var}(F_m)}{\sum_{k\geq 1}(\mathbb{E}(|\Delta_k|))^2}\right].$$

Now, our main task is to estimate $\mathrm{Ent}(\Delta_k^2)$ and $\mathbb{E}(|\Delta_k|)$.

Proposition 3.3. *As $|x|_1$ tends to infinity,*

(i)

$$\sum_{k\geq 1}\mathrm{Ent}(\Delta_k^2) = \mathcal{O}(|x|_1).$$

(ii)

$$\sum_{k\geq 1}(\mathbb{E}(|\Delta_k|))^2 = \mathcal{O}\left(|x|_1^{\frac{5-d}{4}}\right).$$

3.2.1. *Proof of Theorem 1.1 assuming Proposition 3.3.* Since $d \geq 2$, Proposition 3.3 (ii) implies that $\sum_{k\geq 1}(\mathbb{E}(|\Delta_k|))^2 = \mathcal{O}\left(|x|_1^{3/4}\right)$. Therefore, using Propositions 3.1, 3.3 and Lemma 3.2, for any $\varepsilon > 0$, there exists a positive constant C, such that

$$\mathrm{Var}(T(x)) \leq \mathrm{Var}(F_m) + C|x|_1^{3/4+\varepsilon} \leq C\left(|x|_1^{3/4+\varepsilon} + |x|_1\left[\log\left[\frac{\mathrm{Var}(F_m)}{|x|_1^{3/4}}\right]\right]^{-1}\right). \qquad (3.5)$$

If $\mathrm{Var}(F_m) \leq |x|_1^{7/8}$ then $\mathrm{Var}(T(x)) = \mathcal{O}(|x|_1^{7/8})$ and Theorem 1.1 follows. Otherwise, if $\mathrm{Var}(F_m) \geq |x|_1^{7/8}$, using (3.5) we get that $\mathrm{Var}(T(x)) = \mathcal{O}(|x|_1/\log|x|_1)$ and Theorem 1.1 follows. □

3.2.2. *Proof of Proposition 3.3.* By the definition of Δ_k, we have

$$\begin{aligned}|\Delta_k| &= \frac{1}{\#B(m)}\left|\mathbb{E}\left[\sum_{z\in B(m)} T(z,z+x) \mid \mathcal{F}_k\right] - \mathbb{E}\left[\sum_{z\in B(m)} T(z,z+x) \mid \mathcal{F}_{k-1}\right]\right| \\ &\leq \frac{1}{\#B(m)}\sum_{z\in B(m)}\Big|\mathbb{E}\left[T(z,z+x) \mid \mathcal{F}_k\right] - \mathbb{E}\left[T(z,z+x) \mid \mathcal{F}_{k-1}\right]\Big|. \qquad (3.6)\end{aligned}$$

We precise the dependence of passage time on trajectories of SRWs by writing

$$T(u,v) = T(u,v,(S^{x_i}_\cdot)_{i\in\mathbb{N}}).$$

For any k, let us define

$$X_k(u,v) = \mathbb{E}(T(u,v) \mid \mathcal{F}_k).$$

Then $X_k(u,v)$ is a function of trajectories of $(S^{x_i}_\cdot)_{i\le k}$, so we write

$$X_k(u,v) = X_k(u,v)((S^{x_i}_\cdot)_{i<k},(S^{x_k}_\cdot)).$$

Let $(\tilde{S}^x_\cdot)_{x\in\mathbb{Z}^d}$ be an independent copy of $(S^x_\cdot)_{x\in\mathbb{Z}^d}$. We observe that

$$\mathbb{E}(|X_k(u,v) - \mathbb{E}^k(X_k(u,v))|) \le \mathbb{E}^{<k}\mathbb{E}^k\tilde{\mathbb{E}}^k(|X_k(u,v)-\tilde{X}_k(u,v)|), \tag{3.7}$$

where

$$\tilde{X}_k(u,v) = X_k(u,v)((S^{x_i}_\cdot)_{i<k},(\tilde{S}^{x_k}_\cdot)),$$

and $\mathbb{E}^{<k},\mathbb{E}^k$, and $\tilde{\mathbb{E}}^k$ denote the expectations with respect to SRWs $(S^{x_i}_\cdot)_{i<k}$, $(S^{x_k}_\cdot)$ and $(\tilde{S}^{x_k}_\cdot)$ respectively. Then the inequality (3.7) becomes

$$\mathbb{E}\Big|\mathbb{E}\left[T(z,z+x)\mid\mathcal{F}_k\right] - \mathbb{E}\left[T(z,z+x)\mid\mathcal{F}_{k-1}\right]\Big| \le \mathbb{E}\tilde{\mathbb{E}}^k\Big|T(z,z+x)-\tilde{T}_{x_k}(z,z+x)\Big|, \tag{3.8}$$

where for $u,v\in\mathbb{Z}^d$ and $k\ge 1$

$$\tilde{T}_{x_k}(u,v) = T(u,v)((S^{x_i}_\cdot)_{i<k},(\tilde{S}^{x_k}_\cdot),(S^{x_i}_\cdot)_{i>k}).$$

By symmetry,

$$\begin{aligned}&\mathbb{E}\tilde{\mathbb{E}}^k\Big|T(z,z+x)-\tilde{T}_{x_k}(z,z+x)\Big|\\ =\;& 2\mathbb{E}\tilde{\mathbb{E}}^k\left([\tilde{T}_{x_k}(z,z+x)-T(z,z+x)]\mathbb{I}(\tilde{T}_{x_k}(z,z+x)\ge T(z,z+x))\right).\end{aligned}\tag{3.9}$$

For any $u,v\in\mathbb{Z}^d$, we choose an optimal path for $T(u,v)$ with a deterministic rule breaking ties and denote it by $\gamma_{u,v}$. We observe that if $x_k\notin\gamma_{u,v}$ then $\tilde{T}_{x_k}(u,v)\le T(u,v)$. Otherwise, if $x_k\in\gamma_{u,v}$, then

$$T(u,v) = T(u,x_k)+T(x_k,\bar{x}_k)+T(\bar{x}_k,v), \tag{3.10}$$

with $\bar{x}_k$ the next point of x_k in $\gamma_{u,v}$ (recall also that we denote by $y\sim\bar{y}\in\gamma$ if $\bar{y}$ is the next point of y in γ). Due to the subadditivity,

$$\tilde{T}_{x_k}(u,v) \le \tilde{T}_{x_k}(u,x_k)+\tilde{T}_{x_k}(x_k,\bar{x}_k)+\tilde{T}_{x_k}(\bar{x}_k,v). \tag{3.11}$$

It is clear that the optimal path for $T(u,x_k)$ does not use the simple random walk $(S^{x_k}_\cdot)$. Hence,

$$\tilde{T}_{x_k}(u,x_k)\le T(u,x_k). \tag{3.12}$$

In addition, since $\bar{x}_k$ is the next point of x_k in $\gamma_{u,v}$, the optimal path for $T(\bar{x}_k,v)$ does not use the simple random walk $(S^{x_k}_\cdot)$. Thus

$$\tilde{T}_{x_k}(\bar{x}_k,v)\le T(\bar{x}_k,v). \tag{3.13}$$

It follows from (3.10)–(3.13) that

$$\tilde{T}_{x_k}(u,v)-T(u,v)\le\tilde{T}_{x_k}(x_k,\bar{x}_k).$$

Therefore, we have

$$\begin{aligned}&(\tilde{T}_{x_k}(z,z+x)-T(z,z+x))\mathbb{I}(\tilde{T}_{x_k}(z,z+x)\ge T(z,z+x))\\ \le\;&\tilde{T}_{x_k}(x_k,\bar{x}_k)\mathbb{I}(x_k\in\gamma_{z,z+x}).\end{aligned}\tag{3.14}$$

Combining (3.6), (3.8), (3.9) and (3.14), we get

$$\begin{aligned}
\mathbb{E}(|\Delta_k|) &\leq \frac{2}{\#B(m)}\mathbb{E}^{\otimes 2}\left(\sum_{z\in B(m)} \tilde{T}_{x_k}(x_k,\bar{x}_k)\mathbb{I}(x_k\in\gamma_{z,z+x})\right)\\
&= \frac{2}{\#B(m)}\mathbb{E}^{\otimes 2}\left(\sum_{z\in B(m)} \tilde{T}_{x_k-z}(x_k-z,\overline{x_k-z})\mathbb{I}(x_k-z\in\gamma_{0,x})\right)\\
&= \frac{2}{\#B(m)}\mathbb{E}^{\otimes 2}\left(\sum_{y\in x_k-B(m)} \tilde{T}_{y}(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\right)\\
&= \frac{2}{\#B(m)}\sum_{L\geq 0}\mathbb{E}^{\otimes 2}\left(\sum_{y\in x_k-B(m)} \tilde{T}_{y}(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\mathbb{I}(\mathcal{E}_{k,L})\right), \qquad (3.15)
\end{aligned}$$

where $\mathbb{E}^{\otimes 2}$ is the expectation with respect to two independent collections of SRWs $(S^{x_i}_{\cdot})_{i\in\mathbb{N}}$ and $(\tilde{S}^{x_i}_{\cdot})_{i\in\mathbb{N}}$ and let

$$\mathcal{E}_{k,L}=\left\{\sum_{y\in\gamma_{0,x}\cap x_k-B(m)}|y-\bar{y}|_1=L\right\}.$$

Notice that for the second equation, we have used the invariant translation. Let us define

$$T^{[z]}(u,v)=\inf\Big\{\sum_{l=1}^{k}t(y_{l-1},y_l): u=y_0,\dots,y_k=v, y_l\neq z\,\forall\, l\geq 1,\ \text{for some } k\Big\},$$

as the passage time from u to v not using the frog at z, and set

$$T_1(u,v)=\max_{z:\ |z-u|_1=1}T^{[u]}(z,v)+1.$$

Then, it holds that

$$\tilde{T}_u(u,v)\leq T_1(u,v). \qquad (3.16)$$

Using (3.16), we obtain

$$\begin{aligned}
\sum_{y\in x_k-B(m)}\tilde{T}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\mathbb{I}(\mathcal{E}_{k,L}) &\leq \max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset x_k-B(m)\\ |\gamma|_1\leq L}}\sum_{i=1}^{\ell-1}\tilde{T}_{y_i}(y_i,y_{i+1})\mathbb{I}(\mathcal{E}_{k,L})\\
&\leq \max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset x_k-B(m)\\ |\gamma|_1\leq L}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\mathbb{I}(\mathcal{E}_{k,L}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{L=0}^{2dC_1m}\mathbb{E}^{\otimes 2}\left(\sum_{y\in x_k-B(m)}\tilde{T}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\mathbb{I}(\mathcal{E}_{k,L})\right) &\leq \mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset x_k-B(m)\\ |\gamma|_1\leq 2dC_1m}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)\\
&= \mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset B(m)\\ |\gamma|_1\leq 2dC_1m}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)\\
&\leq \mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{2dC_1m}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right), \qquad (3.17)
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{L\geq 2dC_1m+1}\mathbb{E}^{\otimes 2}\left(\sum_{y\in x_k-B(m)}\tilde{T}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\mathbb{I}(\mathcal{E}_{k,L})\right)\\
\leq\quad &\sum_{L\geq 2dC_1m+1}\mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset x_k-B(m)\\ |\gamma|_1\leq L}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\mathbb{I}(\mathcal{E}_{k,L})\right)\\
\leq\quad &\sum_{L\geq 2dC_1m+1}\left[\mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset x_k-B(m)\\ |\gamma|_1\leq L}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_{k,L})^{1/2}\\
\leq\quad &\sum_{L\geq 2dC_1m+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_{k,L})^{1/2},
\end{aligned}\tag{3.18}
$$

where we have used the Cauchy-Schwartz inequality in the second inequality.

These yield that

$$
\begin{aligned}
\mathbb{E}|\Delta_k|\leq\ &\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{2dC_1m}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)\\
&+\sum_{L\geq 2dC_1m+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_{k,L})^{1/2}.
\end{aligned}\tag{3.19}
$$

Using similar arguments for (3.15), (3.17) and (3.18), we can show that

$$
\begin{aligned}
\sum_{k=1}^{\infty}\mathbb{E}(|\Delta_k|)\quad\leq\quad &\frac{2}{\#B(m)}\sum_{k=1}^{\infty}\sum_{z\in B(m)}\mathbb{E}^{\otimes 2}\tilde{T}_{x_k}(x_k,\bar{x}_k)\mathbb{I}(x_k\in\gamma_{z,z+x})\\
=\quad &2\mathbb{E}^{\otimes 2}\left(\sum_{y\in\mathbb{Z}^d}\tilde{T}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\right)\\
\leq\quad &2\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{C_1|x|_1}}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)\\
&+2\sum_{L\geq C_1|x|_1+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_L)^{1/2},
\end{aligned}\tag{3.20}
$$

with

$$\mathcal{E}_L=\{|\gamma_{0,x}|_1=L\}.$$

Lemma 3.4. *There exists a positive constant C, such that for all $L\geq 1$,*

(i)

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)\leq CL.$$

(ii)

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right)^2\leq CL^4.$$

We postpone the proof of this lemma for a while.

Lemma 3.5. *Given a path $\gamma = (y_i)_{i=1}^{\ell} \subset \mathbb{Z}^d$, we define the maximal jump*

$$\mathcal{M}(\gamma) = \max_{1\leq i\leq \ell-1} |y_i - y_{i+1}|_1.$$

Then, for any $L \geq m = |x|_1^{1/4}$,

$$\mathbb{P}(\mathcal{M}(\gamma_{0,x}) \geq L) \leq e^{-L^{\varepsilon}},$$

with some $\varepsilon > 0$.

Proof. We write $\gamma_{0,x} = (y_i)_{i=1}^{\ell}$. If $|y_i - y_{i+1}|_1 \geq L$, then $T(y_i, y_{i+1}) = t(y_i, y_{i+1}) \geq L$. By the union bound, Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned}\mathbb{P}(\mathcal{M}(\gamma_{0,x}) \geq L) &\leq \mathbb{P}(\exists u, v \in B(C_1|x|_1) \text{ s.t. } T(u,v) = t(u,v) \geq L) + \mathbb{P}(|\gamma_{0,x}|_1 \geq C_1|x_1|)\\ &\leq [\#B(C_1|x|_1)]^2\mathbb{P}(T(u,v) = t(u,v) \geq L) + \mathbb{P}(T(0,x) \geq C_1|x_1|)\\ &\leq e^{-L^{\varepsilon}},\end{aligned} \tag{3.21}$$

for some constant $\varepsilon > 0$. □

Proof of Proposition 3.3 (ii). Fix $k \geq 1$. We first estimate $\mathbb{P}(\mathcal{E}_{k,L})$. Assume that $\mathcal{E}_{k,L}$ occurs and $\gamma_{0,x} \cap (x_k - B(m)) = (y_i)_{i=1}^{\ell}$. Then

$$L = \sum_{y \in \gamma_{0,x} \cap x_k - B(m)} |y - \bar{y}|_1 \leq \sum_{i=1}^{\ell-1} t(y_i, y_{i+1}) + t(y_\ell, \bar{y}_\ell) = T(y_1, \bar{y}_\ell).$$

Moreover, $\bar{y}_\ell \in x_k - B(m + \mathcal{M}(\gamma_{0,x}))$, since $|y_\ell - \bar{y}_\ell|_1 \leq \mathcal{M}(\gamma_{0,x})$ and $y_\ell \in x_k - B(m)$. Therefore, using the union bound, Lemma 2.1 and Lemma 3.5,

$$\begin{aligned}\mathbb{P}(\mathcal{E}_{k,L}) &\leq \mathbb{P}\left(\exists u, v \in x_k - B(m + \mathcal{M}(\gamma_{0,x})) \text{ such that } T(u,v) \geq L\right)\\ &\leq \mathbb{P}\left(\exists u, v \in B(m + L) \text{ such that } T(u,v) \geq L\right) + \mathbb{P}(M(\gamma_{0,x}) \geq L)\\ &\leq (2(m+L))^{2d}e^{-L^{\varepsilon}} + e^{-L^{\varepsilon}} \leq (4(m+L))^{2d}e^{-L^{\varepsilon}}.\end{aligned}$$

Combining this inequality with (3.15), (3.17), (3.18) and Lemma 3.4, we obtain that there exists $C > 0$ such that for any $k \geq 1$

$$\begin{aligned}\mathbb{E}(|\Delta_k|) &\leq \frac{C}{\#B(m)}\left(m + \sum_{L \geq 2dC_1 m} L^2(4(m+L))^d e^{-L^{\varepsilon}/2}\right)\\ &= \mathcal{O}(m^{1-d}) = \mathcal{O}(|x|_1^{(1-d)/4}).\end{aligned} \tag{3.22}$$

Since $T(x) \geq |\gamma_{0,x}|_1$, by using Lemma 2.1, for any $L \geq C_1|x|_1$

$$\mathbb{P}(\mathcal{E}_L) \leq \mathbb{P}(T(x) \geq L) \leq e^{-L^{\varepsilon_1}}. \tag{3.23}$$

Using this inequality, (3.20) and Lemma 3.4, we get

$$\begin{aligned}\sum_{k\geq 1}\mathbb{E}(|\Delta_k|) &\leq C\left(|x|_1 + \sum_{L \geq C_1|x|_1} L^2 e^{-L^{\varepsilon}/2}\right)\\ &= \mathcal{O}(|x|_1).\end{aligned} \tag{3.24}$$

Now, Proposition 3.3 (ii) follows from (3.22) and (3.24). □

We now turn to prove Proposition 3.3 (i). To estimate $\mathrm{Ent}(\Delta_k)$, we decompose the simple random walks $(S_\cdot^{x_i})$ into the sum of i.i.d. random variables. More precisely, for any $x_i \in \mathbb{Z}^d$ and $j \geq 1$, we write

$$S_j^{x_i} = x_i + \sum_{r=1}^{j} \omega_{i,r},$$

where $(\omega_{i,r})_{i,r\geq 1}$ is an array of i.i.d. uniform random variables taking value in the set of canonical coordinates in $\mathbb{Z}^d$, denoted by

$$\mathcal{B}_d = \{e_1, \ldots, e_{2d}\}.$$

Therefore, we can view $T(u,v)$ and F_m as a function of $(\omega_{i,r})$, and hence we sometimes write $T(u,v) = T(u,v,\omega)$ to precise the dependence of $T(u,v)$ on ω. We define

$$\Omega = \prod_{i,j\in\mathbb{N}} \Omega_{i,j},$$

where $\Omega_{i,j}$ is a copy of $\mathcal{B}_d$. The measure on Ω is $\pi = \prod_{i,j\in\mathbb{N}} \pi_{i,j}$, where $\pi_{i,j}$ is the uniform measure on $\Omega_{i,j}$. Then we can consider F_m as a random variable on the probability space (Ω,π). Given $\omega \in \Omega, e \in \mathcal{B}_d$ and $i,j \in \mathbb{N}$, we define a new configuration $\omega^{i,j,e}$ as

$$\omega^{i,j,e}_{k,r} = \begin{cases} \omega_{k,r} & \text{if } (k,r) \neq (i,j) \\ e & \text{if } (k,r) = (i,j). \end{cases}$$

We define

$$\Delta_{i,j} f = \left[E\Big(|f(\omega^{i,j,U}) - f(\omega^{i,j,\tilde{U}})|^2 \Big) \right]^{1/2}, \tag{3.25}$$

where the expectation runs over two independent random variables U and $\tilde{U}$, with the same law as the uniform distribution on $\mathcal{B}_d$.

Lemma 3.6. *We have*

$$\sum_{k=1}^{\infty} \mathrm{Ent}(\Delta_k^2) \leq 2d \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j} F_m)^2].$$

Proof. We recall that $\Delta_k = \mathbb{E}(F_m \mid \mathcal{F}_k)$, where

$$\mathcal{F}_k = \sigma((S_j^{x_i}), i \leq k, j \geq 1) = \sigma(\omega_{i,j}, i \leq k, j \geq 1).$$

Notice that $\Delta_k^2 \in L^2$, since $T(x) \in L^4$ by Lemma 2.1. Hence, using the tensorization of entropy (Lemma 2.6), we have for $k \geq 1$,

$$\mathrm{Ent}(\Delta_k^2) = \mathrm{Ent}_\pi(\Delta_k^2) \leq \mathbb{E}_\pi \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathrm{Ent}_{\pi_{i,j}} \Delta_k^2.$$

By Lemma 2.7,

$$\mathrm{Ent}_{\pi_{i,j}} \Delta_k^2 \leq 2d(\Delta_{i,j}\Delta_k)^2.$$

Thus

$$\sum_{k=1}^{\infty} \mathrm{Ent}(\Delta_k^2) \leq 2d \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\Delta_k)^2]. \tag{3.26}$$

Now using the same arguments as in Lemma 6.3 in [8], we can show that

$$\sum_{k=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\Delta_k)^2] = \mathbb{E}_\pi[(\Delta_{i,j}F_m)^2],$$

for any i,j. Combining this equation with (3.26), we get the desired result. □

Proof of Proposition 3.3 (i). Using Lemma 3.6 and the Cauchy-Schwartz inequality, we get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}F_m)^2] \leq \frac{1}{\#B(m)} \sum_{z\in B(m)} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}T(z,z+x))^2]. \tag{3.27}$$

On the other hand,

$$\begin{aligned} & \mathbb{E}_\pi[(\Delta_{i,j}T(z,z+x))^2] = \mathbb{E}_\pi[(E|T(z,z+x,\omega^{i,j,U}) - T(z,z+x,\omega^{i,j,\tilde{U}})|)^2] \\ \leq \; & \mathbb{E}_\pi E[|T(z,z+x,\omega^{i,j,U}) - T(z,z+x,\omega^{i,j,\tilde{U}})|^2] \\ = \; & \mathbb{E}_\pi E[|T(z,z+x,\omega^{i,j,U}) - T(z,z+x)|^2] \\ = \; & 2\mathbb{E}_\pi E[(T(z,z+x,\omega^{i,j,U}) - T(z,z+x))^2 \mathbb{I}(T(z,z+x,\omega^{i,j,U}) \geq T(z,z+x))]. \end{aligned}$$

We observe that if $x_i \notin \gamma_{z,z+x}$, or $x_i \in \gamma_{z,z+x}$ but $T(x_i, \bar{x}_i) < j$, then

$$T(z, z+x, \omega^{i,j,U}) \le T(z, z+x).$$

Otherwise, assume that $x_i \in \gamma_{z,z+x}$ and $T(x_i, \bar{x}_i) \ge j$. Then for any $e \in \mathcal{B}_d$,

$$T(x_i, \bar{x}_i) \ge T(x_i, \bar{x}_i - e + \omega_{i,j}, \omega^{i,j,e}),$$

since if we only replace $\omega_{i,j}$ by e, after $t(x_i, \bar{x}_i)$ (also equals to $T(x_i, \bar{x}_i)$, as $x_i \sim \bar{x}_i \in \gamma_{0,x}$) steps, the simple random walk $(S_\cdot^{x_i})$ arrives at $\bar{x}_i - e + \omega_{i,j}$. Moreover,

$$T(z, x_i, \omega^{i,j,e}) = T(z, x_i), \quad T(\bar{x}_i, z+x, \omega^{i,j,e}) \le T(\bar{x}_i, z+x),$$

and

$$\begin{aligned} T(z, z+x) &= T(z, x_i) + T(x_i, \bar{x}_i) + T(\bar{x}_i, z+x) \\ T(z, z+x, \omega^{i,j,e}) &\le T(z, x_i, \omega^{i,j,e}) + T(x_i, \bar{x}_i - e + \omega_{i,j}, \omega^{i,j,e}) \\ &\quad + T(\bar{x}_i - e + \omega_{i,j}, \bar{x}_i, \omega^{i,j,e}) + T(\bar{x}_i, z+x, \omega^{i,j,e}). \end{aligned}$$

Hence, we reach

$$T(z, z+x, \omega^{i,j,U}) - T(z, z+x) \le T(\bar{x}_i - U + \omega_{i,j}, \bar{x}_i, \omega^{i,j,U}) \le \max_{y:|y-\bar{x}_i|\le 2} T(y, \bar{x}_i, \omega^{i,j,U}).$$

Furthermore, since ω differs from $\omega^{i,j,U}$ only in the trajectory of $(S_\cdot^{x_i})$, for any $u, v \in \mathbb{Z}^d$,

$$T(u, v, \omega^{i,j,U}) \le T^{[x_i]}(u, v) \le T_2(u, v), \tag{3.28}$$

where we define

$$T_2(u, v) = \sup_{z \in \mathbb{Z}^d} T^{[z]}(u, v).$$

Therefore, we have

$$\mathbb{E}_\pi[(\Delta_{i,j} T(z, z+x))^2] \le 2\mathbb{E}\Big[\max_{y:|y-\bar{x}_i|\le 2} T_2(y, \bar{x}_i)^2 \mathbb{I}(x_i \sim \bar{x}_i \in \gamma_{z,z+x}, T(x_i, \bar{x}_i) \ge j)\Big],$$

and thus

$$\begin{aligned} &\sum_{j=1}^\infty \mathbb{E}_\pi(\Delta_{i,j} T(z, z+x))^2 \\ \le\ & 2\mathbb{E}\Big[\sum_{j=1}^\infty \max_{y:|y-\bar{x}_i|\le 2} T_2(y, \bar{x}_i)^2 \mathbb{I}(x_i \sim \bar{x}_i \in \gamma_{z,z+x}, T(x_i, \bar{x}_i) \ge j)\Big] \\ =\ & 2\mathbb{E}\Big[T(x_i, \bar{x}_i) \max_{y:|y-\bar{x}_i|\le 2} T_2(y, \bar{x}_i)^2 \mathbb{I}(x_i \sim \bar{x}_i \in \gamma_{z,z+x})\Big] \\ \le\ & \mathbb{E}\Big[(T(x_i, \bar{x}_i)^2 + \max_{y:|y-\bar{x}_i|\le 2} T_2(y, \bar{x}_i)^4) \mathbb{I}(x_i \sim \bar{x}_i \in \gamma_{z,z+x})\Big]. \end{aligned}$$

This yields that

$$\begin{aligned} &\sum_{i=1}^\infty \sum_{j=1}^\infty \mathbb{E}_\pi(\Delta_{i,j} T(z, z+x))^2 \\ \le\ & \mathbb{E}\Big[\sum_{i=1}^\infty (T(x_i, \bar{x}_i)^2 + \max_{y:|y-\bar{x}_i|\le 2} T_2(y, \bar{x}_i)^4) \mathbb{I}(x_i \sim \bar{x}_i \in \gamma_{z,z+x})\Big] \\ =\ & \mathbb{E}\Big[\sum_{i=1}^\infty (T(x_i, \bar{x}_i)^2 + \max_{y:|y-\bar{x}_i|\le 2} T_2(y, \bar{x}_i)^4) \mathbb{I}(x_i \sim \bar{x}_i \in \gamma_{0,x})\Big] \\ =\ & \mathbb{E}\left(\sum_{y \in \gamma_{0,x}} T(y, \bar{y})^2\right) + \mathbb{E}\left(\sum_{y \in \gamma_{0,x}} \max_{u:|u-y|_1 \le 2} T_2(u, y)^4\right). \end{aligned} \tag{3.29}$$

Now using the same arguments for (3.18) and (3.20), we get

$$\mathbb{E}\left(\sum_{y\in\gamma_{0,x}} \max_{|u-y|_1\le 2} T_2(u,y)^4\right) \tag{3.30}$$

$$\le \mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{C_1|x|_1}} \sum_{i=1}^{\ell} \max_{|u-y_i|_1\le 2} T_2(u,y_i)^4\right)$$

$$+\sum_{L\ge C_1|x|_1+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}\max_{|u-y_i|_1\le 2} T_2(u,y_i)^4\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_L)^{1/2}. \tag{3.31}$$

Lemma 3.7. *As $|x|_1$ tends to infinity,*

$$\mathbb{E}\left(\sum_{y\in\gamma_{0,x}} T(y,\bar{y})^2\right) = \mathcal{O}(|x|_1).$$

Lemma 3.8. *There exists a positive constant C, such that for any $L\ge 1$,*

(i)

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell}\max_{u:|u-y_i|_1\le 2} T_2(u,y_i)^4\right)\le CL.$$

(ii)

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}\max_{u:|u-y_i|_1\le 2} T_2(u,y_i)^4\right)^2\le CL^{10}.$$

We postpone the proofs of the above two lemmas for a while and first complete the proof of Proposition 3.3. Combining (3.23), (3.29), (3.30) and Lemmas 3.7 and 3.8, we get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{E}_{\pi}(\Delta_{i,j}T(z,z+x))^2 = \mathcal{O}(1)\left(|x|_1+\sum_{L\ge C_1|x|_1} L^2 e^{-L^{\varepsilon_1}/2}\right)=\mathcal{O}(|x|_1).$$

This estimate holds with all $z\in B(m)$, so we can conclude the proof of Proposition 3.3 by using (3.27) and Lemma 3.6. □

3.3. **Tessellation estimates.** In this section, we will prove Lemmas 3.4, 3.7 and 3.8. We first observe that the simple union bound is not sharp enough to prove the lemmas and that the main difficulty comes from the correlations of passage times. To overcome this, we establish new techniques combining tessellation arguments and percolation estimates (Lemma 2.5).

Proof of Lemma 3.7. For any $\gamma=(y_i)_{i=1}^{\ell}$, we define

$$A_M^{\gamma} \;=\; \{y_i\in\gamma : T(y_i,y_{i+1})=M\}.$$

Then, we can express

$$\sum_{i=1}^{\ell-1} T(y_i,y_{i+1})^2 = \sum_{k\ge 1} M^2 \# A_M^{\gamma}. \tag{3.32}$$

We decompose

$$\mathbb{E}\left(\sum_{y\in\gamma_{0,x}} T(y,\bar{y})^2\right) \;=\; \mathbb{E}\left[\sum_{y\in\gamma_{0,x}} T(y,\bar{y})^2;\ T(x)\le C|x|_1\right]+\mathbb{E}\left[\sum_{y\in\gamma_{0,x}} T(y,\bar{y})^2;\ T(x)> C|x|_1\right].$$

By a similar argument as in Lemma 2.2, the second term can be bounded from above by

$$\begin{aligned}
&\left(\mathbb{E}\left[\left(\sum_{y\in\gamma_{0,x}} T(y,\bar{y})^2\right)^2\right]\right)^{1/2} \mathbb{P}(|\gamma_{0,x}|_1 > C|x|_1)^{1/2} \\
&\leq \left(\mathbb{E}\left[T(x)^4\right]\right)^{1/2} \mathbb{P}(T(x) > C|x|_1)^{1/2} \\
&\leq C|x|_1^2 e^{-|x|_1^\varepsilon/2},
\end{aligned} \tag{3.33}$$

which goes to 0 as $|x|_1 \to \infty$.

To estimate the first term, under the condition $T(x) \leq C|x|_1$, we will show that for any $M \geq 1$,

$$\mathbb{E}(\#A_M^{\gamma_{0,x}}) \leq C|x|_1 M^{d+1} e^{-M^\varepsilon/d}, \tag{3.34}$$

with some constants $\varepsilon > 0$ and $C > 0$. Then it follows from (3.32) that the first term can be bounded from above by

$$C|x|_1 \sum_{M\geq 1} M^{d+3} e^{-M^\varepsilon/d} = \mathcal{O}(|x|_1).$$

which proves Lemma 3.7.

Now it remains to prove (3.34). The general idea is to cover $\mathbb{Z}^d$ by groups of boxes such that in each group the numbers of two consecutive points in the optimal path having distance M in different boxes are dominated by independent random variables. Then we will apply Lemma 2.5 to get the desired estimate.

We note that

(a) if $T(u,v) = M$ then $|u-v|_1 \leq M$,
(b) if $d_1(\{u,v\},\{x,y\}) \geq 2M$, then the two events $\{T(u,v) = M\}$ and $\{T(x,y) = M\}$ are independent.

In fact, (a) follows from the fact that $T(u,v) \geq |u-v|_1$ and (b) holds since the event $T(u,v) = M$ depends only on simple random walks $(S^w_\cdot)$ with $|w-u|_1 \leq M$.

To each M, we divide $\mathbb{Z}^d$ to 4^d groups of boxes of size $2M$, noted by $\{(B^M_{i,z})_{z\in\mathbb{Z}^d}, i = 1,\dots,4^d\}$, satisfying the following conditions:

(c) For any $u,v \in \mathbb{Z}^d$ with $|u-v|_1 \leq M$, there exists a boxes $B^M_{i,z}$ containining u and v.
(d) For any $i = 1,\dots,4^d$, the boxes in the i-th group, $(B^M_{i,\cdot})$ are totally disjoint, i.e. the distance between two arbitrary boxes is larger than $2M$.

In fact, there are many covers satisfying (c) and (d), and here we choose a simple one defined as follows. To each $x \in M\mathbb{Z}^d$, we define

$$B^M_x = x + [0,2M)^d.$$

We enumerate $\{0,1,2,3\}^d$ by $\{w_i\}_{i=1}^{4^d}$. Given $i \in \{1,\cdots,4^d\}$ and $z \in \mathbb{Z}^d$, we denote

$$B^M_{i,z} = B^M_{M(w_i+4z)}.$$

Lemma 3.9. *The groups of boxes that we have constructed above satisfy (c) and (d).*

Proof. The condition (d) is trivial by construction. We will prove that (c) holds. Assume that $u,v \in \mathbb{Z}^d$ and $|u-v|_1 \leq M$. We consider $x = (x_1,\dots,x_d)$, with $x_j = \min\{M[u_j/M], M[v_j/M]\}$ for $j = 1,\dots,d$. Since $|u_j - v_j| \leq M$, it holds that $|u_j - x_j| < 2M$ and $|v_j - x_j| < 2M$. Thus $u,v \in B^M_x$. In addition, since all the coordinates of x are multiple of M, there exists $(w_i,z) \in \{0,1,2,3\}^d \times \mathbb{Z}^d$, such that $x = M(w_i+4z)$, so $u,v \in B^M_{i,z}$. □

Given $i = 1,\dots,4^d$, we define

$$A^{\gamma_{0,x}}_{M,i,z} = A^{\gamma_{0,x}}_M \cap B^M_{i,z}.$$

Then by (a) and (c),

$$\#A_M^{\gamma_{0,x}} \le \sum_{i=1}^{4^d} \sum_{z\in\mathbb{Z}^d} \#A_{M,i,z}^{\gamma_{0,x}}. \tag{3.35}$$

Notice that if $y \sim \bar{y} \in \gamma_{0,x}$, then $T(y,\bar{y}) = t(y,\bar{y})$. Thus for any (i,z),

$$\#A_{M,i,z}^{\gamma_{0,x}} \le (2M)^d Y_z^i \mathbb{I}(A_{M,i,z}^{\gamma_{0,x}} \neq \emptyset), \tag{3.36}$$

where

$$Y_z^i = \mathbb{I}(\ \exists u,v \in B_{i,z}^M \text{ such that } T(u,v) = t(u,v) = M).$$

Thus for each $i = 1,\ldots,4^d$,

$$\sum_{z\in\mathbb{Z}^d} \#A_{M,i,z}^{\gamma_{0,x}} \le (2M)^d \sum_{z\in\eta_{0,x}^{i,M}} Y_z^i, \tag{3.37}$$

where $\eta_{0,x}^{i,M}$ is the projected path of $\gamma_{0,x}$ defined by

$$\eta_{0,x}^{i,M} = \{z\in\mathbb{Z}^d : A_{M,i,z}^{\gamma_{0,x}} \neq \emptyset\}.$$

Since we assume that $T(x) \le C_1|x|_1$, we have $\gamma_{0,x} \in \mathcal{P}_{C_1|x|_1}$, which implies

$$\eta_{0,x}^{i,M} \in \mathcal{P}_{C_1|x|_1/M}.$$

Hence,

$$\sum_{z\in\eta_{0,x}^M} Y_z^i \le \max_{\eta\in\mathcal{P}_{C_1|x|_1/M}} \sum_{z\in\eta} Y_z^i =: X_{C_1|x|_1,M}^i. \tag{3.38}$$

Combining this inequality with (3.36) and (3.37) yields that

$$\sum_{i=1}^{4^d} \sum_{z\in\mathbb{Z}^d} \#A_{M,i,z}^{\gamma_{0,x}} \le (2M)^d \sum_{i=1}^{4^d} X_{C_1|x|_1,M}^i. \tag{3.39}$$

By (b) and (d), $(Y_z^i)_{z\in\mathbb{Z}^d}$ are i.i.d. Bernoulli random variables. Let $p_M = \mathbb{E}(Y_z^i)$. Then, it follows from the union bound and Lemma 2.4 that

$$\begin{aligned} p_M &= \mathbb{P}(\exists\, u,v \in B_{i,z}^M : T(u,v) = t(u,v) = M) \\ &\le (2M)^{2d} e^{-M^\varepsilon}, \end{aligned} \tag{3.40}$$

with some $\varepsilon > 0$. Now applying Lemma 2.5 to the set of random variables $(Y_z^i)_{z\in\mathbb{Z}^d}$ and the set of paths $\mathcal{P}_{C_1|x|_1/M}$, we get

$$\mathbb{E}(X_{C_1|x|_1,M}^i) \ \le\ C(C_1|x|_1/M) p_M^{1/d}. \tag{3.41}$$

Combining (3.35), (3.39) (3.40) and (3.41) gives that

$$\begin{aligned} \mathbb{E}(\#A_M^{\gamma_{0,x}}) &\le C8^d (C_1|x|_1) M^{d-1} p_M^{1/d} \\ &\le C8^{d+1}(C_1|x|_1) M^{d+1} e^{-M^\varepsilon/d}, \end{aligned} \tag{3.42}$$

which proves (3.34). □

Before going into the rest of the proofs, we will prove the same estimates as in Lemma 2.1 for T_1 and T_2. By repeating the arguments of the proof of Lemma 2.1 ([1, Lemma 4.2]), we can show that there exist positive constants C and ε, such that for any $y,z\in\mathbb{Z}^d$, and $k \ge C|y|_1$,

$$\mathbb{P}\left(T^{[z]}(0,y) \ge k\right) \le e^{-k^\varepsilon}. \tag{3.43}$$

By the union bound, for $k \ge C_2|y|_1$ with $C_2 = 2C$, we have

$$\begin{aligned} \mathbb{P}(T_1(0,y) \ge k) &\le \sum_{z\in\mathbb{Z}^d : |z|_1 = 1} \mathbb{P}(T^{[0]}(z,y) \ge k-1) \\ &\le 2de^{-(k-1)^\varepsilon} \le e^{-k^{\varepsilon_2}}, \end{aligned} \tag{3.44}$$

with some $\varepsilon_2 > 0$, where we have used (3.43) for $k-1 \geq 2C|y|_1 - 1 \geq C|z-y|_1$.

We observe also that if $T(y) \leq k$ then $T^{[z]}(0,y) = T(y)$ for $z \notin B(k)$. Therefore, for $k \geq C_3|y|_1$ with $C_3 = \max\{C_1, C_2\}$,

$$\begin{aligned} \mathbb{P}\left(T_2(0,y) \geq k\right) &\leq \mathbb{P}(T(y) \geq k) + \mathbb{P}(T(y) < k, T_2(0,y) \geq k) \\ &\leq \mathbb{P}(T(y) \geq k) + \sum_{z \in B(k)} \mathbb{P}\left(T^{[z]}(0,y) \geq k\right) \\ &\leq e^{-k^{\varepsilon_1}} + (2k)^d e^{-k^{\varepsilon_2}} \leq e^{-k^{\varepsilon_3}}, \end{aligned} \tag{3.45}$$

with some $\varepsilon_3 > 0$. From now on, for simplicity of notation we use C_1 for all C_1, C_2, C_3, and ε_1 for all $\varepsilon_1, \varepsilon_2, \varepsilon_3$. It means that for $k \geq C_1|y|_1$,

$$\max\{\mathbb{P}(T(0,y) \geq k), \mathbb{P}(T_1(0,y) \geq k), \mathbb{P}(T_2(0,y) \geq k)\} \leq e^{-k^{\varepsilon_1}}. \tag{3.46}$$

Proof of Lemma 3.4. We begin with Part (ii), which is easier than (i). Observe that

$$\max_{\gamma=(y_i)_{i=1}^{\ell} \in \mathcal{P}_L} \sum_{i=1}^{\ell-1} T_1(y_i, y_{i+1}) \leq L \max_{u,v \in B(L)} T_1(u,v)$$

Using the union bound and (3.46), for any $k \geq 2dC_1L$,

$$\mathbb{P}\left(\max_{u,v \in B(L)} T_1(u,v) \geq k\right) \leq (2L)^{2d} e^{-k^{\varepsilon_1}}.$$

The last two inequalities yield that

$$\begin{aligned} \mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell} \in \mathcal{P}_L} \sum_{i=1}^{\ell-1} T_1(y_i, y_{i+1})\right)^2 &\leq CL^4 \left(1 + (2L)^{2d} \sum_{k \geq 2dC_1L} k^2 e^{-k^{\varepsilon_1}}\right) \\ &= \mathcal{O}(L^4). \end{aligned}$$

We now prove (i). For any $\gamma = (y_i)_{i=1}^{\ell} \in \mathcal{P}_L$, we define

$$\begin{aligned} \bar{A}_M^{\gamma} &= \{y_i \in \gamma : |y_i - y_{i+1}|_1 = M\}, \\ \bar{A}_{M,0}^{\gamma} &= \{y_i \in \bar{A}_M^{\gamma} : T_1(y_i, y_{i+1}) \leq C_1 M\}, \\ \bar{A}_{M,k}^{\gamma} &= \{y_i \in \bar{A}_M^{\gamma} : T_1(y_i, y_{i+1}) = C_1 M + k\}, \end{aligned}$$

with C_1 as in (3.46). Then

$$\#\bar{A}_M^{\gamma} = \sum_{k \geq 0} \#\bar{A}_{M,k}^{\gamma}, \quad \sum_{M \geq 1} M \#\bar{A}_M^{\gamma} = |\gamma|_1 \leq L. \tag{3.47}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{\ell-1} T_1(y_i, y_{i+1}) &\leq \sum_{M \geq 1} \left(C_1 M \#\bar{A}_{M,0}^{\gamma} + \sum_{k \geq 1} (C_1 M + k) \#\bar{A}_{M,k}^{\gamma}\right) \\ &\leq C_1 L + \sum_{M \geq 1} \sum_{k \geq 1} k \#\bar{A}_{M,k}^{\gamma}. \end{aligned} \tag{3.48}$$

We shall apply the same arguments as in the proof of Lemma 3.7 to deal with the sum above. For each M, k we tessellate $\mathbb{Z}^d$ to groups of boxes whose size equals $2(C_1M + k)$. Using analogous arguments to prove (3.41) and (3.42), we can show that

$$\mathbb{E}\left(\max_{\gamma \in \mathcal{P}_L} \#\bar{A}_{M,k}^{\gamma}\right) \leq CL(C_1M + k)^{d-1} p_{M,k}^{1/d}, \tag{3.49}$$

where

$$\begin{aligned} p_{M,k} &= \mathbb{P}\left(\exists u, v \in B(C_1M + k) : |u - v|_1 \leq M, T_1(u,v) = C_1M + k\right) \\ &\leq (2(C_1M + k))^{2d} e^{-(C_1M+k)^{\varepsilon_1}}, \end{aligned}$$

by using the union bound and (3.46). Combining (3.48) and (3.49), we have

$$\begin{aligned}\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}T_1(y_i,y_{i+1})\right) &\leq C_1L+\sum_{M\geq 1}\sum_{k\geq 1}k\mathbb{E}\left(\max_{\gamma\in\mathcal{P}_L}\#\bar{A}^{\gamma}_{M,k}\right)\\ &\leq CL\left(1+\sum_{M\geq 1}\sum_{k\geq 1}(C_1M+k)^{d+2}e^{-(C_1M+k)^{\varepsilon_1}/d}\right)\\ &= \mathcal{O}(L),\end{aligned}$$

which proves (i). □

Proof of Lemma 3.8. To show (ii), we notice that

$$\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}\max_{u:|u-y_i|_1\leq 2}T_2(u,y_i)^4\leq L\max_{u,v\in B(L+2)}T_2(u,v)^4. \tag{3.50}$$

Now part (ii) follows from (3.46) and (3.50) by using the same arguments as in Lemma 3.4 (ii).

The proof of (i) is similar to that of Lemma 3.7. For any $M\geq 1$, using (3.46) and the union bound,

$$\begin{aligned}p'_M=\mathbb{P}\left(\exists y\in B(M),\max_{u:|u-y|_1\leq 2}T_2(u,y)^4=M\right) &\leq (2M)^d\mathbb{P}\left(\max_{|u|_1\leq 2}T_2(u,0)^4=M\right)\\ &\leq e^{-M^{\varepsilon}}, \end{aligned}\tag{3.51}$$

for some $\varepsilon>0$ small. As in Lemma 3.7, we define for $\gamma\in\mathcal{P}_L$, and $M\geq 1$

$$A'^{\gamma}_M = \{y\in\gamma:\max_{u:|u-y|_1\leq 2}T_2(u,y)^4=M\}.$$

We also cover $\mathbb{Z}^d$ by groups of boxes whose size equals to $2M$ as in the proof of Lemma 3.7 to verify conditions (c) and (d). Repeating the arguments as in the proof of Lemma 3.7, with A'^{γ}_M, p'_M replacing A^{γ}_M, p_M, we can show that

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell}\max_{u:|u-y_i|_1\leq 2}T_2(u,y_i)^4\right)=\mathcal{O}(L)\left(\sum_{M\geq 1}M^{d-1}e^{-M^{\varepsilon}/d}\right)=\mathcal{O}(L),$$

which proves (i). □

4. Proof of Proposition 1.2

Proof. The upper bound on the length of optimal paths is a consequence of Lemma 2.1. Indeed, if $\gamma\in\mathbb{O}(x)$, then $l(\gamma)\leq T(x)$. Hence, by Lemma 2.1,

$$\mathbb{P}\left(\max_{\gamma\in\mathbb{O}(x)}l(\gamma)>C_1|x|_1\right) \leq \mathbb{P}(T(x)>C_1|x|_1) \tag{4.1}$$

$$\leq e^{-|x|_1^{\varepsilon_1}}, \tag{4.2}$$

with ε_1 and C_1 positive constants as in Lemma 2.1. To show the lower bound, we first recall the definition of A^{γ}_M as in the proof of Lemma 3.7:

$$A^{\gamma}_M=\{y_i\in\gamma:T(y_i,y_{i+1})=t(y_i,y_{i+1})=M\}.$$

Then since $l(\gamma)=\sum_{M\geq 1}\#A^{\gamma}_M$, for any $\gamma\in\mathbb{O}(x)$, and $K\geq 1$

$$\begin{aligned}|x|_1\leq T(x)=\sum_{M\geq 1}M\#A^{\gamma}_M &\leq K\sum_{M=1}^{K}\#A^{\gamma}_M+\sum_{M\geq K}M\#A^{\gamma}_M\\ &\leq Kl(\gamma)+\sum_{M\geq K}M\#A^{\gamma}_M.\end{aligned}\tag{4.3}$$

Rearranging it, we obtain that for any $K \geq 1$,

$$\begin{aligned}\min_{\gamma\in\mathbb{O}(x)} l(\gamma) &\geq \frac{1}{K}\left(|x|_1 - \max_{\gamma\in\mathbb{O}(x)}\sum_{M\geq K} M\#A_M^\gamma\right)\\ &\geq \frac{1}{K}\left(|x|_1 - \sum_{M\geq K} M \max_{\gamma\in\mathbb{O}(x)}\#A_M^\gamma\right).\end{aligned} \tag{4.4}$$

Note that if $T(x) \leq C_1|x|_1$, then $\gamma \in \mathcal{P}_{C_1|x|_1}$ for any $\gamma \in \mathbb{O}(x)$, and thus

$$\sum_{M\geq K} M \max_{\gamma\in\mathbb{O}(x)}\#A_M^\gamma \leq \sum_{M\geq K} M \max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\#A_M^\gamma. \tag{4.5}$$

We define

$$M_x = [|x|_1^{1/2(d+3)}],$$

and

$$\mathcal{E} = \{\forall\, M \geq M_x,\, \forall\gamma \in \mathcal{P}_{C_1|x|_1},\ \#A_M^\gamma = 0\}.$$

Then, by using the union bound and Lemma 2.4, we get

$$\begin{aligned}\mathbb{P}(\mathcal{E}^c) &\leq \mathbb{P}(\exists u, v \in B(C_1|x|_1) \text{ such that } T(u,v) = t(u,v) \geq M)\\ &\leq (2C_1|x|_1)^{2d}\sum_{M\geq M_x} e^{-M^{\varepsilon_1}} \leq Ce^{-|x|_1^\varepsilon},\end{aligned} \tag{4.6}$$

for some positive constants C and ε. We recall the tessellation as in the proof of Lemma 3.7. Based on the groups of boxes $\{(B_{i,z}^M)_{z\in\mathbb{Z}^d}, i = 1,\dots,4^d\}$ whose size equals to $2M$, we have defined

$$A_{M,i,z}^\eta = B_{i,z}^M \cap A_M^\eta.$$

Moreover,

$$\#A_M^\eta \leq \sum_{i=1}^{4^d}\sum_{z\in\mathbb{Z}^d}\#A_{M,i,z}^\eta. \tag{4.7}$$

Using Lemma 2.5 (i) and the same arguments as in Lemma 3.7, we can show that for any $i = 1,\dots,4^d$, $K \leq M \leq M_x$, and $s \geq A_1$ (with A_1 as in Lemma 2.5 (i)),

$$\mathbb{P}\left(\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\sum_{z\in\mathbb{Z}^d}\#A_{M,i,z}^\gamma \geq s\left(\tfrac{C_1|x|_1}{M}\right)(2M)^d p_M^{1/d}\right) \leq \exp\left(-s\left(\tfrac{C_1|x|_1}{M}\right)p_M^{1/d}/2\right),$$

with p_M as in (3.40). Let $s = s_M = p_M^{-1/d}M^{-(d+1)}$. Then

$$\mathbb{P}\left(\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\sum_{i=1}^{4^d}\sum_{z\in\mathbb{Z}^d}\#A_{M,i,z}^\gamma \geq C_d|x|_1M^{-2}\right) \leq 4^d\exp\left(-(C_1|x|_1)/(2M^{d+2})\right),$$

with $C_d = 2^{2d-2}C_1$. Notice that by (3.40), $s_M \to \infty$ as $M \to \infty$, so the condition $s_M \geq A_1$ holds for any $M \geq K$ with K large enough. It follows from (4.7) that

$$\mathbb{P}\left(\sum_{M=K}^{M_x} M\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\#A_M^\gamma \geq C_d|x|_1\sum_{M=K}^{M_x}M^{-2}\right) \leq 4^d\sum_{M=K}^{M_x}M\exp\left(-\tfrac{C_1|x|_1}{2M^{d+2}}\right).$$

Therefore,

$$\mathbb{P}\left(\sum_{M=K}^{M_x} M\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\#A_M^\gamma \geq \tfrac{C_d|x|_1}{K}\right) \leq 4^dM_x^2\exp\left(-\tfrac{C_1|x|_1}{2M_x^{d+2}}\right) \leq e^{-|x|_1^\varepsilon}, \tag{4.8}$$

for some $\varepsilon > 0$. We now only need to take K large enough such that $C_d/K < 1/2$ and $s_M \geq A_1$ for any $M \geq K$. Then by (4.4), (4.5), (4.6) and (4.8), if we take $c > 0$ sufficiently small so that $1 - cK \geq C_d/K$, we get

$$\mathbb{P}\left(\min_{\gamma \in \mathbb{O}(x)} l(\gamma) < c|x|_1\right) \leq \mathbb{P}(T(x) > C_1|x|_1) + \mathbb{P}(\mathcal{E}^c) + \mathbb{P}\left(\sum_{M=K}^{M_x} M \max_{\gamma \in \mathcal{P}_{C_1|x|_1}} \# A_M^{\gamma} > (1-cK)|x|_1\right)$$
$$\leq Ce^{-|x|_1^{\varepsilon}},$$

which completes the proof of Proposition 1.2. □

Acknowledgments. The work of V. H. Can is supported by the fellowship of the Japan Society for the Promotion of Science and the Grant-in-Aid for JSPS fellows Number 17F17319, and by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant 101.03-2017.07.

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