# Optimal control of a crop irrigation model under water scarcity

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**Abstract**

We consider a simple crop irrigation model and study the optimal control which consists of maximizing the biomass production at harvesting time. A specificity of this work is to impose a quota on the water used for irrigation, in a context of limited resources. The model is written as a 2d non-autonomous dynamical system with a state constraint, and a non-smooth right member given by threshold-based soil and crop water stress functions. We show that when the water quota is below the threshold giving the largest possible production, the optimal strategy consists of irrigating once. We then show that the optimal solution can have one or several singular arcs, and therefore be better than simple bang-bang controls, as commonly used. The gains over the best bang-bang controls are illustrated on numerical simulations. These new feedback controls that we obtain are a promising first step towards the concrete application of control theory to the problem of optimal irrigation scheduling under water scarcity.

**Key-words.** Crop irrigation, water management, optimal control, singular arcs, feedback synthesis.

## 1 Introduction

Today, in the context of the climate crisis, tensions around water resources are growing, and agriculture in many countries, particularly in the South, must now be considered as having to be water-saving. One way to cope with this changing context is to impose irrigation quotas. But such quotas could imply yield losses as the best crop water requirements would not necessarily be fulfilled. In these particular but expected situations, optimizing the schedule of irrigation to minimize yield losses becomes crucial. Crop modeling and numerical simulations are handy tools for understanding and adjusting these compromises. More particularly the control theory as irrigation is typically a control variable of a crop model.

Numerous texts and articles have developed systematic approaches to tackle the scheduling optimization problem. We refer to [10, 11, 17, 18, 22] and references therein. However, most of the existing approaches do not allow to have analytical descriptions of an optimal solution's theoretical properties as they are based on the numerical optimization of complex models. Control theoretical tools, such as the Pontryagin's Maximum Principle, allow, as a rule, to discover structures of optimal strategies in terms of state feedback available for large classes of operating conditions and parameter values. Unlike optimization among open-loop controls that apply for precise requirements, closed-loop procedures offer adaptability and robustness for concrete applications, when state variables can be measured in real-time. However, when models have too numerous equations, variables, and parameters or do not present a robust structure, the study of optimal strategies with these techniques is most of the time out of reach of analytical characterizations. This is a motivation to consider models with a reasonable simplicity to benefit from these theoretical approaches (see e.g. [12, 16]). Even when these models are imperfect, optimal policies' derivation is of possible relevance. It generally provides simple but non-intuitive control rules that can be tested in simulation on more realistic detailed models. Sometimes, combinations of analytical description and reduced optimization can be obtained, which still offer advantages over purely numerical solutions in open-loop. This is the spirit in which this work has been conducted.

The aim of this work is to derive feedback strategies in the context of water quotas extending the work initiated in [9], where a simplified crop model is presented to investigate optimal irrigation strategies. The present work goes two steps further, first in relaxing assumptions on the water stress functions of the model, and secondly in providing the complete solution of the optimal control problem as well as associated numerical simulations.

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The organization of the paper is as follows. Section 2 presents the crop model with its assumptions. Section 3 is dedicated to the formulation of the optimal control problem under constraint, along with some preliminary results. In section 4, crucial properties of the optimal solutions are proved. Then, Section 5 is devoted to the application of the Pontryagin Maximum Principle and the synthesis of the optimal irrigation strategy. Finally, in Section 6, we illustrate the theoretical results on numerical simulations and draw comparisons of several control strategies.

## 2 Model description and assumptions

We consider the dynamical model of crop irrigation introduced in [9] and inspired from [14], where $S(t)$ and $B(t)$ stand respectively for the relative soil humidity in the root zone (a quantity between 0 and 1) and the crop biomass at time $t$ in an interval $[0, T]$ representing the crop growth season:

$$\dot{S} = k_1(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)), \tag{1}$$
$$\dot{B} = \varphi(t)K_S(S)f(B), \tag{2}$$

with the initial condition (at the sowing date 0)

$$S(0) = 1, \tag{3}$$
$$B(0) = B_0 > 0, \tag{4}$$

and $T$ is the harvesting date. The control variable $u(t) = F(t)/F_{max} \in [0, 1]$ is the ratio of the input water flow rate $F(t)$ at time $t$ over the maximal flow $F_{max}$ that the irrigation allows.

Eq. (1) represents the variation of a vertically averaged soil moisture as influenced by crop evapotranspiration ($\varphi(t)K_S(S) + (1-\varphi(t))K_R(S)$) and irrigation ($k_2u(t)$). In this model as in [1], crop evapotranspiration is split into crop transpiration ($\varphi(t)K_S(S)$) and soil evaporation ($(1-\varphi(t))K_R(S)$) using the crop radiation interception efficiency $\varphi(t)$. This function $\varphi(t)$, which makes the system non-autonomous, can be seen as a surrogate to the crop coefficient used in the FAO approach [23]. The two functions $K_S$ and $K_R$ (see Assumption 1 and Fig. 1 below) are used to model the regulation of transpiration and evaporation by soil moisture as in [14]. Eq. (2) determines the amount of biomass produced per time unit from the transpiration flux as in [20, 14] and modulated by a normalized growth kinetics function $f(\cdot)$. Note that the present model does not consider a temporal variation of the reference evapotranspiration present in many crop models and does not include either rainfall inputs: it would be more suitable for greenhouse-grown crops.

**Assumption 1.** *The functions $K_S$ and $K_R$ are piecewise linear non decreasing from $[0, 1]$ to $[0, 1]$ with numbers $0 < S_h < S_w < S^\star < 1$ such that*

1. *$K_S$, resp. $K_R$ is null on $[0, S_w]$, resp. $[0, S_h]$, and positive outside this interval.*
2. *$K_S$ is equal to 1 on $[S^\star, 1]$ and concave increasing on $[S_w, S^\star]$.*
3. *$K_R(1) = 1$ and $K_R$ is convex increasing on $[S_w, S^\star]$.*

The value $S_w$ represents the plant wilting point, which is usually higher than the hydroscopic point denoted by $S_h$. $S^\star$ is the minimal threshold on the soil humidity that gives the best biomass production. This assumption generalizes the usual expressions found in the literature (see for instance [14]), given by the following assumption (see Fig. 1).

**Assumption 1bis.** *The functions $K_S$ and $K_R$ are piecewise linear non decreasing from $[0, 1]$ to $[0, 1]$ given by the following expressions*

$$K_S(S) = \begin{cases} 0 & S \in [0, S_w], \\ \dfrac{S - S_w}{S^\star - S_w} & S \in [S_w, S^\star], \\ 1 & S \in [S^\star, 1], \end{cases} \qquad K_R(S) = \begin{cases} 0 & S \in [0, S_h], \\ \dfrac{S - S_h}{1 - S_h} & S \in [S_h, 1], \end{cases} \tag{5}$$

*where $0 < S_h < S_w < S^\star < 1$.*

We shall say that $S$ is a *corner point* of $K_S$, resp. $K_R$ when the function is non differentiable at $S$ (therefore $S^\star$ is necessarily a corner point of $K_S$).

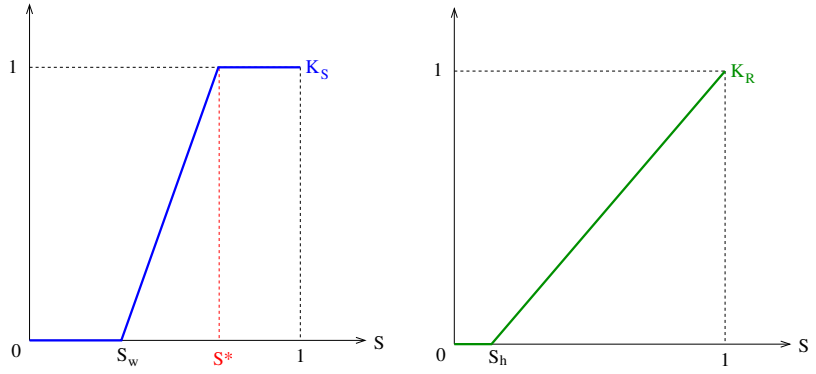



Figure 1: Graphs of the functions $K_S$ and $K_R$ given by expressions (5)

**Assumption 2.** *$\varphi(\cdot)$ is a $C^1$ increasing function from $[0,T]$ to $[0,1]$ with $\varphi(0)=0$ and $\varphi(T)=1$.*

Notice that the dynamics (1)-(2) is non-autonomous and that we consider a mild hypothesis on the function $\varphi$.

**Assumption 3.** *$k_1$, $k_2$ are positive parameters with*

$$k_2 \geq 1.$$

**Assumption 4.** *The function $f$ is a non-negative Lipschitz continuous function with linear growth such that $f(B_0) > 0$.*

Typical instances of growth function $f$ are constants or the logistic law, as considered in [16]

$$f(B) = rB\left(1 - \frac{B}{B_{max}}\right), \tag{6}$$

with $B_{max} > B_0$, but other choices are possible. The condition $k_2 > 1$ is a *controllability* assumption, in the sense that it allows the variable $S$ to stay equal to 1 with the constant control $u = 1/k_2$. However, the dynamics is naturally subject to the state constraint

$$S(t) \leq 1, \quad t \in [0,T]. \tag{7}$$

We shall consider the set $\mathcal{U}$ of *admissible* controls as measurable functions $u(\cdot)$ taking value in $[0,1]$ such that the solution of (1)-(3) verifies the constraint (7). Under the former assumptions, one obtains straightforwardly the following property.

**Lemma 1.** *For any admissible control $u(\cdot)$, the solution $(S(\cdot), B(\cdot))$ of system (1)-(2) with initial condition (3)-(4) verifies*

$$S(t) > S_h, \quad t \geq 0, \tag{8}$$

*and $B(\cdot)$ is uniformly bounded on $[0,T]$.*

**Remark 1.** *Another way to impose the state constraint (7) to be fulfilled is to consider that the extra water than could be brought when the soil is already saturated (i.e. $S=1$) is indeed lost. This amounts to consider the following dynamics of $S$ instead of equation (1):*

$$\dot{S} = k_1\Big(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2\chi(S,u(t))\Big), \tag{9}$$

*where the function $\chi(\cdot)$ is given by*

$$\chi(S,u) := \begin{cases} \min(1/k_2, u) & \text{if } S = 1, \\ u & \text{if } S < 1. \end{cases} \tag{10}$$

*Note that the right member of the o.d.e. (9) remains bounded, continuous w.r.t. $(t,S,u)$ and Lipschitz in $S$, which then gives existence and uniqueness of solution of (9) for any measurable control function $u(\cdot)$ that takes values in $[0,1]$. However, we shall show in Section 4 (Proposition 2) that under water scarcity, an optimal solution never saturates this constraint, so that we no longer need to consider it.*

## 3 The optimization problem

For each control $u(\cdot)$ in $\mathcal{U}$, we associate the total water delivered to the crop during the time interval $[0, T]$ as

$$Q[u(\cdot)] := F_{max} \int_0^T u(t)\, dt \tag{11}$$

and given a quantity of water $\bar{Q} > 0$ available on the time interval $[0, T]$, we define the constraint

$$Q[u(\cdot)] \leq \bar{Q}. \tag{12}$$

Then, we consider the optimal control problem

$$\sup \big\{ B(T),\ u(\cdot) \in \mathcal{U} \text{ that satisfies (12)} \big\}. \tag{13}$$

By the usual argument of compactness of the set of admissible solutions, the dynamics being linear w.r.t. $u$ (see for instance [24]), one concludes about the existence of an optimal solution of Problem (13), that we aim now to characterize. Preliminary results are available in the conference paper [9], which are much proved and generalized in the present work.

Let $\bar{B} = B(T)$ where $B(\cdot)$ is solution of $\dot{B} = \varphi(t) f(B)$ with $B(0) = B_0$, which is an uniform bound of the solutions of (2)-(4) on $[0, T]$. Note that $f$ is necessarily positive on $[B_0, \bar{B})$ and one can consider the change of variable of the biomass

$$B \mapsto \tilde{B} = g(B) := \int_{B_0}^{B} \frac{db}{f(b)}, \quad B \in [B_0, \bar{B}),$$

which gives the simplified dynamics

$$\dot{\tilde{B}} = \varphi(t) K_S(S), \quad \tilde{B} = 0$$

instead of (2) and the true value of the biomass $B(t)$ can be recovered by $B(t) = g^{-1}(\tilde{B}(t))$, the function $g$ being increasing and thus invertible. For instance, for the logistic law (6), one has

$$B(t) = \frac{B_{max}}{1 + \big(\frac{B_{max}}{B_0} - 1\big) e^{-r\tilde{B}(t)}}, \quad t \geq 0.$$

Therefore, maximizing $\tilde{B}(T)$ is equivalent to maximizing $B(T)$ (i.e. an optimal control maximizing $\tilde{B}(T)$ is also optimal for the problem (13)). For sake of simplicity, we shall drop the $f(B)$ term in equation (2) and consider $B_0 = 0$ without any loss of generality.

For convenience, we shall denote, for any $t_0 \in [0, T]$ and $S_0 \in [0, 1]$, $S_{t_0,S_0,0}(\cdot)$, resp. $S_{t_0,S_0,1}(\cdot)$, for the solution of the differential equation (1) with $S(t_0) = S_0$ and the constant control $u = 0$, resp. $u = 1$. We shall see in the following that the corner points, and especially the threshold $S^\star$, are playing a crucial role in the optimal synthesis. The following definitions will be useful in the following.

**Definition 1.** *Denote $\underline{S}(\cdot) := S_{0,1,0}(\cdot)$ and define*

$$\underline{t} := \sup\{t \in [0, T] \ s.t.\ \underline{S}(t) > S^\star\}.$$

*Define also the number*

$$B_T^\star := \int_0^T \varphi(t)\, dt.$$

Straightforwardly, on has the first result.

**Lemma 2.**

*(i) The inequality $B(T) \leq B_T^\star$ is fulfilled for any admissible control $u(\cdot)$.*

*(ii) If $\underline{t} = T$, then any admissible control $u(\cdot)$ gives $B(T) = B_T^\star$ (in particular the control identically null).*

Let us now define *singular controls* as the ones that maintain $S(\cdot)$ constant.

**Definition 2.** *For any $\tilde{S} \in (0,1)$, define the control*

$$\tilde{u}_{\tilde{S}}(t) := \frac{\varphi(t)K_S(\tilde{S}) + (1-\varphi(t))K_R(\tilde{S})}{k_2}, \quad t \in [0,T] \tag{14}$$

*and posit for $\tilde{S} = S^\star$*

$$Q^\star := F_{max} \int_{\underline{t}}^{T} \tilde{u}_{S^\star}(t)\, dt.$$

Note that under Assumption 3, the control (14) is admissible i.e. one has $\tilde{u}_{\tilde{S}}(t) \in [0,1]$ at any $t \in [0,T]$ whatever is $\tilde{S} \in (0,1)$. Moreover one has

$$\tilde{u}_{\tilde{S}}(t) \in (0,1),\ t \in (0,T),\ \tilde{S} \in (S_h, S^\star]. \tag{15}$$

One can easily check that the following Lemma holds.

**Lemma 3.** *Assume $\underline{t} < T$.*

*(i) For any $\bar{Q} \geq Q^\star$, the control*

$$u(t) = \begin{cases} 0 & t \in [0,\underline{t}), \\ \tilde{u}_{S^\star}(t) & t \in [\underline{t},T], \end{cases} \tag{16}$$

*satisfies the constraint (12) and gives $B(T) = B_T^\star$.*

*(ii) For any $\bar{Q} < Q^\star$ and admissible control $u(\cdot)$ satisfying the constraint (12), one has $B(T) < B_T^\star$.*

Consequently, when $\underline{t} = T$ or $\bar{Q} \geq Q^\star$, we know that the maximal biomass production $B_T^\star$ can be reached with the control strategy (16) (other choices could be possible). We shall focus now on the complementary cases that fulfill the following conditions.

**Hypothesis 1.** *$\underline{t} < T$ and $\bar{Q} < Q^\star$.*

This hypothesis corresponds to situations of water scarcity, because there is not any enough water available to maintain the soil humidity constantly above or equal to the level $S^\star$ which provides the maximal production $B_T^\star$ at the harvesting time. Those situations are quite challenging from the control viewpoint because the crop has to suffer from dryness at a certain point and the question amounts to choose, to some extent, how and when, to impact as little as possible the biomass production at final time $T$. We start by investigating the behavior of the optimal solutions above the $S^\star$ level.

# 4 Properties of the optimal solutions with respect to threshold $S^\star$

We introduce below the MRAP (for *Most Rapid Approach Path*) to $S = S^\star$ controls. Such kind of controls have already been considered in several optimal control problems in the plane, characterizing their optimality (e.g. [13, 7, 6] or related to the so-called "turnpike" property (see e.g. [15, 21, 5]). Here, we use it in a different way. We do not pretend that these controls are necessarily optimal (and indeed they are not), but they respect the state constraint (7) and can locally improve the cost, providing then a comparison tool given in Proposition 1 below and used later on. We begin with some definitions.

**Definition 3.** *For $(t_0, S_0) \in [0,T) \times (S^\star, 1]$, we define the number*

$$t^+(t_0,S_0) = \begin{cases} T \quad \text{if } S_{t_0,S_0,0}(t) > S^\star,\ t \in [t_0,T], \\ \inf\{t > t_0\,;\, S_{t_0,S_0,0}(t) = S^\star\} \quad \text{otherwise.} \end{cases}$$

*And for any $(t_0, S_0) \in (0,T] \times (S^\star, 1]$, we define*

$$t^-(t_0,S_0) = \begin{cases} 0 \quad \text{if } S_{t_0,S_0,1}(t) > S^\star,\ t \in [0,t_0], \\ \sup\{t < t_0\,;\, S_{t_0,S_0,1}(t) = S^\star\} \quad \text{otherwise.} \end{cases}$$

**Definition 4.** *For any $(t_1,S_1) \in [0,T) \times [S^\star,1]$ and $(t_2,S_2) \in (t_1,T] \times [S^\star,1]$ such that $S_2$ is attainable from $(t_1,S_1)$ at time $t_2$ with an admissible control, we associate the MRAP control $\tilde{u}(\cdot)$ on the time interval $[t_1,t_2]$ as follows:*

*i) If* $t_-(t_2,S_2) \geq t_+(t_1,S_1)$*:*

$$\tilde{u}(t) := \begin{cases} 0 & \text{if } t \in [t_1, t^+(t_1,S_1)), \\ \tilde{u}_{S^\star}(t) & t \in [t_+(t_1,S_1), t^-(t_2,S_2)], \\ 1 & \text{if } t \in (t_-(t_2,S_2), t_2]. \end{cases} \tag{17}$$

*ii) If* $t^-(t_2,S_2) < t^+(t_1,S_1)$*:*

$$\tilde{u}(t) := \begin{cases} 0 & \text{if } t \in [t_1, \bar{t}(t_1,S_1,t_2,S_2)), \\ 1 & \text{if } t \in (\bar{t}(t_1,S_1,t_2,S_2), t_2]. \end{cases}$$

*where* $\bar{t}(t_1,S_1,t_2,S_2)$ *is the unique* $\bar{t} \in [t_1,t_2]$ *such that* $S_{t_1,S_1,0}(\bar{t}) = S_{t_2,S_2,1}(\bar{t}) > S^\star$ *(one can easily verify that the function* $I(t) := S_{t_1,S_1,0}(t) - S_{t_2,S_2,1}(t)$ *is decreasing on* $[t_1,t_2]$ *and such that* $I(t_1) \geq 0$*,* $I(t_2) \leq 0$*, which gives the existence and uniqueness of* $\bar{t}(t_1,S_1,t_2,S_2)$*).*

These particular trajectories are depicted on Fig. 2 and 3.

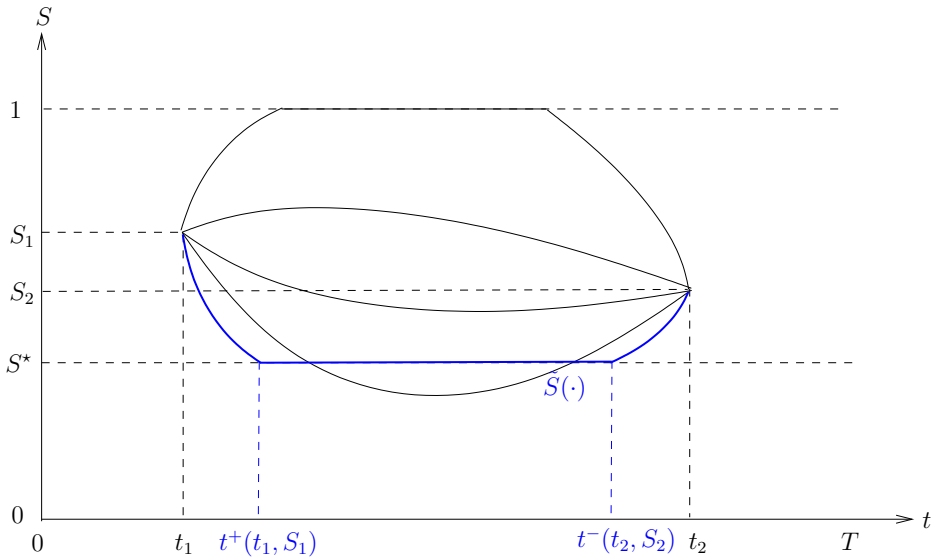



Figure 2: The MRAP trajectory $\tilde{S}(\cdot)$ (in blue) compared to other admissible trajectories $S(\cdot)$ when $t_-(t_2,S_2) > t_+(t_1,S_1)$

Then, one has the following comparison result.

**Proposition 1.** *Let* $S(\cdot)$ *be a solution of* (1) *on* $[t_1,t_2]$ *(with* $0 \leq t_1 < t_2 \leq T$*) for an admissible control* $u(\cdot)$ *such that* $S(t) \geq S^\star$ *for any* $t \in [t_1,t_2]$*. Denote* $S_1 = S(t_1)$ *and* $S_2 = S(t_2)$*. Then, the solution* $\tilde{S}(\cdot)$ *of* (1) *on* $[t_1,t_2]$ *with* $\tilde{S}(t_1) = S_1$ *and the MRAP control* $\tilde{u}(\cdot)$ *(given in Definition 4) satisfies the following properties:*

$$\tilde{S}(t_2) = S_2. \tag{18}$$

$$\tilde{S}(t) \leq S(t), \quad t \in [t_1,t_2]. \tag{19}$$

$$\int_{t_1}^{t_2} \tilde{u}(t)\,dt \leq \int_{t_1}^{t_2} u(t)\,dt. \tag{20}$$

*Moreover, the last inequality is strict when* $S(\cdot)$ *and* $\tilde{S}(\cdot)$ *are not identical.*

*Proof.* By construction, the solution $\tilde{S}(\cdot)$ verifies $\tilde{S}(t_1) = S(t_1)$ and $S(t_2) = S(t_2)$. Thus, property (18) is verified.

From standard comparison results of scalar differential equation with right hand sides that are Lipschitz continuous w.r.t. the state variable (see e.g. [25]), one has for any solution $S(\cdot)$ of (1) with $S(t_0) = S_0$ and any admissible control function $u(\cdot)$, the following frame

$$S_{t_0,S_0,0}(t) \leq S(t) \leq S_{t_0,S_0,1}(t), \quad t \in [t_0,T]. \tag{21}$$

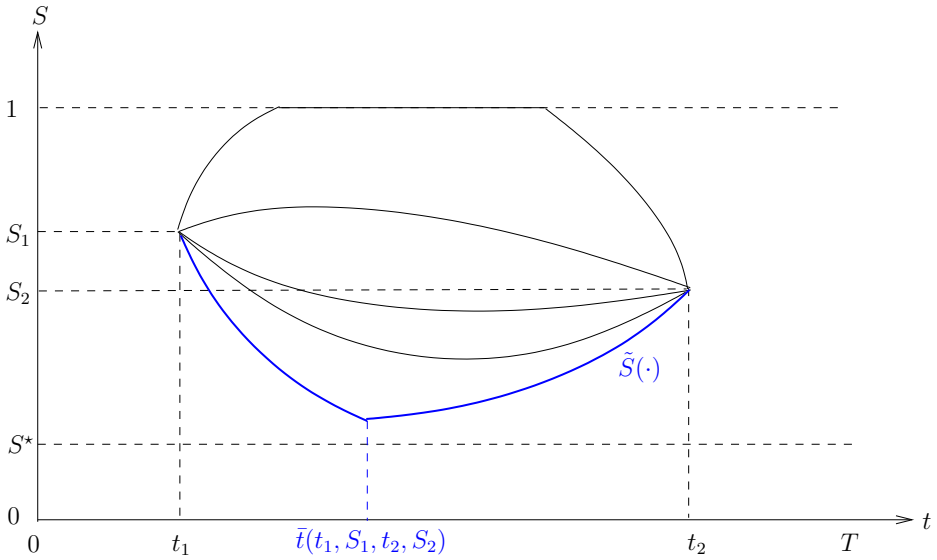



Figure 3: The MRAP trajectory $\tilde{S}(\cdot)$ (in blue) compared to other admissible trajectories $S(\cdot)$ when $t_-(t_2, S_2) < t_+(t_1, S_1)$

Therefore, property (19) is verified.

Consider then the function $\delta(t) := S(t) - \tilde{S}(t)$. From expression (1), one can write

$$d\delta = -k_1 \Big( F(t, S(t)) - F(t, \tilde{S}(t)) \Big)\, dt + k_1 k_2 \left( u(t) - \tilde{u}(t) \right) dt, \tag{22}$$

where we posit

$$F(t, S) = \varphi(t) K_S(S) + (1 - \varphi(t)) K_R(S).$$

Integrating (22) between $t = t_1$ and $t = t_2$, one obtains

$$\begin{aligned} \delta(t_2) - \delta(t_1) = -k_1 \int_{t_1}^{t_2} \Big( F(t, S(t)) - F(t, \tilde{S}(t)) \Big)\, dt \\ + k_1 k_2 \left( \int_{t_1}^{t_2} u(t)\, dt - \int_{t_1}^{t_2} \tilde{u}(t)\, dt \right). \end{aligned}$$

As $F$ is non-decreasing w.r.t. $S$ and $S(t) \geq \tilde{S}(t)$ for $t \in [t_1, t_2]$, one obtains

$$\int_{t_1}^{t_2} u(t)\, dt - \int_{t_1}^{t_2} \tilde{u}(t)\, dt \geq \frac{\delta(t_2) - \delta(t_1)}{k_1 k_2} = 0$$

which proves property (20). □

This result leads to the following properties of the optimal solutions.

**Proposition 2.** *Assume that Hypothesis 1 is satisfied (water scarcity). Then, any optimal solution satisfies the following properties.*

*(i)* $u(t) = 0$ *for a.e.* $t \in [0, \underline{t}]$,

*(ii)* $S(t) \leq S^\star$ *for any* $t \in [\underline{t}, T]$,

*(iii)* $Q[u(\cdot)] = \bar{Q}$.

*Proof.* Let $\tilde{u}(\cdot)$ be the MRAP control for $(t_1, S_1) = (0, 1)$ and $(t_2, S_2) = (T, S^\star)$ (see Definition 4).

Consider any $S(\cdot)$ solution of (1),(3) for an admissible control $u(\cdot)$ satisfying the constraint (12). Notice first that the set

$$E := \{t \in [0,T] \text{ s.t. } S(t) < S^\star\}$$

is non-empty, otherwise one would have $B(T) = B_T^\star$, which is excluded by Lemma 3.ii. Let $t^\star := \inf E < T$. By continuity of $S(\cdot)$, one has necessarily $S(t^\star) = S^\star$ and by Proposition 1 one has

$$\int_0^{t^\star} \tilde{u}(t)\,dt \leq \int_0^{t^\star} u(t)\,dt. \tag{23}$$

Notice that one has $\tilde{u}(t) = \tilde{u}_{S^\star}(t)$ for $t \in [t^\star, T]$. From Hypothesis 1, the inequality

$$Q[u(\cdot)] = F_{max} \int_0^T u(t)\,dt < Q^\star = F_{max} \int_0^T \tilde{u}(t)\,dt \tag{24}$$

is fulfilled. Consequently, (23) and (24) give the inequality

$$\int_{t^\star}^T u(t)\,dt < \int_{t^\star}^T \tilde{u}_{S^\star}(t)\,dt,$$

where $\tilde{u}_{S^\star}(t) < 1$ for $t \in [t^\star, T)$ (cf property (15)). Therefore, the set

$$E_1 := \{t \in [t^\star, T] \text{ s.t. } u(t) < 1\}$$

is necessarily of non-null measure. Moreover, the set $E \cap E_1$ is also of non-null measure (otherwise one would have $u(t) = 1$ for a.e. $t \in E$ that would imply that $S(\cdot)$ is increasing on $E$, which contradicts $S(t^\star) = S^\star$).

If $t^\star > \underline{t}$, inequality (23) is strict (by Proposition 1), and one can consider a control $v(\cdot)$ such that

$$\begin{cases} v(t) = \tilde{u}(t), & t \in [0, t^\star], \\ v(t) = u(t), & t \in [t^\star, T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1, \end{cases}$$

with

$$0 < \int_{E \cap E_1} \big(v(t) - u(t)\big)\,dt \leq \int_0^{t^\star} \big(u(t) - \tilde{u}(t)\big)\,dt.$$

Then, one has

$$Q[v(\cdot)] \leq Q[u(\cdot)] \leq \bar{Q}$$

which guarantees that $v(\cdot)$ satisfies the constraint (12). Its associated solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies then $S_v(t) \geq S(t)$ for any $t \in [0,T]$ with

$$\int_{E \cap E_1} S_v(t)\,dt > \int_{E \cap E_1} S(t)\,dt.$$

As $S(t) < S^\star$ for $t \in E \cap E_1$, one obtains under Assumption 1 the inequality

$$\int_{E \cap E_1} \varphi(t) K_S(S_v(t))\,dt > \int_{E \cap E_1} \varphi(t) K_S(S(t))\,dt, \tag{25}$$

which yields

$$B_v(T) = \int_0^T \varphi(t) K_S(S_v(t))\,dt > \int_0^T \varphi(t) K_S(S(t)) = B(T). \tag{26}$$

We conclude that an optimal solution has to verify $t^\star = \underline{t}$, that is such that

$$S(t) = \underline{S}(t), \quad t \in [0, \underline{t}]$$

or equivalently that $u(t) = 0$ for $t \in [0, \underline{t}]$ is optimal.

Consider now a solution $S(\cdot)$, $B(\cdot)$ with an admissible control $u(\cdot)$ that is null on $[0, \underline{t}]$ and satisfies the constraint (12), and the set

$$F := \{t \in [\underline{t}, T] \text{ s.t. } S(t) > S^\star\}$$

is non empty. From Proposition 1, one has

$$\int_F \tilde{u}(t)\,dt < \int_F u(t)\,dt.$$

Let us consider an admissible control $v(\cdot)$ such that

$$\left\{\begin{array}{ll} v(t) = \tilde{u}(t), & t \in F, \\ v(t) = u(t), & t \in [0,T] \setminus (F \cup (E \cap E_1)), \\ v(t) \in [u(t), 1], & t \in E \cap E_1, \end{array}\right.$$

with

$$0 < \int_{E\cap E_1} \big(v(t) - u(t)\big)\,dt \leq \int_F \big(u(t) - \tilde{u}(t)\big)\,dt.$$

Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) = S^\star$ for $t \in F$ and $S_v(t) \geq S^\star$ for $t \in [0,T] \setminus F$ with

$$\int_{E\cap E_1} S_v(t)\,dt > \int_{E\cap E_1} S(t)\,dt.$$

As before, we obtain inequalities (25), (26), and conclude that an optimal solution has to verify $F = \emptyset$, that is such that $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$.

Finally, consider an admissible control $u(\cdot)$ that is null on $[0,\underline{t}]$ with $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$ and $Q[u(\cdot)] < \bar{Q}$. As previously, one can consider another admissible control $v(\cdot)$ such that:

$$\left\{\begin{array}{ll} v(t) = u(t), & t \in [0,T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1, \end{array}\right.$$

with

$$0 < F_{max} \int_{E\cap E_1} \big(v(t) - u(t)\big)\,dt \leq \bar{Q} - Q[u(\cdot)].$$

Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) \geq S(t)$ for $t \in [0,T]$ with

$$\int_{E\cap E_1} S_v(t)\,dt > \int_{E\cap E_1} S(t)\,dt.$$

One obtains again inequality (26), which shows that the control $u(\cdot)$ cannot be optimal. Therefore, an optimal control $u(\cdot)$ has to satisfy $Q[u(\cdot)] = \bar{Q}$. □

# 5 An optimal synthesis

Note first that one can write equivalently the optimization problem (13) as an (non-autonomous) scalar optimal control problem

$$\max_{u(\cdot)} \int_0^T \varphi(t) K_S(S(t))\,dt, \tag{27}$$

where $S(\cdot)$ is solution of (1), under constraints (7) and (12), or equivalently as a optimal control in the plane for the dynamics

$$\dot{S} = k_1\Big(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)\Big), \quad S(0) = 1, \tag{28}$$

$$\dot{V} = u(t), \quad V(0) = 0, \tag{29}$$

with the target

$$V(T) \leq \bar{V} := \frac{\bar{Q}}{F_{max}} \tag{30}$$

and the criterion (27). Moreover, we know from Proposition 2 that under Hypothesis 1 the state constraint (7) is never saturated for any optimal solution.

### 5.1 Application of the Maximum Principle

Let us write the Hamiltonian associated to this optimal control problem:

$$H(t,S,\lambda_S,\lambda_V,u) := \\ \lambda_S k_1\Big(k_2 u - (\varphi(t)K_S(S) + (1-\varphi(t))K_R(S))\Big) + \lambda_V u + \lambda_0\varphi(t)K_S(S), \tag{31}$$

and its adjoint equations:

$$\dot\lambda_S \in \varphi(t)\big(\lambda_S k_1 - \lambda_0\big)\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S k_1 \partial_C K_R(S(t)), \tag{32}$$

$$\dot\lambda_V = 0, \tag{33}$$

where $\partial_C K_S$, $\partial_C K_R$ denote the Clarke generalized gradients of the Lipschitz maps $K_S$, $K_R$. Therefore, $\lambda_V$ is constant. The (non-smooth) Maximum Principle of Pontryagin (see for instance [3]) states that for any optimal solution $S(\cdot)$, $V(\cdot)$, $u(\cdot)$, there exists an adjoint vector $\lambda(\cdot) = (\lambda_S(\cdot),\lambda_V(\cdot))$ which is an absolutely continuous solution of the adjoint system (32)-(33) and a scalar $\lambda_0$ equal to 0 or 1 such that

$$\lambda_0 + |\lambda_S(t)| + |\lambda_V(t)| \neq 0, \quad t\in[0,T], \tag{34}$$

which satisfy the transversality condition

$$\lambda_S(T) = 0, \tag{35}$$

(remind that $S(T)$ is free and $V(T) = \bar V$ by Proposition 2), along with the maximization condition

$$H(t,S(t),\lambda_S(t),\lambda_V(t),u(t)) = \max_{v\in[0,1]} H(t,S(t),\lambda_S(t),\lambda_V(t),v), \quad \text{a.e. } t\in[0,T]. \tag{36}$$

Defining the *switching function*

$$\phi(t) := \lambda_S(t)k_1k_2 + \lambda_V, \tag{37}$$

the maximization (36) gives, for a.e. $t\in[0,T]$

$$\begin{cases} u(t) = 1 & \text{if } \phi(t) > 0, \\ u(t)\in[0,1] & \text{if } \phi(t) = 0, \\ u(t) = 0 & \text{if } \phi(t) < 0. \end{cases} \tag{38}$$

We first show that that an optimal solution cannot be abnormal.

**Lemma 4.** *For any optimal solution, one has $\lambda_0 = 1$.*

*Proof.* If $\lambda_0 = 0$, the only solution of (32) for the terminal condition (35) is $\lambda_S(t) = 0$ for $t\in[0,T]$. Moreover, the constant value of $\lambda_V$ has to be negative to fulfill the conditions (35) and (34). This implies that the $\phi(t)$ is negative for any $t\in[0,T]$ and by (38), one has $u(t) = 0$ for a.e. $t\in[0,T]$ i.e. $\underline{S}(\cdot)$ is the optimal trajectory. Let $\underline{t}\in[0,T]$ be such that $\underline{S}(\underline{t}) = S^\star$. Then the control $v(\cdot)$ defined by

$$\bar v(t) = \begin{cases} 0, & t\in[0,\underline{t}), \\ \dfrac{\bar Q}{T-\underline{t}}, & t\in[\underline{t},T], \end{cases}$$

is admissible, and its associated solution $S_v(\cdot)$ verifies

$$S_v(t) = \underline{S}(t),\ t\in[0,\underline{t}), \quad S_v(t) > \underline{S}(t),\ t\in[\underline{t},T]$$

which implies the inequality

$$\int_0^T \varphi(t)K_S(S_v(t))\,dt > \int_0^T \varphi(t)K_S(\underline{S}(t))\,dt,$$

and thus is a contradiction with the optimality of $\underline{S}(\cdot)$. □

We prove now sign properties of the adjoint variables, that will play a crucial role in the following.

**Proposition 3.** *For any optimal solution, one has $\lambda_S(t)\geq 0$ for $t\in[0,T)$. Moreover, one has $\lambda_V < 0$.*

*Proof.* Let us consider the set

$$E := \{t \in [0,T) \text{ s.t. } \lambda_S(t) < 0\}$$

and assume by contradiction that $E$ is non-empty. As one has $S(t) > S_h$ for any $t \in [0,T]$ (cf Lemma 1), and the functions $K_S$ and $K_R$ are respectively non-decreasing and increasing on $[S_h, 1]$ (by Assumption 1), one obtains from equation (32) that $\lambda_S$ is non-increasing on $E$. Therefore, one has $\sup E = T$ and $\lambda_S(T) < 0$, which is a contradiction with the transversality condition (35).

If the constant $\lambda_V$ is non negative, then one has $\phi(t) > 0$ for any $t \in [0,T)$ which implies by (38) that one has $u(t) = 1$ for almost any $t \in [0,T]$. Then one obtains

$$V(T) > \int_0^T \tilde{u}_{OT}(t)\, dt = \frac{Q[\tilde{u}_{OT}(\cdot)]}{F_{max}} > \bar{V},$$

which is in contradiction with the target condition (30). □

Then, Proposition 3 and the transversality condition (35) imply that one has $\phi(T) < 0$, which gives straightforwardly the following property of the optimal solutions.

**Corollary 4.** *For any optimal solution, there exists $\bar{t} < T$ such that $u(t) = 0$ for a.e. $t \in [\bar{t}, T]$. Moreover, one has $S(T) < S^\star$.*

## 5.2 Study of singular arcs

Let us now study the possibilities of singular arcs (we recall that a singular arc is a part of an optimal trajectory such that the switching function $\phi$ remains equal to zero).

**Proposition 5.** *A singular arc on a closed interval $I$ not reduced to a singleton satisfies $S(t) = \tilde{S}$, $t \in I$ where $\tilde{S}$ is a corner point of $K_S$ or $K_R$ that belongs to $(S_w, S^\star]$.*

*Proof.* A singular arc occurs when the switching function $\phi$ is equal to zero on a closed interval $I$ of non-null measure. This amounts to have $\lambda_S$ constant equal to $\lambda_S^\star := -\lambda_v/(k_1 k_2) > 0$ on such an interval. If $K_S$ and $K_R$ are differentiable at $S(t_1)$ with $t_1 \in I$, $K_S'(S(t))$ and $K_R'(S(t))$ are constant equal to $K_S'(S(t_1))$ and $K_R'(S(t_1))$ on a neighborhood $(t_1+\epsilon, t_1+\epsilon)$ of $t_1$ (as the functions $K_S$, $K_R$ are piecewise linear by Assumption 1). Then, from equation (32), one gets

$$\varphi(t)\left[(\lambda_S^\star k_1 - 1)K_S'(S(t_1)) - \lambda_S^\star k_1 K_R'(S(t_1)\right] = -\lambda_S^\star k_1 K_R'(S(t_1)), \quad t \in I \cap (t_1+\epsilon, t_1+\epsilon). \tag{39}$$

From Lemma 1, one has $S(t_1) > S_h$ and thus $K_R'(S(t_1)) > 0$ (by Assumption 1). Finally, as the function $\varphi$ is strictly increasing, we deduce that (39) cannot be fulfilled. We deduce that a singular arc can occur only for constant $S = \tilde{S}$ that are non-differential points of $K_S$ or $K_R$.

From Proposition 2, we known that a singular arc cannot be optimal at values $\tilde{S}$ above $S^\star$, and from Lemma 1, that it cannot occur for $\tilde{S} \le S_h$. From equation (32), one should have

$$0 \in \varphi(t)\big(\lambda_S^\star k_1 - \lambda_0\big)\partial_C K_S(\tilde{S}) + (1-\varphi(t))\lambda_S^\star k_1 \partial_C K_R(\tilde{S}) \text{ a.e. } t \in I. \tag{40}$$

At $\tilde{S} \in (S_h, S_w)$, one has $\partial_C K_S(\tilde{S}) = \{0\}$ and any element in $\partial_C K_R(\tilde{S})$ is positive. Therefore condition (40) cannot be fulfilled. Let us show now that a singular arc with $\tilde{S} = S_w$ cannot be part of an optimal solution. If it is, let $I = [t_1, t_2]$ and define

$$\delta_I = \int_I u(t)dt := \int_I \tilde{u}_{S_w}(t)dt < t_2 - t_1.$$

From Corollary 4, one has $t_2 < T$. If $S(t) \le S_w$ for any $t > t_2$, one would have $B(T) = B(t_1)$. Consider then the control

$$\check{u}(t) = \begin{cases} u(t), & t \in [0, t_1), \\ 1, & t \in [t_1, t_1 + \delta_I), \\ 0, & t \in [t_1 + \delta_I, T], \end{cases}$$

and the associated solution $\check{S}(\cdot)$, $\check{B}(\cdot)$. This control would be admissible (i.e. $Q[\check{u}(\cdot)] \le Q[u(\cdot)] = \bar{Q}$), and one would have $\check{B}(T) > \check{B}(t_1 + \delta_I) > \check{B}(t_1) = B(T)$ (because $\check{S}(t) > S_w$ for $t \in [t_1, t_1 + \delta I]$). Then, the control $u(\cdot)$ would not be optimal. Therefore $S(\cdot)$ has to take values above $S_w$ in the time interval $[t_2, T]$ and from what precedes for a.e. $t > t_2$ with $S(t) > S_w$, either $\dot{S}(t) > 0$ (with $u(t) = 1$) or $\dot{S} < 0$ (with $u(t) = 0$) or $\dot{S}(t) = 0$ (with

a singular control). Then, from Corollary 4 we deduce that there exists a sub-interval $J = [\bar{t}_1, \bar{t}_2] \subset (t_2, T]$ with $\bar{t}_1 < \bar{t}_2$ such that $S(t) > S_w$ for any $t \in J$ and $u(t) = 0$ for a.e. $t \in J$.

Let us now consider a control with $t^\dagger \in I$, defined as follows

$$u^\dagger(t) = \begin{cases} 0 & \text{if } t \in [t_1, t^\dagger), \\ 1 & \text{if } t \in [t^\dagger, t_2], \\ u(t) & \text{otherwise.} \end{cases}$$

Denote by $S^\dagger$, $B^\dagger$ the corresponding solution. For $t^\dagger = t_1$, one has $S^\dagger(t_2) > S_w$ and for $t^\dagger = t_2$, $S^\dagger(t_2) < S_w$. By the intermediate value theorem, there exists $t^{\dagger_0} \in (t_1, t_2)$ such that $S^{\dagger_0}(t_2) = S_w$. Then, one has $S^{\dagger_0}(t) < S_w$ at any $t \in (t_1, t_2)$ and $S^{\dagger_0}(t) = S(t)$ for $t \in [0, T] \setminus (t_1, t_2)$. Moreover one has $B^{\dagger_0}(t) = B(t)$ for any $t \in [0, T]$ (due to $\dot{B}(t) = \dot{B}^{\dagger_0}(t) = 0$ at $t \in (t_1, t_2)$). As one has $S^{\dagger_0}(t_1) = S^{\dagger_0}(t_2) = S_w$, one gets from the integration of equation (1) on $I$ and $K_R$ increasing on $(S_h, S_w)$, the inequality

$$\int_I u^{\dagger_0}(t)dt = \frac{1}{k_2}\int_I (1-\varphi(t))K_R(S^{\dagger_0}(t))dt < \frac{1}{k_2}\int_I (1-\varphi(t))K_R(S_w)dt = \int_I u(t)dt.$$

Therefore, the control $u^{\dagger_0}(\cdot)$ gives the same cost $B^{\dagger_0}(T) = B(T)$ but with a lower water consumption i.e. $Q[u_0^\dagger(\cdot)] < Q[u(\cdot)] = \bar{Q}$. Let $\delta = \bar{Q} - Q[u_0^\dagger(\cdot)] > 0$ and consider the control

$$u^\#(t) = \begin{cases} u^{\dagger_0}(t), & t \in [0, \bar{t}_1) \cup [\min(\bar{t}_2, \bar{t}_1 + \delta), T], \\ 1, & t \in [\bar{t}_1, \min(\bar{t}_2, \bar{t}_1 + \delta)), \end{cases}$$

which satisfies also the constraint (12). Denote $S^\#(\cdot)$, $B^\#(\cdot)$ the solution with the control $u^\#(\cdot)$. One has $S^\#(\bar{t}_1) = S(\bar{t}_1)$ and $B^\#(\bar{t}_1) = B(\bar{t}_1)$ and then $S^\#(t) > S(t)$ for any $t > \bar{t}_1$, which implies $B^\#(t) > B(t)$ for any $t > \bar{t}_1$ (because $K_S(S^\#(t)) > K_S((t))$ for $t \in J$). Therefore, the control $u(\cdot)$ cannot optimal, which shows that $\tilde{S} = S_w$ cannot be a singular arc. □

### 5.3 The SMS strategy

For convenience, we denote sub-sets of corner points of $K_S$ or $K_R$ as below.

**Definition 5.** *For any $S \in (S_w, S^\star]$, let $\mathcal{C}(S)$ be the set of corner points $\tilde{S} \geq S$ in $(S_w, S^\star]$ and $n(S) = card\,\mathcal{C}(S)$. Define then the increasing sequence of corner points $\{\tilde{S}_i(S)\}_{i=1\cdots n(S)}$ such that $\mathcal{C}(S) = \{\tilde{S}_1(S), \cdots, \tilde{S}_{n(S)}(S)\}$.*

Note that for any $S \in (S_w, S^\star]$, the set $\mathcal{C}(S)$ is non empty. It contains at least $S^\star$ as the largest element, i.e. $\tilde{S}_{n(S)} = S^\star$. We define now the *saturated multiple shots* (SMS) strategy.

**Definition 6.** *For $S_m \in (S_h, S^\star]$ and a sequence of non decreasing numbers $V_i \in (0, \bar{V}]$, $i \in \{1, \cdots, n(S_m)\}$ with at least one equal to $\bar{V}$, define the time-varying feedback control*

$$\psi^{SMS}_{S_m,\{V_i\}}(t, S, V) := \begin{cases} 0 & \text{if } V = \bar{V} \text{ or } S > S_m \text{ with } V = 0, \\ \tilde{u}_S(t) & \text{if } S = \tilde{S}_i(S_m) \text{ for } i \in \{1, \cdots, n(S_m)\} \text{ with } V < V_i, \\ 1 & \text{otherwise.} \end{cases} \tag{41}$$

Note that this control strategy is admissible because it guarantees $V(T) \leq \bar{V}$. This strategy consists of starting the irrigation when the moisture $S$ reaches an *irrigation trigger threshold* $S_m$, with one or several stages. If the humidity rate $S(t)$ reaches a level $\tilde{S}_i(S_m)$ for some $i \in \{1, \cdots, n(S_m)\}$ the flow rate is saturated to maintain $S$ constant at this *level value* as long as the used volume $V(t)$ stays below the value $V_i$. This is what we call a "saturated shot". Note that if $V_i$ is too small or if $\tilde{S}_i(S_m)$ cannot be reached, there is no saturation of the flow rate for this value i.e. the trajectory does not present a step at this value. The generated trajectory has then at most $n(S_m)$ increasing saturated slots. This is why we call this feedback control a "saturated multiple shots" (see Section 6 for an illustration). Remark also that once $S(\cdot)$ has reached $S_m$ then $S(t) \leq S^\star$ for any future time. We now give our main result about the optimality of the SMS strategy.

**Theorem 1.** *Under Assumptions 1, 2, 3, 4 and Hypothesis 1, there exists a value $S_m \in (S_h, S^\star]$ and a sequence of non decreasing numbers $V_i$, $i = 1, \cdots, n(S_m)$ with at least one equal to $\bar{V}$ such that the SMS feedback (41) with*

$$\int_0^{t_M} \psi^{SMS}_{S_m,\{V_i\}}(t, S(t), V(t))\, dt = \bar{V} \quad \text{for some } t_M < T \tag{42}$$

*is optimal.*

*Proof.* Let $u(\cdot)$ be an optimal control, $S(\cdot)$, $V(\cdot)$ the associated solution of (28), (29) and $\lambda_S(\cdot)$, $\lambda_V$ the adjoint variables given by the Maximum Principle (see Section 5.1). Recall first that under Hypothesis 1, $S(\cdot)$ satisfies $S(t) \geq \underline{S}(t)$ for any $t \in [0,T]$ with $S(t) \leq S^\star$ for $t \in [\underline{t},T]$ (cf Proposition 2.ii).

As $u(t) = 0$ for a.e. $t \in [0,\underline{t}]$ (Proposition 2.i), the switching function $\phi(\cdot)$ has to be non-positive on $[0,\underline{t}]$, or equivalently one should have $\lambda_S(t) \leq \lambda_{S^\star} = -\lambda_v/(k_1k_2) > 0$ for $t \in [0,\underline{t}]$. On the interval $[0,\underline{t}]$, $S(\cdot)$ is thus decreasing with $S(\underline{t}) = S^\star$. As $K_R$ and $K_S$ are respectively increasing and constant on the interval $[S^\star,1]$, we deduce from (32) that $\lambda_S(t) < \lambda_{S^\star}$ for any $t \in [0,\underline{t})$.

Consider the set

$$C := \{t \in [\underline{t},T] \text{ s.t. } \lambda_S(t) \geq \lambda_{S^\star}\}$$

which is of non empty interior (otherwise $u(t) = 0$ would be optimal for any $t \in [\underline{t},T]$, which is not possible by Proposition 2.iii). Let us show that the set $C$ is a connected. If not, consider a time interval $(t_1,t_2)$ in-between two consecutive connected components of $C$, that is such that $\lambda_S(t_1) = \lambda_S(t_2) = \lambda_{S^\star}$ and $\lambda_S(t) < \lambda_{S^\star}$ for any $t \in (t_1,t_2)$. Note that $\lambda_S(t) < \lambda_{S^\star}$ for $t \in (t_1,t_2)$ implies that $u(t) = 0$ for almost any $t \in (t_1,t_2)$ and consequently $S(\cdot)$ is decreasing on $(t_1,t_2)$.

The function $\lambda_S$ attains necessarily its minimum on $(t_1,t_2)$, say at $\hat{t}$, and one has then

$$0 \in \varphi(\hat{t})(\lambda_S(\hat{t})k_1 - 1)\partial_C K_S(S(\hat{t})) + (1-\varphi(\hat{t}))\lambda_S(\hat{t}))k_1\partial_C K_R(S(\hat{t})). \tag{43}$$

If $S(\hat{t}) < S_w$, one has $\partial_C K_S(S(\hat{t})) = \{0\}$ and $\partial_C K_R(S(\hat{t})) \subset \mathbb{R}_+^\star$ (by Assumption 1 and Lemma 1), and thus (43) cannot be satisfied. So, one has necessarily $S(\hat{t}) \geq S_w$. Note that one has also $\lambda_S(\hat{t})k_1 - 1 < 0$.

As $S(\cdot)$ is decreasing on $(t_1,t_2)$, $K_S'(S(t))$ and $K_R'(S(t))$ exist for almost any $t \in (t_1,t_2)$. The function $K_S$, resp. $K_R$, being concave, resp. convex, on $(S_w,S^\star)$, and as $S(t) \in (S_w,S^\star)$ for any $t \in (t_1,\hat{t})$, one has the property

$$S(t) > S(\hat{t}) \;\Rightarrow\; \{\xi \geq K_S'(S(t)) > 0,\ \forall \xi \in \partial_C K_S(S(\hat{t})) \text{ and } K_R'(S(t)) \geq \zeta > 0,\ \forall \zeta \in \partial_C K_R(S(\hat{t})),\} \tag{44}$$

for almost any $t \in (t_1,\hat{t})$. Now, $\dot{\lambda}_S(t)$ exists for a.e. $t \in (t_1,\hat{t})$ and one can write

$$\begin{array}{rcl} \dot{\lambda}_S(t) & = & \varphi(t)(\lambda_S(t)k_1 - 1)K_S'(S(t)) + (1-\varphi(t))\lambda_S(t))k_1K_R'(S(t)) \\ & \geq & \varphi(t)(\lambda_S(\hat{t})k_1 - 1)K_S'(S(t)) + (1-\varphi(t))\lambda_S(\hat{t}))k_1K_R'(S(t)), \end{array}$$

for almost any $t \in (t_1,\hat{t})$. Finally, as the function $\varphi$ is increasing and positive, one obtains with (44) the inequality

$$\dot{\lambda}_S(t) \geq \varphi(\hat{t})(\lambda_S(\hat{t})k_1 - 1)\xi + (1-\varphi(\hat{t}))\lambda_S(\hat{t}))k_1\zeta, \quad \forall \xi \in \partial_C K_S(S(\hat{t})),\ \forall \zeta \in \partial_C K_R(S(\hat{t})),$$

for almost any $t \in (t_1,\hat{t})$. From (43), one gets $\dot{\lambda}_S(t) \geq 0$ for almost any $t \in (t_1,\hat{t})$, which contradicts $\lambda_S(\hat{t})$ being a minimum of $\lambda_S$ on $(t_1,t_2)$. The set $C$ is thus connected.

The set $C$ being non-empty connected and $\lambda_S(\cdot)$ continuous, $C$ is an closed interval $[t_m,t_M]$ and $u(t) = 0$ for a.e. $t \notin C$. From Proposition 2 and Lemma 1, we have that $t_m \geq \underline{t}$ and $S_m = S(t_m)$ belongs to $(S_h,S^\star]$. Moreover, from Corollary 4, we have $t_M < T$. Then, at any $t \in C$, the switching function $\phi(t)$ is non-negative. Therefore, for a.e. $t \in C$, either $u(t) = 1$ or $u(t) = \tilde{u}_{\tilde{S}}(t)$ with $\tilde{S} \in (S_w,S^\star)$ a corner point (Proposition 5)). Consequently, $V(\cdot)$ is increasing on $C$ and $S(\cdot)$ non decreasing on $C$, composed of increasing parts $u = 1$ and possibly singular ones $S = \tilde{S}_i$ with $\tilde{S}_i \in \mathcal{C}(S_m)$. As $V(\cdot)$ is increasing, the time $t_i$ when the solution $S(\cdot)$ leaves $S = \tilde{S}_i$ is equivalently defined by the value $V_i = V(t_i)$ reached by the variable $V(\cdot)$. If $\tilde{S}_j$ is not reached, we can set by convention $V_j = \bar{V}$.

At $t = t_M$, Proposition 2 gives $S(t_M) \leq S^\star$ and $V(t_M) = \bar{V}$. Then, either one has $S(t_M) \notin \mathcal{C}(S_m)$ and $S^\star = \tilde{S}_{n(S_m)}$ is not reached, or $S(t_M) = \tilde{S}_j \in \mathcal{C}(S_m)$ with $j \in \{1,\cdots,n(S_m)\}$ and $V_j = \bar{V}$. In any case there exists $i \in \{1,\cdots,n(S_m)\}$ such that $V_i = \bar{V}$. Finally, one can easily check that $u(\cdot)$ fulfills $u(t) = \psi^{SMS}_{S_m,\{V_i\}}(t,S(t),V(t))$ for a.e. $t \in [0,T]$, and property iii of Proposition 2 imposes that the $S_m$ and the $V_i$ are such that the equality (42) is satisfied. □

Let us underline that the structure of the feedback (41) does not depend on the shape of the radiation interception efficiency function $\varphi$, although the optimal switching times do depend on the values of this function.

### 5.4 The SOS strategy

When the functions $K_S$, $K_R$ have no corner on the interval $(S_w, S^\star)$, as it is the case for Assumption 1bis, the SMS strategy takes a simpler expression, that we define as a "saturated one slot" (SOS) strategy as follows.

**Definition 7.** *For $S_m \in (S_h, S^\star]$, define the time-varying feedback control*

$$\psi^{SOS}_{S_m}(t,S,V) := \begin{cases} 0 & \text{if } V = \bar{V} \text{ or } S > S_m \text{ with } V = 0, \\ \tilde{u}_{S^\star}(t) & \text{if } S = S^\star \text{ with } V < \bar{V}, \\ 1 & \text{otherwise.} \end{cases} \tag{45}$$

This strategy consists of irrigating crops at once when the humidity rate $S(t)$ gets equal to $S_m$. The water is delivered at the maximal flow rate ($u = 1$) as long as the humidity rate $S(\cdot)$ is below $S^\star$, or maintain it $S = S^\star$ (with the singular control $\tilde{u}_{S^\star}(\cdot)$), until the entire water budget $\bar{V}$ has been used up. This strategy is parameterized by the single value $S_m$ or equivalently the time when irrigation starts. This explains the wording "saturated one shot". One has the following result about the optimality of this strategy.

**Theorem 2.** *Under Assumptions 1bis, 2, 3, 4 and Hypothesis 1, there exists a value $S_m \in (S_h, S^\star]$ such that the SOS strategy (45) with*

$$\int_0^{t_M} \psi^{SOS}_{S_m}(t, S(t), V(t))\, dt = \bar{V} \quad \text{for some } t_M < T, \tag{46}$$

*is optimal.*

*Proof.* Under Assumption 1bis, one has $n(S) = 1$ for any $S \in (S_h, S^\star)$ with $V_1 = \bar{V}$. Therefore, the SMS feedback (41) has only one parameter $S_m$ which gives the simpler expression (45). □

## 6 Numerical simulations and discussion

In this section, we compare numerically three irrigation strategies on two examples: the SMS and SOS previously introduced and the "One Shot" (OS) strategy which consists of delivering water at maximum flow rate during a single irrigation period at a triggering humidity level $S_m$.

**Definition 8.** *For $S_m \in (S_h, S^\star]$, we denote by $t_S$ the irrigation triggering time associated to an humidity level $S_m$ ($t_S = \underline{S}^{-1}(S_m)$) and define the OS open-loop control as follows :*

$$u^{OS}_{S_m}(t) := \begin{cases} 0 & \text{if } t < t_S \text{ or } t > \min(t_S + \bar{Q}/F_{max}, T)), \\ 1 & \text{if } t \in [t_S, \min(t_S + \bar{Q}/F_{max}, T)). \end{cases}$$

The OS strategy is a pure "bang-bang" control that is implemented in open-loop (once the irrigation starts, it lasts $\bar{Q}/F_{max}$)). It represents a class of widely used irrigation strategies, typically when drip irrigation is not available. For this strategy, we shall look for the triggering humidity level $S_m$ that give the best biomass production, to be compared with the biomass obtained with the best SMS and SOS strategies.

The optimal SMS (and SOS) strategy can be numerically determined, characterizing the parameters $(S_m, V_1, \cdots, V_n)$ in the set

$$\mathcal{P} := (S_w, S^\star] \times \{V \in \mathbb{R}^n;\, 0 \le V_1 \le V_2 \le \cdots \le V_n = \bar{V}\}$$

where $n \ge 1$ is the number of distinct corner points $\tilde{S}$ of the functions $K_S$ or $K_R$ in the interval $(S_w, S^\star]$. For each $(S_m, V_1, \cdots, V_n)$ in $\mathcal{P}$, the SMS feedback (41) is well defined and generates an unique solution of equations (1)-(2), for which we associate the cost $B(T)$, even though the constraint (42) might be not satisfied. Indeed, when looking for values of the parameters in $\mathcal{P}$ that give the largest $B(T)$, we know that the optimal ones necessarily satisfy the equality (42), according to Proposition 2.

In order to compare different irrigation strategies by their triggering level, we also introduce the partially optimized control SMS$^*$, parameterized by a humidity level $S_m$, which corresponds to the best SMS strategy for a given triggering level $S_m$. This strategy can thus be obtained numerically by the approach presented above but optimizing with respect to the only parameters $(V_1, \cdots, V_n)$. This allows us the compare the crop productions provided by the three strategies SMS, SOS, OS for the same triggering level $S_m$. For each of them, their optimal production is then obtained for their best value $S_m$.

The performances of these strategies are compared for two kinds of configurations: one with the simplest $K_s$, $K_r$ functions that fulfill Assumption 1bis, (see Fig.1) and a second one when the function $K_s$ exhibits a more complex shape (see Fig. 4). For illustrative purposes only, we have considered dimensionless parameters (by normalizing the units), functions $\varphi$ in the family of $t \mapsto (t/T)^\alpha$ ($\alpha > 0$) and functions $f$ as logistic laws $B \mapsto rB(1 - \frac{B}{B_{max}})$ parameterized by $(r, B_{max}, B_0)$. The optimal solutions have been verified with the Bocop-HJB solver [2] that provided with a very good accuracy the same optimal trajectories, but in open-loop.

## 6.1 Under Assumption 1bis

We recall that under this assumption, the SMS and SOS strategies coincide. We present in Fig. 5 the simulations performed with irrigation strategies SOS and OS and with inputs data given in Table 1.

| $T$ | $k_1$ | $k_2$ | $S^\star$ | $S_w$ | $S_h$ | $F_{max}$ | $\bar{Q}$ | $\alpha$ | $r$ | $B_{max}$ | $B_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.1 | 5 | 0.7 | 0.4 | 0.2 | 1.2 | 0.1 | 4 | 25 | 1 | 0.0005 |

Table 1: Normalized parameters used for the simulations under Assumption 1bis

The best OS strategy was obtained for a triggering level $S_m = 0.338$ and produced a biomass $B(T) = 0.022$. The corresponding humidity dynamics are plotted in Fig. 5c. It can be seen that some values of $S$ are above $S^\star$. It can be therefore concluded from the application of Proposition 2 that an OS strategy cannot be optimal. This is further illustrated by applying the SOS strategies for the same input data. We find that the optimal SOS strategy gives a final biomass $B(T) = 0.0388$ which is 77% higher than what gives the best OS strategy. The associated control is a bang-singular-bang (see Fig. 5b).

## 6.2 Under Assumption 1 without Assumption 1bis

In this example, we consider that the function $K_s$ presents an additional corner point $S_c$ in between $S_w$ and $S^\star$ (see Fig. 4). We present in Fig. 6 the simulations performed with irrigation strategies SMS, SOS and OS and with

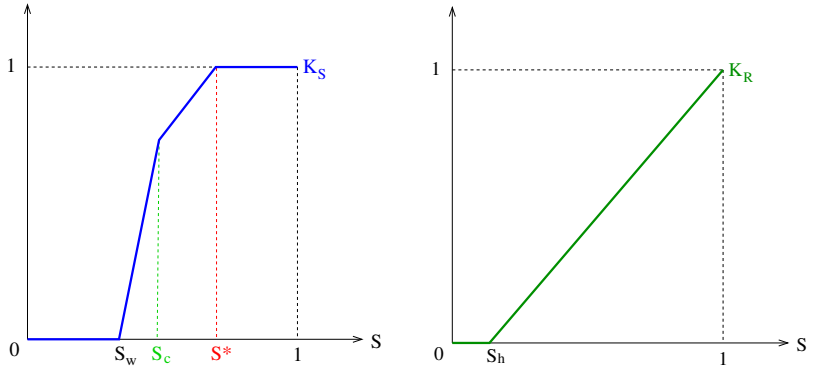



Figure 4: Graphs of the functions $K_S$ and $K_R$ that fulfill Assumption 1 but not Assumption 1bis

inputs data given in Table 2. The biomass levels obtained when optimizing these strategies were 0.111 for SMS, 0.094 for SOS and 0.052 for OS. The best production is obtained with the SMS strategy (as expected by Theorem 1) and corresponds to the highest triggering humidity level ($S = 0.43$) among the three tested strategies (Fig. 6a). This level corresponds to the earliest irrigation triggering time ($t_S = 0.392$, see Fig. 6b).

We have shown that the optimal solution may have a singular arc corresponding to intermediate corner points. When looking at the humidity profile in Fig. 6c, we can see that indeed two arcs do occur in this setting: one at $S = S_c$ and another one later at $S = S^\star$.

## 6.3 Discussion

The structure of SMS strategies differs from the structure of solutions obtained by the authors in [16] with a comparable methodological approach but different model, criterion and constraint: their optimal irrigation policy

| $T$ | $k_1$ | $k_2$ | $S^\star$ | $S_w$ | $S_h$ | $F_{max}$ | $\bar{Q}$ | $\alpha$ | $r$ | $B_{max}$ | $B_0$ | $S_c$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.5 | 5 | 0.7 | 0.4 | 0.2 | 1.2 | 0.09 | 3 | 25 | 1 | 0.0005 | 0.43 |

Table 2: Normalized parameters used for the simulations under Assumption 1.

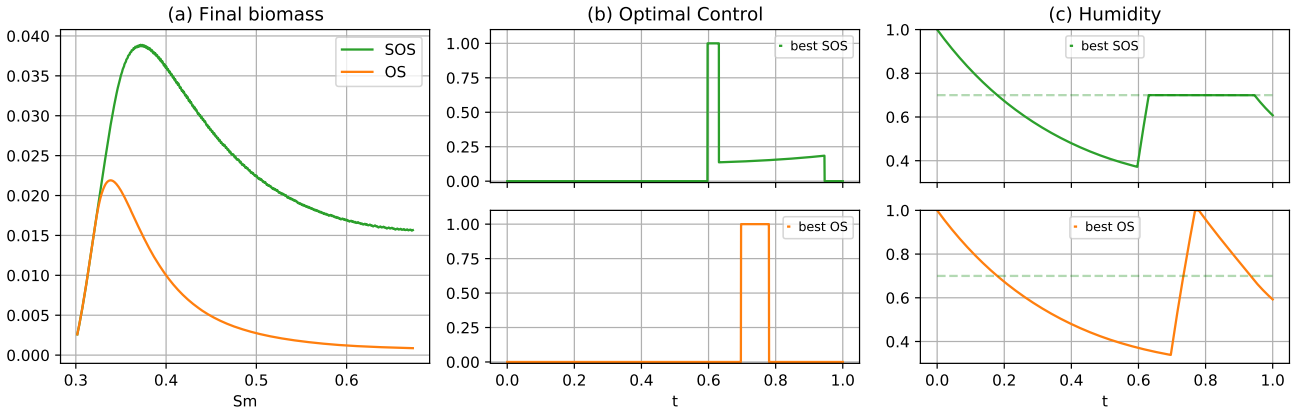



Figure 5: Comparison of OS and $SOS$ controls strategies with model parameters given in Table 1.

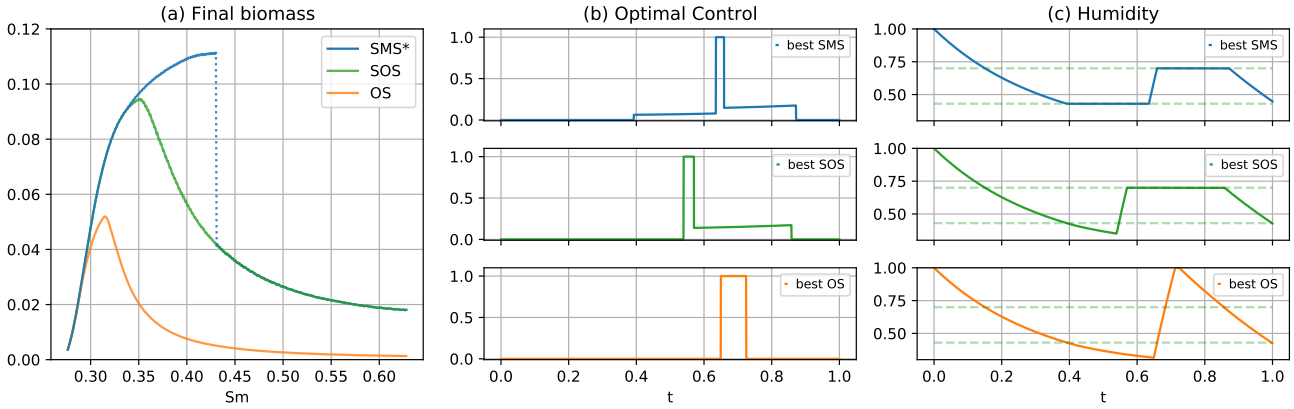



Figure 6: Comparison of the SMS control with OS and SOS controls with $K_s$ having three corner points. Model parameters used are given in Table 2.

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consists of bringing the soil moisture from its initial level to an optimal target value as fast as possible and maintaining it until harvesting time when irrigation ceases. In a different context (with rainfall inputs and no water quota), the authors of [12] study on a simple irrigation model the optimal control minimizing the total quantity of water to ensure the soil humidity to remain above a given threshold $S_{min}$. They found an optimal control policy whose structure is an SOS strategy with $\bar{S} = S_{min}$ but with a triggering level equal to $\bar{S}$. We believe that these differences between these models, their criterion, and their optimal solutions are of interest and make our mathematical analysis worth of interest.

Note finally that the practical implementation of an SMS (OR SOS) strategy on a real irrigation system requires an adaptive controller to maintain the humidity level constant at the values of the singular arcs (differently to the OS strategy), which needs the on-line measurement of the variable S. Moreover, differently to the OS strategy, it gradually changes the input flow rate during the singular arc phase. Therefore, the SMS (or SOS) strategy can be considered more "sophisticated" than the OS one, as it requires a humidity sensor for its concrete application.

# 7 Conclusion

We have introduced a simple crop irrigation model in order to study optimal irrigation scheduling using mathematical analysis. We have first shown, using a comparison tool, that the state constraint of this model is never activated for the optimal control problem solutions.

Moreover, we have demonstrated that, under water scarcity, an optimal trajectory has to reach as fast as possible the domain for which the relative humidity is below or equal to the threshold of maximal crop transpiration, and then to maintain it in this domain until the harvesting time. However, due to water scarcity, it has to be strictly below this threshold at some stage. We have then proved that the optimal strategy consists of irrigating once but with possible multiple steps ("SMS strategy") and not necessarily with a single shot ("OS strategy") as commonly used in practice. Moreover, we have shown that when the water-stress functions do not present any corner in between their extreme values, the SMS strategy has at most one step which is necessary at the maximal crop transpiration threshold (that we called "SOS strategy"). The SOS strategy is simpler to apply than the SMS one (with more than one step) as the soil moisture has simply to be maintained at the transpiration threshold until the water quota is reached. A remarkable feature is that the structure of the optimal strategy does not depend on the radiation interception efficiency function (although optimal trigger threshold and step values do rely on this function).

We have then compared the three control strategies: the open-loop one-shot (OS), commonly used in practice, the feedback saturated one-shot (SOS), that could exhibit a singular arc, and the more sophisticated feedback strategy with multiple increasing shots (SMS), that could exhibit several singular arcs. We have shown numerically the superiority of this last strategy. We guess that the SMS strategy (that coincides with the SOS one for simple water-stress functions) could also be the best one in real situations. This would be a promising result since SMS/SOS irrigation schemes are not such intuitive control strategies, that could also be tested on simulations of more detailed models. This shall be the matter of future work.

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# References

[1] Bertrand, N., Roux, S., Forey, O., Guinet M. and Wery, J., *Simulating plant water stress dynamics in a wide range of bi-specific agrosystems in a region using the BISWAT model*, European Journal of Agronomy, vol. 99, 116–128, 2018

[2] Bonnans, F., Giorgi, D., Heymann, B., Martinon, P. and Tissot, O. *Bocop HJB 1.0.1 - User Guide.* Technical Report INRIA 0467, 2015.

[3] Clarke, F., *Functional Analysis, Calculus of Variations and Optimal Control*, Springer 2013.

[4] Cheviron, B., and Vervoort, R. W., Albasha, R., Dairon, R., Le Priol, C. and Mailhol, J., *A framework to use crop models for multi-objective constrained optimization of irrigation strategies*, Environmental Modelling & Software, 86, 145–157, 2016.

[5] Faulwasser, T., Korda, M., Jones, C. and Bonvin, *On turnpike and dissipativity properties of continuous-time optimal control problems*, Automatica, 81, 297–304, 2017.

[6] Hartl, R.F. and Feichtinger, G., *A new sufficient condition for most rapid approach paths*, Journal of Optimization Theory and Applications, Vol; 54 (2), pp. 402–411, 1987.

[7] Hermes, H. and Lasalle, J.P. *Functional analysis and time optimal control*, Mathematics in Science and Engineering, Vol. 56, Elsevier, 1969.

[8] Hiriart-Urruty, J.B. and Lemaréchal, C. *Fundamentals of Convex Analysis*, Springer 2001.

[9] Kalboussi, N., Roux, S., Boumaza, K., Sinfort, C. and Rapaport, A., *About modeling and control strategies for scheduling crop irrigation.* IFAC Workshop on Control Methods for Water Resource Systems CMWRS, Delft (Netherlands), IFAC-PapersOnLine, Vol. 52(23), pp. 43-48, 2019.

[10] Kloss, S., Pushpalatha, R., Kamoyo, K. J., SchÃ¼tze, N., *Evaluation of crop models for simulating and optimizing deficit irrigation systems in arid and semi-arid countries under climate variability* Water resources management 26(4), pp. 997-1014, 2012.

[11] Li, Z., Cournède, P.H. and Mailhol, J.C., *Irrigation Optimization by Modeling of Plant-Soil Interaction*, 2011 Proceedings of the Conference on Applied Simulation and Modelling, 2011.

[12] Lopes, S. O., Fontes, F. A. C. C., Pereira, R. M. S., de Pinho, MdR., and Gonçalves A. M., *Optimal Control Applied to an Irrigation Planning Problem*, Mathematical Problems in Engineering, 2016.

[13] Miele, A., *Extremization of Linear Integrals by Green's Theorem*, Optimization Techniques, Edited by G. Leitmann, Academic Press, New York, New York, pp. 69-98, 1962.

[14] Pelak, N., Revellia, R. and Porporato, A. *A dynamical systems framework for crop models: Toward optimal fertilization and irrigation strategies under climatic variability*, Ecological Modelling, Vol. 365, pp. 80–92, 2017.

[15] Rapaport, R. and Cartigny, P., *Turnpike theorems by a value function approach*, ESAIM Control, Optimizationa and Calculus of variations, 10, 123–141, 2004.

[16] Shani, U, Tsur, Y and Zemel, A. *Optimal dynamics irrigation schemes*, Optimal Control, Applications and Methods, 25, 91–106, 2004.

[17] Schütze, N., Schmitz, G. H., *New planning tool for optimal climate change adaption strategies in irrigation*, Journal of Irrigation and Drainage Engineering, 136(12), pp. 836-846, 2010.

[18] Schütze, N., de Paly, M., and Shamir, U., *Novel simulation-based algorithms for optimal open-loop and closed-loop scheduling of deficit irrigation systems*, Journal of Hydroinformatics, Vol. 14(1), 136-151, 2011.

[19] Spence, M. and Starrett, D. *Most Rapid Approach Paths in Accumulation Problems*, International Economic Review, Vol. 16, pp. 388–403, 1975.

[20] Steduto, P., Hsiao, H., Raes D. and Fereres E., *AquaCrop, The FAO crop model to simulate yield response to water: I. Concepts and underlying principles*, Agronomy Journal, 101(3), 426–437, 2009.

[21] Trélat, E. and Zuazua, E., *The turnpike property in finite-dimensional nonlinear optimal control*, Journal of Differential Equations, 258, 81–114, 2015.

[22] Vico, G. and Porporato, A., *Probabilistic description of crop development and irrigation water requirements with stochastic rainfall*, Water Resources Research, 49, 3, 1466–1482, 2013.

[23] Allen, Richard G. and Pereira, Luis S. and Raes, Dirk and Smith, Martin et al., *Crop evapotranspiration - Guidelines for computing crop water requirements - FAO Irrigation and drainage paper 56*, Fao, Rome, 1998.

[24] Vinter, R. *Optimal Control*, Systems & Control: Foundations & Applications, Birkhäuser, 2000.

[25] Walter, W. *Ordinary differential equations*, Springer, 1998.