On a,b-edge-weightings of bipartite graphs with odd a,b

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On $\{a,b\}$-edge-weightings of bipartite graphs with odd a,b*

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Abstract

For any $S \subset \mathbb{Z}$ we say that a graph G has the S-property if there exists an S-edge-weighting $w : E(G) \to S$ such that for any pair of adjacent vertices u, v we have $\sum_{e \in E(v)} w(e) \neq \sum_{e \in E(u)} w(e)$, where $E(v)$ and $E(u)$ are the sets of edges incident to v and u, respectively. This work focuses on $\{a, a+2\}$-edge-weightings where $a \in \mathbb{Z}$ is odd. We show that a 2-connected bipartite graph has the $\{a, a+2\}$-property if and only if it is not a so-called odd multi-cactus. In the case of trees, we show that only one case is pathological. That is, we show that all trees have the $\{a, a+2\}$-property for odd $a \neq -1$, while there is an easy characterization of trees without the $\{-1, 1\}$-property.

1 Introduction

Let G be an undirected graph. For an S-edge-weighting $w : E(G) \to S$ of G, where $S \subset \mathbb{Z}$, each vertex $v \in V(G)$ has *weighted degree* equal to the sum of the weights of its incident edges. We call w *neighbour sum-distinguishing* if no two adjacent vertices of G have the same weighted degree. For a set S of weights, we say that G has the *S-property* if it admits neighbour sum-distinguishing S-edge-weightings. The study of graphs having or not having the S-property for some sets S is highly related to the well-known **1-2-3 Conjecture** raised by Karonski, Łuczak, and Thomason in 2004 [6]. That conjecture states that every connected graph different from K_2[1] has the $\{1, 2, 3\}$-property. A particular case of a list version of the 1-2-3 Conjecture (introduced by Bartnicki, Grytczuk, and Niwczyk [2]), even states that every graph should have the $\{a, b, c\}$-property for every distinct $a, b, c \in \mathbb{N}$. For more details on the progress towards the 1-2-3 Conjecture (and variants of it), please refer to [11] for a survey on this topic.

For any smaller set $S \subset \mathbb{Z}$ of weights, *i.e.*, with $|S| = 2$, one can easily come up with examples showing that there do exist graphs not having the S-property (complete graphs are such examples).

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[1]This requirement is mandatory for any graph to be weightable; throughout this work, it is thus implicit, unless stated otherwise, that every considered graph does not have K_2 as a connected component.

A natural question that has been investigated is about the existence of a good characterization of graphs that have the S-property for such smaller sets S. Here and further on, by a "good characterization" we mean a description in terms of a graph class whose members can be recognized in polynomial time. Dudek and Wajc [5] settled the question in the negative, as they proved that, unless P=NP, there is no good characterization of graphs with the $\{1,2\}$-property, and similarly for the $\{0,1\}$-property. Later on, noticing that, for any two distinct sets $S, S' \subset \mathbb{Z}$ of weights with $|S|, |S'| = 2$, any neighbour sum-distinguishing S-edge-weighting of a regular graph yields a neighbour sum-distinguishing S'-edge-weighting, Ahadi, Dehghan, and Sadeghi [1] proved that there is no good characterization of graphs with the $\{a,b\}$-property for any two distinct $a, b \in \mathbb{Z}$.

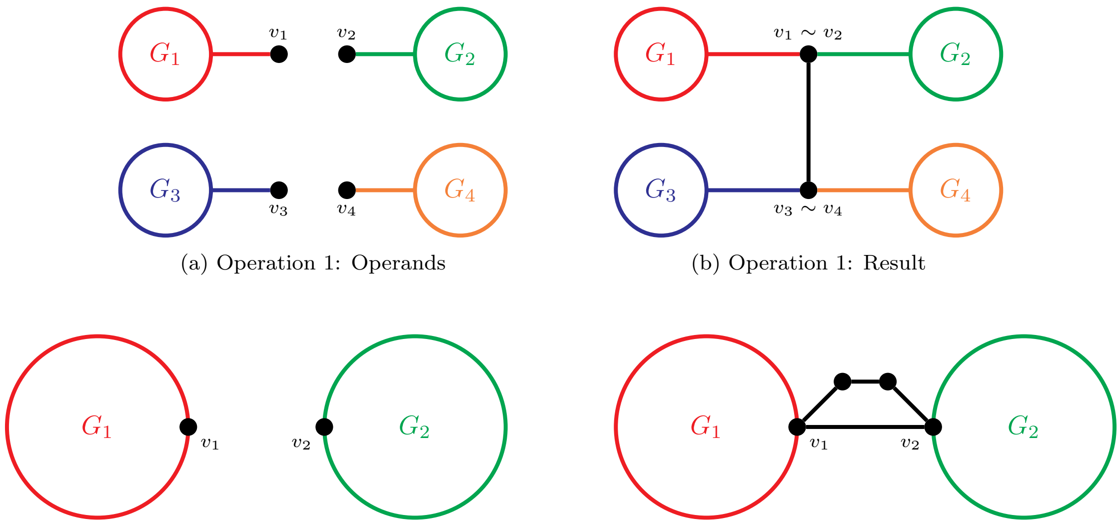
From this point on, it thus made sense investigating, for any two distinct $a, b \in \mathbb{Z}$, sufficient conditions for graphs to have the $\{a,b\}$-property. A special focus has been dedicated to bipartite graphs, as 1) the aforementioned NP-completeness results were not known to hold in the bipartite context, and 2) bipartite graphs form one of the rare graph classes for which the 1-2-3 Conjecture is relatively well understood (see [6]). As a first step, several works [3, 4, 7, 8, 9] investigated whether there is a good characterization of bipartite graphs with the $\{1,2\}$-property. Back then, it was believed that such a good characterization should exist, as, notably, all 3-connected bipartite graphs were proved to have the $\{1,2\}$-property [9]. It was not until quite recently that Thomassen, Wu, and Zhang proved that, indeed, bipartite graphs without the $\{1,2\}$-property are easy to describe [13]. Namely, only so-called **odd multi-cacti** are bipartite and do not have the $\{1,2\}$-property. These graphs are defined as follows (the comprehensive definition is from [10]; refer to Figure 2 later on for an illustration):

"Take a collection of cycles of length 2 modulo 4, each of which has edges coloured alternately red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a tree-like fashion along green edges; the resulting graph is an odd multi-cactus. The graph with one green edge and two vertices (K_2) is also an odd multi-cactus. When replacing a green edge of an odd multi-cactus by a green edge of any multiplicity, we again obtain an odd multi-cactus."

One main ingredient behind Thomassen *et al.*'s result is the nice observation, already made back in [3], that, when a and b are integers with distinct parity, every bipartite graph G with bipartition (X,Y) such that at least one of X and Y has even cardinality has the $\{a,b\}$-property. This is because, in such a case, one can easily construct $\{a,b\}$-edge-weightings of G where all vertices in X have odd weighted degree while those in Y have even weighted degree. These observations also imply that, for a and b with distinct parity, bipartite graphs without the $\{a,b\}$-property have their two partite sets of odd cardinality, and they thus have even order.

Reusing some of Thomassen *et al.*'s ideas, Lyngsie later considered the $\{0,1\}$-property for bipartite graphs [10]. His main result is a good characterization of 2-edge-connected bipartite graphs without the $\{0,1\}$-property, which turns out to be nothing but the class of odd multi-cacti. This result was established, in particular, through aforementioned tools and results for cases where a and b have different parities. However, both Thomassen *et al.* and Lyngsie observed that there exist infinitely many separable (*i.e.*, with cut-vertices) bipartite graphs without the $\{0,1\}$-property.

Although they are far from covering all the cases of a and b, the previous series of results show two things. First, that, when considering 2-connected bipartite graphs without the $\{a,b\}$-property, one should pay attention to odd multi-cacti. Second, that separable bipartite graphs without the



(a) Operation 1: Operands
(b) Operation 1: Result
(c) Operation 2: Operands
(d) Operation 2: Result

Figure 1: Constructing graphs without the $\{-1,1\}$-property from graphs without that property.

$\{a,b\}$-property and those without the $\{a',b'\}$-property may differ for different pairs a,b and a',b'. This is already well illustrated by the class of trees: while they all have the $\{1,2\}$-property [3], infinitely many of them do not have the $\{0,1\}$-property [10].

This paper is mainly devoted to studying $\{a,b\}$-properties where both a and b are odd. As a first step, we focus on the cases where $b = a+2$. We introduce mechanisms that are reminiscent of the ones mentioned above (for a and b with distinct parity), which allow us to study the $\{a,a+2\}$-property for bipartite graphs and odd $a \in \mathbb{Z}$. One of the main results we get from these is that, for any odd a, 2-connected bipartite graphs without the $\{a,a+2\}$-property are precisely odd multi-cacti again.

Theorem 1.1. *Let $a,b \in \mathbb{Z}$ be odd integers with $b = a+2$. A 2-connected bipartite graph G does not have the $\{a,b\}$-property if and only if G is an odd multi-cactus.*

Similarly as for the $\{0,1\}$-property, the structure of separable bipartite graphs without the $\{a,b\}$-property for odd a and b does not appear obvious. As a first step, we give a special focus on the case $\{a,b\} = \{-1,1\}$. In that case, we can already point out two operations that, given bipartite graphs without the $\{-1,1\}$-property, clearly provide more separable bipartite graphs without the $\{-1,1\}$-property (see Figure 1):

- Let G_1,G_2,G_3,G_4 be four bipartite graphs without the $\{-1,1\}$-property, and let v_1,v_2,v_3,v_4 be any four degree-1 vertices of G_1,G_2,G_3,G_4, respectively. The operation (see Figure 1 (a) and (b)) consists in considering the disjoint union $G_1 \cup G_2 \cup G_3 \cup G_4$, identifying the vertices v_1 and v_2, identifying the vertices v_3 and v_4, and adding an edge joining the two vertices resulting from these identifications (*i.e.*, $v_1 \sim v_2$ and $v_3 \sim v_4$).
- Let G_1,G_2 be two bipartite graphs without the $\{-1,1\}$-property, and let v_1,v_2 be any two vertices of G_1,G_2, respectively. The operation (see Figure 1, (c) and (d)) consists in consid-

ering the disjoint union $G_1 + G_2$, adding the edge v_1, v_2, and further joining v_1, v_2 by a path with odd length at least 3.

In the case of trees, when a and b are any two non-zero integers that are both positive (or negative), it is easy to see that K_2 is the only tree without the $\{a, b\}$-property: consider a vertex v whose all neighbours $u_1, ..., u_{d-1}$ but one u_d (if any) are leaves, remove $u_1, ..., u_{d-1}$, apply induction to deduce a neighbour sum-distinguishing $\{a, b\}$-edge-weighting, and extend the weighting to the edges $vu_1, ..., vu_{d-1}$ so that the conflict vu_d is avoided. Thus, when $b = a + 2$ and a, b are odd, only the case $a = -1$, $b = 1$ is potentially non-trivial. In Section 3, we show that trees without the $\{-1, 1\}$-property can all be constructed through the first operation above (illustrated in Figure 1, (a) and (b)) performed on K_2's.

Theorem 1.2. *A tree does not have the $\{-1, 1\}$-property if and only if it can be constructed from a disjoint union of K_2's through repeated applications of the first operation above.*

In particular, the structure of trees without the $\{-1, 1\}$-property is very different and simpler than that of trees without the $\{0, 1\}$-property (for more on the structure of these trees, see [10]). Recall that all trees have the $\{1, 2\}$-property, as was shown, e.g., in [3].

Terminology and notation. Let G be a connected graph. For a given vertex v of G we denote by $E(v)$ the set of edges incident to v. A *bridge* in G is an edge whose removal results in two components. Let w be an edge-weighting of G. Abusing the notation, the weighted degree of v in G by w will sometimes be denoted $w(v)$ for convenience. We say that an edge uv of G is a *conflict* by w if $w(u) = w(v)$. In other words, w is neighbour sum-distinguishing if no edge is a conflict. In what follows, we will instead use the term *proper* in place of neighbour sum-distinguishing to lighten the writing. By an *x-edge* of G (by w), we mean an edge assigned weight x by w.

2 Proof of Theorem 1.1

In this section, we prove that for every odd integer $a \in \mathbb{Z}$, the class of 2-connected bipartite graphs without the $\{a, a + 2\}$-property is exactly that of odd multi-cacti. Another way to define these graphs is as follows. Start from K_2, the simple connected graph on two vertices, having its only edge coloured green. Then, repeatedly apply an arbitrary number of the following operation (see Figure 2 for an illustration). Consider any green edge uv of the current graph, and join u, v by a new path P of length $\ell \geq 1$ congruent to 1 modulo 4 whose edges are coloured red and green as follows:

- if $\ell = 1$, *i.e.*, P has a unique edge, then this edge is green;
- if $\ell \geq 5$, then the edges of P are coloured red and green properly (*i.e.*, no two subsequent edges have the same colour) so that the two end-edges are red.

Figure 2 notably shows that performing this operation multiple times for a same green edge is allowed, and that adding paths of length 1 is similar to increasing the multiplicity of a green edge. Note also that it is not possible to get two adjacent green edges with distinct ends at any point of the process. Furthermore, every obtained graph is bipartite. An odd multi-cactus is any graph

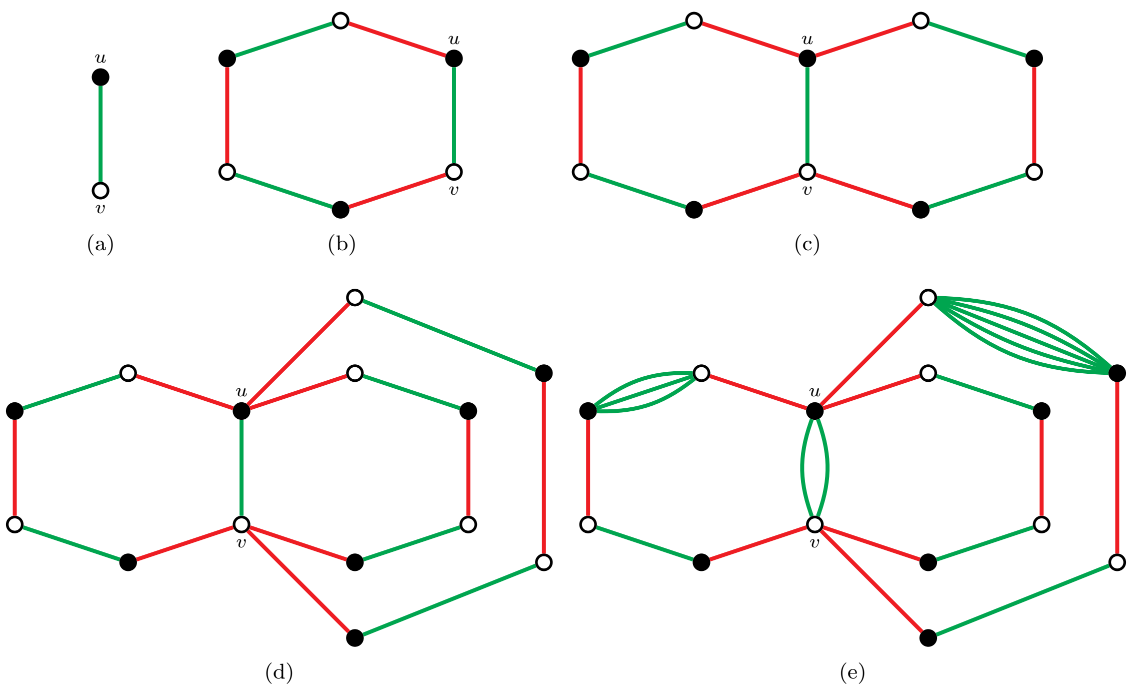


Figure 2: Constructing an odd multi-cactus through several steps, from K_2 (a). Red-green paths with length at least 5 congruent to 1 modulo 4 are being attached onto the green edge uv through steps (b) to (d). In step (e), (green) paths of length 1 are added, which corresponds to increasing the multiplicity of some green edges.

that can be obtained during this process, no matter how many times the operation is applied. In particular, K_2 itself is regarded as an odd multi-cactus.

In this section, we will implicitly use several properties of odd multi-cacti, such as:

Observation 2.1. *Let M be an odd multi-cactus with its edges being coloured red and green as described above. Then:*

- *M is 2-connected;*
- *when replacing every (green) edge of M by an edge with multiplicity 1, a 2-degenerate graph (i.e., a graph in which every subgraph has a vertex of degree at most 2) is obtained;*
- *for every green edge uv of M, we have $d_M(u) = d_M(v)$.*

Having the structure of odd multi-cacti in mind, it can be proved that the following holds true.

Lemma 2.2. *If G is not an odd multi-cactus and was obtained from an odd multi-cactus M by replacing a red edge with an edge of multiplicity at least 2 or by replacing a green edge by a path of length $k \geq 5$ with $k \equiv 1 \mod 4$, then G has the $\{a, b\}$-property for any two distinct integers $a, b \in \mathbb{Z}$.*

Proof. The proof is by induction on the order of M. So suppose that G is obtained from an odd multi-cactus M by replacing an edge e with either an edge of multiplicity at least 2 (if e is red in

M) or a path of length $k \geq 5$ with $k \equiv 1 \mod 4$ (if e is green in M). It is easy to check that the statement is true if M is just a cycle with some multiple green edges. So we can focus on cases where M was obtained in the general way, *i.e.*, by pasting together cycles with length 2 modulo 4 having possibly green edges of any multiplicity. Furthermore, it is easy to check that the statement is true if M was constructed by only pasting cycles together along one single green edge e' as in Figures 2 (c), (d) and (e) where there have only been pasted cycles together along the edge uv: in this case, since G is not an odd multi-cactus, G is either obtained from M by replacing a red edge with an edge of multiplicity at least 2, or $e = e'$ must be a simple edge and G is obtained by replacing that edge e with a path of length $k \geq 5$ with $k \equiv 1 \mod 4$ and $k \geq 5$. In both cases it is easy to check that G has the $\{a, b\}$-property.

Thus, we can assume that M was obtained by pasting together at least three cycles and that there are at least two disjoint edges to which other cycles have been pasted to. This implies that there are at least two disjoint cycles of length congruent to 2 modulo 4 where all vertices except two which are adjacent have exactly two neighbours. One of these cycles $C = v_1v_2...v_nv_1$ does not contain e. By possibly relabelling the vertices we can assume that all vertices of C except v_1 and v_2 only have two distinct neighbours. By induction the graph G' obtained from G by replacing the path $v_2v_3...v_n$ with an edge e'' has the $\{a, b\}$-property, but any proper $\{a, b\}$-edge-weighting of G' can be converted to a proper $\{a, b\}$-edge-weighting of G by assigning the same weight to an edge in G as in G' and assigning the weight assigned to e'' in G' to the edges v_2v_3 and $v_{n-1}v_n$, and finally assigning the weights of the remaining edges $v_2v_3, ..., v_{n-2}v_{n-1}$ in a way avoiding conflicts inside C. □

We now introduce or recall results that will be needed during the course of our main proof below. The following first observation is obvious and implies that studying the $\{a, b\}$-property only makes sense when $\gcd(a, b) = 1$.

Observation 2.3. *Let w be a proper $\{a, b\}$-edge-weighting of a graph G. If we multiply all edge weights of w by a non-zero integer α, then we get a proper $\{a\alpha, b\alpha\}$-edge-weighting of G.*

In what follows, given a graph G and a mapping $f : V(G) \to \mathbb{Z}_k$, by an *$f$-factor modulo k* we mean a spanning subgraph H of G such that, for every $v \in V(G)$, we have $d_H(v) \equiv f(v) \pmod k$.

Lemma 2.4 (Thomassen [12]). *Let G be a connected graph. If $f : V(G) \to \mathbb{Z}_2$ is a mapping satisfying $\sum_{v \in V(G)} f(v) \equiv 0 \mod 2$, then G contains an f-factor modulo 2.*

When dealing with bipartite graphs with a bipartition set of even cardinality, and when a and b have distinct parity, f-factors modulo 2 can be employed as a convenient tool to deduce proper $\{a, b\}$-edge-weightings in quite an easy way (see [10, 13]). More precisely, let (X, Y) be the bipartition of a bipartite graph G where $|X|$ is even. Lemma 2.4 (when applied onto the function f where $f(x) = 1$ for $x \in X$ and $f(y) = 0$ for $y \in Y$) implies that G has a spanning subgraph H where all of the vertices in X have odd degree, while all of the vertices in Y have even degree. From this, it is easy to see that, assuming a is odd and b is even, assigning weight a to all of the edges in $E(H)$ and weight b to all of the edges in $E(G) \setminus E(H)$ yields a proper $\{a, b\}$-edge-weighting of G.

The upcoming new tools and concepts (in particular that of *mod-4 vertex-colourings*) are the key to generalize this approach to odd $a, b \in \mathbb{Z}$ when $|a - b| = 2$.

Definition 2.5. *A mod-*4 *vertex-colouring of a graph* G *is a vertex-colouring* $c : V(G) \to \{1, 2\}$ *of* G *satisfying the following conditions for any* $uv \in E(G)$ *where* $d(u)$ *and* $d(v)$ *have the same parity:*

1. $d(u) \equiv d(v) \mod 4 \Rightarrow c(u) \neq c(v)$.
2. $d(u) \not\equiv d(v) \mod 4 \Rightarrow c(u) = c(v)$.

In the next result, we prove that every bipartite graph G admits a mod-4 vertex-colouring c. It is important to point out that, in general, c might be far from fitting with the bipartition of G. Actually, G might have many edges whose two ends have the same colour by c.

Lemma 2.6. *Every bipartite graph has a mod-*4 *vertex-colouring.*

Proof. It suffices to prove the lemma for connected bipartite graphs where all vertices have odd degree or where all vertices have even degree (as otherwise we can consider, still in the whole graph, the vertices with even degree first, and then those with odd degree). So let G be a connected bipartite graph where all vertex degrees have the same parity. Let v be a vertex in G and let $D_0, D_1, ..., D_m$ denote the distance classes of G from $v \in D_0$. Since G is bipartite, each D_i is an independent set. Now give v colour 1 and colour the distance classes in the given order starting with D_1, then D_2 and so on until we reach a vertex $v' \in D_{i'}$ we cannot assign a colour without violating conditions 1 or 2 in Definition 2.5. If this happens one or both of the following two cases have occurred:

1. there are two neighbours $v_1, v_2 \in D_{i'-1}$ of v' with $d(v_1) \equiv d(v_2) \mod 4$ and $c(v_1) \neq c(v_2)$;
2. there are two neighbours $v_1, v_2 \in D_{i'-1}$ of v' with $d(v_1) \not\equiv d(v_2) \mod 4$ and $c(v_1) = c(v_2)$.

Let us first assume that we are in the first case and let P_1, P_2 be two internally disjoint shortest paths towards v starting with $v'v_1$ and $v'v_2$ respectively and ending in a common vertex $v'' \in D_{i''}$. That is, v'' is the first vertex on both P_1 and P_2 that is encountered when going from v_1 towards v along P_1; possibly $v'' = v$. All the vertices of P_1 and P_2 except v' are coloured without violating conditions 1 and 2 in Definition 2.5, and P_1 and P_2 have the same length. The parity of the number of times the degree modulo 4 changes when walking from v'' to v_1 on P_1 is the same as the parity of the number of times the degree modulo 4 changes when walking from v'' to v_2 on P_2. Thus, the parity of the number of times the degree modulo 4 does not change when walking from v'' to v_1 on P_1 is the same as the parity of the number of times the degree modulo 4 does not change when walking from v'' to v_2 on P_2. Since conditions 1 and 2 in Definition 2.5 are not violated, this implies that the parity of the number of times the colour changes when walking from v'' towards v' is the same when walking along P_1 as when walking along P_2. Thus, $c(v_1) = c(v_2)$, a contradiction. The second case above can be dealt with in a similar way. □

Let $a, b \in \mathbb{Z}$ be two odd integers with $b = a + 2$. Let G be a graph and X, Y be two disjoint subsets of its vertices. By an (X, Y)*-a-parity* $\{a, b\}$*-edge-weighting* of G, we mean an $\{a, b\}$-edge-weighting where all vertices in X are incident to an odd number of a-edges and all vertices in Y are incident to an even number of a-edges. (X, Y)*-b-parity* $\{a, b\}$*-edge-weightings* are defined similarly, but with respect to the incident b-edges. In the following result, we establish a crucial connection between mod-4 vertex-colourings and (X, Y)-parity $\{a, b\}$-edge-weightings, leading to the existence of proper $\{a, b\}$-edge-weightings.

Lemma 2.7. *Let G be a connected bipartite graph and let $a, b \in \mathbb{Z}$ be odd integers with $b = a + 2$. If G has a mod-4 vertex-colouring where at least one of the two colour classes has even size, then G has the $\{a, b\}$-property. Consequently, if G does not have the $\{a, b\}$-property, then, in every mod-4 vertex-colouring, the two colour classes have odd size.*

Proof. Let G be a connected bipartite graph, and c a mod-4 vertex-colouring of G. We denote by X and Y the sets of vertices with colour 1 and 2, respectively. Assume $|X|$ is even. By Lemma 2.4 there is an $\{a, b\}$-edge-weighting $w : E(G) \to \{a, b\}$ such that all vertices in X are incident to an odd number of b-edges and all vertices in Y are incident to an even number of b-edges. This corresponds to our notion of an (X, Y)-b-parity $\{a, b\}$-edge-weighting. The possible weighted degrees of a vertex v of even degree and colour 1 induced by such an edge-weighting are $\{a(d(v) - 1) + b, a(d(v) - 1) + b + 4, a(d(v) - 1) + b + 8, ..., a + b(d(v) - 1)\}$ and the possible weighted degrees of a vertex v' of even degree and colour 2 induced by such an edge-weighting are $\{ad(v'), ad(v') + 4, ad(v') + 8, ..., bd(v')\}$. The possible weighted degrees of a vertex u of odd degree and colour 1 induced by such an edge-weighting are $\{a(d(u) - 1) + b, a(d(u) - 1) + b + 4, a(d(u) - 1) + b + 8, ..., bd(u)\}$ and the possible weighted degrees of a vertex u' of odd degree and colour 2 induced by such an edge-weighting are $\{ad(u'), ad(u') + 4, ad(u') + 8, ..., a + b(d(u') - 1)\}$. Let $xy \in E(G)$. We will show that $w(x) \neq w(y)$. To do this we distinguish two distinct cases (note that we can assume that x and y have the same degree parity, as otherwise $w(x)$ cannot be equal to $w(y)$):

1. x and y have the same colour by c.

2. x and y have distinct colours by c.

First assume that x and y have the same colour. Since c is a mod-4 vertex-colouring we have that $d(x) \not\equiv d(y) \mod 4$. Note that by the above it suffices to show that $ad(x) \not\equiv ad(y) \mod 4$ and this is trivially true since $\gcd(a, 4) = 1$. Now assume that x and y have distinct colours. Since c is a mod-4 vertex-colouring we have that $d(x) \equiv d(y) \mod 4$. Note that by the above it suffices to show that $ad(x) \equiv ad(y) \mod 4$, and as mentioned above this follows since $\gcd(a, 4) = 1$. □

From the previous proof, we can also extract the following:

Observation 2.8. *Let $a, b \in \mathbb{Z}$ be odd integers with $b = a + 2$ and let uv be an edge in a graph G whose edges are weighted with a and b. If either*

1. $d(u)$ and $d(v)$ have distinct parity, or

2. $d(u) \equiv d(v) \mod 4$ and v is incident to an odd number of a-edges while u is incident to an even number of a-edges, or

3. $d(u) \not\equiv d(v) \mod 4$ and both v and u are incident to an odd or even number of a-edges,

then u and v have distinct weighted degrees. This is also true if one considers the parity of the numbers of incident b-edges instead of the parity of the numbers of incident a-edges.

Let G be a graph and w an $\{a, b\}$-edge-weighting of G. By *swapping* (the weight of) an edge, we mean changing its weight to a if it is a b-edge, or changing its weight to b otherwise. By swapping a path or a cycle, we mean swapping all of its edges. For a vertex v in a cycle C of G, it can be observed that the parity of the number of a-edges (and similarly b-edges) incident to v is not altered upon swapping C. In the proof of our main result below, this fact will be used a lot to get rid of conflicts in the following way.

Let X, Y be the two colour classes of a mod-4 vertex-colouring of G and assume that, for some vertex $v \in X$, w is an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting, *i.e.*, all vertices in $X \setminus \{v\}$ are incident to an odd number of a-edges and all vertices in $Y\{v\}$ are incident to an even number of a-edges. According to Observation 2.8, all conflicts (if any) involve v. So let uv be a conflict. To get rid of this conflict while controlling the possible creation of new conflicts, we will swap particular cycles of G. Let C be a cycle of G going through u using two edges e, e' incident to u. If e, e' are assigned the same weight by w, then C is called *u-changing*. C will be called *v-avoiding* if it does not go through v.

Observation 2.9. *Let G be a graph, X, Y be the two colour classes of a mod-4 vertex-colouring of G, and let w be an $(X \setminus \{v\}, Y, \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting for some vertex v, where $a, b \in \mathbb{Z}$ are odd integers with $b = a + 2$. If uv is a conflict, then, by swapping a u-changing v-avoiding cycle C of G, we get rid of this conflict. Furthermore, any remaining/arising conflicts involve v.*

Proof. According to Observation 2.8, all original conflicts by w must involve v. When swapping C, the weighted degree of u is altered since C is u-changing, while the weighted degree of v is unaltered since C is v-avoiding. So we get rid of the conflict uv. Furthermore, it can be noticed that, upon swapping any cycle of G, the parities of the number of a-edges (and similarly b-edges) incident to the vertices are unaltered. Therefore, we get another $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting, and Observation 2.8 indicates that, after the swapping of C, all conflicts (if any) in the resulting $\{a, b\}$-edge-weighting must involve v. □

We finish with a few general lemmas to be used in particular cases of our upcoming main proof.

Lemma 2.10. *Let G be a 2-connected bipartite graph, X, Y be the two colour classes of a mod-4 vertex-colouring of G, and let $a, b \in \mathbb{Z}$ be odd integers with $b = a + 2$. If both X and Y have odd size and $v \in X$ is such that $G - v - N(v)$ is connected, then there is an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all edges incident to v have weight b and every vertex $u \in N(v)$ is incident to at most $1 + M(uv)$ b-edges, where $M(uv)$ denotes the multiplicity of the edge uv.*

Proof. Suppose $G' = G - v - N(v)$ is connected. Let G'' be obtained from $G - v$ by, for every vertex $u \in N(v)$, removing all edges but one incident to u in $G - v$. For each $u \in N(v)$, let e_u be the unique edge incident to u in G'' and let $n(u)$ denote the unique neighbour of u in G''. Note that since G' is connected, then so is G''. Let S denote the set of edges in G not incident to v and not in G''. That is, S is the set of edges removed from $G - v$ to obtain G''. Let $G[S]$ denote the subgraph of G induced by the edges in S and let Z denote the vertices of odd degree in $G[S]$. Clearly $|Z|$ is even, so, since $X \setminus \{v\}$ has even size, the set $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$ also has even size. Thus, Lemma 2.4 implies that there is an $(X', V(G'') \setminus X')$-$a$-parity $\{a, b\}$-edge-weighting of G''. We now extend this weighting to G by assigning weight a to all edges in S and weight b to all edges in $E(v)$; this results in a desired edge-weighting of G. □

Lemma 2.11. *Let G be a 2-connected bipartite graph. If there is a vertex $v \in V(G)$ of degree at least 4 and with $|N(v)| \geq 3$ such that $G - v - N(v)$ is connected, then G has the $\{-1, 1\}$-property.*

Proof. By Lemma 2.10, there is an $(X\setminus\{v\}, Y\cup\{v\})$-$(-1)$-parity $\{-1,1\}$-edge-weighting of G, where all edges incident to v have weight 1 and any vertex $u \in N(v)$ is incident to at most $1+M(uv)$ 1-edges, where $M(uv)$ denotes the multiplicity of the edge uv. Observation 2.8 implies that the only potential conflicts are between v and its neighbours. But the weighted degree of v is $d(v)$ and since $|N(v)| \geq 3$, we have for any $u \in N(u)$ that the multiplicity of uv is less than $d(v)-1$. Thus, the weighted degree of any $u \in N(v)$ is less than $d(v)$ and therefore, there can be no conflicts. □

Lemma 2.12 (Thomassen, Wu, Zhang [13])**.** *Let q be a natural number such that $q \geq 4$. Let G be a connected graph and let A be an independent set of at most q vertices such that each vertex in A has degree at least $q-1$, or, each vertex in A, except possibly one, has degree at least q. Assume that no vertex in A is adjacent to a bridge in G. Then, for each vertex a of A, there is an edge e_a incident with a such that the deletion of all e_a, $a \in A$, results in a connected graph unless $|A| = q = 4$, all vertices of A have degree 3, and $G - A$ has six components each of which is joined to two distinct vertices of A.*

Let a and b be two odd integers with $b = a+2$. In some cases Lemmas 2.4 and 2.12 work well together when trying to construct a proper $\{a,b\}$-edge-weighting of a connected bipartite graph G: suppose $c : V(G) \to \{1,2\}$ is a mod-4 vertex-colouring of G and let X and Y denote the sets of vertices in G of colour 1 and 2 respectively and assume that both X and Y have odd size. Furthermore, suppose that the degree of a vertex $v \in X$ is at least 4 and no vertex in $N(v)$ has degree strictly larger than 4. Let A be the vertices in $N(v)$ with the same degree as v and suppose that no vertex in A is incident to a bridge in $G - v$, the graph $G - v$ is connected, and we are not in the exceptional case of Lemma 2.10, that is, for each $u \in A$ there is an edge e_u such that $G - \cup_{u\in A}\{e_u\}$ is connected. Define $S = \cup_{u\in A}\{e_u\}$ and let Z denote the set of vertices in G which have odd degree in the subgraph of G induced by S (note that $A \subset Z$). Since $X \setminus \{v\}$ and Z have even size, the set $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$ also has even size. Thus, Lemma 2.4 implies that there is an $(X', V(G-v) \setminus X')$-$a$-parity $\{a,b\}$-edge-weighting of $G - v$. We can extend this edge-weighting to G by assigning weight a to all edges in S and weight b to all edges in $E(v)$ to obtain an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a,b\}$-edge-weighting of G where all edges incident to v have weight b and every vertex $u \in N(v)$ is incident to at least one a-edge. Thus, the weighted degree of v is greater than that of its neighbours and Observation 2.8 implies that the edge-weighting is proper.

We are now ready to prove the main result of this section. Let us emphasize that the main steps in the proof follow the lines of those in the proofs of the main results in [10] and [13]. In particular, Claims 2, 3, 4, 5, as stated below, can also be found in [10]. The proofs are rather different, though, as most arguments used to deal with the $\{0,1\}$-property do not apply immediately for the $\{a,b\}$-property when $a, b \in \mathbb{Z}$ are odd and $b = a+2$. Instead, some of the tools and results we have introduced earlier are used. Also, the end of the proof in our case is more straightforward than those for the $\{1,2\}$- and $\{0,1\}$-properties.

Proof of Theorem 1.1. Suppose the theorem is false and, for some odd $a \in \mathbb{Z}$ and $b = a+2$, let G be a counterexample which has smallest order possible. By possibly multiplying the weights by -1 we can assume $b > 0$. Let c be a mod-4 vertex-colouring of G (such a colouring exists by Lemma

2.6), and let X denote the set of vertices with colour 1 and Y denote the set of vertices with colour 2. By Lemma 2.7, we can assume that both X and Y have odd size.

Claim 1. *G has no multiple edge uv where both u and v have only two distinct neighbours.*

Proof of the claim. Suppose uv is a multiple edge and both u and v have only two distinct neighbours. By the minimality of G, Lemma 2.2, and the fact that G is not an odd multi-cactus, the graph obtained from G by replacing uv with one non-multiple edge has a proper $\{a, b\}$-edge-weighting w. But since the multiplicity of uv in G is at least 2 and since u and v can each be in only one conflict distinct from uv, we can obtain a proper $\{a, b\}$-edge-weighting of G from w by weighting the edges joining u and v to avoid the conflicts involving u and v (such an edge-weighting exists because there are at least three possible sums for the weight of edges joining u and v). ◇

Since G is 2-connected, it has minimum degree at least 2. In what follows, by a *suspended path* of G, we mean a path $v_1x_1...x_kv_2$ where all internal vertices $x_1, ..., x_k$ have degree 2 and v_1 and v_2 have degree at least 3.

Claim 2. *G has no suspended path of length 2.*

Proof of the claim. Suppose the claim is false and let v_1xv_2 be a suspended path in G, where $d(x) = 2$ and $d(v_1), d(v_2) \geq 3$. We can assume $x \in X$. Define $G' = G - x$. Recall that G' is connected since G is 2-connected. Lemma 2.4 implies that there is an $(X \setminus \{x\}, Y \cup \{x\})$-$a$-parity $\{a, b\}$-edge-weighting w of G, where $w(v_1x) = w(v_2x) = a$. More precisely, w can be obtained as follows (recall that $|V(G') \cap X|$ is even):

- If $v_1, v_2 \in X$, then, according to Lemma 2.4, there is an $(X \setminus \{v_1, v_2, x\}, Y \cup \{v_1, v_2\})$-$a$-parity $\{a, b\}$-edge-weighting of G'. Then, assigning weight a to v_1x and v_2x gives the desired weighting of G.

- If $v_1, v_2 \in Y$, then the same conclusion can be reached when applying Lemma 2.4 so that we start from an $(X - \{x\} \cup \{v_1, v_2\}, Y \setminus \{v_1, v_2\})$-$a$-parity $\{a, b\}$-edge-weighting of G'.

- If $v_1 \in X$ and $v_2 \in Y$ (resp. $v_2 \in X$ and $v_1 \in Y$), then, again, we can get the same conclusion after applying Lemma 2.4 from an $(X \setminus \{v_1, x\} \cup \{v_2\}, Y \cup \{v_1\} \setminus \{v_2\})$-$a$-parity $\{a, b\}$-edge-weighting (resp. $(X \setminus \{v_2\} \cup \{v_1\}, Y \cup \{v_2\} \setminus \{v_1\})$-$a$-parity $\{a, b\}$-edge-weighting) of G'.

Observation 2.8 implies that the only conflicts that can arise are xv_1 and xv_2. So we can assume that x and v_1 are in conflict, *i.e.*, both x and v_1 have weighted degree $2a$. This implies that v_1 has even degree at least 4 (by the definition of a suspended path, and only vertices with degree of the same parity can be in conflict) and $a < 0 < b$ and hence $a = -1$ and $b = 1$. Let u_1, u_2 be two neighbours of v_1 in G' such that u_1v_1 and u_2v_1 have weight -1 and let C be a cycle in G' using the edges u_1v_1 and v_1u_2. Such a cycle exists as, because G is 2-connected, there is a path from u_1 to u_2 in $G - v_1$. Because C is v_1-changing and x-avoiding, if we swap all weights on C we do not create new conflicts in G' and we lose the conflict xv_1 (recall Observation 2.9). In particular, x remains of weighted degree -2 while v_1 becomes of weighted degree 2.

Thus, we can now assume that xv_2 is a conflict. This implies that v_2 also has even degree at least 4. We can now get rid of this conflict in the same way as we got rid of the conflict v_1x, unless all

v_2-changing cycles in G' (that thus use two edges in $E(v_2)$ having the same weight) all use two edges in $E(v_1)$ both having weight 1 (in this case we only move the conflict from xv_2 to xv_1). Since xv_2 is a conflict, v_2 must be incident to at least two -1-edges in G' and at least one 1-edge. Furthermore, as mentioned above, we can assume that any v_2-changing cycle in G' contains two edges incident to v_1 having weight 1.

Assume that v_1 is incident to a -1-edge e in G'. Since G is 2-connected, there is, in $G - v_1$, a path from v_2 to the end of e different from v_1. From the existence of that path, we get that there is a path P in G' from v_1 to v_2 using e. If the weight on the last edge e' of P (the one incident to v_2) is 1, then swapping the weights on the cycle $P \cup v_1x \cup xv_2$ yields a proper edge-weighting; so we can assume e' has weight -1. Since xv_2 is a conflict and v_2 has even degree at least 4, vertex v_2 must be incident to a -1-edge $e'' \neq e'$ in G'. Now, because G is 2-connected, the graph $G - v_2$ has a path P' joining the end of e' different from v_2 and the end of e'' different from v_2. Note that if P' does not contain v_1, then we would get a cycle whose weights can be swapped to immediately deduce a proper edge-weighting of G. The same conclusion holds if P' and P intersect for the first time on a vertex different from v_1. So v_1 is the first intersection point between P and P', in which case we deduce a cycle of G containing v_2 as well as all of e, e', e'' (first go from v_2 to v_1 along P', before going back to v_2 along P); when swapping the weights along that cycle, we get rid of all conflicts between x and v_1, v_2.

We are left with the case where v_1 is not incident to a -1-edge e in G'. Thus, we deduce that v_1 has degree 4 and, by symmetry, v_2 also has degree 4. Note that this implies that $v_1, v_2 \in X$ and furthermore, that all four edges incident to v_1 except v_1x have weight 1. Also, according to all hypotheses so far, v_1 has weighted degree 2, v_2 has weighted degree -2, and so two edges incident to v_2 in G' have weight -1 while the last edge incident to v_2 in G' has weight 1. We now consider the graph $G'' = G' - v_1 - v_2$. If G'' is connected, then we can find in G' a cycle including the two -1-edges incident to v_2 (thus v_2-changing), and not passing through v_1; then, as earlier, we can swap the weights along this cycle to get rid of the conflict v_2x. So we may assume the two -1-edges incident to v_2 in G' are incident to two distinct components of G''. This leaves us with the following three cases to consider:

Case 1: *G'' has two components C_1, C_2, such that v_1 has two neighbours in C_1 and one neighbour in C_2, and v_2 has two neighbours in C_2 and one neighbour in C_1.*

Let $e_{1,a}$ and $e_{1,b}$ denote the two edges incident to v_1 going to C_1. Recall that $e_{1,a}, e_{1,b}$ have weight 1. Since C_1 is connected, there is a path from the end of $e_{1,a}$ different from v_1 to the end of $e_{1,b}$ different from v_1. We swap all weights along the cycle formed by this path and $e_{1,a}, e_{1,b}$ to get another edge-weighting of G where v_1 has weighted degree -2; so, now, both xv_1 and xv_2 are conflicts.

Now let $e_{1,c}$ denote the edge incident to v_1 going to C_2, and $e_{2,a}$ denote the 1-edge incident to v_2 going to C_2. Both these edges are weighted 1. Since C_2 is connected, there is a path P from the end of $e_{1,c}$ different from v_1 to the end of $e_{2,a}$ different from v_2. Now consider the cycle of G starting in x, going through xv_2 and $e_{2,a}$, then going along P, and finally going through $e_{1,c}$ and v_1x. When swapping all weights along this cycle, note that v_1, v_2 remain of weighted degree -2, while x becomes of weighted degree 2. So the $\{-1, 1\}$-edge-weighting of G becomes proper according to Observation 2.9.

Case 2: *G'' has two components C_1, C_2, such that both v_1 and v_2 have two neighbours in C_1 and one neighbour in C_2.*

Let $v_{1,a}, v_{1,b}$ denote the two neighbours of v_1 in C_1, and let $v_{1,c}$ denote the neighbour of v_1 in C_2. Note that for one of $v_{1,a}, v_{1,b}$, say $v_{1,a}$, the graph $G''' = G - v_1 - v_{1,a} - v_{1,c}$ is connected, for if G' is disconnected, then it must be the case that $v_{1,a}$ is a cut-vertex in $G - v_1$ and then it is easy to see that $G - v_1 - v_{1,b} - v_{1,c}$ is connected, and we can just rename $v_{1,a}$ and $v_{1,b}$ accordingly. Note that, by Lemma 2.11, we can assume that $G''' - v_{1,b}$ is disconnected. Let $L_1, ..., L_n$ denote the components of $G''' - v_{1,b}$, where $v_2 \in V(L_1)$. Since $v_{1,b}$ is not a cut-vertex in G, it follows that $v_{1,a}$ has a neighbour in each of the components L_i for $i \geq 2$.

Let us now consider the graph obtained from G by removing the vertex v_1, and, for each of $v_{1,a}, v_{1,c}$, removing all remaining incident edges but one. Note that, in that graph, x also has degree 1. Lemma 2.4 implies that this graph has an $(X \setminus \{v_1\}, Y)$-(-1)-parity $\{-1, 1\}$-edge-weighting. By then assigning weight -1 to all removed edges incident to v_1 and weight 1 to all remaining edges (incident to one of $v_{1,a}, v_{1,c}$), we get that G has an $(X \setminus \{v_1\}, Y)$-(-1)-parity $\{-1, 1\}$-edge-weighting where all four edges incident to v_1 are weighted -1, and each of $v_{1,a}, v_{1,c}$ is incident to at most two -1-edges. Now the only possible conflict is $v_1v_{1,b}$, so we can assume this is indeed a conflict, and hence, both v_1 and $v_{1,b}$ have weighted degree -4. We can also assume that we cannot swap the weights in a cycle in G''' containing two edges incident to $v_{1,b}$ having the same weight, and hence, $v_{1,b}$ has at most two neighbours in each of L_i for $i = 1, ..., n$. Since the weighted degree of $v_{1,b}$ is -4, there must be some components in $G''' - v_{1,b}$ which are incident to strictly more -1-edges in $E(v_{1,b})$ than 1-edges in $E(v_{1,b})$. Again, since we can assume that there is no cycle in G''' containing two edges incident to $v_{1,b}$ having the same weight, we can also deduce that no two edges incident to $v_{1,b}$ having the same weight go to the same component in $G''' - v_{1,b}$. Thus, there are at least three components L'_1, L'_2, L'_3 in $G''' - v_{1,b}$, each of which is incident to only one edge in $E(v_{1,b})$ and each of these edges has weight -1. We can assume that L'_1 and L'_2 are distinct from L_1, and, since $v_{1,b}$ is not a cut-vertex in G, the vertex $v_{1,a}$ has a neighbour u'_i in each L'_i for $i = 1, 2$. There is now a cycle C in $G''' + v_{1,a}$ containing two edges incident to $v_{1,b}$ having weight -1 and containing the two edges $v_{1,a}u'_1$ and $v_{1,a}u'_2$. If we swap the weights on C, then the only possible conflict is $v_1v_{1,a}$ in the case where $v_{1,a}$ is a vertex of degree 4 and $v_1v_{1,a}$ has weight -1, both $v_{1,a}u'_1$ and $v_{1,a}u'_2$ have weight 1, and $v_{1,a}$ is incident to some fourth -1-edge $v_{1,a}u'$. We can assume that the component L' to which u' belongs in $G''' - v_{1,b}$ is not incident to a -1-edge of $v_{1,b}$, since otherwise, we could modify C to contain the edge $v_{1,a}u'$. Note that this also implies that $L'_3 = L_1$. Since $v_{1,b}$ had weighted degree -4 this implies that there is another component L'_4 distinct from all of L'_1, L'_2, L'_3, which is incident to an edge in $E(v_{1,b})$ having weight -1. The vertex $v_{1,a}$ must have a neighbour u'' in this component L'_4 and, since we can assume that we cannot modify C to contain $v_{1,a}u'$, we must have $u'' \neq u'$. This contradicts $v_{1,a}$ having degree 4.

Case 3: *G'' has three components C_1, C_2, C_3, such that v_1 and v_2 have one neighbour in each of these three components.*

In that case, G has the $\{-1, 1\}$-property according to Lemma 2.11 as v_1 has even degree 4, and it can be checked that $G - v_1 - N_G(v_1)$ remains connected due to the 2-connectedness of G. In particular, for $i = 1, 2, 3$, note that the edge incident to v_1 going to C_i and the edge incident to v_2 going to C_i cannot share an end. A contradiction. $\diamond$

Claim 3. *G has no suspended path of length 4.*

Proof of the claim. Suppose the claim is false and let $v_1x_1x_2x_3v_2$ be a suspended path in G, where $d(x_1) = d(x_2) = d(x_3) = 2$ and $d(v_1), d(v_2) \geq 3$. We can assume $x_2 \in X$, which implies that $x_1, x_3 \in Y$. Define $G' = G - x_1 - x_2 - x_3$. Using Lemma 2.4 similarly as in the proof of Claim 2, we can come up with an $(X \setminus \{x_1\}, Y \setminus \{x_2, x_3\})$-$a$-parity $\{a, b\}$-edge-weighting w of G where $w(v_1x_1) = w(x_3v_2) = a$ and $w(x_1x_2) = w(x_2x_3) = b$. One can check that, by slightly modifying the exact same arguments used in the proof of Claim 2, we can eventually remove all conflicts from w, or deduce another proper $\{a, b\}$-edge-weighting of G. ◇

Claim 4. *G has no suspended path of length at least 5.*

Proof of the claim. Suppose the claim is false and let $v_1x_1x_2x_3x_4v_2$ be a path in G, where x_1, x_2, x_3, x_4 all have degree 2, and v_1, v_2 here might be of degree 2. Let G' be obtained from G by replacing $v_1x_1x_2x_3x_4v_2$ by an edge $e = v_1v_2$ even if that edge is already there. If G' has the $\{a, b\}$-property, then so does G. Indeed, assume there is a proper $\{a, b\}$-edge-weighting of G where the weight of e is, say a, and consider that weighting back in G. We start the extension to the five edges by assigning weight a to v_1x_1 and x_4v_2, so that v_1 and v_2 keep the same weighted degree as in G'. Since v_1v_2 is an edge in G', note that v_1 and v_2 have different weighted degrees. From this, we deduce that either v_1 has weighted degree different from $2a$ and v_2 has weighted degree different from $a + b$, or conversely. Assume the first situation holds. Then, we can achieve the weighting of G by assigning weight a to x_1x_2 and weight b to x_2x_3 and x_3x_4.

So we can assume that G' does not have the $\{a, b\}$-property and is thus an odd multi-cactus by the minimality of G. The edge e cannot be red in G', since then G would also be an odd multi-cactus. Thus e is green and Lemma 2.2 implies that G has the $\{a, b\}$-property. ◇

By Claims 1, 2, 3, 4, all degree-2 vertices in G (if any) lie on suspended paths of length 3. In G we replace all suspended paths of length 3 by edges (even if the two ends were already adjacent) to form a bipartite multigraph G^*. Edges arising from suspended paths of length 3, we call *blue edges*. Every other edge of G^*, *i.e.*, which was already present in G, we call a *white edge*.

Note that G^* is bipartite, 2-connected, has minimum degree at least 3, and it may have more multiple edges than G has. Also, note that for every vertex v in G^*, we have $d_{G^*}(v) = d_G(v)$. In general, it is not easy to deduce a proper $\{a, b\}$-edge-weighting of G from one of G^* (typically because of blue edges); however, information on the structure of G can be deduced from that of G^*. In particular, we will study the existence of paths or cycles in G^* to deduce that of corresponding paths or cycles in G (where any traversed blue edge in G^* is replaced by the corresponding path of length 3 in G).

If the deletion of some pair of adjacent vertices u, v disconnects G^*, then let $z_0y_0 \in E(G^*)$ be such that $G - z_0 - y_0$ is disconnected and such that some component H of $G^* - z_0 - y_0$ has smallest possible order. The union of that component H and z_0, y_0 together with all edges connecting them is denoted B. In case G has no pair of adjacent vertices whose removal disconnects the graph, we define $H = B = G^*$, and y_0, z_0 do not exist.

Claim 5. *For every vertex v of H, we have $d_{G^*}(v) = 3$.*

Proof of the claim. Suppose the claim is false and let w_0 be a vertex in H of maximum degree $d = d(w_0)$ at least 4. Without loss of generality, we can suppose that $w_0 \in X$. Assume first that w_0 is adjacent to none of z_0, y_0 (this is the case if these two vertices do not exist). By the remark following Lemma 2.12, we can assume that we are in the exceptional case when considering $G - w_0$ and defining A to be the set of vertices in $N(v)$ which have the same degree as w_0. Thus, w_0 and all vertices in $N(w_0)$ have degree 4 and $G - w_0 - N(w_0)$ has exactly six components. We can now choose another vertex of degree d as w_0 by choosing w_0 such that the order of the component of $G - w_0 - N(w_0)$ containing z_0 is maximum and avoid the exceptional case in Lemma 2.12. Again, the remark following Lemma 2.12 shows how to find a proper $\{a, b\}$-edge-weighting of G.

So we can assume w_0 is adjacent to, say, z_0 and that all the neighbours of w_0 in H which have the same degree as w_0 are adjacent to y_0 (due to the bipartiteness of G^*). Since y_0, z_0 thus exist, we have $G \neq H$. We can also assume that all vertices in H having maximum degree are adjacent to z_0 or y_0 (as otherwise, the previous situation would apply). Note that this implies that we can never be in the exceptional case in Lemma 2.12 when we delete a vertex v in H of maximum degree and define A to be the neighbours of v with the same degree as v.

As pointed out earlier, there is an $(X \setminus \{w_0\}, Y)$-a-parity $\{a, b\}$-edge-weighting of G where all edges incident to w_0 are weighted b and every neighbour of w_0 with degree d is incident to at least one a-edge. This edge-weighting is proper unless z_0w_0 is a conflict, which occurs only if the degree of z_0 is strictly greater than that of w_0. Note that we can assume that z_0 is incident to exactly one edge going to each component other than H in $G - y_0 - z_0$, since otherwise, we could deduce an $\{a, b\}$-edge-weighting of G as above with the extra condition that two edges e, e' incident to z_0 going to a component C of $G - y_0 - z_0$ other than H are weighted b. Then, if z_0w_0 is a conflict, we could get rid of it by swapping the weights along a cycle going through z_0 and C via e, e' (so that it is z_0-changing) and not going through H (so that it is w_0-avoiding). So z_0 is incident to exactly one edge going to each component other than H in $G - y_0 - z_0$. We can also assume that there is at most one component C other than H in $G - y_0 - z_0$, since otherwise, we could reach the exact same conclusion by deducing an $\{a, b\}$-edge-weighting of G as before with the extra condition that two edges e, e' incident to z_0 going to two different components C, C' distinct from H are weighted b. In case z_0w_0 is a conflict, we could again get rid of it by swapping the weights along a cycle going through z_0, in C via e, back to y_0, in C', and back to z_0 via e'. This would be correct since such a cycle would not go through H, and thus, would be w_0-avoiding. Similarly, we can assume that the multiplicity of z_0y_0 is 1.

Let us denote by z_1 the unique neighbour of z_0 in C. By swapping the weights along a cycle through C containing z_0, y_0 and the edge z_0z_1, and not going through H, we can further assume that the edge z_0z_1 is weighted a. For a similar reason, we can assume that the edge y_0z_0 is weighted b. Recall that w_0 and all the neighbours of w_0 with degree d are all incident to an even number of a-edges; thus, each of the neighbours of w_0 with degree d is incident to at least two a-edges.

To get rid of the conflict z_0w_0, we would like to swap the weights along a z_0-changing cycle in $G - w_0$ (thus, w_0-avoiding). According to Observation 2.9, recall that this would not alter the parity of the number of incident a's of any vertex in $V(G) \setminus \{w_0\}$. Furthermore, this would get rid of the conflict z_0w_0. However, this swapping process can create a conflict between w_0 and a neighbour v of w_0 with degree d; but such a conflict can only arise when the cycle goes through the only two a-edges incident to v. For a neighbour v of w_0 with degree d that is incident to only two a-edges, we call this pair of edges a *forbidden pair*. Our goal in what follows is to show that $G - w_0$ has a z_0-changing cycle not containing any forbidden pair of edges.

Let us denote by $v_1, ..., v_m$ the neighbours of w_0 with degree d. As mentioned earlier, recall that the v_i's are all adjacent to y_0. Since z_0w_0 is a conflict, recall that $d(z_0)$ and $d(w_0)$ have the same parity. Furthermore, since $d(z_0) > d(w_0) > 3$, it follows that $d(z_0) \geq 6$, and, because the only neighbours of z_0 outside H are y_0 and z_1, there is a vertex $z_2 \neq w_0$ in $N(z_0) \cap V(H)$. Note that, to find the desired cycle through z_0 in $G - w_0$, it suffices to find a path P from y_0 to a vertex z' in $N(z_0) \cap V(H)$ in the connected graph $G - z_0 - w_0$ (which is connected by the minimality of H) not using any forbidden pair of edges. Indeed, if the weight on z_0z' is b, then we can define our cycle to be $P \cup \{z_0y_0, z'z_0\}$, while, if the weight on z_0z' is a, then we can define our cycle to be $P \cup P_c$, where P_c is a path from z_0 to y_0 in $G - H - z_0y_0$ (thus, through C). Since the graph $G - z_0 - w_0$ is connected, there is a path P_1 from z_2 to y_0. We can assume that P_1 uses forbidden pairs of edges. Without loss of generality, let pv_1 and v_1q be the first forbidden pair of edges P_1 used when going from z_2 to y_0. Since v_1 is adjacent to y_0, it follows that $q = y_0$, since otherwise, we have found a path from y_0 to z_2 not using any forbidden pair of edges. Thus, we can assume that all paths from y_0 to a vertex in $N(z_0) \cap V(H)$ use exactly one forbidden pair of edges. Now we look at all such paths using only one pair of forbidden edges y_0v_i and v_ip (for $i \in \{1, ..., m\}$) and consider one such path P that goes through the most neighbours of w_0. Let y_0v_i and v_ip be the pair of forbidden edges that P contains.

First suppose that v_i has a neighbour v_i' distinct from y_0, p, w_0. The edge v_iv_i' must have weight b. Since $G - w_0 - v_i$ is connected, it has a path P' from v_i' to a vertex in $N(z_0) \cap V(H)$. The path P' must use a forbidden pair of edges, as otherwise, the graph induced by $E(P) \cup E(P')$ would contain a desired path from y_0 to a vertex in $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Let the first pair of forbidden edges P' used when starting from v_i' be qv and vr. The subpath P_1' of P' from v_i' to v must be disjoint from P, since otherwise, the graph induced by $E(P) \cup E(P_1')$ contains a desired path from y_0 to $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Furthermore, we must have that $r = y_0$, since otherwise, the path P'' defined to be y_0v together with the subpath of P' from v to v_i' followed by $v_i'v_i$ and the subpath of P from v_i to $N(z_0) \cap V(H)$ is a desired path from y_0 to $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Now the path P'' contradicts the maximality of P.

So we can assume that $N(v_i) = \{y_0, p, w_0\}$, which means, because v_i has degree $d > 3$, that some of the edges v_iy_0, v_ip, v_iw_0 have multiplicity more than 1. The multiplicity of both y_0v_i and v_ip must be 1, since otherwise, we would get a desired path from z_2 to y_0 avoiding the forbidden pair of edges y_0v_i, v_ip. So the multiplicity of v_iw_0 is at least 2. If the only neighbours of w_0 are v_i and z_0, then we can swap the weights on two edges between w_0 and v_i to avoid the conflict z_0w_0 and obtain a proper $\{a, b\}$-edge-weighting of G; so we can assume that w_0 is incident to a vertex v' in H distinct from v_i. Since $d(v_i) = d(w_0)$, the edges w_0z_0 and w_0v' are not multiple and since z_0 has degree at least 6, there are two edges z_0z_2', z_0z_2'' incident with z_0 having the same weight and where $z_2', z_2'' \in H$. Possibly $z_2' = z_2''$ or $z_2' = z_2$. The graph $B - w_0 - z_0$ is connected, so it contains a path P_1' from z_2' to y_0 and a path P_2' from from z_2'' to y_0. These paths P_1' and P_2' must be internally disjoint, since otherwise, there would be a z_0-changing cycle in B not containing any pair of forbidden edges. We can assume that both P_1' and P_2' contain a pair of forbidden edges, since otherwise, there is a desired path from y_0 to $(N(z_0) - w_0) \cap V(H)$. Hence, we can assume that

P_1' contains y_0v_i and v_ip and P_2' contains a pair of forbidden edges qv', $v'r$ incident to v'. Since this implies that y_0 and v' are adjacent, we can assume that $y_0 = q$. The vertex v' must have a neighbour s distinct from y_0, w_0, r and since the graph $B - w_0 - v'$ is connected, there is a path P'' in $B - w_0 - v'$ from s to y_0. If P'' contains the forbidden pair of edges y_0v_i and v_ip, then the graph $P_1' \cup P_2' \cup P''$ contains a z_0-changing cycle in B containing no forbidden pair of edges. Thus, we can assume that P'' contains no pair of forbidden edges. Now $P_2' \cup P''$ contains a desired path from w_0 to $(N(z_0) - w_0) \cap V(H)$. ◇

Claim 6. *There is no vertex $v \in V(H)$ such that $G - v - N(v)$ is connected.*

Proof of the claim. Suppose $v \in V(H)$ and that $G' = G - v - N(v)$ is connected. We can assume $v \in X$. By Lemma 2.10, there is an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a,b\}$-edge-weighting of G, where all edges incident to v have weight b and any vertex $u \in N(v)$ is incident to at most $1 + M(uv)$ b-edges, where $M(uv)$ denotes the multiplicity of the edge uv. Note that Claim 1 implies that if vv' is a multiple edge, then $v' \in \{z_0, y_0\}$. Thus, the only potential conflict is between v and one of z_0, y_0, say y_0. This implies that y_0 must have odd degree at least 5. But since G' is connected there must then be a y_0-changing cycle in $G - (N(v) \setminus \{y_0\})$ and swapping the weights on this cycle yields a proper $\{a,b\}$-edge-weighting of G. ◇

Claim 7. *There are no multiple edges between two vertices in H.*

Proof of the claim. Suppose uv is a multiple edge in H. We can assume $v \in X$. Since u and v have degree 3 in G^* (by Claim 5), the multiplicity of uv is exactly 2. Let e and e' be the two edges between u and v. By Claim 1, e, e' are not both white. Thus, at least one of e, e', say e, is a blue edge in H. Let v' denote, in G^*, the unique neighbour of v different from u.

Let G' be obtained from $G - v$ by removing all edges but one edge e'' incident to v'. Clearly G' is connected since $G - v - v'$ is connected by the minimality of H. Let $S = E(v') \setminus \{vv', e''\}$. Let X' denote the set of vertices in $G - v$ which are incident to an odd number of edges in S. Note that S has even size. Thus, $X' = (X \setminus S \cup (S \cap Y)$ has even size and Lemma 2.4 implies that there is an $(X', V(G') \setminus X')$-a-parity $\{a,b\}$-edge-weighting of G'. We now extend this weighting to G by assigning weight a to all edges in S and weight b to all edges incident to v. This gives an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a,b\}$-edge-weighting of G where all vertices in $N(v)$ are incident to at most two b-edges (the edge in the suspended path of length 3 joining u and v incident to u must have weight a). Observation 2.8 implies that the only conflict can be between v and its neighbours, so the only possible conflict is vv' in the case where $v' \in \{z_0, y_0\}$, say $v' = y_0$, and y_0 has degree at least 5. But since $G - v$ is connected there must then be a y_0-changing cycle avoiding v and u and swapping the weights on this cycle yields a proper $\{a,b\}$-edge-weighting of G. ◇

Claim 8. *Every vertex of H is incident to at most one blue edge.*

Proof of the claim. Recall that every vertex v of H has degree 3, by Claim 5. If v is incident to three blue edges, then $G - v - N(v)$ is connected, which contradicts Claim 6. So now assume v is incident to two blue edges. Let uv denote the white (third) edge incident to v. Still by Claim 6, the graph $G - v - N(v)$ cannot be connected, which means that u, v is a pair contradicting the choice of y_0, z_0. ◇

We now have all the tools in hand for finishing the proof. Two cases must be considered:

Case 1: *There is a vertex $v \in V(H)$ not adjacent to any of z_0, y_0.*

Recall that, according to Claim 6, whenever removing from G a vertex of H and its neighbourhood, we get a disconnected graph. Let v be a vertex not adjacent to z_0, y_0 such that the component K of $G' = G - v - N(v)$ containing z_0 and y_0 has maximum order. We can assume that $v \in X$. Note that there must be a vertex $v' \in V(H)$ distinct from v with $N(v') = N(v)$ such that the components of G' are exactly K and the isolated vertex v'. This is because otherwise there would be a vertex $v'' \neq v$ in H such that $G - v'' - N(v'')$ has a bigger K, a contradiction to our choice of v.

Let $e_1 = vv_1, e_2 = vv_2, e_3 = vv_3$ denote the three edges incident to v. Since v, v', v_1, v_2, v_3 all belong to H, by Claim 7, all these vertices are distinct. Furthermore, since $N(v) = N(v')$, we can also assume that none of vv_1, vv_2, vv_3 are blue. Hence, $v, v' \in X$ and $v_1, v_2, v_3 \in Y$. The graph $G - v$ is connected and so is the graph G' obtained from it by removing the edges $v'v_1, v'v_2$. Lemma 2.4 implies that there is an $\{a, b\}$-edge-weighting of G' where the vertices in $X \setminus \{v\} \cup \{v_2, v_3\}$ are incident to an odd number of a-edges and the vertices in $Y \setminus \{v_2, v_3\}$ are incident to an even number of a-edges. In particular, in G' the only a-edge incident to v' is $v'v_1$. We extend this weighting to G by assigning weight a to $v'v_2, v'v_3$, and weight b to all three edges incident to v. That way, we get an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all three edges incident to v have weight b and all three edges incident to v' have weight a. Observation 2.8 implies that the only potential conflicts are between v and its neighbours, but since all vertices in $N(v)$ are incident to at least one a-edge (the one incident to v') this edge-weighting is proper.

Case 2: *All vertices in H are adjacent to z_0 or y_0.*

By Claim 6, we can assume that, for every vertex $v \in V(H)$, the graph $G' = G - v - N(v)$ is disconnected. First, suppose z_0 is joined in G^* to some vertex $v \in V(H)$ by an edge of multiplicity 2. Let e' and e'' be the two edges joining z_0 and v in G^*. Claim 6 implies that not both of e', e'' are blue; say e' is white. If e'' is blue, then by Claim 8, the third edge e''' incident to v in G^* must be white. If e'' is white, then Claim 6 implies that e''' is white, so the edge $e''' = vu$ must be white. We can assume $v \in X$ and hence $u \in Y$. Let Z denote the set of vertices in $G - v$ which are incident to exactly one edge incident to u. Note that either Z is empty or Z has size 2. Note that $X' = (X \setminus (Z \cup \{v\})) \cup (Y \cap Z)$ has even size, so by Lemma 2.4, there is an $(X', V(G - v - u) \setminus X')$-$a$-parity $\{a, b\}$-edge-weighting of $G - v - u$. We now extend this edge-weighting to all of G by assigning weight b to all edges in $E(v)$ and weight a to the two edges incident to u distinct from uv. Thus, we have obtained an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all edges incident to v have weight b and u is incident to exactly one b-edge. Observation 2.8 implies that the only potential conflict is vz_0 in the case where z_0 has degree at least 5. We can also assume that there is no z_0-changing cycle in G avoiding v and u. Hence, z_0 must have degree 2 in $G - H$ and be joined by an edge in G^* to at least one vertex v' in H distinct from v. We can now find a desired z_0-changing cycle unless the only neighbours of v' in B are z_0 and u. But in this case $G - u - N(u)$ is connected, contradicting Claim 6.

Thus, we can assume that z_0, and similarly y_0, is not joined to any vertex in H by a multiple edge in G^*. Claim 7 now implies that any vertex in H has three distinct neighbours in G^*. Let $v \in X$ be any vertex in H incident to z_0. The graph $G - v - N(v)$ is disconnected by Claim 6, so there must be a vertex v' in H which in G^* has the same neighbourhood as v. Since all vertices in H have degree 3 (Claim 5) this implies that H only has four vertices: two joined to z_0 and two joined to y_0. The graph $G - v - (N(v) \setminus \{z_0\})$ is connected, so as above, there is an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all edges incident to v have weight b, and the neighbours of v distinct from z_0 which have degree 3 are incident to exactly one b-edge. Again, Observation 2.8 implies that the only possible conflict is vz_0 in the case where z_0 has degree at least 5. In this case, it is easy to see that there is a z_0-changing cycle avoiding H, thus, we can get rid of this conflict and obtain a proper $\{a, b\}$-edge-weighting of G. □

3 Proof of Theorem 1.2

Before describing the structure of trees without the $\{-1, 1\}$-property, we first introduce the following two general lemmas which will be used in the proof. Some of the tools and results used here were introduced in Section 2.

Lemma 3.1. *If G is a simple connected bipartite graph without the $\{-1, 1\}$-property and e is a bridge in G, then the deletion of e results in two components each containing an odd number of vertices.*

Proof. Suppose the lemma is false. Let G be a connected bipartite graph without the $\{-1, 1\}$-property and let $e = uv \in E(G)$ be a bridge in G. Let $c : V(G) \to \{1, 2\}$ be a mod-4 vertex-colouring of G (such a colouring exists by Lemma 2.6) and let X, Y denote the sets of vertices coloured 1 and 2 respectively. By Lemma 2.7, both colour classes of c have odd size. Let C_1, C_2 denote the two components of $G - e$ with $u \in V(C_1)$ and $v \in V(C_2)$. For a contradiction, assume that both $|V(C_1) \cap X|$ and $|V(C_1) \cap Y|$ are odd and both $|V(C_2) \cap X|$ and $|V(C_2) \cap Y|$ are even. Recall that two vertices with degree of distinct parity cannot have the same weighted degree by a $\{-1, 1\}$-edge-weighting (Observation 2.8). There are thus four cases to be considered:

Case 1: *Both u and v have odd degree and colour 1.*

By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$-$(-1)$-parity $\{-1, 1\}$-edge-weighting of C_1 and a $(Y \cap V(C_2), X \cap V(C_2))$-1-parity $\{-1, 1\}$-edge-weighting of C_2. It follows from Observation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper edge-weighting of the whole G.

Case 2: *Both u and v have even degree and colour 1.*

By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$-$(-1)$-parity $\{-1, 1\}$-edge-weighting of C_1 and an $(X \cap V(C_2), Y \cap V(C_2))$-1-parity $\{-1, 1\}$-edge-weighting of C_2. It follows from Observation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper edge-weighting of the whole G.

Case 3: *Both u and v have odd degree, u has colour 1 and v has colour 2.*

By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$-$(-1)$-parity $\{-1, 1\}$-edge-weighting of C_1 and a $(Y \cap V(C_2), X \cap V(C_2))$-1-parity $\{-1, 1\}$-edge-weighting of C_2. It follows from Observation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper edge-weighting of the whole G.

Case 4: *Both u and v have even degree, u has colour 1 and v has colour 2.*

By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$-$(-1)$-parity $\{-1,1\}$-edge-weighting of C_1 and an $(X \cap V(C_2), Y \cap V(C_2))$-1-parity $\{-1,1\}$-edge-weighting of C_2. It follows from Observation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper edge-weighting of the whole G. □

Lemma 3.2. *If G is a connected bipartite graph without the $\{-1,1\}$-property and e is a bridge in G, then there is a $\{-1,1\}$-edge-weighting of G such that e is the only conflict.*

Proof. Let G be a connected bipartite graph without the $\{-1,1\}$-property, e be a bridge in G, and C_1, C_2 be the two components of $G - e$. Let c be a mod-4 vertex-colouring of G (such a colouring exists by Lemma 2.6), and let X denote the set of vertices with colour 1 and Y denote the set of vertices with colour 2. By Lemma 2.7, both X and Y have odd size and by Lemma 3.1, we can assume that $|X \cap V(C_1)|$ and $|Y \cap V(C_2)|$ are even and $|Y \cap V(C_1)|$ and $|X \cap V(C_2)|$ are odd. Now Lemma 2.4 implies that there is an $(X \cap V(C_1), Y \cap V(C_1))$-1-parity $\{-1,1\}$-edge-weighting of C_1 and an $(X \cap V(C_2), Y \cap V(C_2))$-1-parity $\{-1,1\}$-edge-weighting of C_2. Observation 2.8 implies that these two edge-weightings, together with assigning weight -1 to the edge e, is a $\{-1,1\}$-edge-weighting of G where e is the only potential conflict. □

We can now prove Theorem 1.2. When referring to Operation 1, we mean the first operation described at the end of Section 1 (illustrated in Figure 1, (a) and (b)).

Proof of Theorem 1.2. As mentioned in the introduction, it is straightforward to check that any graph constructed with Operation 1 from four graphs without the $\{-1,1\}$-property does not have the $\{-1,1\}$-property itself. An easy argument is that all five edges incident to the vertices $v_1 \sim v_2$ and $v_3 \sim v_4$ should have the same weight (in which case a conflict arises), as otherwise, the proper $\{-1,1\}$-edge-weighting would yield one of at least one of the four graphs used in the construction, a contradiction. Thus, it suffices to prove that any tree without the $\{-1,1\}$-property is constructed from a disjoint union of K_2's through repeated (possibly none) applications of Operation 1. Suppose this is false and let T be a minimum counterexample. Note that Lemma 3.1 implies that, for any vertex $v \in V(T)$ and any edge $e \in E(v)$, the component C_e not containing v in $T - e$ has an odd number of vertices. Since we can write $|V(T)| = 1 + \sum_{e \in E(v)} |V(C_e)|$ for any vertex $v \in V(T)$ and since $|V(T)|$ is even, this implies that all vertices in T have odd degree. A consequence of this is that if $S \subset T$ is a subtree of T, then $T - S$ has no components isomorphic to K_2 as otherwise T would have vertices with degree 2.

Let $P = v_1 ... v_m$ be a longest path in T. Clearly, v_m is a leaf and, since all vertices have odd degree, v_{m-1} is incident to an even number of leaves. Suppose v_{m-1} is incident to an even number $n \geq 4$ of leaves $u_1, ..., u_n$, with $u_1 = v_m$. Recall that Lemma 2.7 implies that $|V(T)|$ is even; now, since $T' = T - \{u_1, ... u_{n-1}\}$ has an odd number of vertices, Lemma 2.7 implies that T' has a proper $\{-1,1\}$-edge-weighting. We can now extend this edge-weighting to a proper $\{-1,1\}$-edge-weighting of T by assigning the same weight to all the edges $v_{m-1}u_1, ..., v_{m-1}u_n$ (we choose whether it is 1 or -1 so that we avoid the conflict $v_{m-2}v_{m-1}$). Thus, v_{m-1} has degree exactly 3 and, from these arguments and the maximality of P, any neighbour of v_{m-2} distinct from v_{m-3} and v_{m-1} is either a leaf or a vertex of degree 3 adjacent to two leaves. Let $U' = \{u'_1, ..., u'_p\}$ be the set of leaves adjacent to v_{m-2} and let $U'' = \{u''_1, ..., u''_q\}$ be the set of neighbours of v_{m-2} distinct from v_{m-1} and v_{m-3} which have degree 3. Possibly $p = 0$ or $q = 0$, but $p + q > 0$ since v_{m-2} has odd degree and hence

$p+q$ is odd. Let T_1 and T_2 be the two components of $T - v_{m-3}v_{m-2}$ such that $v_{m-2} \in V(T_2)$. By Lemma 3.2, there is a $\{-1,1\}$-edge-weighting w of T such that the only potential conflict is $v_{m-3}v_{m-2}$. By possibly multiplying all edge weights by -1, we can assume that the weight of $v_{m-3}v_{m-2}$ is 1. We look at three separate cases:

Case 1: $p+q \geq 5$.

By possibly modifying the weights of the edges in $E(T_2)$ such that they all have weight 1 or -1, the vertex v_{m-2} can obtain weighted degree $2+p+q$ and $-p-q$. Now we simply pick the one of these two options such that $v_{m-3}v_{m-2}$ is not a conflict. Since all vertices in T_2 except v_{m-2} have degree at most 3, this gives a proper $\{-1,1\}$-edge-weighting of T.

Case 2: $p+q=3$.

As in Case 1, we can modify the edge weights such that the vertex v_{m-2} can obtain weighted degree $2+p+q$ and $-p-q$. Furthermore, since $2+p+q=5$ and all vertices in T_2 except v_{m-2} have degree at most 3, we can in this way find a proper $\{-1,1\}$-edge-weighting of T, unless v_{m-3} has weight 5. So we can assume that v_{m-3} has weight 5. If $p \geq 1$ and p is odd (resp. even), then we modify the weights in T_2 such that all edges incident with $u'_1, ..., u'_p$ have weight 1 (resp. -1) and all other edges in T_2 have weight -1 (resp. 1). This yields a proper $\{-1,1\}$-edge-weighting of T, so we can assume $p=0$ and $q=3$. In this case, we modify the weights in T_2 such that all edges incident to v_{m-1} and u''_1 have weight 1 and all other edges in T_2 have weight -1. This yields a proper $\{-1,1\}$-edge-weighting of T.

Case 3: $p+q=1$.

First suppose $q=1$ and $p=0$. If we modify the edge weights in T_2 such that they all have weight -1, then we obtain a proper $\{-1,1\}$-edge-weighting of T, unless v_{m-3} has weight -1. In this case, we change the weights of the three edges incident to v_{m-1} to 1 to obtain a proper $\{-1,1\}$-edge-weighting of T. Thus, we can assume $p=1$ and $q=0$. We can assume that $T''' = T - v_m - v_{m-1} - u_2 - u'_1$ has a proper $\{-1,1\}$-edge-weighting w, since otherwise, the minimality of T implies that T''' is constructed from a disjoint union of K_2's through repeated (possibly none) applications of Operation 1, and then so is T. By possibly multiplying all edge weights of w by -1, we can assume that $v_{m-3}v_{m-2}$ has weight 1. Now assigning weight 1 to all edges incident to v_{m-1} and weight -1 to $v_{m-2}u'_1$ yields a proper $\{-1,1\}$-edge-weighting of T. □

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