Cevian operations on distributive lattices

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Abstract. We construct a completely normal bounded distributive lattice D in which for every pair (a,b) of elements, the set $\{x \in D \mid a \leq b \vee x\}$ has a countable coinitial subset, such that D does not carry any binary operation $\smallsetminus$ satisfying the identities $\mathsf{x} \leq \mathsf{y} \vee (\mathsf{x} \smallsetminus \mathsf{y})$, $(\mathsf{x} \smallsetminus \mathsf{y}) \wedge (\mathsf{y} \smallsetminus \mathsf{x}) = 0$, and $\mathsf{x} \smallsetminus \mathsf{z} \leq (\mathsf{x} \smallsetminus \mathsf{y}) \vee (\mathsf{y} \smallsetminus \mathsf{z})$. In particular, D is not a homomorphic image of the lattice of all finitely generated convex ℓ-subgroups of any (not necessarily Abelian) ℓ-group. It has $\aleph_2$ elements. This solves negatively a few problems stated by Iberkleid, Martínez, and McGovern in 2011 and recently by the author. This work also serves as preparation for a forthcoming paper in which we prove that *for any infinite cardinal λ, the class of Stone duals of spectra of all Abelian ℓ-groups with order-unit is not closed under $\mathscr{L}_{\infty\lambda}$-elementary equivalence.*

1. Introduction

It has been known since the seventies that for any Abelian lattice-ordered group (from now on *ℓ-group*) G, the distributive lattice $\mathrm{Id_c}\, G$ of all finitely generated (equivalently principal) ℓ-ideals of G is *completely normal*, that is, it satisfies the statement

$$(\forall a, b)(\exists x, y)(a \vee b = a \vee y = x \vee b \text{ and } x \wedge y = 0)\,.$$

Delzell and Madden found in [7] an example of a completely normal bounded distributive lattice which is not isomorphic to $\mathrm{Id_c}\, G$ for any Abelian ℓ-group G. Since then, the problem of characterizing all lattices of the form $\mathrm{Id_c}\, G$ has been widely open, possibly under various equivalent forms, one of which being the *MV-spectrum problem* (cf. Mundici [16, Problem 2]). The author's paper [18] settles the *countable* case, by proving that complete normality is then sufficient. However, moving to the uncountable case, we prove in [18] that *the class of all lattices of the form* $\mathrm{Id_c}\, G$*, for Abelian ℓ-groups G with order-unit, is not closed under $\mathscr{L}_{\infty\omega}$-elementary equivalence.*

A remarkable additional property of lattices of the form $\mathrm{Id_c}\, G$, for Abelian ℓ-groups G, was coined, under different names, on the one hand in Cignoli *et al.* [6], where it was denoted by $(\mathrm{Id}\,\omega)$, on the other hand in Iberkleid *et al.* [14], where it was called "σ-Conrad". In [18] we express that property by an $\mathscr{L}_{\omega_1\omega_1}$ sentence of lattice theory that we call *having countably based differences* (cf. Subsection 2.1). This property is trivially satisfied in the countable case, but fails for various uncountable examples such as Delzell and Madden's.

In this paper we prove (cf. Theorem 7.2) that requiring countably based differences, together with complete normality, is not sufficient to characterize distributive

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lattices of the form $\mathrm{Id_c}\, G$ for Abelian ℓ-groups G. It turns out that our counterexample also gives a strong negative answer to [14, Question 4.3.1], by proving that "σ-Conrad does not imply Conrad" (it was proved in [14] that normal-valued Conrad implies σ-Conrad). It also proves that the implication (4)⇒(5), in [14, § 4], is strict ([14] achieved a partial result in that direction). Our main counterexample has cardinality $\aleph_2$.

The proof of our main result is achieved in several steps. We observe (cf. Proposition 5.5) that for any (not necessarily Abelian) ℓ-group G, the (completely normal, distributive) lattice $\mathrm{Cs_c}\, G$ of all finitely generated convex ℓ-subgroups of G carries a binary operation $\smallsetminus$ satisfying the identities $\mathsf{x} \leq \mathsf{y} \vee (\mathsf{x} \smallsetminus \mathsf{y})$, $(\mathsf{x} \smallsetminus \mathsf{y}) \wedge (\mathsf{y} \smallsetminus \mathsf{x}) = 0$, and $\mathsf{x} \smallsetminus \mathsf{z} \leq (\mathsf{x} \smallsetminus \mathsf{y}) \vee (\mathsf{y} \smallsetminus \mathsf{z})$. We call such operations *Cevian operations* and we call such lattices *Cevian lattices* (Definition 5.1).

We thus need to construct a non-Cevian completely normal distributive lattice with zero and countably based differences. In order to achieve this, we first solve the problem at *diagram level*, by constructing (cf. Lemma 4.3) a $\{0,1\}^3$-indexed commutative diagram, of countable completely normal distributive lattices with zero, which is a counterexample to a diagram analogue of a "local" form of the main question. This diagram is obtained by applying the functor $\mathrm{Id_c}$ to a certain *non-commutative*[1] diagram of Abelian ℓ-groups, which we denote by $\vec{A}$ (cf. Section 4).

The proof of Lemma 4.3 rests on a lattice-theoretical interpretation, established in Proposition 3.1, of the configuration underlying Ceva's Theorem in elementary plane geometry.

Our final line of argument relies on the results of the monograph Gillibert and Wehrung [11], which sets up a machinery making it possible to turn certain *diagram counterexamples* to *object counterexamples*, *via* constructs called *condensates*, infinite combinatorial objects called *lifters*, and a technical result called the *Armature Lemma*. We summarize the required machinery in Section 6, and we embark on our main result's final proof in Section 7.

Since, as mentioned above, having countably based differences is an $\mathscr{L}_{\omega_1\omega_1}$ sentence, this is related to the question, stated as [18, Problem 1], whether there exists an infinite cardinal λ such that the class of all lattices of the form $\mathrm{Id_c}\, G$, for Abelian ℓ-groups G (equivalenly, Stone dual lattices of spectra of Abelian ℓ-groups), can be characterized by some class of $\mathscr{L}_{\infty\lambda}$ sentences. In our subsequent paper [20] we prove that this is not so, by establishing the even stronger result that *the class of Stone duals of spectra of all Abelian ℓ-groups with order-unit is not closed under $\mathscr{L}_{\infty\lambda}$-elementary equivalence.*

2. Notation, terminology, and basic concepts

2.1. **Sets, posets.** Following standard set-theoretical notation, we denote by ω the first infinite ordinal and also the set of all nonnegative integers. For a natural number n, we denote by $\aleph_n$ the nth transfinite cardinal number, also denoted by ω_n in case it should be viewed as an ordinal. The set of all finite subsets of a set X will be denoted by $[X]^{<\omega}$. We denote by $\mathfrak{P}(X)$, or just $\mathfrak{P}X$, the powerset of a set X. By "countable" we will always mean "at most countable".

[1]We need the non-commutativity of the diagram $\vec{A}$—otherwise, by definition, $\mathrm{Id_c}\, \vec{A}$ would not be a diagram counterexample! This will be strongly illustrated in Proposition 5.5.

For any element a in a poset (i.e., partially ordered set) P, we set

$$P \downarrow a \stackrel{\text{def}}{=} \{p \in P \mid p \leq a\} , \quad P \uparrow a \stackrel{\text{def}}{=} \{p \in P \mid p \geq a\} .$$

A subset X of P is

— a *lower subset* (resp., *upper subset*) of P if $P \downarrow x \subseteq X$ (resp., $P \uparrow x \subseteq X$) whenever $x \in X$;
— an *ideal* of P if it is a nonempty, upward directed lower subset of P;
— *coinitial* in P if $P = \bigcup(P \uparrow x \mid x \in X)$.

For posets P and Q, a map $f \colon P \to Q$ is *isotone* if $x \leq y$ implies that $f(x) \leq f(y)$ whenever $x, y \in P$. We let $\mathbf{2} \stackrel{\text{def}}{=} \{0, 1\}$, ordered by $0 < 1$.

We refer to Grätzer [13] for standard facts on lattice theory. A distributive lattice D with zero is *completely normal* if for all $x, y \in D$ there are $u, v \in D$ such that $x \leq y \vee u$, $y \leq x \vee v$, and $u \wedge v = 0$. Equivalently (replacing u by $u \wedge x$ and v by $v \wedge y$), $x \vee y = x \vee v = u \vee y$ and $u \wedge v = 0$. By a result from Monteiro [15], this is equivalent to saying that the specialization order in the Stone dual of D is a root system (see also Cignoli *et al.* [6]).

For any elements a and b in a join-semilattice S, we set, following the notation in the author's paper [18],

$$a \ominus_S b \stackrel{\text{def}}{=} \{x \in S \mid a \leq b \vee x\} . \tag{2.1}$$

Following [18], we say that S has *countably based differences* if $a \ominus_S b$ has a countable coinitial subset whenever $a, b \in S$.

Following [18], we define a join-homomorphism $f \colon A \to B$, between join-semilattices, to be *closed* if for all $a, a' \in A$ and $b \in B$, if $b \in f(a) \ominus_B f(a')$, there exists $x \in a \ominus_A a'$ such that $f(x) \leq b$. In particular, if X is a coinitial subset of $a \ominus_A a'$, then $f[X]$ is a coinitial subset of $f(a) \ominus_B f(a')$. We thus get the following lemma.

Lemma 2.1. *Let A and B be join-semilattices and let $f \colon A \to B$ be a closed join-homomorphism. For all $a, a' \in A$, if $a \ominus_A a'$ has a countable coinitial subset, then $f(a) \ominus_B f(a')$ has a countable coinitial subset.*

For ℓ-groups we refer to Bigard *et al.* [5], Anderson and Feil [2]. All our ℓ-groups will be written additively (even in the non-commutative case), with the lattice operations $\wedge$ and $\vee$ being given higher precedence than the group operations (e.g., $u + x \wedge y - v = u + (x \wedge y) - v$). For any ℓ-group G, the lattice $\mathrm{Cs}\, G$ of all convex ℓ-subgroups of G is a distributive algebraic lattice, of which the collection $\mathrm{Cs_c}\, G$ of all finitely generated convex ℓ-subgroups is a sublattice; moreover, $\mathrm{Cs_c}\, G$ is completely normal. The elements of $\mathrm{Cs_c}\, G$ are exactly those of the form

$$\langle x \rangle_G \stackrel{\text{def}}{=} \{y \in G \mid (\exists n < \omega)(|y| \leq n|x|)\} , \quad \text{for } x \in G \text{ (equivalently, for } x \in G^+) .$$

We refer the reader to Iberkleid *et al.* [14, § 1.2] for a more detailed overview of the matter.

The lattice $\mathrm{Id}\, G$ of all ℓ-ideals (i.e., normal convex ℓ-subgroups) of G is a distributive algebraic lattice, isomorphic to the congruence lattice of G. The $(\vee, 0)$-semilattice $\mathrm{Id_c}\, G$ of all finitely generated ℓ-ideals of G may not be a lattice (cf. Remark 5.7 for further explanation). Its elements are exactly those of the form

$$\langle x \rangle_G^\ell \stackrel{\text{def}}{=} \{y \in G \mid \text{there are } n < \omega \text{ and conjugates } x_1, \dots, x_n \text{ of } |x| \text{ such that } |y| \leq x_1 + \cdots + x_n\} , \quad \text{for } x \in G \text{ (equivalently, for } x \in G^+) .$$

As observed in [18, Subsection 2.2], the assignment $\mathrm{Id_c}$ naturally extends to a *functor* from Abelian ℓ-groups and ℓ-homomorphisms to completely normal distributive lattices with zero and closed 0-lattice homomorphisms. In a similar manner, the assignment $\mathrm{Cs_c}$ naturally extends to a *functor* from ℓ-groups and ℓ-homomorphisms to completely normal distributive lattices with zero and closed 0-lattice homomorphisms. Of course, if G is Abelian, then $\mathrm{Cs}\,G = \mathrm{Id}\,G$, $\langle x\rangle_G = \langle x\rangle^{\ell}_G$, and so on.

For any ℓ-group G and any $x, y \in G^+$, let $x \propto y$ hold if $x \leq ny$ for some positive integer n, and let $x \asymp y$ hold if $x \propto y$ and $y \propto x$.

2.2. **Open polyhedral cones.** Throughout the paper we will denote by $\mathbb{Q}$ the ordered field of all rational numbers and by $\mathbb{Q}^+$ its positive cone. For every positive integer n and every n-ary term t in the similarity type $(0, +, -, \vee, \wedge)$ of ℓ-groups (in short *ℓ-term*), we set

$$[\![t(\mathsf{x}_1, \dots, \mathsf{x}_n) \neq 0]\!]_n \overset{\text{def}}{=} \left\{(x_1, \dots, x_n) \in (\mathbb{Q}^+)^n \mid t(x_1, \dots, x_n) \neq 0\right\} ,$$

and similarly for $[\![t(\mathsf{x}_1, \dots, \mathsf{x}_n) > 0]\!]_n$. In particular, for every positive integer n and all rational numbers $\lambda_1, \dots, \lambda_n$, we get

$$[\![\lambda_1\mathsf{x}_1 + \cdots + \lambda_n\mathsf{x}_n > 0]\!]_n = \left\{(x_1, \dots, x_n) \in (\mathbb{Q}^+)^n \mid \lambda_1 x_1 + \cdots + \lambda_n x_n > 0\right\} ;$$

we will call such sets *open half-spaces*[2] of $(\mathbb{Q}^+)^n$. Define a *basic open polyhedral cone* of $(\mathbb{Q}^+)^n$ as the intersection of a finite, nonempty collection of open half-spaces, and define a *strict open polyhedral cone* of $(\mathbb{Q}^+)^n$ as a finite union of basic open polyhedral cones. Observe that no strict open polyhedral cone of $(\mathbb{Q}^+)^n$ contains 0 as an element. For $n \geq 2$, the lattice $\mathcal{O}_n$ of all strict open polyhedral cones of $(\mathbb{Q}^+)^n$ is a bounded distributive lattice, with zero the empty set and with unit $(\mathbb{Q}^+)^n \setminus \{0\}$.

2.3. **Non-commutative diagrams.** Several sections in the paper will involve the concept of a "non-commutative diagram". A (*commutative*) *diagram*, in a category $\mathcal{S}$, is often defined as a functor D from a category $\mathcal{P}$ (the "indexing category" of the diagram) to $\mathcal{S}$. Allowing any morphism in $\mathcal{P}$ to be sent to more than one morphism in $\mathcal{S}$, we get D as a kind of "non-deterministic functor". Specializing to the case where $\mathcal{P}$ is the category naturally assigned to a poset P, we get the following definition.

Definition 2.2. Let P be a poset and let $\mathcal{S}$ be a category. A *P-indexed diagram in $\mathcal{S}$* is an assignment D, sending each element p of P to an object $D(p)$ (or D_p) of $\mathcal{S}$ and each pair (p, q) of elements of P, with $p \leq q$, to a *nonempty set* $D(p, q)$ of morphisms from $D(p)$ to $D(q)$, such that

(1) $\mathrm{id}_{D(p)} \in D(p, p)$ for every $p \in P$,
(2) Whenever $p \leq q \leq r$, $u \in D(p, q)$, and $v \in D(q, r)$, $v \circ u$ belongs to $D(p, r)$.

We say that D is a *commutative diagram* if each $D(p, q)$, for $p \leq q$ in P, is a singleton.

We will often write poset-indexed commutative diagrams in the form

$$\vec{D} = (D_p, \delta_p^q \mid p \leq q \text{ in } P) ,$$

[2] This will include "degenerate" cases such as the one where all λ_i are zero (resp., positive) and should not cause any problem in the sequel.

where all D_p are objects and all $\delta_p^q \colon D_p \to D_q$ are morphisms subjected to the usual commutation relations (i.e., $\delta_p^p = \mathrm{id}_{D_p}$, $\delta_p^r = \delta_q^r \circ \delta_p^q$ whenever $p \leq q \leq r$); hence $\vec{D}(p,q) = \{\delta_p^q\}$. If P is a directed poset we will say that $\vec{D}$ is a *direct system.*

The following construction will be briefly mentioned in Proposition 4.1, which will play a prominent role in our forthcoming paper [20].

Definition 2.3. Let I be a set, let $\mathcal{S}$ be a category with all I-indexed products, let P be a poset, and let D be a P-indexed diagram in $\mathcal{S}$. Denoting by P^I the I-th cartesian power of the poset P, we define a P^I-indexed diagram D^I in $\mathcal{S}$ by setting

(1) $D^I(p_i \mid i \in I) \overset{\text{def}}{=} \prod_{i \in I} D(p_i)$;
(2) whenever $p = (p_i \mid i \in I)$ and $q = (q_i \mid i \in I)$ in P^I with $p \leq q$, $D^I(p,q)$ consists of all morphisms of the form $\prod_{i \in I} f_i$ where each $f_i \in D(p_i\,, q_i)$.

3. A lattice-theoretical version of Ceva's Theorem

The goal of this section is to establish Proposition 3.1. This result solves a problem, mostly of lattice-theoretical nature, on open polyhedral cones in dimension three; its proof involves the main configuration underlying Ceva's Theorem in elementary plane geometry.

Although we will only need to apply Proposition 3.1 to the ordered field $\mathbb{Q}$ of all rational numbers, it does not bring any additional complexity to state it over an arbitrary totally ordered division ring $\Bbbk$. For such a ring, we set

$$\Bbbk^+ \overset{\text{def}}{=} \{x \in \Bbbk \mid x \geq 0\}\,,$$
$$\Bbbk^{++} \overset{\text{def}}{=} \{x \in \Bbbk \mid x > 0\}\,,$$
$$\overline{\Bbbk}^+ \overset{\text{def}}{=} \Bbbk^+ \cup \{\infty\}\,, \quad \text{where we declare that } x < \infty \text{ whenever } x \in \Bbbk^+\,.$$

For all $x, y \in \overline{\Bbbk}^+$, we write

$$[x,y] \overset{\text{def}}{=} \left\{t \in \overline{\Bbbk}^+ \mid x \leq t \leq y\right\}\,, \quad [x,y[\overset{\text{def}}{=} \left\{t \in \overline{\Bbbk}^+ \mid x \leq t < y\right\}\,,$$

and so on. We denote by $\mathcal{O}(\overline{\Bbbk}^+)$ the set of all finite unions of intervals of $\overline{\Bbbk}^+$ of one of the forms $[0,x[\,$, $]y,\infty]$, or $]x,y[$ with $x, y \in \overline{\Bbbk}^+$. For a nonzero pair (x,y) of elements of $\Bbbk^+$, the expression $x^{-1}y$ is given its usual meaning in $\Bbbk$ if $x > 0$, and extended to the case where $x = 0$ (thus $y > 0$) by setting $0^{-1}y = \infty$.

Proposition 3.1. *Let $\Bbbk$ be a totally ordered division ring. For all integers i, j with $1 \leq i < j \leq 3$, let $U_{ij} \in \mathcal{O}(\overline{\Bbbk}^+)$ and set*

$$C_{ij} \overset{\text{def}}{=} \left\{(x_1,x_2,x_3) \in (\Bbbk^+)^3 \mid (x_i\,, x_j) \neq (0,0) \text{ and } x_i^{-1}x_j \in U_{ij}\right\}\,.$$

Suppose that the following statements hold:

(1) $0 \in U_{12} \cap U_{23} \cap U_{13}$ *;*
(2) $[0,\infty[\, \not\subseteq U_{12}$ *and* $[0,\infty[\, \not\subseteq U_{23}$ *;*
(3) $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$ *.*

Then there are $x, y \in \Bbbk^{++}$ such that $U_{12} = [0,x[\,$, $U_{23} = [0,y[\,$, and $U_{13} = [0,xy[\,$.

The conclusion of Proposition 3.1 is represented in Figures 3.1 and 3.2. The configuration represented in Figure 3.1 will be called a *Ceva configuration* [for the sets C_{ij}]. The sets C_{ij} are emphasized with a gray shade in Figure 3.2. The sets U_{ij} are marked in thick black lines, on the boundary of the main triangle, on both pictures.

In all the figures involved in Section 3, open polyhedral cones of $(\Bbbk^+)^3$ will be represented by their intersection with the 2-simplex

$$\left\{(x_1, x_2, x_3) \in (\Bbbk^+)^3 \mid x_1 + x_2 + x_3 = 1\right\} ,$$

and points will be represented by their *homogeneous coordinates*, so

$$\langle x, y, z\rangle = \{(\lambda x, \lambda y, \lambda z) \mid \lambda \in \Bbbk \setminus \{0\}\} .$$

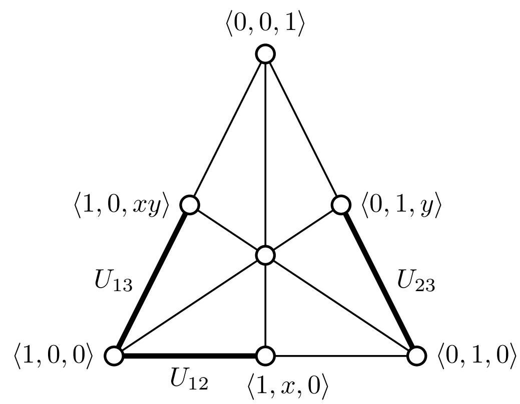


FIGURE 3.1. A Ceva configuration

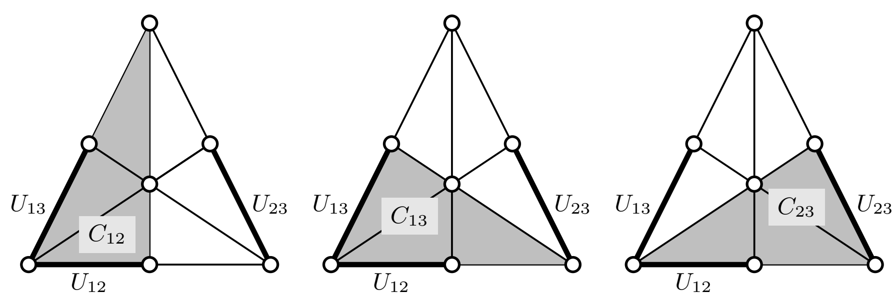


FIGURE 3.2. The sets C_{12}, C_{13}, and C_{23}

Proof of Proposition 3.1. Say that a member of $\mathcal{O}(\overline{\Bbbk}^+)$ is *initial* if it has the form $[0, z[$ for some $z \in \Bbbk^{++}$.

Claim 1. *The set U_{23} is initial.*

Proof of Claim. (cf. Figure 3.3). From Assumption (1) it follows that the leftmost interval of U_{12} has the form $[0, x[$, where $0 < x \leq \infty$. From Assumption (2) it follows that $x < \infty$, so $x \in \Bbbk^{++}$. A similar argument applies to U_{23}.

Now suppose that U_{23} is not initial. From Assumptions (1) and (2) it follows that the second leftmost interval of U_{23} has one of the the forms $]y, y'[$ or $]y, y']$

where $0 < y < y' \leq \infty$ (and $y' = \infty$ in the second case). Pick $v \in]y, y'[$; observe that $v \in U_{23}$. The element $u \overset{\text{def}}{=} xyv^{-1}$ belongs to $]0, x[$, thus to U_{12}; whence $(1, u, xy) \in C_{12}$. Moreover, the element $u^{-1}xy = v$ belongs to U_{23}, thus $(1, u, xy) \in C_{23}$. Using Assumption (3), follows that $(1, u, xy) \in C_{13}$, that is, $xy \in U_{13}$. It follows that $(1, x, xy) \in C_{13}$, thus, by Assumption (3), either $(1, x, xy) \in C_{12}$ or $(1, x, xy) \in C_{23}$. In the first case, $x \in U_{12}$, a contradiction. In the second case, $y \in U_{23}$, a contradiction. □ Claim 1.

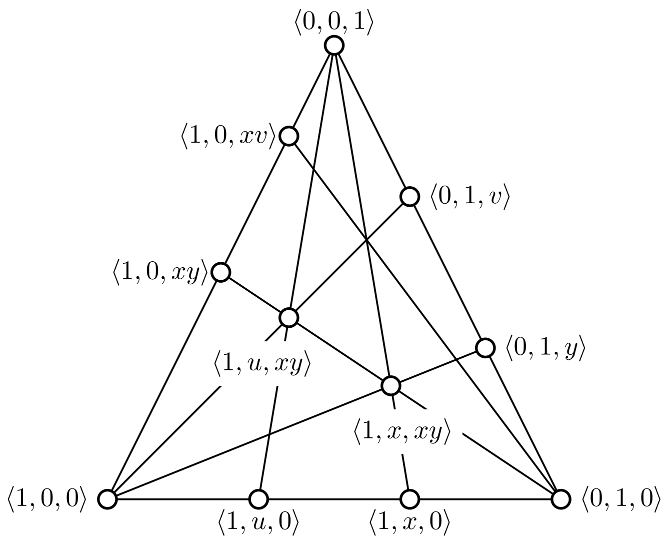


FIGURE 3.3. Illustrating the proof of Proposition 3.1, Claim 1

From now on we shall write $U_{23} = [0, y[$ where $y \in \Bbbk^{++}$.

Claim 2. *The set U_{12} is initial.*

Proof of Claim. (cf. Figure 3.4). Suppose that U_{12} is not initial. From Assumptions (1) and (2) it follows that the second leftmost interval of U_{12} has one of the forms $]x, x'[$ or $]x, x']$ where $0 < x < x' \leq \infty$ (and $x' = \infty$ in the second case). Pick any $u \in]x, x'[$; observe that $u \in U_{12}$, that is, $(1, u, xy) \in C_{12}$. The element $v \overset{\text{def}}{=} u^{-1}xy$ belongs to $]0, y[$, thus to U_{23}; that is, $(1, u, xy) \in C_{23}$. Using Assumption (3), it follows that $(1, u, xy) \in C_{13}$, whence also $(1, x, xy) \in C_{13}$. By Assumption (3) again, it follows that either $(1, x, xy) \in C_{12}$ or $(1, x, xy) \in C_{23}$. In the first case, $x \in U_{12}$, a contradiction. In the second case, $y \in U_{23}$, a contradiction. □ Claim 2.

From now on we shall write $U_{12} = [0, x[$ where $x \in \Bbbk^{++}$.

Claim 3. *The set U_{13} contains $[0, xy[$.*

Proof of Claim. (cf. Figure 3.5). We need to prove that every element $t \in [0, xy[$ belongs to U_{13}. Let first $t = 0$. We need to prove that $(1, 0, 0) \in C_{13}$, which holds owing to Assumption (1). Suppose from now on that $t > 0$. There are $u \in]0, x[$ and $v \in]0, y[$ such that $t = uv$. Observe that $u \in U_{12}$ and $v \in U_{23}$. It follows

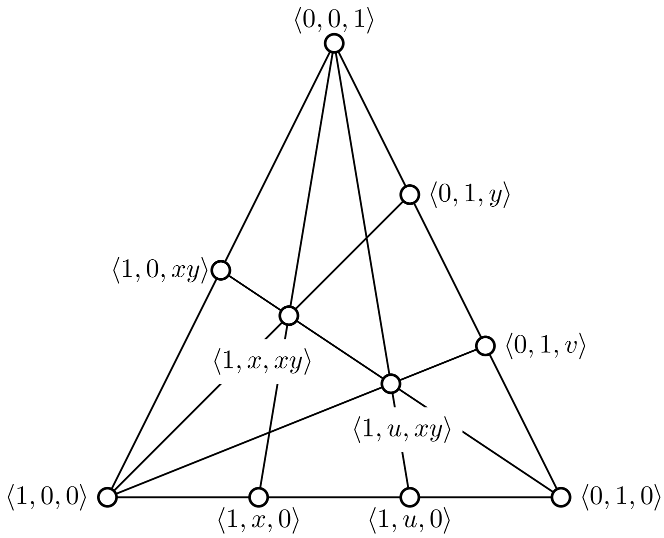


FIGURE 3.4. Illustrating the proof of Proposition 3.1, Claim 2

that $(1, u, uv) \in C_{12} \cap C_{23}$, thus, by Assumption (3), $(1, u, uv) \in C_{13}$, that is, $t \in U_{13}$. □ Claim 3.

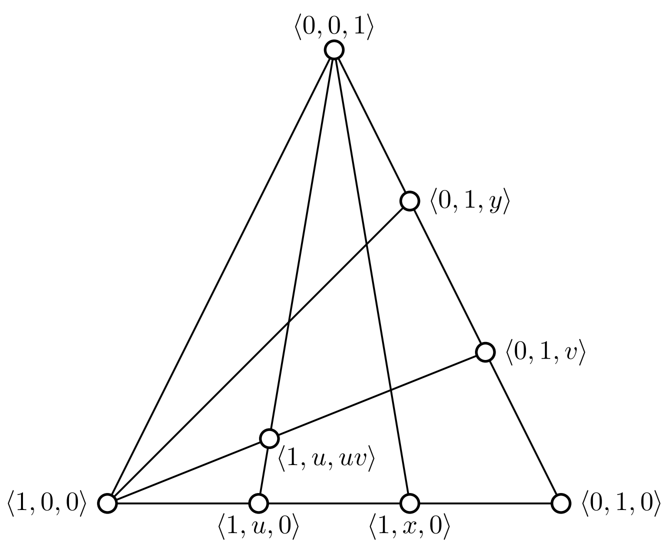


FIGURE 3.5. Illustrating the proof of Proposition 3.1, Claim 3

Claim 4. $U_{13} = [0, xy[$.

Proof of Claim. (cf. Figure 3.6). Suppose that there exists $z \in U_{13} \cap [xy, \infty[$. Then $(1, x, z)$ belongs to C_{13}, thus, by Assumption (3), either to C_{12} or to C_{23}. In the first case, $x \in U_{12}$, a contradiction. In the second case, $x^{-1}z \in U_{23}$, thus

$x^{-1}z < y$, a contradiction. This completes the proof that $U_{13} \subseteq [0, xy[$. Now apply Claim 3. □ Claim 4.

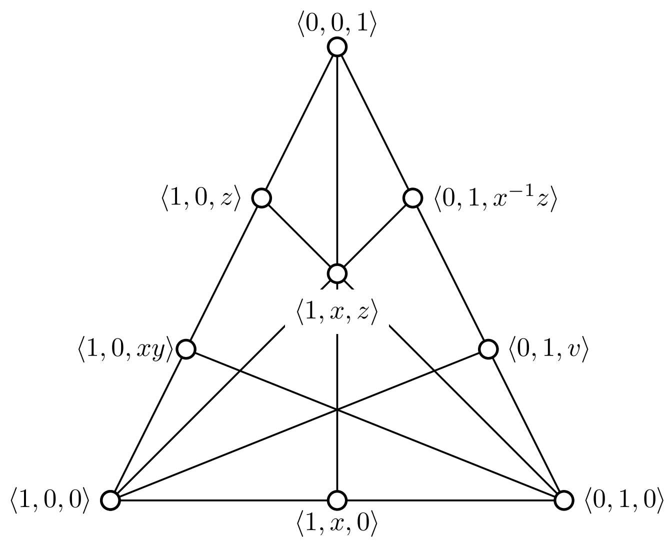


FIGURE 3.6. Illustrating the proof of Proposition 3.1, Claim 4

The combination of Claims 1–4 entails the conclusion of Proposition 3.1. □

4. THE NON-COMMUTATIVE DIAGRAM $\vec{A}$

In this section we shall introduce a non-commutative diagram (cf. Subsection 2.3), denoted by $\vec{A}$, of Abelian ℓ-groups and ℓ-homomorphisms, indexed by the cube

$$\mathfrak{P}[3] = \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}\,, \quad \text{endowed with set inclusion} \tag{4.1}$$

(where $[3] = \{1, 2, 3\}$, $12 = 21 = \{1, 2\}$, and so on).

We define A_{123} as the Abelian ℓ-group defined by the generators a, a', b, c subjected to the relations

$$0 \leq a \leq a' \leq 2a\,; \quad 0 \leq b\,; \quad 0 \leq c\,.$$

Further, we define the following ℓ-subgroups of A_{123} :

- A_{12} is the ℓ-subgroup of A_{123} generated by $\{a, b\}$;
- A_{13} is the ℓ-subgroup of A_{123} generated by $\{a', c\}$;
- A_{23} is the ℓ-subgroup of A_{123} generated by $\{b, c\}$;
- A_1 is the ℓ-subgroup of A_{123} generated by $\{a\}$;
- A_2 is the ℓ-subgroup of A_{123} generated by $\{b\}$;
- A_3 is the ℓ-subgroup of A_{123} generated by $\{c\}$;
- $A_\varnothing = \{0\}$.

It is easy to see that each A_p , for $p \in \mathfrak{P}[3]$, can also be defined by generators and relations in a natural way; for example A_{12} is the Abelian ℓ-group defined by the generators a, b subjected to the relations $0 \leq a$ and $0 \leq b$, and so on. In particular,

$A_1 \cong A_2 \cong A_3 \cong \mathbb{Z}$. The diagram $\vec{A}$ is the $\mathfrak{P}[3]$-indexed diagram of Abelian ℓ-groups, whose vertices are the A_p where $p \in \mathfrak{P}[3]$ and whose arrows are the loops at every vertex together with the following ℓ-embeddings:

- For every $p \in \mathfrak{P}[3]$, $\vec{A}(\varnothing, p)$ consists of the zero map $\alpha_\varnothing^p$ from $A_\varnothing$ to A_p.
- For all distinct $i, j \in [3]$, $\vec{A}(i, ij)$ consists of the single map α_i^{ij}, defined as the inclusion map from A_i into A_j, *except in case* $i = 1$ *and* $j = 3$, in which case $\alpha_i^{ij} = \alpha_1^{13}$ is the unique ℓ-homomorphism sending a to a'. We emphasize this by marking the arrow α_1^{13} with a thick line on Figure 4.1.
- For all distinct $i, j \in [3]$, $\vec{A}(ij, 123)$ consists of the single map α_{ij}^{123}, which is the inclusion map from A_{ij} into A_{123}.
- $\vec{A}(2, 123)$ consists of the single map $\alpha_2^{123} = \alpha_{12}^{123} \circ \alpha_2^{12} = \alpha_{23}^{123} \circ \alpha_2^{23}$, which is also the inclusion map from A_2 into A_{123}.
- $\vec{A}(3, 123)$ consists of the single map $\alpha_3^{123} = \alpha_{13}^{123} \circ \alpha_3^{13} = \alpha_{23}^{123} \circ \alpha_3^{23}$, which is also the inclusion map from A_3 into A_{123}.
- $\vec{A}(1, 123)$ consists of the two *distinct* maps $\alpha_{12}^{123} \circ \alpha_1^{12}$ (which is also the inclusion map) and $\alpha_{13}^{123} \circ \alpha_1^{13}$ (which sends a to a') from A_1 into A_{123}.

The diagram $\vec{A}$ is partly represented in Figure 4.1. On that picture, each node is highlighted by its canonical generators: for example, A_{123} is marked $A_{123}(a, a', b, c)$, A_{13} is marked $A_{13}(a', c)$, and so on.

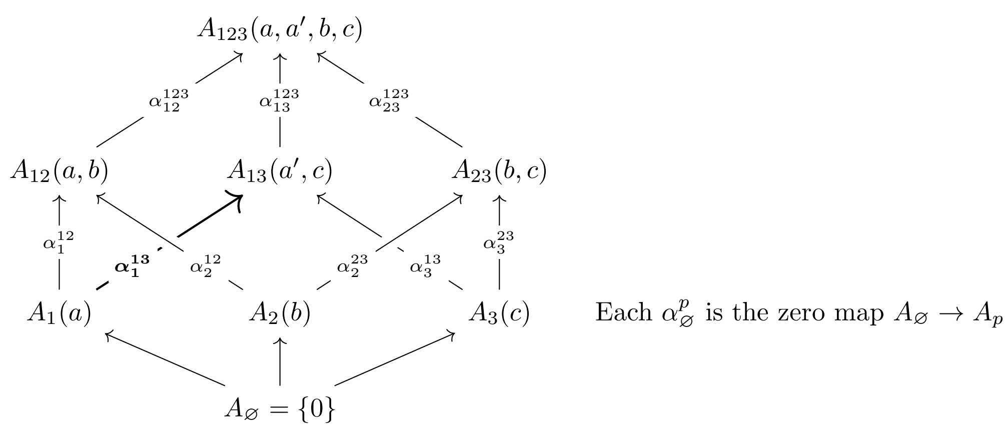


FIGURE 4.1. The non-commutative diagram $\vec{A}$

We emphasize that the diagram $\vec{A}$ of Abelian ℓ-groups is *not* commutative (for $\vec{A}(1, 123)$ has two elements). However, it has the following remarkable property, involving the construction of the I-th power of a diagram (cf. Definition 2.3), which we will fully bring to use in [20].

Proposition 4.1. *For every set I, the $\mathfrak{P}[3]$-indexed diagram $\mathrm{Id_c}\,\vec{A}^I$ is a commutative diagram of completely normal distributive lattices with zero and closed 0-lattice homomorphisms.*

Proof. Although $\vec{A}$ is not a commutative diagram, it only barely fails to be so since its only non-commutative square is $(1, 12, 13, 123)$, and then the inequalities

$a \leq a' \leq 2a$ yield the statement

$$(\alpha_{12}^{123} \circ \alpha_1^{12})(x) \leq (\alpha_{13}^{123} \circ \alpha_1^{13})(x) \leq 2(\alpha_{12}^{123} \circ \alpha_1^{12})(x) \quad \text{whenever } x \in A_1 .$$

It follows that for all $p \leq q$ in $\mathfrak{P}[3]$ and all $f, g \colon A_p \to A_q$ in $\vec{A}$, the statement

$$f(x) \leq 2g(x) \text{ and } g(x) \leq 2f(x) , \quad \text{for every } x \in A_p ,$$

which we shall denote by $f \asymp_2 g$, holds. It follows easily that if $p_i \leq q_i$ in $\mathfrak{P}[3]$ and $f_i , g_i \colon A_{p_i} \to A_{q_i}$ whenever $i \in I$, then $\prod_{i\in I} f_i \asymp_2 \prod_{i\in I} g_i$. Hence, $\operatorname{Id_c}\bigl(\prod_{i\in I} f_i\bigr) = \operatorname{Id_c}\bigl(\prod_{i\in I} g_i\bigr)$, so there is exactly one arrow from $\prod_{i\in I} A_{p_i}$ to $\prod_{i\in I} A_{q_i}$ in $\vec{A}^I$.

Finally, every $\vec{A}^I(p)$, for $p \in P^I$, is an Abelian ℓ-group, thus $\operatorname{Id_c} \vec{A}^I(p)$ is a completely normal distributive lattice with zero. Whenever $p \leq q$ in P^I, every member of $\vec{A}^I(p, q)$ is an ℓ-homomorphism, thus, by [18, Proposition 2.6], every member of $\operatorname{Id_c} \vec{A}^I(p, q)$ is a closed 0-lattice homomorphism. □

In the present paper we will only need the case where I is a singleton:

Corollary 4.2. *The diagram* $\vec{\boldsymbol{A}} \overset{\text{def}}{=} \operatorname{Id_c} \vec{A}$ *is commutative diagram of completely normal distributive lattices with zero and closed* 0*-lattice homomorphisms.*

For $p \leq q$ in $\mathfrak{P}[3]$, we shall denote by $\boldsymbol{\alpha}_p^q$ the unique arrow from $\boldsymbol{A}_p$ to $\boldsymbol{A}_q$ in $\vec{\boldsymbol{A}}$. For example, $\boldsymbol{\alpha}_1^{123} = \operatorname{Id_c}(\alpha_{12}^{123} \circ \alpha_1^{12}) = \operatorname{Id_c}(\alpha_{13}^{123} \circ \alpha_1^{13})$. The elements

$$\boldsymbol{a}_1 \overset{\text{def}}{=} \langle a \rangle_{A_{123}} = \langle a' \rangle_{A_{123}} , \quad \boldsymbol{a}_2 \overset{\text{def}}{=} \langle b \rangle_{A_{123}} , \quad \boldsymbol{a}_3 \overset{\text{def}}{=} \langle c \rangle_{A_{123}} \tag{4.2}$$

all belong to $\boldsymbol{A}_{123}$.

Our main technical lemma is the following.

Lemma 4.3. *There is no family* $(\boldsymbol{c}_{ij} \mid i \neq j \text{ in } [3])$ *of elements of* $\boldsymbol{A}_{123}$ *satisfying the following statements:*

(1) *Each* $\boldsymbol{c}_{ij}$ *belongs to the range of* $\boldsymbol{\alpha}_{ij}^{123}$.
(2) $\boldsymbol{a}_i \leq \boldsymbol{a}_j \vee \boldsymbol{c}_{ij}$ *whenever* $\{i, j\}$ *is either* $\{1, 2\}$ *or* $\{2, 3\}$.
(3) $\boldsymbol{c}_{ij} \wedge \boldsymbol{c}_{ji} = 0$ *whenever* $\{i, j\}$ *is either* $\{1, 2\}$ *or* $\{2, 3\}$.
(4) $\boldsymbol{c}_{12} \wedge \boldsymbol{c}_{23} \leq \boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}$.

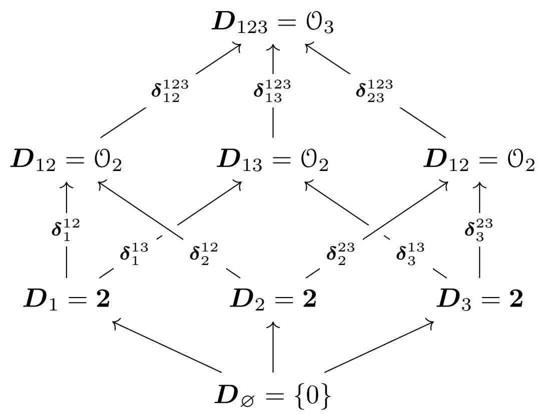
Proof. We shall introduce a $\mathfrak{P}[3]$-indexed commutative diagram $\vec{\boldsymbol{D}}$ of bounded distributive lattices with 0-lattice homomorphisms. We set $\boldsymbol{D}_\varnothing \overset{\text{def}}{=} \{0\}$, $\boldsymbol{D}_1 = \boldsymbol{D}_2 = \boldsymbol{D}_3 = \boldsymbol{2} \overset{\text{def}}{=} \{0, 1\}$, and $\boldsymbol{D}_p \overset{\text{def}}{=} \mathcal{O}_k$ (cf. Subsection 2.2) whenever $p \in \{12, 13, 23, 123\}$ has k elements. Each $\boldsymbol{\delta}_p^p$ is the identity map on $\boldsymbol{D}_p$. For $p < q$ in $\mathfrak{P}[3]$, the map $\boldsymbol{\delta}_p^q \colon \boldsymbol{D}_p \to \boldsymbol{D}_q$ is defined as follows:

- If $p = \varnothing$ we have no choice, namely $\boldsymbol{\delta}_\varnothing^p = 0$.
- $\boldsymbol{\delta}_1^{12}(1) = \boldsymbol{\delta}_1^{13}(1) = \boldsymbol{\delta}_2^{23}(1) \overset{\text{def}}{=} [\![\mathsf{x}_1 > 0]\!]_2$.
- $\boldsymbol{\delta}_2^{12}(1) = \boldsymbol{\delta}_3^{13}(1) = \boldsymbol{\delta}_3^{23}(1) \overset{\text{def}}{=} [\![\mathsf{x}_2 > 0]\!]_2$.
- $\boldsymbol{\delta}_{ij}^{123}(X) \overset{\text{def}}{=} \bigl\{(x_1, x_2, x_3) \in (\mathbb{Q}^+)^3 \mid (x_i , x_j) \in X\bigr\}$, whenever $1 \leq i < j \leq 3$ and $X \in \mathcal{O}_2$.
- $\boldsymbol{\delta}_i^{123}(1) = [\![\mathsf{x}_i > 0]\!]_3$ whenever $i \in [3]$.

We represent the diagram $\vec{\boldsymbol{D}}$ in Figure 4.2.

The verification of the following claim is straightforward.

Claim 1. $\vec{\boldsymbol{D}}$ *is a commutative diagram of bounded distributive lattices with* 0*-lattice homomorphisms.*



Each $\delta_\varnothing^p$ is the zero map $D_\varnothing \to D_p$

FIGURE 4.2. The diagram $\vec{D}$

Define 0-lattice homomorphisms $\eta_p \colon A_p \to D_p$, for $p \in \mathfrak{P}[3]$, as follows:

- Whenever $p \in \{\varnothing, 1, 2, 3\}$, η_p is the unique isomorphism from A_p onto D_p.
- Now we describe η_p for $p \in \{12, 13, 23\}$:

$$\eta_{12}\big(\langle t(a,b)\rangle_{A_{12}}\big) = \eta_{13}\big(\langle t(a',c)\rangle_{A_{13}}\big) = \eta_{23}\big(\langle t(b,c)\rangle_{A_{23}}\big) \overset{\text{def}}{=} [\![t(\mathsf{x}_1,\mathsf{x}_2) \neq 0]\!]_2\,,$$

  for every binary ℓ-term t.
- We finally describe η_{123}:

$$\eta_{123}\big(\langle t(a,a',b,c)\rangle_{A_{123}}\big) \overset{\text{def}}{=} [\![t(\mathsf{x}_1,\mathsf{x}_1,\mathsf{x}_2,\mathsf{x}_3) \neq 0]\!]_3\,,$$

  for every 4-ary ℓ-term t. This makes sense because for all $x_1, x_2, x_3 \in \mathbb{Q}^+$, the quadruple (x_1, x_1, x_2, x_3) satisfies the defining relations of A_{123}.

Claim 2. *The family $\vec{\eta} \overset{\text{def}}{=} (\eta_p \mid p \in \mathfrak{P}[3])$ is a natural transformation from $\vec{A}$ to $\vec{D}$. Furthermore, η_p is an isomorphism whenever $p \neq 123$.*

Proof of Claim. The statement about isomorphisms easily follows from the Baker-Beynon duality for finitely presented Abelian ℓ-groups (cf. Baker [3], Beynon [4]).

Now in order to verify that $\vec{\eta}$ is a natural transformation, it suffices to prove that $\eta_q \circ \alpha_p^q = \delta_p^q \circ \eta_p$ whenever p is a lower cover of q in $\mathfrak{P}[3]$. This is trivial if $p = \varnothing$, in which case both composed maps are zero. If $p = 1$ and $q = 13$, we compute

$$(\eta_{13} \circ \alpha_1^{13})\big(\langle a\rangle_{A_1}\big) = \eta_{13}\big(\langle a'\rangle_{A_{13}}\big) = [\![\mathsf{x}_1 > 0]\!]_2\,,$$
$$(\delta_1^{13} \circ \eta_1)\big(\langle a\rangle_{A_1}\big) = \delta_1^{13}(1) = [\![\mathsf{x}_1 > 0]\!]_2\,,$$

so we are done in that case. The other cases, where p has one element and q two elements, are handled similarly.

If $p = 13$ and $q = 123$, we compute, for every binary ℓ-term t,

$$(\eta_{123} \circ \alpha_{13}^{123})\big(\langle t(a',c)\rangle_{A_{13}}\big) = \eta_{123}\big(\langle t(a',c)\rangle_{A_{123}}\big) = [\![t(\mathsf{x}_1,\mathsf{x}_3) \neq 0]\!]_3\,,$$
$$(\delta_{13}^{123} \circ \eta_{13})\big(\langle t(a',c)\rangle_{A_{13}}\big) = \delta_{13}^{123}\big([\![t(\mathsf{x}_1,\mathsf{x}_2) \neq 0]\!]_2\big) = [\![t(\mathsf{x}_1,\mathsf{x}_3) \neq 0]\!]_3\,,$$

so we are done in that case. The other cases, where $p \in \{12, 23\}$ and $q = 123$, are handled similarly. □ Claim 2.

Now we argue by contradiction, by supposing that the $\boldsymbol{c}_{ij}$ satisfy Conditions (1)–(4) from the statement of Lemma 4.3. For $i \neq j$ in $[3]$, it follows from Condition (1) that $\boldsymbol{c}_{ij} = \langle c_{ij} \rangle_{A_{123}}$ for some $c_{ij} \in A_{ij}^{+}$. The set $C'_{ij} \overset{\text{def}}{=} \eta_{ij}\bigl(\langle c_{ij} \rangle_{A_{ij}}\bigr)$ belongs to $\boldsymbol{D}_{ij} = \mathcal{O}_2$, thus it is determined by its intersection with the segment (1-simplex) $\bigl\{(x,y) \in (\mathbb{Q}^{+})^2 \mid x+y=1\bigr\}$, which is a finite union of relatively open intervals of that segment. Hence, there exists $U_{ij} \in \mathcal{O}(\overline{\mathbb{Q}}^{+})$ such that

$$C'_{ij} = \bigl\{(x,y) \in (\mathbb{Q}^{+})^2 \setminus \{(0,0)\} \mid x^{-1}y \in U_{ij}\bigr\}\ . \tag{4.3}$$

Setting $C_{ij} \overset{\text{def}}{=} \eta_{123}(\boldsymbol{c}_{ij}) = \boldsymbol{\delta}_{ij}^{123}(C'_{ij})$, we get

$$C_{ij} = \begin{cases} \bigl\{(x_1,x_2,x_3) \in (\mathbb{Q}^{+})^3 \mid (x_i,x_j) \neq (0,0) \text{ and } x_i^{-1}x_j \in U_{ij}\bigr\} & \text{if } i<j\,,\\ \bigl\{(x_1,x_2,x_3) \in (\mathbb{Q}^{+})^3 \mid (x_i,x_j) \neq (0,0) \text{ and } x_j^{-1}x_i \in U_{ij}\bigr\} & \text{if } i>j\,. \end{cases} \tag{4.4}$$

By applying the homomorphism η_{123} to Conditions (2)–(4), we thus obtain, setting $P_i \overset{\text{def}}{=} [\![\mathsf{x}_i > 0]\!]_3$, the following relations:

(C1) $P_i \subseteq P_j \cup C_{ij}$ whenever $\{i,j\}$ is either $\{1,2\}$ or $\{2,3\}$.
(C2) $C_{ij} \cap C_{ji} = \varnothing$ whenever $\{i,j\}$ is either $\{1,2\}$ or $\{2,3\}$.
(C3) $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$.

By (C1) and since $(1,0,0) \in P_1 \setminus P_2$, we get $(1,0,0) \in C_{12}$, that is (cf. (4.4)), $0 \in U_{12}$. Similar arguments yield the relations $0 \in U_{23}$ and $0 \in U_{13}$. Similarly, since $(0,1,0) \in P_2 \setminus P_1$ and by (C1), we get $(0,1,0) \in C_{21}$, that is (cf. (4.4)), $\infty \in U_{21}$. Since $U_{21} \in \mathcal{O}(\overline{\mathbb{Q}}^{+})$, it follows that U_{21} contains an interval of the form $[z,\infty]$. From (C2) it follows that $U_{12} \cap U_{21} = \varnothing$, thus U_{12} is a bounded subset of $\mathbb{Q}^{+}$. We thus have proved that U_{12} *is a bounded subset of* $\mathbb{Q}^{+}$ *containing* 0 *as an element*. By a similar argument, U_{23} *is a bounded subset of* $\mathbb{Q}^{+}$ *containing* 0 *as an element*. Therefore, the assumptions of Proposition 3.1 are satisfied, so there are $\lambda, \mu \in \mathbb{Q}^{++}$ such that

$$U_{12} = [0,\lambda[\ , \quad U_{23} = [0,\mu[\ , \text{ and } U_{13} = [0,\lambda\mu[\ . \tag{4.5}$$

From $U_{12} = [0,\lambda[$ and (4.3) it follows that

$$\begin{aligned} \eta_{12}\bigl(\langle c_{12} \rangle_{A_{12}}\bigr) &= \bigl\{(x_1,x_2) \in (\mathbb{Q}^{+})^2 \setminus \{(0,0)\} \mid x_1^{-1}x_2 < \lambda\bigr\}\\ &= [\![\lambda\mathsf{x}_1 - \mathsf{x}_2 > 0]\!]\\ &= \eta_{12}\bigl(\langle (\lambda a - b)^{+} \rangle_{A_{12}}\bigr)\,. \end{aligned}$$

Since η_{12} is an isomorphism (cf. Claim 2), it follows that $\langle c_{12} \rangle_{A_{12}} = \langle (\lambda a - b)^{+} \rangle_{A_{12}}$, that is,

$$c_{12} \asymp (\lambda a - b)^{+} \text{ within } A_{12}\,. \tag{4.6}$$

Similar arguments, using (4.5), yield the relations

$$c_{23} \asymp (\mu b - c)^{+} \text{ within } A_{23} \quad \text{and} \quad c_{13} \asymp (\lambda\mu a' - c)^{+} \text{ within } A_{13}\,. \tag{4.7}$$

Condition (4), together with (4.6) and (4.7), thus yields

$$(\lambda a - b)^{+} \wedge (\mu b - c)^{+} \propto (\lambda\mu a' - c)^{+} \propto (\lambda a - b)^{+} \vee (\mu b - c)^{+} \text{ within } A_{123}\,. \tag{4.8}$$

Since the quadruple $(1,2,\lambda,\lambda\mu)$ of rational numbers satisfies the defining relations of A_{123}, there exists a unique ℓ-homomorphism $f\colon A_{123} \to \mathbb{Q}$ sending (a,a',b,c) to $(1,2,\lambda,\lambda\mu)$. By applying f to the right hand side inequality of (4.8), we obtain that $\lambda\mu = (2\lambda\mu - \lambda\mu)^{+} \propto 0$, a contradiction. □

5. Cevian operations

In this section we shall define *Cevian operations* on certain distributive lattices with zero. The existence of a Cevian operation is a strong form of complete normality. It will turn out that such operations exist on all lattices of the form $\mathrm{Cs_c}\, G$ (cf. Proposition 5.5) or $\mathrm{Id_c}\, G$ where the ℓ-group G is representable (cf. Proposition 5.10).

Definition 5.1. Let D be a distributive lattice with zero. A binary operation $\smallsetminus$ on D is *Cevian* if the following conditions hold:

(Cev1) $x \leq y \vee (x \smallsetminus y)$ for all $x, y \in D$;
(Cev2) $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$ for all $x, y \in D$;
(Cev3) $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z)$ for all $x, y, z \in D$.

A distributive lattice with zero is *Cevian* if it has a Cevian operation.

Obviously, every Cevian lattice is completely normal. The main example of Wehrung [19, § 6] shows that the converse fails at cardinality $\aleph_2$. We will shortly see that there is no such counterexample in the countable case (cf. Corollary 5.6). We will also find a new completely normal non-Cevian example of cardinality $\aleph_2$, with additional features, in Theorem 7.2.

Lemma 5.2. *Let $\smallsetminus$ be a binary operation on a distributive lattice D with zero, satisfying both* (Cev2) *and* (Cev3). *Then $(x \smallsetminus y) \wedge (y \smallsetminus z) \leq x \smallsetminus z$ for all $x, y, z \in D$.*

Proof. It follows from (Cev3) that

$$x \smallsetminus y \leq (x \smallsetminus z) \vee (z \smallsetminus y)\,. \tag{5.1}$$

Further, it follows from (Cev2) that $(y \smallsetminus z) \wedge (z \smallsetminus y) = 0$. Therefore, meeting (5.1) with $y \smallsetminus z$ and using the distributivity of D, we obtain

$$(x \smallsetminus y) \wedge (y \smallsetminus z) \leq (x \smallsetminus z) \wedge (y \smallsetminus z) \leq x \smallsetminus z\,. \qquad \square$$

Lemma 5.3.

(1) *Any product of a family of Cevian lattices is Cevian.*
(2) *Any homomorphic image of a Cevian lattice is Cevian.*
(3) *Any ideal of a Cevian lattice is Cevian.*

Proof. *Ad* (1). Let $\smallsetminus_i$ be a Cevian operation on D_i for each $i \in I$. On the product $D \overset{\mathrm{def}}{=} \prod_{i\in I} D_i$, set $x \smallsetminus y \overset{\mathrm{def}}{=} (x_i \smallsetminus_i y_i \mid i \in I)$.

Ad (2). Let $f \colon D \twoheadrightarrow E$ be a surjective lattice homomorphism and let $\smallsetminus_D$ be a Cevian operation on D. Then E is also a distributive lattice with zero. For each $x \in E$, pick a preimage $\overline{x}$ of x under f, and set $x \smallsetminus_E y \overset{\mathrm{def}}{=} f\bigl(\overline{x} \smallsetminus_D \overline{y}\bigr)$ for all $x, y \in E$. Then $\smallsetminus_E$ is a Cevian operation on E.

Ad (3). Say that a Cevian operation $\smallsetminus$ on D is *normalized* if $x \smallsetminus y \leq x$ for all $x, y \in D$. For every Cevian operation $\smallsetminus$, the variant operation $\smallsetminus'$ defined by

$$x \smallsetminus' y \overset{\mathrm{def}}{=} x \wedge (x \smallsetminus y)\,, \quad \text{for all } x, y \in D\,,$$

is a normalized Cevian operation on D. In particular, every ideal I of D is closed under $\smallsetminus'$, thus $\smallsetminus'$ defines, by restriction, a (normalized) Cevian operation on I. $\square$

For any elements x and y in an ℓ-group G, set $x \smallsetminus y \overset{\mathrm{def}}{=} (x - y)^+ = x - x \wedge y$; write $x \smallsetminus_G y$ instead of $x \smallsetminus y$ if G needs to be specified.

Lemma 5.4. *The operation* $\smallsetminus_G$*, defined on* G*, satisfies the statements* (Cev2) *and* (Cev3)*, for any* (*not necessarily Abelian*) ℓ*-group* G*.*

Proof. (Cev2) is easy: $(x \smallsetminus y) \wedge (y \smallsetminus x) = (x - x \wedge y) \wedge (y - x \wedge y) = x \wedge y - x \wedge y = 0$. For the right hand side inequality of (Cev3), observe that $x \leq (x \smallsetminus y) + y$ and $y \leq (y \smallsetminus z) + z$, thus $x \leq (x \smallsetminus y) + (y \smallsetminus z) + z$, and thus $x - z \leq (x \smallsetminus y) + (y \smallsetminus z)$. Since $0 \leq (x \smallsetminus y) + (y \smallsetminus z)$, it follows that $x \smallsetminus z = (x - z)^+ \leq (x \smallsetminus y) + (y \smallsetminus z)$. □

Proposition 5.5. *Let* G *be an* ℓ*-group. Then* $\mathrm{Cs_c}\, G$ *is a Cevian lattice.*

Proof. For any $\boldsymbol{x} \in \mathrm{Cs_c}\, G$, pick $\gamma(\boldsymbol{x}) \in G^+$ such that $\boldsymbol{x} = \langle \gamma(\boldsymbol{x}) \rangle$ and set

$$\boldsymbol{x} \smallsetminus \boldsymbol{y} \stackrel{\mathrm{def}}{=} \langle \gamma(\boldsymbol{x}) \smallsetminus_G \gamma(\boldsymbol{y}) \rangle\,, \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathrm{Cs_c}\, G\,.$$

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathrm{Cs_c}\, G$ with respective images x, y, z under γ. It follows from the equation $x = (x \smallsetminus_G y) + (x \wedge y)$ that $\langle x \rangle \subseteq \langle y \rangle \vee \langle x \smallsetminus_G y \rangle$, that is, $\boldsymbol{x} \subseteq \boldsymbol{y} \vee (\boldsymbol{x} \smallsetminus \boldsymbol{y})$; whence (Cev1) holds.

Using Lemma 5.4 together with Bigard *et al.* [5, Proposition 2.2.11], we get

$$(\boldsymbol{x} \smallsetminus \boldsymbol{y}) \wedge (\boldsymbol{y} \smallsetminus \boldsymbol{x}) = \langle x \smallsetminus_G y \rangle \cap \langle y \smallsetminus_G x \rangle = \langle (x \smallsetminus_G y) \wedge (y \smallsetminus_G x) \rangle = 0\,;$$

whence (Cev2) holds. By using Lemma 5.4, we also get

$$\boldsymbol{x} \smallsetminus \boldsymbol{z} = \langle x \smallsetminus_G z \rangle \subseteq \langle x \smallsetminus_G y \rangle \vee \langle y \smallsetminus_G z \rangle = (\boldsymbol{x} \smallsetminus \boldsymbol{y}) \vee (\boldsymbol{y} \smallsetminus \boldsymbol{z})\,;$$

whence (Cev3) holds. □

The main result of the author's paper [18] states that every countable completely normal distributive lattice with zero is isomorphic to $\mathrm{Id_c}\, G$ for some Abelian ℓ-group G. Consequently, by Proposition 5.5, we get:

Corollary 5.6. *A countable distributive lattice with zero is Cevian iff it is completely normal.*

Remark 5.7. The result of Proposition 5.5 cannot be extended to $\mathrm{Id_c}\, G$ for arbitrary ℓ-groups G. Indeed, we proved in Růžička *et al.* [17, Theorem 6.3] that *every countable distributive* $(\vee, 0)$*-semilattice is isomorphic to* $\mathrm{Id_c}\, G$ *for some* ℓ*-group* G*.* In particular, $\mathrm{Id_c}\, G$ may fail to be a lattice, and even if it is a lattice, it may fail to be completely normal (consider a square with a new zero element added).

Recall that an ℓ-group is *representable* if it is a subdirect product of totally ordered groups. Equivalently (cf. Bigard *et al* [5, Proposition 4.2.9]), G satisfies the identity $(2\mathsf{x}) \wedge (2\mathsf{y}) = 2(\mathsf{x} \wedge \mathsf{y})$. We will see shortly (cf. Proposition 5.10) that the kind of counterexample following from the results of [17, § 6] does not occur within the class of representable ℓ-groups.

Incidentally, it follows from [17, Corollary 3.9] that *not every distributive* $(\vee, 0)$*-semilattice is isomorphic to* $\mathrm{Id_c}\, G$ *for an* ℓ*-group* G*.*

Lemma 5.8. *Let* x*,* y*,* u*,* v *be elements in a representable* ℓ*-group* G*. Then*

$$(u + x - u) \wedge (v + y - v) \leq (u + x \wedge y - u) \vee (v + x \wedge y - v)\,. \tag{5.2}$$

Proof. It suffices to consider the case where G is totally ordered; so, by symmetry, we may assume that $x \leq y$. Then the right hand side of (5.2) is equal to $(u+x-u) \vee (v+x-v)$, which lies above $u+x-u$, thus above $(u+x-u) \wedge (v+y-v)$. □

Lemma 5.9. *Let* G *be a representable* ℓ*-group and let* $x, y \in G^+$*. Then* $\langle x \rangle^\ell \cap \langle y \rangle^\ell = \langle x \wedge y \rangle^\ell$*. Consequently,* $\mathrm{Id_c}\, G$ *is a distributive lattice.*

Proof. We prove the nontrivial containment. Any element of $\langle x\rangle^{\ell}$ lies, in absolute value, below a finite sum of conjugates of x; and similarly for $\langle y\rangle^{\ell}$ and y. By Bigard *et al.* [5, Théorème 1.2.16], it thus suffices to prove that $(u+x-u)\wedge(v+y-v)$ belongs to $\langle x\wedge y\rangle^{\ell}$ whenever $u,v\in G$. This follows immediately from Lemma 5.8. □

Proposition 5.10. *Let G be a representable ℓ-group. Then $\operatorname{Id_c} G$ is naturally (in the functorial sense) a homomorphic image of $\operatorname{Cs_c} G$. In particular, it is a Cevian lattice.*

Proof. By Lemma 5.3 and Proposition 5.5, it suffices to prove that $\operatorname{Id_c} G$ is a homomorphic image of $\operatorname{Cs_c} G$. By Lemma 5.9, the assignment $\langle x\rangle\mapsto\langle x\rangle^{\ell}$ defines a meet-homomorphism from $\operatorname{Cs_c} G$ onto $\operatorname{Id_c} G$, and this naturally in G. It is obviously a surjective join-homomorphism. □

Remark 5.11. We observed in [18, Example 10.6] that the class of all lattices of the form $\operatorname{Id_c} G$, with G an Abelian ℓ-group, is not closed under homomorphic images. Since every Abelian ℓ-group is representable and by Proposition 5.10, it follows that *not every Cevian distributive lattice with zero is isomorphic to $\operatorname{Id_c} G$ for some Abelian ℓ-group G.*

The following result shows that the non-commutativity of the diagram $\vec{A}$ can be read on the commutative diagram $\operatorname{Id_c}\vec{A}$.

Theorem 5.12. *Let $\vec{G}=(G_p,\gamma_p^q\mid p\leq q\text{ in }\mathfrak{P}[3])$ be a $\mathfrak{P}[3]$-indexed commutative diagram of ℓ-groups and ℓ-homomorphisms and let $\vec{\eta}=(\eta_p\mid p\in\mathfrak{P}[3])$ be a natural transformation from $\operatorname{Cs_c}\vec{G}$ to $\operatorname{Id_c}\vec{A}$. Then $\eta_i=0$ for some $i\in\{1,2,3\}$.*

Proof. Suppose otherwise. For each $i\in[3]$, there exists $c_i\in G_i^+$ such that $\eta_i(\langle c_i\rangle_{G_i})\neq 0$. Since $\operatorname{Id_c} A_i\cong\{0,1\}$, it follows that $\eta_i(\langle c_i\rangle_{G_i})=1$. Set $b_i\overset{\text{def}}{=}\gamma_i^{123}(\langle c_i\rangle_{G_i})$ and

$$\boldsymbol{c}_{ij}\overset{\text{def}}{=}\eta_{123}(\langle b_i\smallsetminus_{G_{123}} b_j\rangle_{G_{123}})$$

(an element of $\operatorname{Id_c} A_{123}$), for all distinct $i,j\in[3]$. Hence, the element

$$\begin{aligned}\boldsymbol{c}_{ij}&=(\eta_{123}\circ\operatorname{Cs_c}\gamma_{ij}^{123})\big(\langle\gamma_i^{ij}(g_i)\smallsetminus_{G_{ij}}\gamma_j^{ij}(g_j)\rangle_{G_{ij}}\big)\\&=(\boldsymbol{\alpha}_{ij}^{123}\circ\eta_{ij})\big(\langle\gamma_i^{ij}(g_i)\smallsetminus_{G_{ij}}\gamma_j^{ij}(g_j)\rangle_{G_{ij}}\big)\end{aligned}$$

belongs to the range of $\boldsymbol{\alpha}_{ij}^{123}$. Furthermore, for each $i\in[3]$,

$$\begin{aligned}\eta_{123}(\langle b_i\rangle_{G_{123}})&=(\eta_{123}\circ\operatorname{Cs_c}\gamma_i^{123})(\langle b_i\rangle_{G_i})\\&=(\boldsymbol{\alpha}_i^{123}\circ\eta_i)(\langle b_i\rangle_{G_i})\\&=\boldsymbol{\alpha}_i^{123}(1)\\&=\boldsymbol{a}_i\end{aligned}$$

(we defined the $\boldsymbol{a}_i$ in (4.2)). By using (the argument of) Proposition 5.5, together with Lemma 5.2, it follows that the elements $\boldsymbol{c}_{ij}$, where $i\neq j$ in [3], satisfy Assumptions (2)–(4) of the statement of Lemma 4.3; a contradiction. □

By using Proposition 5.10, we thus obtain

Corollary 5.13. *There is no $\mathfrak{P}[3]$-indexed commutative diagram $\vec{G}$ of ℓ-groups (resp., representable ℓ-groups) and ℓ-homomorphisms such that $\operatorname{Cs_c}\vec{G}\cong\operatorname{Id_c}\vec{A}$ (resp., $\operatorname{Id_c}\vec{G}\cong\operatorname{Id_c}\vec{A}$).*

6. A crash course on P-scaled Boolean algebras and condensates

In order to turn the diagram counterexample (Lemma 4.3) to an object counterexample (Theorem 7.2), we will need to apply a complex, technical result of category theory called the *Armature Lemma*, introduced in Gillibert and Wehrung [11, Lemma 3.2.2]. In order to help the reader get a feel of the machinery underlying that tool, we shall devote this section to giving a flavor of that machinery.

6.1. **P-scaled Boolean algebras, normal morphisms, $\mathbf{2}[p]$.** For an arbitrary poset P, a *P-scaled Boolean algebra* is a structure

$$\boldsymbol{A} = \bigl(A, (A^{(p)} \mid p \in P)\bigr)\,,$$

where A is a Boolean algebra, every $A^{(p)}$ is an ideal of A, $A = \bigvee(A^{(p)} \mid p \in P)$ within the ideal lattice of A, and $A^{(p)} \cap A^{(q)} = \bigvee(A^{(r)} \mid r \geq p, q)$ whenever $p, q \in P$. For P-scaled Boolean algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a *morphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a homomorphism $f \colon A \to B$ of Boolean algebras such that $f[A^{(p)}] \subseteq B^{(p)}$ for every $p \in P$. If f is surjective and $f[A^{(p)}] = B^{(p)}$ for every p, we say that f is *normal.* The category of all P-scaled Boolean algebras is denoted by $\mathbf{Bool}_P$. It has all small directed colimits and all finite products.

We prove in [11, Corollary 4.2.7] that a P-scaled Boolean algebra $\boldsymbol{A}$ is finitely presented (in the sense of Gabriel and Ulmer [8], Adámek and Rosický [1]) within $\mathbf{Bool}_P$ iff A is finite and for every atom a of A, the ideal $\|a\|_{\boldsymbol{A}} \overset{\text{def}}{=} \bigl\{p \in P \mid a \in A^{(p)}\bigr\}$ has a largest element, then denoted by $|a|_{\boldsymbol{A}}$. Finitely presented members of $\mathbf{Bool}_P$ approximate well the whole class:

— (cf. [11, Proposition 2.4.6]) Every member of $\mathbf{Bool}_P$ is a monomorphic directed colimit of a direct system of finitely presented members of $\mathbf{Bool}_P$.

— (cf. [11, Proposition 2.5.5]) Every normal morphism in $\mathbf{Bool}_P$ is a directed colimit, within the category $\mathbf{Bool}_P^{\mathbf{2}}$ of all arrows of $\mathbf{Bool}_P$, of a direct system of normal morphisms between finitely presented members of $\mathbf{Bool}_P$.

For every $p \in P$, we introduced in [11, Definition 2.6.1] the P-scaled Boolean algebra

$$\mathbf{2}[p] \overset{\text{def}}{=} \bigl(\mathbf{2}, (\mathbf{2}[p]^{(q)} \mid q \in P)\bigr)$$

where we define $\mathbf{2}[p]^{(q)}$ as $\{0,1\}$ if $q \leq p$, $\{0\}$ otherwise.

6.2. **Norm-coverings, ω-lifters, $\mathbf{F}(X)$, π_x^X.** Following [11, Definition 2.1.2], we say that a poset P is

— a *pseudo join-semilattice* if the set U of all upper bounds of any finite subset X of P is a finitely generated upper subset of P; then we denote by $\triangledown X$ the (finite) set of all minimal elements of U;

— *supported* if it is a pseudo join-semilattice and every finite subset of P is contained in a finite subset Y of P which is *$\triangledown$-closed*, that is, $\triangledown Z \subseteq Y$ whenever Z is a finite subset of Y;

— an *almost join-semilattice* if it is a pseudo join-semilattice in which every principal ideal $P \downarrow a$ is a join-semilattice.

We observed in [11, § 2.1] the non-reversible implications

$$\text{join-semilattice} \Rightarrow \text{almost join-semilattice} \Rightarrow \text{supported} \Rightarrow \text{pseudo join-semilattice}\,.$$

Following [11, § 2.6], a *norm-covering* of a poset P is a pair (X, ∂) where X is a pseudo join-semilattice and $\partial\colon X \to P$ is an isotone map. For such a norm-covering, we denote by $\mathrm{F}(X)$ the Boolean algebra defined by generators $\tilde{u}$, where $u \in X$, and relations

$$\begin{aligned} \tilde{v} &\leq \tilde{u}\,, && \text{whenever } u \leq v \text{ in } X\,;\\ \tilde{u} \wedge \tilde{v} &= \bigvee(\tilde{w} \mid x \in \bigtriangledown\{u,v\})\,, && \text{whenever } u, v \in X\,;\\ 1 &= \bigvee(\tilde{w} \mid w \in \bigtriangledown\varnothing)\,. \end{aligned}$$

Furthermore, for every $p \in P$, we denote by $\mathrm{F}(X)^{(p)}$ the ideal of $\mathrm{F}(X)$ generated by $\{\tilde{u} \mid u \in X\,,\ p \leq \partial u\}$. The structure $\mathbf{F}(X) \overset{\mathrm{def}}{=} \bigl(\mathrm{F}(X), (\mathrm{F}(X)^{(p)} \mid p \in P)\bigr)$ is a P-scaled Boolean algebra [11, Lemma 2.6.5].

In [11] we also introduce, for every $x \in X$, the unique morphism $\pi_x^X\colon \mathbf{F}(X) \to \mathbf{2}[\partial x]$ that sends every $\tilde{u}$, where $u \in X$, to 1 if $u \leq x$ and 0 otherwise. This morphism is normal (cf. [11, Lemma 2.6.7]).

We will also need here a specialization of the concept of λ-*lifter* (obtained by setting $\lambda = \aleph_0$ and $\boldsymbol{X}$ = set of all principal ideals of X) introduced in [11, § 3.2].

Definition 6.1. A *principal ω-lifter* of a poset P is a norm-covering (X, ∂) of P such that

(1) the set $X^{=} \overset{\mathrm{def}}{=} \{x \in X \mid \partial x \text{ is not maximal in } P\}$ is lower finite;
(2) X is supported;
(3) For every map $S\colon X^{=} \to [X]^{<\omega}$, there exists an isotone section σ of ∂ such that $S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$ for all $p < q$ in P.

6.3. **The construction $\boldsymbol{A} \otimes \vec{S}$.** Let a category $\mathcal{S}$ have all nonempty finite products and all small directed colimits. Let $\vec{S} = (S_p, \sigma_p^q \mid p \leq q \text{ in } P)$ be a P-indexed direct system in $\mathcal{S}$. The functor $\_ \otimes \vec{S}\colon \mathbf{Bool}_P \to \mathcal{S}$ is first defined on all finitely presented members of $\mathbf{Bool}_P$, as follows. If $\boldsymbol{A}$ is finitely presented, we set

$$\boldsymbol{A} \otimes \vec{S} \overset{\mathrm{def}}{=} \prod(S_{|a|_{\boldsymbol{A}}} \mid a \in \mathrm{At}\, A)\,.$$

In particular, $\mathbf{2}[p] \otimes \vec{S} = S_p$. For a morphism $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ between finitely presented P-scaled Boolean algebras and an atom b of B, we define b^φ as the unique atom of A such that $b \leq \varphi(b^\varphi)$. Then the product morphism $\varphi \otimes \vec{S} \overset{\mathrm{def}}{=} \prod\Bigl(\sigma_{|b^\varphi|_{\boldsymbol{A}}}^{|b|_{\boldsymbol{B}}} \mid b \in \mathrm{At}\, B\Bigr)$ goes from $\boldsymbol{A} \otimes \vec{S}$ to $\boldsymbol{B} \otimes \vec{S}$. This defines a functor from the finitely presented members of $\mathbf{Bool}_P$ to $\mathcal{S}$. Since every member of $\mathbf{Bool}_P$ is a small directed colimit of a direct system of finitely presented objects, it follows, using [11, Proposition 1.4.2], that this functor can be uniquely extended, up to isomorphism, to a functor from $\mathbf{Bool}_P$ to $\mathcal{S}$ that preserves all small directed colimits. This functor will also be denoted by $\_ \otimes \vec{S}$. For a P-scaled Boolean algebra $\boldsymbol{A}$, the object $\boldsymbol{A} \otimes \vec{S}$ will be called a *condensate* of $\vec{S}$.

In the particular case where φ is a normal morphism, $\varphi \otimes \vec{S}$ is a directed colimit of projection morphisms (i.e., canonical morphisms $X \times Y \to X$). Now in all the cases we will be interested in, $\mathcal{S}$ will be a category of models of first-order languages, so projection morphisms are surjective, thus so are their directed colimits. Hence, in all those cases, if φ is a *normal* morphism, then $\varphi \otimes \vec{S}$ is *surjective*.

7. A non-Cevian lattice with countably based differences

Throughout this section, we shall consider P-scaled Boolean algebras with

$$P = \mathfrak{P}[3] = \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}$$

(cf. (4.1)). Since P has exactly three join-irreducible elements (viz. 1, 2, 3), it follows from Gillibert and Wehrung [12, Proposition 4.2] that the relation denoted there by $(\aleph_2, {<}\aleph_0) \rightsquigarrow P$ holds. This means that for every mapping $F\colon \mathfrak{P}(\omega_2) \to [\omega_2]^{<\omega}$, there exists a one-to-one map $f\colon P \hookrightarrow \omega_2$ such that $F(f[P{\downarrow}x]) \cap f[P{\downarrow}y] \subseteq f[P{\downarrow}x]$ whenever $x \leq y$ in P.

Now define X as the poset denoted by $P\langle\omega_2\rangle$ in the proof of [11, Lemma 3.5.5], together with the canonical isotone map $\partial\colon X \to P$. That is,

$$X = \Big\{(a, u) \mid a \in P\,,\ u\colon U \to \omega_2 \text{ with } a = \bigvee U\Big\}$$

with componentwise ordering ($\leq$ on the first component, extension ordering on the second one), and $\partial(a, u) = a$ whenever $(a, u) \in X$. It follows from (the proof of) [11, Lemma 3.5.5] that X is lower finite and that furthermore, it is, together with the map ∂, a *principal ω-lifter* of P (cf. Definition 6.1).

We apply the Armature Lemma to the following data:

- $\mathcal{S}$ is the category of all distributive lattices with zero with 0-lattice homomorphisms;
- $\mathcal{A}$ is the subcategory of $\mathcal{S}$ whose objects are the completely normal members of $\mathcal{S}$ with countably based differences, and whose morphisms are the closed 0-lattice homomorphisms;
- Φ is the inclusion functor from $\mathcal{A}$ into $\mathcal{S}$.
- $\vec{S} \overset{\text{def}}{=} \boldsymbol{A} = \mathrm{Id_c}\,\vec{A}$, where $\vec{A}$ is the diagram introduced in Section 4.

Lemma 7.1. *$\mathcal{A}$ is a subcategory of $\mathcal{S}$, closed under all small directed colimits and finite products.*

Proof. The statement about finite products is straightforward. Now let

$$(D, \delta_i \mid i \in I) = \varinjlim \vec{D}$$

within the category of all distributive lattices with zero and 0-lattice homomorphisms, where $\vec{D} = (D_i\,, \delta_i^j \mid i \leq j \text{ in } I)$ is a direct system in $\mathcal{A}$. Hence,

$$D = \bigcup(\delta_i[D_i] \mid i \in I) \text{ (directed union)}\,; \tag{7.1}$$

$$\text{For all } i \in I \text{ and all } x, y \in D_i\,,\ \delta_i(x) = \delta_i(y) \Rightarrow (\exists j \geq i)\big(\delta_i^j(x) = \delta_i^j(y)\big)\,. \tag{7.2}$$

It is straightforward to verify, using (7.1) and (7.2), that D is a completely normal distributive lattice with zero (for this we do not need the assumption that the δ_i^j are closed maps).

Let $i \in I$, let $x, x' \in D_i$ and $\overline{y} \in D$ such that $\delta_i(x) \leq \delta_i(x') \vee \overline{y}$. By (7.1), there are $j \in I$ and $y \in D_j$ such that $\overline{y} = \delta_j(y)$; since I is directed, we may assume that $j \geq i$. By (7.2), there exists $k \geq j$ such that $\delta_i^k(x) \leq \delta_i^k(x') \vee \delta_j^k(y)$. Since the map δ_i^k is closed, there is $z \in D_i$ such that $x \leq x' \vee z$ and $\delta_i^k(z) \leq \delta_j^k(y)$. Now the latter inequality implies that $\delta_i(z) \leq \delta_j(y) = \overline{y}$, thus completing the proof that the map δ_i is closed.

Now let $\overline{x}, \overline{y} \in D$. We claim that $\overline{x} \ominus_D \overline{y}$ has a countable coinitial subset. By (7.1), there are $i \in I$ and $x, y \in D_i$ such that $\overline{x} = \delta_i(x)$ and $\overline{y} = \delta_i(y)$. Since D_i has countably based differences, $x \ominus_{D_i} y$ has a countable coinitial subset. Since, by the paragraph above, δ_i is closed and by Lemma 2.1, it follows that $\overline{x} \ominus_D \overline{y}$ has a countable coinitial subset. Therefore, D has countably based differences. □

By Lemma 7.1, $\mathbf{F}(X) \otimes \vec{\boldsymbol{A}}$ (cf. Section 6) denotes the same object in $\mathcal{A}$ as in $\mathcal{S}$. Denote it by $\boldsymbol{B}$.

Theorem 7.2. *The structure $\boldsymbol{B}$ is a non-Cevian completely normal distributive lattice with zero and countably based differences. It has cardinality $\aleph_2$.*

Proof. Since $\boldsymbol{B}$ is an object of $\mathcal{A}$, it is a completely normal distributive lattice with zero and countably based differences. Since X has cardinality $\aleph_2$, so does $\mathbf{F}(X)$, and so $\mathbf{F}(X)$ is the directed colimit of a diagram, indexed by a set of cardinality $\aleph_2$, of finitely presented P-scaled Boolean algebras; since all $\boldsymbol{A}_p$ are countable, it follows that $\boldsymbol{B} = \mathbf{F}(X) \otimes \vec{\boldsymbol{A}}$ has cardinality at most $\aleph_2$.

We claim that $\boldsymbol{B}$ has cardinality exactly $\aleph_2$. Indeed, for every $\xi < \omega_2$, denote by u_ξ the constant function on the singleton $\{123\}$ with value ξ. The pair $v_\xi \overset{\mathrm{def}}{=} (123, u_\xi)$ belongs to X with $p \leq 123 = \partial v_\xi$ whenever $p \in P$; thus $\tilde{v}_\xi \in \mathrm{F}(X)^{(p)}$. Hence, the Boolean subalgebra $V_\xi \overset{\mathrm{def}}{=} \{0, \tilde{v}_\xi, \neg\tilde{v}_\xi, 1\}$ of $\mathrm{F}(X)$, endowed with the ideals

$$V_\xi^{(p)} \overset{\mathrm{def}}{=} \begin{cases} \{0, \tilde{v}_\xi\}\,, & \text{if } p \neq \varnothing\,, \\ V_\xi\,, & \text{if } p = \varnothing\,, \end{cases} \qquad \text{for } p \in P\,,$$

defines a finitely presented P-scaled Boolean algebra $\boldsymbol{V}_\xi$, and the inclusion map from $\boldsymbol{V}_\xi$ into $\mathrm{F}(X)$ defines a morphism $\boldsymbol{V}_\xi \to \mathbf{F}(X)$ in $\mathbf{Bool}_P$, which in turns yields a morphism $e_\xi \colon \boldsymbol{V}_\xi \otimes \vec{\boldsymbol{A}} \to \mathbf{F}(X) \otimes \vec{\boldsymbol{A}}$ in $\mathcal{A}$. Now pick any $\boldsymbol{u} \in \boldsymbol{A}_{123} \setminus \{0\}$. Using the canonical isomorphisms $\boldsymbol{V}_\xi \otimes \vec{\boldsymbol{A}} \cong \boldsymbol{A}_{123} \times \boldsymbol{A}_\varnothing \cong \boldsymbol{A}_{123}$, it can be verified that the elements $e_\xi(\boldsymbol{u})$, for $\xi < \omega_2$, are pairwise distinct. (Think of $e_\xi(\boldsymbol{u})$ as the constant map with value $\boldsymbol{u}$ on the clopen subset of the Stone space of $\mathbf{F}(X)$ associated to v_ξ.)

Finally, towards a contradiction, we shall suppose that $\boldsymbol{B}$ has a Cevian operation $\smallsetminus$. Set

$$X_{(k)} \overset{\mathrm{def}}{=} \{x \in X \mid \partial x \text{ has height } k \text{ within } P\}\,,$$

for every nonnegative integer k. In particular, $X_{(k)}$ is nonempty iff $k \in \{0,1,2,3\}$, so X is the disjoint union of $X_{(0)}$, $X_{(1)}$, $X_{(2)}$, $X_{(3)}$. Further, $X_{(1)} = \partial^{-1}\{1,2,3\}$.

For each $x \in X$, the map $\rho_x \overset{\mathrm{def}}{=} \pi_x^X \otimes \vec{\boldsymbol{A}}$ is a surjective lattice homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{A}_{\partial x}$ (cf. 2.6.7, 3.1.2, and 3.1.3 in [11]). In particular, if $x \in X_{(1)}$, then $\partial x \in \{1,2,3\}$, thus $\boldsymbol{A}_{\partial x} \cong \{0,1\}$, and we may pick $\boldsymbol{b}_x \in \boldsymbol{B}$ such that $\rho_x(\boldsymbol{b}_x) = 1$. For those x, $\boldsymbol{B}_x \overset{\mathrm{def}}{=} \{0, \boldsymbol{b}_x\}$ is a 0-sublattice of $\boldsymbol{B}$.

Now for each $x \in X_{(2)} \cup X_{(3)}$, it follows from the lower finiteness of X that the 0-sublattice $\boldsymbol{B}_x$ of $\boldsymbol{B}$ generated by all elements of $\boldsymbol{B}$ of the form either $\boldsymbol{b}_u$ or $\boldsymbol{b}_u \smallsetminus \boldsymbol{b}_v$, where $u, v \in X_{(1)} \downarrow x$, is finite[3]. For any $x \in X$, $\boldsymbol{B}_x$ is thus a finite 0-sublattice of $\boldsymbol{B}$. Denote by $\varphi_x \colon \boldsymbol{B}_x \hookrightarrow \boldsymbol{B}$ the inclusion map, and by $\varphi_x^y \colon \boldsymbol{B}_x \hookrightarrow \boldsymbol{B}_y$ the inclusion map in case $x \leq y$. Hence $\vec{\boldsymbol{B}} \overset{\mathrm{def}}{=} (\boldsymbol{B}_x, \varphi_x^y \mid x \leq y \text{ in } X)$ is an X-indexed commutative diagram in $\mathcal{S}$.

[3]In the original statement of the Armature Lemma, we need $\boldsymbol{B}_{\boldsymbol{x}}$ to be defined whenever $\boldsymbol{x}$ is a certain kind of *ideal* of X. However, since (X, ∂) is a principal lifter of P, it suffices here to consider the case where $\boldsymbol{x}$ is a principal ideal, which is then identified to its largest element.

Since all $\boldsymbol{B}_x$ are finite, they are finitely presented within $\mathcal{S}$, thus we can apply the Armature Lemma [11, Lemma 3.2.2] to those data, with the $\boldsymbol{B}_x$ in place of the required S_x and the identity of $\boldsymbol{B}$ in place of χ. We get an isotone section $\sigma\colon P \hookrightarrow X$ of ∂ such that the family

$$\vec{\chi} = (\chi_p \mid p \in P) \overset{\text{def}}{=} (\rho_{\sigma(p)}{\restriction}_{\boldsymbol{B}_{\sigma(p)}} \mid p \in P)$$

is a natural transformation from $\vec{\boldsymbol{B}}\sigma \overset{\text{def}}{=} (\boldsymbol{B}_{\sigma(p)}, \varphi_{\sigma(p)}^{\sigma(q)} \mid p \leq q \text{ in } P)$ to $\vec{\boldsymbol{A}}$. This means that for all $p \leq q$ in P, the square represented in Figure 7.1 is commutative.

$$\begin{array}{ccc}
\boldsymbol{B}_{\sigma(q)} & \xrightarrow{\chi_q} & \boldsymbol{A}_q \\
{\scriptstyle \varphi_{\sigma(p)}^{\sigma(q)}}\uparrow & & \uparrow{\scriptstyle \boldsymbol{\alpha}_p^q} \\
\boldsymbol{B}_{\sigma(p)} & \xrightarrow{\chi_p} & \boldsymbol{A}_p
\end{array}$$

FIGURE 7.1. The natural transformation $\vec{\chi}$

For each $p \in \{1, 2, 3\}$, $\sigma(p) \in X_{(1)}$ thus $\boldsymbol{B}_{\sigma(p)} = \bigl\{0, \boldsymbol{b}_{\sigma(p)}\bigr\}$ and

$$\chi_p(\boldsymbol{b}_{\sigma(p)}) = \rho_{\sigma(p)}(\boldsymbol{b}_{\sigma(p)}) = 1\,,$$

so, using the commutativity of the diagram of Figure 7.1 with $q \overset{\text{def}}{=} 123$,

$$\chi_{123}(\boldsymbol{b}_{\sigma(p)}) = \bigl(\chi_{123} \circ \varphi_{\sigma(p)}^{\sigma(123)}\bigr)(\boldsymbol{b}_{\sigma(p)}) = (\boldsymbol{\alpha}_p^{123} \circ \chi_p)(\boldsymbol{b}_{\sigma(p)}) = \boldsymbol{\alpha}_p^{123}(1) = \boldsymbol{a}_p \tag{7.3}$$

(we defined the $\boldsymbol{a}_i$ in (4.2)).

Now we set

$$\boldsymbol{c}_{ij} \overset{\text{def}}{=} \chi_{123}(\boldsymbol{b}_{\sigma(i)} \smallsetminus \boldsymbol{b}_{\sigma(j)})\,, \quad \text{for all distinct } i, j \in [3]\,.$$

Since $\sigma(i)$ and $\sigma(j)$ both lie below $\sigma(ij)$, $\boldsymbol{b}_{\sigma(i)} \smallsetminus \boldsymbol{b}_{\sigma(j)}$ belongs to $\boldsymbol{B}_{\sigma(ij)}$, thus

$$\boldsymbol{c}_{ij} = \bigl(\chi_{123} \circ \varphi_{\sigma(ij)}^{\sigma(123)}\bigr)(\boldsymbol{b}_{\sigma(i)} \smallsetminus \boldsymbol{b}_{\sigma(j)}) = \bigl(\boldsymbol{\alpha}_{ij}^{123} \circ \chi_{ij}\bigr)(\boldsymbol{b}_{\sigma(i)} \smallsetminus \boldsymbol{b}_{\sigma(j)})$$

belongs to the range of $\boldsymbol{\alpha}_{ij}^{123}$.

Since $\smallsetminus$ is a Cevian operation, the inequality $\boldsymbol{b}_{\sigma(i)} \leq \boldsymbol{b}_{\sigma(j)} \vee (\boldsymbol{b}_{\sigma(i)} \smallsetminus \boldsymbol{b}_{\sigma(j)})$ holds, thus, applying the homomorphism χ_{123} and using (7.3), we get

$$\boldsymbol{a}_i \leq \boldsymbol{a}_j \vee \boldsymbol{c}_{ij}\,.$$

Similarly, the equation

$$(\boldsymbol{b}_{\sigma(i)} \smallsetminus \boldsymbol{b}_{\sigma(j)}) \wedge (\boldsymbol{b}_{\sigma(j)} \smallsetminus \boldsymbol{b}_{\sigma(i)}) = 0$$

holds, thus, applying χ_{123}, we get

$$\boldsymbol{c}_{ij} \wedge \boldsymbol{c}_{ji} = 0\,.$$

Finally, the inequalities

$$(\boldsymbol{b}_{\sigma(1)} \smallsetminus \boldsymbol{b}_{\sigma(2)}) \wedge (\boldsymbol{b}_{\sigma(2)} \smallsetminus \boldsymbol{b}_{\sigma(3)}) \leq \boldsymbol{b}_{\sigma(1)} \smallsetminus \boldsymbol{b}_{\sigma(3)} \leq (\boldsymbol{b}_{\sigma(1)} \smallsetminus \boldsymbol{b}_{\sigma(2)}) \vee (\boldsymbol{b}_{\sigma(2)} \smallsetminus \boldsymbol{b}_{\sigma(3)})$$

hold (use Lemma 5.2), thus, applying χ_{123}, we get

$$\boldsymbol{c}_{12} \wedge \boldsymbol{c}_{23} \leq \boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}\,.$$

We have thus proved that the elements $\boldsymbol{c}_{ij}$, for $i \neq j$ in [3], satisfy Conditions (1)–(4) in the statement of Lemma 4.3; a contradiction. □

We obtain the following object (as opposed to diagram) version of Corollary 5.13.

Corollary 7.3. *There exists a non-Cevian bounded completely normal distributive lattice with countably based differences, of cardinality* $\aleph_2$ *. Hence,* $\boldsymbol{B}'$ *is not a homomorphic image of* $\mathrm{Cs_c}\, G$ *for any* ℓ*-group* G *or of* $\mathrm{Id_c}\, G$ *for any representable* ℓ*-group* G*.*

Proof. Let $\boldsymbol{B}'$ be obtained by adding a new top element to $\boldsymbol{B}$. Since $\boldsymbol{B}$ is an ideal of $\boldsymbol{B}'$ and $\boldsymbol{B}$ is not Cevian, neither is $\boldsymbol{B}'$ (cf. Lemma 5.3).

The last part of the statement of Corollary 7.3 follows immediately, using Lemma 5.3, from Propositions 5.5 and 5.10. □

Remark 7.4. A blunt application of [11, Lemma 3.4.2] (called there CLL) to the non-lifting result obtained in Corollary 5.13 would have yielded, in the statement of Corollary 7.3, a counterexample of cardinality $\aleph_3$ (as opposed to $\aleph_2$). In order to get around that difficulty, we are applying here the Armature Lemma to the result of Lemma 4.3, which can be viewed as a "local" version of Corollary 5.13. This technique was first put to full use in Gillibert [10]. It was instrumental in the proof, in Gillibert deep paper [9], that *the congruence class of any finitely generated lattice variety* $\mathcal{V}$ *determines the pair consisting of* $\mathcal{V}$ *and its dual variety.*

Remark 7.5. In the sequel [20] to the present paper, we investigate an apparently innocuous extension of the construction $\boldsymbol{A} \otimes \vec{S}$, denoted there by $\boldsymbol{A} \otimes_{\Phi}^{\lambda} \vec{S}$. This construction, applied to $\vec{S} \overset{\mathrm{def}}{=} \vec{A}$ (the diagram introduced in Section 4) yields for example that *for any infinite cardinal* λ*, the class of principal ideal lattices of all Abelian* ℓ*-groups with order-unit is not closed under* $\mathscr{L}_{\infty\lambda}$*-elementary equivalence.*

Problem. Let D be a Cevian lattice with zero and with countably based differences. Is there an Abelian ℓ-group G such that $D \cong \mathrm{Id_c}\, G$?

The counterexample given in Remark 5.11 shows that "Cevian" alone is not sufficient to get representability as $\mathrm{Id_c}\, G$, while Corollary 7.3 shows that "countably based differences" alone is also not sufficient. On the other hand, both "Cevian" and "countably based differences" are preserved under retracts, while it is not known whether any retract of a lattice of the form $\mathrm{Id_c}\, G$, for G an Abelian ℓ-group, has this form (cf. [18, Problem 2]). This would rather suggest a negative answer to the Problem above.

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