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Abstract. We study the asymptotic behaviour for asymmetric neuronal dynamics in a network of linear Hopfield neurons. The randomness in the network is modelled by random couplings which are centered i.i.d. random variables with finite moments of all orders. We prove that if the initial condition of the network is a set of i.i.d random variables with finite moments of all orders and independent of the synaptic weights, each component of the limit system is described as the sum of the corresponding coordinate of the initial condition with a centered Gaussian process whose covariance function can be described in terms of a modified Bessel function. This process is not Markovian. The convergence is in law almost surely w.r.t. the random weights. Our method is essentially based on the CLT and the method of moments.

AMS Subject of Classification (2010):
60F10, 60H10, 60K35, 82C44, 82C31, 82C22, 92B20 proc

1. Introduction

We revisit the problem of characterizing the limit of a network of Hopfield neurons. Hopfield [6] defined a large class of neuronal networks and characterized some of their computational properties [7, 8], i.e. their ability to perform computations. Inspired by his work Sompolinsky and co-workers studied the thermodynamic limit of these networks when the interaction term is linear [3] using the dynamic mean-field theory developed in [11] for symmetric spin glasses. The method they use is a functional integral formalism used in particle physics and produces the self-consistent mean-field equations of the network. This was later extended to the case of a nonlinear interaction term, the nonlinearity being an odd sigmoidal function [10]. Using the same formalism the authors established the self-consistent mean-field equations of the network and the dynamics of its solutions which featured a chaotic behaviour for some values of the network parameters. A little later the problem was picked up again by mathematicians. BenArous and Guionnet applied large deviation techniques to study the thermodynamic limit of a network of spins interacting linearly with i.i.d. centered Gaussian weights. The intrinsic spin dynamics (without interactions) is a stochastic differential equation where the drift is the gradient of a potential. They prove that the annealed (averaged) law of the empirical measure satisfies a large deviation principle and that the good rate function of this large deviation principle achieves its minimum value at a unique measure which is not Markovian [4, 1, 5]. They also prove averaged propagation of chaos results. Moynot and Samuelides [9] adapt their work to the case of a network of Hopfield neurons with a nonlinear interaction term, the nonlinearity being a sigmoidal function, and prove similar results in the case of discrete time. The intrinsic neural dynamics is the gradient of a quadratic potential. Our work is in-between that of BenArous-Guionnet and Moynot-Samuelides: we consider a network of Hopfield network, hence the intrinsic dynamics is simpler than the one in BenArous-Guionnet's case, with linear interaction between the neurons, hence simpler than the one in Moynot-Samuelides' work. We do not make the hypothesis that the interaction (synaptic) weights are Gaussian unlike the previous authors. The equations of our network are linear and therefore their solutions can be expressed analytically. Thanks to this we are able to use variants of the CLT and the moments method to characterise in a simpler way the Thermodynamic limit of our network without the tools of the theory of large deviations. Our main result is that the solution to the network equations converges in law toward a non Markovian process, sum of the initial condition and a centered Gaussian process whose covariance is characterized by a modified Bessel function.

2. Network model

We consider a network of N neurons in interaction. Each neuron $i \in \{1, \cdots, N\}$ is characterized by its membrane potential $(V_t^{i,(N)})_t$ where $t \in \mathbb{R}_+$ represents the time. The membrane potential of neuron i is described by the stochastic differential equation

$$\begin{cases} \mathrm{d}V_t^{i,(N)} = -\lambda V_t^{i,(N)} \mathrm{d}t + \dfrac{1}{\sqrt{N}} \displaystyle\sum_{j=1}^{N} J_{i,j}^{(N)} V_t^{j,(N)} \mathrm{d}t + \gamma \mathrm{d}B_t^i \\ \mathcal{L}(V_0^{(N)}) = \nu_0^{\otimes N}, \end{cases} \tag{2.1}$$

where $V_0^{(N)} = \left(V_0^1, \cdots, V_0^N\right)$ is the vector of initial conditions. The matrix $J^{(N)}$ is a square matrix of size N and contains the synaptic weights. The parameters λ and γ are real constants and $(B_t^i)_t$ models the internal noise of the neuron i. The $(B_t^i)_t$, $i \in \{1, \cdots, N\}$ are N independent standard Brownian motions. The initial condition is a random vector with *i.i.d.* coordinates, each of distribution ν_0. Each initial condition V_0^i is fixed and does not depend on the size N of the network. For the sake of simplicity, we remove the size exponent N in $J^{(N)}$, i.e. J is the matrix of the synaptic weights, for all $N \in \mathbb{N}$.

We denote $V_t^{(N)} = (V_t^{1,(N)}, \cdots, V_t^{N,(N)})$ and write system (2.1) in matrix form:

$$\begin{cases} \mathrm{d}V_t^{(N)} = -\lambda V_t^{(N)} \mathrm{d}t + \dfrac{J}{\sqrt{N}} V_t^{(N)} \mathrm{d}t + \gamma \mathrm{d}B_t \\ \mathcal{L}\left(V_0^{(N)}\right) = \nu_0^{\otimes N}. \end{cases} \tag{2.2}$$

System (2.2) can be solved explicitly, its solution $V_t^{(N)}$ being given by,

$$V_t^{(N)} = e^{-\lambda t} \left[\exp\left(\frac{J}{\sqrt{N}} t\right) V_0^{(N)} + \gamma \int_0^t e^{\lambda s} \exp\left(\frac{J}{\sqrt{N}} (t-s)\right) \mathrm{d}B_s \right], \quad \forall t \in \mathbb{R}_+. \tag{2.3}$$

We make the following hypotheses

(H1) : ν_0 has finite moments of all positive orders:

$$\left| \int_{\mathbb{R}} x^p \, d\nu_0(x) \right| < +\infty \quad \text{for all } p \in \mathbb{N}$$

we note

$$\mu_0 := \int_{\mathbb{R}} x \, d\nu_0(x) \quad \text{and} \quad \phi_0 = \int_{\mathbb{R}} x^2 \, d\nu_0(x)$$

(H2) : The elements of the random matrix J are *i.i.d.* centered random variables of variance σ^2 and with finite moments of all orders. They are independent of the initial condition.

3. Convergence of the particle system without noise

3.1. **Mean field limit.** In this section, we consider the case $\gamma = 0$ (no noise). We have the following result on the convergence of the coordinates of vector $(V_t^{(N)})_{t\in\mathbb{R}_+}$, for $N \to +\infty$ to a Gaussian process whose covariance is determined by a Bessel function.

Theorem 3.1. *Under the hypothesis (H1) and (H2), for each $k \in \mathbb{N}$, the process $(V_t^{k,(N)})_{t\in\mathbb{R}_+}$ converges in law to $(V_t^{k,(\infty)})_{t\in\mathbb{R}_+}$ where, for all $t \in \mathbb{R}_+$,*

$$V_t^{k,(\infty)} = e^{-\lambda t}\left[V_0^k + Z_t^k\right], \qquad \forall t \in \mathbb{R}_+, \quad \mathcal{L}(Z_t^k) = \mathcal{N}\left(0, \phi_0 \widetilde{I}_0(2\sigma t)\right),$$

with

$$\widetilde{I}_0(z) = \sum_{\ell\geq 1} z^{2\ell}/(2^{2\ell}(\ell!)^2). \tag{3.1}$$

Moreover, for all $t \in \mathbb{R}_+$, Z_t^k is independent of V_0^k.

The limiting process is then entirely determined by its initial condition and the Gaussian process Z_t^k independent of V_0^k.

Remark 3.2. *The modified Bessel function of the first kind I_0 is defined as being the solution of the ordinary differential equation $z^2y'' + zy' - z^2y = 0$. This function is the sum of the series $(z^{2\ell}/(2^{2\ell}(\ell!)^2))_{\ell\geq 0}$ which is absolutely convergent for all $z \in \mathbb{C}$, i.e.: $I_0(z) = \sum_{\ell\geq 0} z^{2\ell}/(2^{2\ell}(\ell!)^2)$, so that we have*

$$\widetilde{I}_0(z) = I_0(z) - 1.$$

The hypothesis of independence between J and $V_0^{(N)}$ is crucial in the proof of Theorem 3.1. Therefore the proof is not valid if we start the system at a time $t_1 > 0$, in others words, we can not establish the existence of a process $(\bar{Z}_t^k)_{t\in\mathbb{R}_+}$ which has the same distribution as $(Z_t^k)_{t\in\mathbb{R}_+}$ and independent of $V_{t_1}^k$ such that

$$V_{t_1+t}^{k,(\infty)} = e^{-\lambda t}\left(V_{t_1}^{k,(\infty)} + \bar{Z}_t^k\right). \tag{3.2}$$

Remark 3.3. *The proof of Theorem 3.1 shows that the limiting process Z_t^k is an infinite sum of independent Gaussian variables. Let $(G_\ell)_{\ell\geq 1}$ be a infinite sequence of independent standard Gaussian random variables, then the following equality holds in law:*

$$Z_t^k \overset{\mathcal{L}}{=} \sqrt{\phi_0} \sum_{\ell=1}^{\infty} \frac{t^\ell \sigma^\ell}{\ell!} G_\ell^k, \qquad \forall t \in \mathbb{R}_+. \tag{3.3}$$

To obtain the proof of Theorem 3.1 as well as the covariance of the limiting process, we need the following lemma

Lemma 3.4. *Let $(Y_{\ell,j})_{(j,\ell)\in\mathbb{N}^2}$ be a sequence random variables independent of J, whose coordinates are not necessarily independent and identically distributed and such that, for all $p \in \mathbb{N}$, and for $\ell \in \mathbb{N}$ fixed,*

$$\lim_{N\to+\infty} \mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{N}(Y_{\ell,j})^2\right)^p\right] = \phi_{\ell,p} < +\infty. \tag{3.4}$$

For all $\ell \in \mathbb{N}$, *let* $U_\ell^{(N)} = \frac{1}{\sqrt{N}^\ell} \sum_{j=1}^N (J^\ell)_{1,j} Y_{\ell,j}$, *we have for all* $p \in \mathbb{N}$

i- $\lim_{N\to+\infty} \mathbb{E}\left[(U_\ell^{(N)})^{2p}\right] = \phi_{\ell,p} \dfrac{(2p)!}{p!\,2^p} \sigma^{2\ell p}$ *and* $\lim_{N\to+\infty} \mathbb{E}\left[(U_\ell^{(N)})^{2p+1}\right] = 0$

ii- Given any family of m *integers* $(b_i)_{1\le i\le m}$, $m \in \mathbb{N}$, *if at least one of the* b_i *is odd, then for all integers* $0 \le \ell_1 < \cdots < \ell_m$,

$$\lim_{N\to+\infty} \mathbb{E}\left[(U_{\ell_1}^{(N)})^{b_1} \cdots (U_{\ell_m}^{(N)})^{b_m}\right] = 0.$$

iii- Given any family of m *integers* $(a_i)_{1\le i\le m}$, $m \in \mathbb{N}$, *with* $\sum_{i=1}^m a_i = p$, *then for for all integers* $0 \le \ell_1 < \cdots < \ell_m$,

$$\lim_{N\to+\infty} \mathbb{E}\left[(U_{\ell_1}^{(N)})^{2a_1} \cdots (U_{\ell_m}^{(N)})^{2a_m}\right] = \frac{1}{2^p} \prod_{i=1}^m \phi_{\ell_i,a_i} \frac{(2a_i)!}{a_i!} \sigma^{2\ell_i a_i}.$$

The proof of Lemma 3.4 is given in Section 5. It allows us to state the following result on the covariance of the process $\left(V_t^{k,(\infty)}\right)_{t\in\mathbb{R}_+}$

Corollary 3.5. *For all* $k \in \mathbb{N}$, *the covariance of the limiting process* $(V_t^{k,(\infty)})_{t\in\mathbb{R}_+}$ *is given by*

$$\mathrm{Cov}\left(V_t^{k,(\infty)}, V_s^{k,(\infty)}\right) = e^{-\lambda(t+s)} \left(\phi_0 \widetilde{I}_0(2\sigma\sqrt{ts}) + \phi_0 - \mu_0^2\right), \qquad t, s \in \mathbb{R}_+, \tag{3.5}$$

Remark 3.6. *The increments of the process* $\left(V_t^{k,(\infty)}\right)_{t\in\mathbb{R}_+}$ *are not independent. Indeed by Corollary 3.5, for all* $0 \le t_1 < t_2 < t_3 < t_4$,

$$\mathrm{Cov}\left[V_{t_2}^{k,(\infty)} - V_{t_1}^{k,(\infty)}, V_{t_4}^{k,(\infty)} - V_{t_3}^{k,(\infty)}\right] \neq 0.$$

Proof of Corollary 3.5: Because of Theorem 3.1 Z_t^k is centered for all $t \in \mathbb{R}_+$ implying that $\mathbb{E}\left[V_t^{k,(\infty)}\right] = e^{-\lambda t}\mu_0$. It follows that

$$\begin{aligned}
\mathrm{Cov}\left(V_t^{k,(\infty)}, V_s^{k,(\infty)}\right) &= \mathbb{E}\left[\left(V_t^{k,(\infty)} - e^{-\lambda t}\mu_0\right)\left(V_s^{k,(\infty)} - e^{-\lambda s}\mu_0\right)\right] \\
&= \mathbb{E}\left[V_t^{k,(\infty)} V_s^{k,(\infty)}\right] - e^{-\lambda(t+s)}\mu_0^2 \\
&= e^{-\lambda(t+s)}\Big\{\mathbb{E}\left[(V_0^k)^2\right] + \mathbb{E}\left[V_0^k\left(Z_t^{k,(\infty)} + Z_s^{k,(\infty)}\right)\right] \qquad (3.6)\\
&\qquad + \mathbb{E}\left[Z_t^{k,} Z_s^k\right] - \mu_0^2\Big\}.
\end{aligned}$$

Then, let $e_k^{(N)}$ the k-th unit vector of $\mathbb{R}^N$, we remark that

$$\begin{aligned}
V_0^k \left[\left(\exp\left(\frac{J}{\sqrt{N}}s\right) - \mathrm{Id}\right) V_0^{(N)}\right]_k &= V_0^k (e_k^{(N)})^t \sum_{\ell\ge1} \frac{J^\ell s^\ell}{\ell!\sqrt{N}^\ell} V_0^{(N)} \\
&= (e_k^{(N)})^t \sum_{\ell\ge1} \frac{J^\ell s^\ell}{\ell!\sqrt{N}^\ell} V_0^k V_0^{(N)},
\end{aligned}$$

where $(e_k^{(N)})^t$ is the row vector transposed of the column vector $e_k^{(N)}$. As V_0 and J are independent,

$$\begin{aligned}\mathbb{E}\left[(e_k^{(N)})^t \sum_{\ell\geq 1}\frac{J^\ell s^\ell}{\ell!\sqrt{N}^\ell}V_0^k V_0^{(N)}\right] &= \mathbb{E}\left[(e_k^{(N)})^t \sum_{\ell\geq 1}\frac{J^\ell s^\ell}{\ell!\sqrt{N}^\ell}\right]\mathbb{E}\left[V_0^k V_0^{(N)}\right]\\ &= \mathbb{E}\left[(e_k^{(N)})^t \sum_{\ell\geq 1}\frac{J^\ell s^\ell}{\ell!\sqrt{N}^\ell}\right]Y\\ &= \mathbb{E}\left[(e_k^{(N)})^t \sum_{\ell\geq 1}\frac{J^\ell s^\ell}{\ell!\sqrt{N}^\ell}Y\right]\end{aligned}$$

where Y is the deterministic vector with all coordinates equal to μ_0^2 except the k-th coordinate which is equal to ϕ_0. Y satisfies

$$\lim_{N\to+\infty}\left(\frac{1}{N}\sum_{j=1}^N (Y_j)^2\right)^p = \mu_0^{4p}.$$

It follows from Lemma 3.4.i and the proof of Theorem 3.1 that

$$(e_k^{(N)})^t \sum_{\ell\geq 1}\frac{J^\ell s^\ell}{\ell!\sqrt{N}^\ell}Y \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, \mu_0^4\widetilde{I}_0(2\sigma s)\right).$$

This convergence yields that

$$\mathbb{E}\left[V_0^k Z_s^k\right] = 0. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\operatorname{Cov}\left(V_t^{k,(\infty)}, V_s^{k,(\infty)}\right) = e^{-\lambda(t+s)}\left(\phi_0 - \mu_0^2 + \mathbb{E}\left[Z_t^k Z_s^k\right]\right).$$

Moreover, by (3.3) and (3.1),

$$\mathbb{E}\left[Z_t^k Z_s^k\right] = \phi_0\sum_{\ell\geq 1}\frac{\sigma^{2\ell}(ts)^\ell}{(\ell!)^2} = \phi_0\widetilde{I}_0(2\sigma\sqrt{ts}). \tag{3.8}$$

Combining (3.6) and (3.8) we obtain (3.5). Note that the proof of (3.7) implies that $\mathbb{E}\left[V_0^m Z_s^k\right] = 0$, $m\in\mathbb{N}$. This yields the last point of Theorem 3.1. $\square$

Using Remark 3.3 and Corollar 3.5, we can write the EDS which is verified by the process $(V_t^{k,(\infty)})_{t\in\mathbb{R}_+}$:

$$\begin{cases}\mathrm{d}V_t^{k,(\infty)} = -\lambda V_t^{k,(\infty)}\mathrm{d}t + H_t^k\mathrm{d}t\\ \mathcal{L}(V_0^k) = \nu_0.\end{cases} \tag{3.9}$$

with $\left(H_t^k\right)_{t\in\mathbb{R}_+}$ a Gaussian process such that for all $t\in\mathbb{R}_+$,

$$H_t^k = \sqrt{\phi_0}\sigma\sum_{\ell\geq 0}\frac{(\sigma t)^\ell}{\ell!}G_{\ell+1}^k$$

where the G_ℓ^k are the same Gaussian random variables than in (3.3). Moreover, we can proceed exactly as in the proof of Corollary 3.5 to obtain the covariance of the process $\left(H_t^k\right)_{t\in\mathbb{R}_+}$ which gives us

$$\operatorname{Cov}\left(H_t^k, H_s^k\right) = \phi_0\sigma^2 I_0\left(2\sigma\sqrt{st}\right).$$

The proof of Theorem 3.1 rests upon the use of the method of moments (see [2, Th. 30.2]) applied to the exponential of J in (2.3). The method of moments consists in proving the weak convergence by establishing that the moments converge. This requires that the limiting distribution is uniquely determined by it moments. This is the case of the Gaussian distributions.

Proof of Theorem 3.1. Without loss of generality we assume $k = 1$. By expanding the exponential of the matrix J in (2.3) (with $\gamma = 0$) we express the first coordinate of $V_t^{(N)}$ as

$$\begin{aligned} V_t^{1,(N)} &= e^{-\lambda t}\left(V_0^1 + \sum_{\ell\geq 1}\frac{t^\ell}{\ell!}\frac{1}{\sqrt{N}^\ell}\sum_{j=1}^N (J^\ell)_{1,j}V_0^j\right) \\ &= e^{-\lambda t}\left(V_0^1 + \sum_{\ell\geq 1}\frac{t^\ell}{\ell!}\frac{1}{\sqrt{N}^\ell}\sum_{j_1,\cdots,j_\ell=1}^N J_{1,j_1}J_{j_1,j_2}\cdots J_{j_{\ell-1},j_\ell}V_0^{j_\ell}\right). \end{aligned} \tag{3.10}$$

We fix n and define the partial sum $Z_n^{(N)}(t)$ of order n of (3.10):

$$Z_n^{(N)}(t) = \sum_{\ell=1}^n \frac{t^\ell}{\ell!}U_\ell^{(N)}, \qquad U_\ell^{(N)} = \frac{1}{\sqrt{N}^\ell}\sum_{j_1,\cdots,j_\ell=1}^N J_{1,j_1}\cdots J_{j_{\ell-1},j_\ell}V_0^{j_\ell}. \tag{3.11}$$

In the first part of the proof we apply the method of moments to $Z_n^{(N)}(t)$. We will identify the limit of the moments of $Z_n^{(N)}(t)$ for $N\to\infty$ with the ones of a centred Gaussian process, noted $Z_n(t)$, of variance $\phi_0\widetilde{I}_{0,n}(2\sigma t)$ where

$$\widetilde{I}_{0,n}(z) = \sum_{\ell=1}^n \frac{z^{2\ell}}{2^{2\ell}(\ell!)^2}, \tag{3.12}$$

is the partial sum of order n of $\widetilde{I}_0(z)$ defined in (3.1). The method of moments then guarantees that, for all $n\in\mathbb{N}$, $Z_n^{(N)}(t)$ converges in law to $Z_n(t)$ when $N\to\infty$.

We apply Lemma 3.4 to $U_\ell^{(N)}$ defined in (3.11) with $Y_{\ell,j} = V_0^j$ not depending on ℓ. We define

$$\phi_p = \lim_{N\to+\infty}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^N (V_0^j)^2\right)^p\right],$$

and observe that

$$\frac{1}{N^p}\left(\sum_{j=1}^N (V_0^j)^2\right)^p = \frac{1}{N^p}p!\sum_{1\leq j_1<\cdots<j_p\leq N}(V_0^{j_1})^2\cdots(V_0^{j_p})^2 + \mathcal{O}\left(\frac{1}{N}\right).$$

By hypothesis (H2) the V_0^j are *i.i.d.* random variables, we then have for all $p \in \mathbb{N}$,

$$\begin{aligned}\phi_p &= \lim_{N\to+\infty} \frac{1}{N^p} p! \mathbb{E}\left[\sum_{1\le j_1<\cdots<j_p\le N} (V_0^{j_1})^2 \cdots (V_0^{j_p})^2\right] + \mathcal{O}\left(\frac{1}{N}\right) \\ &= \frac{p!}{p!}\phi_0^p = \phi_0^p.\end{aligned}$$

Lemma 3.4.i and the method of moments imply that, for all $\ell \in \mathbb{N}$, the sequence $\left(U_\ell^{(N)}\right)_{N\in\mathbb{N}}$ converges in law when $N \to +\infty$ to a centered Gaussian of variance $\phi_0\sigma^{2\ell}$.

Using Lemma 3.4 again we show that $Z_n^{(N)}(t)$ converges in law to $Z_n(t)$ for all $n \in \mathbb{N}$ fixed. Let $p \in \mathbb{N}$, $(Z_n^{(N)}(t))^p$ is the sum of terms of the form

$$C_{p,b_1,\cdots,b_m} \sum_{1\le\ell_1<\cdots<\ell_m\le n} \left(\frac{t^{\ell_1}}{\ell_1!}U_{\ell_1}^{(N)}\right)^{b_1} \cdots \left(\frac{t^{\ell_m}}{\ell_m!}U_{\ell_m}^{(N)}\right)^{b_m} \tag{3.13}$$

over $1 \le m \le p$ and for families of positive integers $(b_i)_{1\le i\le m} \in \Sigma_{p,m}$ where

$$\Sigma_{p,m} = \{(b_1,\cdots,b_m) \in \mathbb{N}_*^m \mid \sum_{i=1}^m b_i = p\}. \tag{3.14}$$

$C_{p,(b_i)_{1\le i\le m}}$ is the multinomial coefficient :

$$C_{p,b_1,\cdots,b_m} = \frac{p!}{b_1!\cdots b_m!}.$$

By the second statement of Lemma 3.4, when taking the expected value of $\left(Z_n^{(N)}(t)\right)^p$ the terms with at least one of the b_i, $1 \le i \le m$, odd vanish in the limit in $N \to \infty$. It directly yields that the odd moments of $Z_n^{(N)}(t)$ go to zero as $N \to \infty$:

$$\mathbb{E}\left[\left(Z_n^{(N)}(t)\right)^{2p+1}\right] \underset{N\to+\infty}{\longrightarrow} 0, \qquad \forall p \in \mathbb{N}. \tag{3.15}$$

We next consider the even moments of $Z_n^{(N)}(t)$. Let $p \in \mathbb{N}$, applying the second statement of Lemma 3.4, we obtain

$$\begin{aligned}\lim_{N\to+\infty} \mathbb{E}\left[(Z_n^{(N)}(t))^{2p}\right] = \lim_{N\to+\infty} \sum_{m=1}^p \sum_{(a_i)_{1\le i\le m}\in\Sigma_{p,m}} & C_{2p,2a_1,\cdots,2a_m} \\ & \times \sum_{1\le\ell_1<\cdots<\ell_m\le n} \left(\frac{t^{\ell_1}}{\ell_1!}U_{\ell_1}^{(N)}\right)^{2a_1} \cdots \left(\frac{t^{\ell_m}}{\ell_m!}U_{\ell_m}^{(N)}\right)^{2a_m}\end{aligned}$$

where $\Sigma_{p,m}$ is defined in (3.14) Applying the third statement of Lemma 3.4, we obtain

$$\mathbb{E}\left[(Z_n^{(N)}(t))^{2p}\right] = \sum_{m=1}^{p} \sum_{(a_i)_i \in \Sigma_{p,m}} \frac{(2p)!}{(2a_1)!\cdots(2a_m)!} \sum_{1\le \ell_1 < \cdots < \ell_m \le n} \frac{t^{2\sum a_i \ell_i}}{\prod_{i=1}^m (\ell_i!)^{2a_i}} \mathbb{E}\left[\prod_{i=1}^m (U_{\ell_i}^{(N)})^{2a_i}\right]. \tag{3.16}$$

We can now apply the first statement of Lemma 3.4 to obtain

$$\mathbb{E}\left[(Z_n^{(N)}(t))^{2p}\right] \underset{N\to+\infty}{\longrightarrow} \sum_{m=1}^{p} \sum_{(a_i)_i \in \Sigma_{p,m}} \frac{(2p)!\, \phi_0^p}{2^p (a_1)!\cdots(a_m)!} \sum_{1\le \ell_1 < \cdots < \ell_m \le n} \prod_{i=1}^m \frac{t^{2a_i\ell_i}}{(\ell_i!)^{2a_i}} \prod_{i=1}^m \sigma^{2a_i\ell_i}, \tag{3.17}$$

which is precisely the moment of order $2p$ of a centered Gaussian random variable of variance $\phi_0 \widetilde{I}_{0,n}(2\sigma t)$. It yields that for all $n\in\mathbb{N}$, and for $t\in\mathbb{R}_+$, $Z_n^{(N)}(t)$ converges in law when $N\to\infty$ to $Z_n(t)$ with

$$\mathcal{L}\left(Z_n(t)\right) = \mathcal{N}\left(0, \phi_0 \widetilde{I}_{0,n}(2\sigma t)\right),$$

where $\widetilde{I}_{0,n}(2\sigma t)$ is defined in (3.12). Moreover, the function $\widetilde{I}_{0,n}$ converges pointwise to $\widetilde{I}_0$ defined in (3.1).It implies that $Z_n(t)$ converges in law, for $n\to+\infty$ to a $\mathcal{N}\left(0, \widetilde{I}_0(2\sigma t)\right)$.

The second part of the proof consists in adding the initial condition V_0^1 to our analysis. We compute the characteristic function of the pair $\left(V_0^1, Z_n^{(N)}(t)\right)$. The main difficulty in computing this characteristic function is that V_0^1 appears in $Z_n^{(N)}(t)$, indeed we can write $Z_n^{(N)}(t)$ as the sum

$$Z_n^{(N)}(t) = \widetilde{Z}_n^{(N)}(t) + \sum_{\ell=1}^{n} \frac{t^\ell}{\sqrt{N}^\ell \ell!} \sum_{j_1,\cdots j_{\ell-1}=1}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},1} V_0^1, \tag{3.18}$$

where $\widetilde{Z}_n^{(N)}(t)$ is the restriction of $Z_n^{(N)}(t)$ over $j_\ell \ge 2$

$$\widetilde{Z}_n^{(N)}(t) = \sum_{\ell=1}^{n} \frac{t^\ell}{\sqrt{N}^\ell \ell!} \sum_{j_1,\cdots j_{\ell-1}=1}^{N} \sum_{j_\ell=2}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},j_\ell} V_0^{j_\ell}.$$

First, we show that this dependence disappears in the limit $N\to+\infty$.

Let $(u_1,u_2)\in\mathbb{R}^2$, by (3.18) we have

$$\left|\exp\left(iu_1V_0^1\right)\exp\left(iu_2 Z_n^{(N)}(t)\right) - \exp\left(iu_1V_0^1\right)\exp\left(iu_2 \widetilde{Z}_n^{(N)}(t)\right)\right| = \left|\exp\left(iu_1V_0^1\right)\exp\left(iu_2 \widetilde{Z}_n^{(N)}(t)\right)\right| \times \left|\exp\left(iu_2 \sum_{\ell=1}^{n} \frac{t^\ell}{\sqrt{N}^\ell \ell!} \sum_{j_1,\cdots j_{\ell-1}=1}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},1} V_0^1\right) - 1\right|.$$

Then, using the fact that $|e^z - 1| \le |z| e^{|z|}$,

$$\begin{aligned}
&\left|\exp\left(iu_1V_0^1\right)\exp\left(iu_2 Z_n^{(N)}(t)\right) - \exp\left(iu_1V_0^1\right)\exp\left(iu_2\widetilde{Z}_n^{(N)}(t)\right)\right| \le \left|\exp\left(iu_1V_0^1\right)\exp\left(iu_2\widetilde{Z}_n^{(N)}(t)\right)\right| \\
&\qquad\times\left|u_2\sum_{\ell=1}^{n}\frac{t^\ell}{\sqrt{N}^\ell \ell!}\sum_{j_1,\cdots j_{\ell-1}=1}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},1}V_0^1\right| \exp\left(\left|u_2\sum_{\ell=1}^{n}\frac{t^\ell}{\sqrt{N}^\ell \ell!}\sum_{j_1,\cdots j_{\ell-1}=1}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},1}V_0^1\right|\right)
\end{aligned}$$

(3.19)

$$\le \left|u_2\sum_{\ell=1}^{n}\frac{t^\ell}{\sqrt{N}^\ell \ell!}\sum_{j_1,\cdots j_{\ell-1}=1}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},1}V_0^1\right| \exp\left(\left|u_2\sum_{\ell=1}^{n}\frac{t^\ell}{\sqrt{N}^\ell \ell!}\sum_{j_1,\cdots j_{\ell-1}=1}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},1}V_0^1\right|\right).$$

Using the method of moment on

$$\sum_{\ell=1}^{n}\frac{t^\ell}{\ell!\sqrt{N}^\ell}\sum_{j_1,\cdots j_{\ell-1}=1}^{N} J_{1,j_1}\cdots J_{j_{\ell-1},1}V_0^1$$

gives us that all moments are zero hence it converges in law, and then in probability to 0 as $N \to +\infty$. The limit of (3.19), as $N \to +\infty$ is zero. It follows that

$$\begin{aligned}
&\lim_{N\to+\infty}\mathbb{E}\left[\exp\left(iu_1V_0^1\right)\exp\left(iu_2 Z_n^{(N)}(t)\right)\right] \\
&\qquad= \lim_{N\to+\infty}\mathbb{E}\left[\exp\left(iu_1V_0^1\right)\exp\left(iu_2\widetilde{Z}_n^{(N)}(t)\right)\right] \\
&\qquad= \mathbb{E}\left[\exp\left(iu_1V_0^1\right)\right]\lim_{N\to+\infty}\mathbb{E}\left[\exp\left(iu_2\widetilde{Z}_n^{(N)}(t)\right)\right]
\end{aligned}$$

and taking $j_\ell \ge 2$ does not change anything in the previous calculus on the moments, which yields that

$$\lim_{N\to+\infty}\mathbb{E}\left[\exp\left(iu_2\widetilde{Z}_n^{(N)}(t)V_0^{j_\ell}\right)\right] = \lim_{N\to+\infty}\mathbb{E}\left[\exp\left(iu_2 Z_n^{(N)}(t)\right)\right].$$

It shows the independence between $Z_n(t)$ and V_0^1.

Hence we have the convergence in law as $N \to +\infty$ of the pair $\left(V_0^1, Z_n^{(N)}(t)\right)$ to $\left(V_0^1, Z_n(t)\right)$, and it follows that

$$V_0^1 + Z_n^{(N)}(t) \xrightarrow[N\to+\infty]{\mathcal{L}} V_0^1 + Z_n(t). \tag{3.20}$$

It remains to let $n \to +\infty$ and to show that, if $Z^{(N)}(t) = \sum_{\ell\ge1}\frac{t^\ell}{\ell!}U_\ell^{(N)}$, then $V_0^1 + Z^{(N)}(t) \xrightarrow[N\to+\infty]{\mathcal{L}} V_0^1 + Z(t)$, where $\mathcal{L}\left(Z(t)\right) = \mathcal{N}(0, \widetilde{I}_0(2\sigma t))$. We are in the following situation:

$$\begin{array}{ccc}
V_0^1 + Z_n^{(N)}(t) & \xrightarrow[N\to\infty]{\mathcal{L}} & V_0^1 + Z_n(t) \\
 & & \mathcal{L}\Big\downarrow n\to\infty \\
 & & V_0^1 + Z(t)
\end{array}$$

According to [2, Th. 25.5], to obtain the weak convergence of $V_0^1 + Z^{(N)}(t)$ to $V_0^1 + Z(t)$ as $N \to +\infty$, it is sufficient to show that for all $t \in \mathbb{R}_*$

$$\lim_{n\to+\infty} \limsup_{N\to+\infty} \mathbb{P}\left[\mid (V_0^1 + Z_n^{(N)}(t)) - (V_0^1 + Z^{(N)}(t)) \mid \geq \varepsilon\right] = 0, \qquad \forall \varepsilon > 0. \tag{3.21}$$

By Markov inequality, we have

$$\mathbb{P}\left[\mid Z_n^{(N)}(t) - Z^{(N)}(t) \mid \geq \varepsilon\right] \leq \frac{1}{\varepsilon}\mathbb{E}\left[\left|Z_n^{(N)}(t) - Z^{(N)}(t)\right|\right], \tag{3.22}$$

and

$$\begin{aligned}
\mathbb{E}\left[\mid Z_n^{(N)}(t) - Z^{(N)}(t) \mid\right] &= \mathbb{E}\left[\left|\sum_{\ell\geq n+1} \frac{t^\ell}{\ell!} U_\ell^{(N)}\right|\right] \\
&\leq \sum_{\ell\geq n+1} \frac{t^\ell}{\ell!} \mathbb{E}\left[\left|U_\ell^{(N)}\right|\right].
\end{aligned}$$

Moreover, by statement i) of Lemma 3.4 and the moment method, $(U_\ell^{(N)})_N$ converges in law as $N \to \infty$ to a Gaussian random variable of zero mean and variance $\phi_0 \sigma^{2\ell}$. Then as $|\cdot|$ is continuous, for all $K \in \mathbb{R}$

$$\lim_{N\to+\infty} \mathbb{E}\left[\left|U_\ell^{(N)}\right| \mathbb{1}_{|U_\ell^{(N)}|<K}\right] = \mathbb{E}\left[|U_\ell| \mathbb{1}_{|U_\ell|<K}\right],$$

with, $\forall \ell \in \mathbb{N}$, $\mathcal{L}(U_\ell) = \mathcal{N}(0, \phi_0 \sigma^{2\ell})$. It follows that the law of $|U_\ell|$ is a half normal distribution, and hence that

$$\begin{aligned}
\lim_{K\to+\infty} \mathbb{E}\left[|U_\ell| \mathbb{1}_{|U_\ell|<K}\right] &= \mathbb{E}\left[|U_\ell|\right] \\
&= \frac{\sqrt{2\phi_0}}{\sqrt{\pi}} \sigma^\ell.
\end{aligned}$$

Then,

$$\limsup_{N\to+\infty} \mathbb{E}\left[\mid Z_n^{(N)}(t) - Z^{(N)}(t) \mid\right] \leq \frac{\sqrt{2\phi_0}}{\sqrt{\pi}} \sum_{\ell\geq n+1} \frac{t^\ell}{\ell!} \sigma^\ell.$$

The right hand side of this inequality goes to zero when $n \to \infty$ and (3.21) follows. This concludes the proof of Theorem 3.1. □

3.2. **Propagation of chaos.** An important consequence of Lemma 3.4 is that the propagation of chaos is satisfied by the mean field limit.

Theorem 3.7. *For any pair $(k_1, k_2) \in \mathbb{N}^2$, $k_1 \neq k_2$, and for all $t \in \mathbb{R}_+$, we have independence between $V_t^{k_1,(\infty)}$ and $V_t^{k_2,(\infty)}$.*

Proof. Consider $k_1 = 1$ and $k_2 = 2$ and we denote by

$$U_\ell^{k,(N)} = \frac{1}{\sqrt{N}^\ell} \sum_{j=1}^{N} (J^\ell)_{k,j} Y_0^j, \qquad k = 1, 2.$$

The scheme of the proof consist on taking k and $\ell \in \mathbb{N}$ not necessarily different and consider $(u_1, u_2) \in \mathbb{R}^2$. The proof consist again of using the method of moment to establish that for all $(u_1, u_2) \in \mathbb{R}^2$,

$$U_\ell^{1,(N)} u_1 + U_k^{2,(N)} u_2 \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, \phi_0\left((\sigma^\ell u_1)^2 + (\sigma^k u_2)^2\right)\right). \tag{3.23}$$

It will imply that the couple $(U_\ell^{1,(N)}, U_k^{2,(N)})$ converges in law to a Gaussian vector with a diagonal covariance matrix, hence the limits in law of $U_{ell}^{1,(N)}$ and $U_k^{2,(N)}$ will be independent. Moreover, it is equivalent to the independence of the limit in law of $Z_1^{1,(N)}$ and $Z_1^{2,(N)}$. Then we will proceed recursively by admitting that for a $n \in \mathbb{N}^*$,

$$Z_{n-1}^{1,(N)} u_1 + Z_{n-1}^{2,(N)} u_2 \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, 2\phi_0 \widetilde{I}_{0,n-1}(2\sigma t)(u_1^2 + u_2^2)\right) \tag{3.24}$$

(which is the case of $Z_1^{1,(N)} u_1 + Z_1^{2,(N)} u_2$) and we will show that it implies that $Z_n^{1,(N)} u_1 + Z_n^{2,(N)} u_2$ converges in law to a $\mathcal{N}\left(0, 2\phi_0 \widetilde{I}_{0,n}(2\sigma t)(u_1^2 + u_2^2)\right)$. The result will follow.

First, let (u_1, u_2) be in $\mathbb{R}^2$. Then we compute the moments of $U_\ell^{1,(N)} u_1 + U_k^{2,(N)} u_2$. As in the proof of Lemma 3.4, the odd moments are zero. Consider $p \in \mathbb{N}$,

$$\left(U_\ell^{1,(N)} u_1 + U_k^{2,(N)} u_2\right)^{2p} = \sum_{m=0}^{2p} \frac{(2p)!}{m!(2p-m)!} u_1^m u_2^{2p-m} (U_\ell^{1,(N)})^m (U_k^{1,(N)})^{2p-m}.$$

By Lemma 3.4 ii-, the expectation of the sum over the m odd converges to 0, hence,

$$\lim_{N\to+\infty} \mathbb{E}\left[\left(U_\ell^{1,(N)} u_1 + U_k^{2,(N)} u_2\right)^{2p}\right] = \lim_{N\to+\infty} \sum_{m=0}^{p} \frac{(2p)!}{(2m)!(2(p-m))!} u_1^{2m} u_2^{2(p-m)} \mathbb{E}\left[(U_\ell^{1,(N)})^{2m} (U_k^{2,(N)})^{2(p-m)}\right] \tag{3.25}$$

by Lemma 3.4 iii- (the proof is also valid for $k = \ell$)

$$\mathbb{E}\left[(U_\ell^{1,(N)})^{2m} (U_k^{2,(N)})^{2(p-m)}\right] \underset{N\to+\infty}{\to} \phi_0^p \frac{(2m)!(2(p-m))!}{2^p m!(p-m)!} \sigma^{2\ell m} \sigma^{2k(p-m)}. \tag{3.26}$$

Combining (3.25) and (3.26) yields convergence of law

$$(U_\ell^{1,(N)} u_1 + U_k^{2,(N)} u_2) \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, (u_1^2 + u_2^2)\phi_0(\sigma^{2\ell} + \sigma^{2k})\right)$$

and then,

$$\left(U_\ell^{1,(N)}, U_k^{2,(N)}\right) \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, \Sigma_{\ell,k}\right), \text{ with } \Sigma_{\ell,k} \text{ a diagonal matrix.} \tag{3.27}$$

the independence between the limit in law of $U_\ell^{1,(N)}$ and $U_k^{2,(N)}$. It yields, taking $\ell = 1 = k$, that $Z_1^1(t)$ and $Z_1^2(t)$ are independent and that, for all $(u_1, u_2) \in \mathbb{R}^2$,

$$Z_1^{1,(N)}(t) u_1 + Z_1^{2,(N)}(t) u_2 \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, 2\phi_0(\sigma t)^2(u_1^2 + u_2)^2\right). \tag{3.28}$$

Now, let $n \in \mathbb{N}$ and assume (3.24) is satisfied for at rank $n-1$, then, let $p \in \mathbb{N}$;

$$\begin{aligned}\left(Z_n^{1,(N)}(t)u_1 + Z_n^{2,(N)}(t)u_2\right)^p &= \left((Z_{n-1}^{1,(N)}(t)u_1 + Z_{n-1}^{2,(N)}(t)u_2) + \frac{t^n}{n!}(U_n^{1,(N)}u_1 + U_n^{2,(N)}u_2)\right)^p \\ &= \sum_{m=0}^{p} \frac{p!}{m!(p-m)!}(Z_{n-1}^{1,(N)}u_1 + Z_{n-1}^{2,(N)}u_2)^m(\frac{t^n}{n!})^{p-m}(U_n^{1,(N)}u_1 + U_n^{2,(N)}u_2)^{p-m}\end{aligned}$$

If p is odd, then, either m or $2p+1-m$ is odd and consequently, the expectation converges to zero as $N \to +\infty$. It yields, combining with the independence (3.27) between the limit in law of $U_\ell^{1,(N)}$ and $U_k^{2,(N)}$ that the odd moment of $\left(Z_n^{1,(N)}u_1 + Z_n^{2,(N)}u_2\right)$ converges to 0. Consider now the even case, let $p \in \mathbb{N}$,

$$\left(Z_n^{1,(N)}u_1 + Z_n^{2,(N)}u_2\right)^{2p} = \sum_{m=0}^{2p} \frac{(2p)!}{m!(2p-m)!}(Z_{n-1}^{1,(N)}u_1 + Z_{n-1}^{2,(N)}u_2)^m(\frac{t^n}{n!})^{2(p-m)}(U_n^{1,(N)}u_1 + U_n^{2,(N)}u_2)^{2p-m},$$

as before, if m is odd then the expectation will converges to zero. It yields that

$$\left(Z_n^{1,(N)}u_1 + Z_n^{2,(N)}u_2\right)^{2p} = \sum_{m=0}^{p} \frac{(2p)!}{(2m)!(2(p-m))!}(Z_{n-1}^{1,(N)}u_1 + Z_{n-1}^{2,(N)}u_2)^{2m}(\frac{t^n}{n!})^{2(p-m)}(U_n^{1,(N)}u_1 + U_n^{2,(N)}u_2)^{2(p-m)}$$

By (3.24) and by (3.27), it implies the convergence

$$\left(Z_n^{1,(N)}u_1 + Z_n^{2,(N)}u_2\right)^{2p} \to_{N\to+\infty} (u_1^2+u_2^2)^p \phi_0^p \frac{(2p)!}{2^p p!} 2^p(\widetilde{I}_{0,n}(2\sigma t))^p.$$

We have the convergence in law of $\left(Z_n^{1,(N)}u_1 + Z_n^{2,(N)}u_2\right)$ to a $\mathcal{N}(0, 2(u_1^2+u_2^2)\phi_0\widetilde{I}_{0,n}(2\sigma t))$ with $N \to +\infty$ and hence the couple $(Z_n^{1,(N)}(t), Z_n^{2,(N)}(t))$ converges in law to Gaussian vector with diagonal covariance matrix which lead to the independence between $Z_n^1(t)$ and $Z_n^2(t)$. The independence with the initial condition has already been shown, which allow us to conclude that V_t^1 and V_t^2 are independent. □

4. Convergence of the particle system with noise

In this section, we consider the case $\gamma \neq 0$. Then the explicit solution of (2.2) is

$$V_t^{(N)} = e^{-\lambda t}\exp\left(\frac{Jt}{\sqrt{N}}\right)\left[V_0^{(N)} + \gamma\int_0^t e^{-\lambda s}\exp\left(-\frac{Js}{\sqrt{N}}\right)\mathrm{d}B_s\right]. \tag{4.1}$$

Theorem 4.1. *Under hypothesis* (H2) *on the matrix J and the initial random vector $V_0^{(N)}$ we have the convergence in law as $N \to +\infty$ of $V_t^{k,(N)}$ to $V_t^{k,(\infty)}$ where*

$$V_t^{k,(\infty)} = e^{-\lambda t}\left[V_0^k + \hat{G}_t^{k,(\infty)} + \gamma\int_0^t e^{\lambda s}\mathrm{d}B_s^k\right] \tag{4.2}$$

Where, for all $t \in \mathbb{R}_+$, $\hat{G}_t^{k,(\infty)}$ has Gaussian distribution with zero mean and variance $\phi_0\widetilde{I}_0(2\sigma t) + \gamma^2\Psi(\sigma,t)$ with

$$\Psi(\sigma,t) = \sum_{\ell\geq 1}\frac{\sigma^{2\ell}}{(\ell!)^2}v(t,\ell), \qquad v(t,\ell) = \int_0^t e^{2\lambda s}(t-s)^{2\ell}\mathrm{d}s.$$

Proof. We again prove the result for $k=1$. The proof follows exactly the same steps as that of Theorem 3.1. First, the explicit solution of (2.1) has the form

$$V_t^{(N)} = e^{-\lambda t}\left[\exp\left(\frac{J}{\sqrt{N}}t\right)V_0^{(N)} + \gamma\int_0^t e^{\lambda s}\exp\left(\frac{J}{\sqrt{N}}(t-s)\right)\mathrm{d}B_s\right]. \tag{4.3}$$

Expanding the exponential, we obtain

$$V_t^{(N)} = e^{-\lambda t}\sum_{\ell\geq 0}\frac{J^\ell}{\ell!\sqrt{N}^\ell}\left[t^\ell V_0^{(N)} + \gamma\int_0^t e^{\lambda s}(t-s)^\ell \mathrm{d}B_s\right].$$

Taking the first coordinate and separate $\ell=0$ from the sum

$$V_t^1 = e^{-\lambda t}\left[V_0^1 + \gamma\int_0^t e^{\lambda s}\mathrm{d}B_s^1 + \sum_{\ell\geq 1}\sum_{j=1}^N \frac{(J^\ell)_{1,j}u_{\ell,j}(t)}{\sqrt{N}^\ell \ell!}\right] \tag{4.4}$$

with

$$u_{\ell,j}(t) := t^\ell V_0^j + \gamma\int_0^t e^{\lambda s}(t-s)^\ell \mathrm{d}B_s^j. \tag{4.5}$$

Then we can follows the same scheme than for the proof of Theorem 3.1. The sequence $(u_{\ell,j}(t)$ satisfies the hypothesis of Lemma (3.4) for $t\in\mathbb{R}_+$ fixed. Then applying Lemma 3.4 with $Y_{j,\ell} = u_{j,\ell}(t)$, for $t\in\mathbb{R}_+$ fixed. It yields that ϕ_0 in Theorem 3.1 is replaced by $\mathbb{E}\left[u(1,\ell)^2\right] = t^{2\ell}\phi_0 + \gamma^2\int_0^t (t-s)^{2\ell}\mathrm{d}s$ by independence between $V_0^{(N)}$ and the Brownian motion. Theorem 4.1 follows. $\square$

5. Proof of Lemma 3.4

Proof of Lemma 3.4. i) We start the proof of the first statement of Lemma 3.4 by considering the case $\ell=2$. First, remark that

$$U_2^{(N)} = \frac{1}{N}J_{1,1}\sum_{j_2=1}^N J_{1,j_2}Y2,j_2 + \frac{1}{N}\sum_{j_1=2,j_2=1}^N J_{1,j_1}J_{j_1,j_2}Y_{2,j_2}. \tag{5.1}$$

By the strong law of large numbers, the first term of the sum in the right hand side goes to zero almost surely as $N\to+\infty$ by hypothesis (H2). Next we note that the products $J_{1,j_1}J_{j_1,j_2}$ that appear in the second term are always made up of two different elements. In order to deal with this second term we define

$$H_{\ell,j_1} := \sum_{j_2=1}^N J_{j_1,j_2}Y_{\ell,j_2},$$

and

$$\widetilde{U}_2^{(N)} = \frac{1}{N}\sum_{j_1=2}^N J_{1,j_1}H_{2,j_1}.$$

Then, for all $k\in\mathbb{N}_*$, we can expand $\left(\widetilde{U}_2^{(N)}\right)^k$ as the sum over m from 1 to k and over the families $(a_i)_{i\in\Sigma_{k,m}}$ (defined in (3.14)) of the terms

$$\frac{1}{\sqrt{N}^{2k}}C_{k,a_1,\cdots,a_m}\sum_{2\leq j_1^{(1)}<\cdots<j_1^{(m)}\leq N} J^{a_1}_{1,j_1^{(1)}}H^{a_1}_{2,j_1^{(1)}}\cdots J^{a_m}_{1,j_1^{(m)}}H^{a_m}_{2,j_1^{(m)}}. \tag{5.2}$$

Recall that $C_{k,a_1,\cdots,a_m}$ is the multinomial coefficient

$$C_{k,a_1,\cdots,a_m} := \frac{k!}{a_1!\cdots a_m!}.$$

As the $(j_1^{(\ell)})_{1\le\ell\le m}$ are higher than 2, all the terms $\left(J_{1,j_1^{(\ell)}}\right)_{1\le\ell\le m}$, are independent from all the $\left(H_{2,j_1^{(\ell)}}\right)_{1\le\ell\le m}$. Moreover, as the Js are centered, if at least one of the a_ℓ is equal to one, the expectation of the sum over the $(j_1^{(\ell)})_{1\le\ell\le m}$ vanishes. This is the case when $m > \lfloor k/2\rfloor$: at least one of the a_i is equal to one.

Thus we may restrict our attention to the sum for which for all $\ell\in\{1,\cdots,m\}$, $a_\ell\ge 2$, i.e the sum over $m\le\lfloor k/2\rfloor$. We remark that the family $(H_{2,j_1^{(\ell)}})_{1\le\ell\le m}$ is not independent. Indeed, all the coordinates of $(Y_{2,i})_{1\le i\le N}$ appear in each $H_{2,j_1^{(\ell)}}$, $1\le\ell\le N$. Furthermore the dependence is only between the coordinates of $Y_{2,\cdots}$ which are independent of the matrix J. Consider one of the $H^{a_\ell}_{2,j_1^{(\ell)}}$, $1\le\ell\le m$. We can expand it in the same way as (5.2) for a fixed value of ℓ:

$$H^{a_\ell}_{2,j_1^{(\ell)}} = \sum_{m_\ell=1}^{a_\ell}\sum_{\Sigma_{a_\ell,m_\ell}} C^{(\ell)}_{a_\ell,a_1^{(\ell)},\cdots,a_{m_\ell}^{(\ell)}} \sum_{1\le j_2^{(1)}<\cdots<j_2^{(m_\ell)}\le N} \left(J_{j_1^{(\ell)},j_2^{(1)}}Y_{2,j_2^{(1)}}\right)^{a_1^{(\ell)}}\cdots\left(J_{j_1^{(\ell)},j_2^{(m_\ell)}}Y_{2,j_2^{(m_\ell)}}\right)^{a_{m_\ell}^{(\ell)}}.$$

Hence, with the same argument as before, if at least one of the $(a_j^{(\ell)})_j$ is equal to one, the expectation vanishes. It follows that for each $\ell\in\{1,\cdots m\}$, $m_\ell\le\lfloor a_\ell/2\rfloor$. The consequence is that for a fixed m and a family (a_ℓ), in (5.2), using the hypothesis that Y and J have finite moments (hypotheses (H1) and (H2),

$$\mathbb{E}\left[\sum_{2\le j_1^{(1)}<\cdots<j_1^{(m)}\le N} J^{a_1}_{1,j_1^{(1)}}H^{a_1}_{2,j_1^{(1)}}\cdots J^{a_m}_{1,j_1^{(m)}}H^{a_m}_{2,j_1^{(m)}}\right] = \mathcal{O}\left(N^{m+\sum_{\ell=1}^m\lfloor a_\ell/2\rfloor}\right). \tag{5.3}$$

We first look at the odd case and take $k = 2p+1$. As previously noted, $m\le\lfloor k/2\rfloor$, i.e. $m\le p$, and

$$\sum_{\ell=1}^m\left\lfloor\frac{a_\ell}{2}\right\rfloor \le \left\lfloor\frac{1}{2}\sum_{\ell=1}^m a_\ell\right\rfloor \le \left\lfloor\frac{2p+1}{2}\right\rfloor = p.$$

By (5.3), it follows that in (5.2) we have at most N^{2p} terms. As all the moments of the coordinates of J and Y are finite, we obtain that, in this case, (5.2) is an $\mathcal{O}(1/N)$, and then

$$\lim_{N\to+\infty}\mathbb{E}\left[\left(\widetilde{U}_2^{(N)}\right)^{2p+1}\right] = 0. \tag{5.4}$$

Next we consider the even case, $k = 2p$. As previously, $m \leq p$. Considering $1 \leq m \leq p-1$ and proceeding exactly as for the odd case, we have that

$$\frac{1}{N^{2p}}\mathbb{E}\left[\sum_{m=1}^{p-1}\sum_{(a_1,\cdots,a_m)\in\Sigma_{k,m}} C_{k,a_1,\cdots,a_m} \underbrace{\sum_{2\leq j_1^{(1)}<\cdots<j_1^{(m)}\leq N} J^{a_1}_{1,j_1^{(1)}} H^{a_1}_{2,j_1^{(1)}} \cdots J^{a_m}_{1,j_1^{(m)}} H^{a_m}_{2,j_1^{(m)}}}_{\text{contains at most } N^{m+\lfloor\sum_{\ell=1}^m a_\ell/2\rfloor}\leq N^{2p-1} \text{ terms}}\right] = \mathcal{O}\left(\frac{1}{N}\right).$$

It follows that the only surviving term in (5.2) is the one such that $m = p$. Hence, the only choice for the values of the $(a_\ell)_{1\leq m\leq p}$ is $a_1 = \cdots = a_p = 2$. It yields that

$$\mathbb{E}\left[(\widetilde{U}_2^{(N)})^{2p}\right] = \mathcal{O}\left(\frac{1}{N}\right) + \frac{1}{N^{2p}} \sum_{2\leq j_1^{(1)}<\cdots<j_1^{(p)}\leq N} \frac{(2p)!}{2^p}\sigma^{2p}\mathbb{E}\left(H^2_{2,j_1^{(1)}}\cdots H^2_{2,j_1^{(p)}}\right) + 0.$$

It remains to control $\mathbb{E}\left(H^2_{2,j_1^{(1)}}\cdots H^2_{2,j_1^{(p)}}\right)$. As $j_1^{(1)} < \cdots < j_1^{(p)}$, by Hypothesis (H2):

$$\begin{aligned}
\mathbb{E}\left[H^2_{2,j_1^{(1)}}\cdots H^2_{2,j_1^{(p)}}\right] &= \mathbb{E}\left[\left(\sum_{j_2^{(1)}=1}^N J_{j_1^{(1)},j_2^{(1)}} Y_{2,j_2^{(1)}}\right)^2 \cdots \left(\sum_{j_2^{(p)}=1}^N J_{j_1^{(p)},j_2^{(p)}} Y_{2,j_2^{(p)}}\right)^2\right] \\
&= \mathbb{E}\left[\sum_{1\leq j_2^{(1)},\cdots j_2^{(p)}\leq N} (J_{j_1^{(1)},j_2^{(1)}} Y_{2,j_2^{(1)}})^2 \cdots (J_{j_1^{(p)},j_2^{(p)}} Y_{2,j_2^{(p)}})^2\right] \\
&\quad + 2\mathbb{E}\left[\sum_{1\leq j_2^{(1)}<i_2^{(1)}<\cdots<j_2^{(p)}<i_2^{(p)}\leq N} J_{j_1^{(1)},j_2^{(1)}} Y_{2,j_2^{(1)}} J_{j_1^{(1)},i_2^{(1)}} Y_{2,i_2^{(1)}} \cdots J_{j_1^{(p)},j_2^{(p)}} Y_{2,j_2^{(p)}} J_{j_1^{(p)},i_2^{(p)}} Y_{2,i_2^{(p)}}\right] \\
&= \sigma^{2p}\mathbb{E}\left(\sum_{1\leq j_2^{(1)},\cdots,j_2^{(p)}\leq N}\left(Y_{2,j_2^{(1)}}\cdots Y_{2,j_2^{(p)}}\right)^2\right) + 0. \qquad (5.5)
\end{aligned}$$

We note that since the indexes $j_2^{(\ell)}$, $1 \leq \ell \leq p$ can be equal, the $Y_{2,j_2^{(\ell)}}$s are not necessary independent from each other.

This motivates the following decomposition. Let

$$\begin{aligned}
\Delta_N^{\neq,p} &= \{1 \leq j_2^{(1)},\cdots,j_2^{(p)} \leq N| \quad \forall 1\leq \ell,r\leq p,\, \ell\neq r,\, j_2^{(\ell)} \neq j_2^{(r)}\}, \qquad (5.6)\\
\Delta_N^{=,p} &= \{1 \leq j_2^{(1)},\cdots,j_2^{(p)} \leq N| \quad \exists(\ell_1,\ell_2),\, \ell_1\neq\ell_2,\, j_2^{(\ell_1)} = j_2^{(\ell_2)}\}.
\end{aligned}$$

The sets $\Delta_N^{\neq,p}$ and $\Delta_N^{=,p}$ are disjoint and

$$\{\{j_2^{(1)},\cdots j_2^{(p)}\} \in \{1,\cdots,N\}^p\} = \Delta_N^{\neq,p} \cup \Delta_N^{=,p}.$$

Because the cardinal of $\Delta_N^{=,p}$ is of order N^{p-1}, and since because of hypothesis (H1), all the moments of the coordinates of Y are finite, it follows that the only surviving term in the right hand side of (5.5) is the sum over $\Delta_N^{\neq,p}$. Then, we have that

$$\mathbb{E}\left[(\widetilde{U}_2^{(N)})^{2p}\right] = \frac{1}{N^{2p}}\frac{(2p)!}{2^p}(\sigma^2)^{2p}\sum_{(j_2^{(i)})\in\Delta_N^{\neq,p}}\mathbb{E}\left[(Y_{2,j_2^{(1)}})^2\right]\cdots\mathbb{E}\left[(Y_{2,j_2^{(p)}})^2\right]+\mathcal{O}\left(\frac{1}{N}\right), \tag{5.7}$$

and, thanks to (3.4),

$$\lim_{N\to+\infty}\frac{1}{N^p}\sum_{(j_2^{(i)})\in\Delta_N^{\neq,p}}\mathbb{E}\left[(Y_{2,j_2^{(1)}})^2\right]\cdots\mathbb{E}\left[(Y_{2,j_2^{(p)}})^2\right] = \lim_{\mathbb{N}\to\infty}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^N(Y_{2,j})^2\right)^p\right]=\phi_{2,p}. \tag{5.8}$$

Then, combining (5.7) and (5.8) we finally obtain that

$$\lim_{N\to+\infty}\mathbb{E}\left[(\widetilde{U}_2^{(N)})^{2p}\right] = \frac{(2p)!}{2^p\,p!}(\sigma^2)^{2p}\phi_{2,p}. \tag{5.9}$$

This yields statement i) of the Lemma in the special case $\ell=2$.

We now extend it to the general case and proceed by recurrence. Let $\ell\geq 3$, as before, we denote by

$$\widetilde{U}_\ell^{(N)} = \frac{1}{\sqrt{N}^\ell}\sum_{j_1=2}^N\sum_{j_2,\cdots,j_\ell=1}^N J_{j_1,j_2}\cdots J_{j_{\ell-1},j_\ell}Y_{\ell,j_\ell}.$$

We suppose that, for all $p\in\mathbb{N}$:

$$\lim_{N\to+\infty}\mathbb{E}\left[\left(U_{\ell-1}^{(N)})^{2p}\right)\right] = \lim_{N\to+\infty}\mathbb{E}\left[\left(\widetilde{U}_{\ell-1}^{(N)}\right)^{2p}\right] = \frac{2p!}{2^p p!}\sigma^{2p(\ell-1)}\phi_{\ell-1,p} \tag{5.10}$$

$$\lim_{N\to+\infty}\mathbb{E}\left[\left(U_{\ell-1}^{(N)})^{2p+1}\right)\right] = \lim_{N\to+\infty}\mathbb{E}\left[\left(\widetilde{U}_{\ell-1}^{(N)}\right)^{2p+1}\right] = 0. \tag{5.11}$$

We then decompose $U_\ell^{(N)}$ over $j_1=1$ and $j_1\geq 2$

$$\begin{aligned} U_\ell^{(N)} &= \frac{1}{\sqrt{N}^\ell}J_{11}\sum_{j_2,\cdots,j_\ell=1}^N J_{1,j_2}\cdots J_{j_{\ell-1},j_\ell}Y_{\ell,j_\ell} + \frac{1}{\sqrt{N}^\ell}\sum_{j_1=2}^N\sum_{j_2,\cdots,j_\ell=1}^N J_{1,j_1}J_{j_1,j_2}\cdots J_{j_{\ell-1},j_\ell}Y_{\ell,j_\ell} \\ &= \frac{1}{\sqrt{N}}\widetilde{U}_{\ell-1}^{(N)} \quad + \quad \widetilde{U}_\ell^{(N)}. \end{aligned} \tag{5.12}$$

By the recurrence hypothesis, it is easy to see that all the moments of the first term of the right hand side of (5.12) converge to zero. It yields that the latter converges to 0 in probability as $N\to+\infty$. It remains to show that, for all $p\in\mathbb{N}$,

$$\mathbb{E}\left[(\widetilde{U}_\ell^{(N)})^{2p}\right]\underset{N\to+\infty}{\longrightarrow}\frac{(2p)!}{2^p p!}(\sigma^2)^{\ell p}\phi_{\ell,p} \qquad \text{and} \qquad \mathbb{E}\left[(\widetilde{U}_\ell^{(N)})^{2p+1}\right]\underset{N\to+\infty}{\longrightarrow}0.$$

To this effect, similar to the decomposition (5.6), let us define the following three disjoint sets. $\Delta_N^{<,\ell}$ is the subset of ℓ-tuples of indexes in $\{1,\cdots,N\}^\ell$ having between 2 and $\lfloor \ell/2\rfloor+1$ indexes which are equal. Let $\Delta_N^{\neq,\ell}$ be the subset of $\{1,\cdots,N\}^\ell$ with all indexes different, and let $\Delta_N^{>,\ell}$ be the complement of the union of the other two: at least $\lfloor \ell/2 \rfloor + 2$ indexes are equal. Decomposing $\widetilde{U}_\ell^{(N)}$ on the disjoint sets $\Delta_N^{<,\ell}$, $\Delta_N^{>,\ell}$, and $\Delta_N^{\#,\ell}$, we write

(5.13)

$$\widetilde{U}_\ell^{(N)} = \frac{1}{\sqrt{N}^\ell}\Big(\sum_{\substack{2 \le j_1 \le N \\ (j_1,\dots,j_\ell)\in \Delta_N^{<,\ell}}} J_{1,j_1} J_{j_1,j_2} \cdots J_{j_{\ell-1},j_n} Y_{\ell,j_\ell} + \sum_{\substack{2 \le j_1 \le N \\ (j_1,\dots,j_\ell)\in \Delta_N^{>,\ell}}} J_{1,j_1} J_{j_1,j_2} \cdots J_{j_{\ell-1},j_n} Y_{\ell,j_\ell} + \sum_{\substack{2 \le j_1 \le N \\ (j_1,\dots,j_\ell)\in \Delta_N^{\neq,\ell}}} J_{1,j_1} J_{j_1,j_2} \cdots J_{j_{\ell-1},j_n} Y_{\ell,j_\ell} \Big)$$

The sum over $\Delta_N^{<,\ell}$ contain at least one entry of the matrix J independent of the others and of degree 1. By the zero mean hypothesis of J, the first term on the right hand side of (5.13) has mean zero. On $\Delta_N^{>,\ell}$, the sums have of the order of $N^{\ell-(i+1)} < N^{\ell/2}$ terms, $i \ge \lfloor l/2 \rfloor + 2$, and hence disappear in the limit $N \to \infty$ when taking the expectation of (5.13). As before, the only surviving term is the one with the sum over $\Delta_N^{\neq,\ell}$. We then apply the same arguments as for $\ell = 2$ to this term to obtain that, for all $p \in \mathbb{N}$

$$\lim_{N\to+\infty} \mathbb{E}\left[\left(\widetilde{U}_\ell^{(N)}\right)^{2p}\right] = \frac{(2p)!}{2^p p!}\sigma^{2\ell p}\phi_{\ell,p},$$
$$\lim_{N\to+\infty} \mathbb{E}\left[\left(\widetilde{U}_\ell^{(N)}\right)^{2p+1}\right] = 0.$$

It yields the same limits for $U_\ell^{(N)}$. This conclude the proof statement i), for all $\ell \in \mathbb{N}$.

We now prove statements ii) and iii), which are generalizations of statement i). First of all, for $i \in \{1, \cdots, N\}$ and for $\ell \in \mathbb{N}$, we define

$$H_{\ell,j} = \sum_{1 \le j_2, \cdots, j_\ell \le N} J_{i,j_2} \cdots J_{j_{\ell-1},j_\ell} Y_{\ell,j_\ell}.$$

Then in the case $m = 2$, let $\ell_1 < \ell_2$ and $b_1, b_2 \in \mathbb{N}$: $\mathbb{E}\left[(U_{\ell_1}^{(N)})^{b_1}(U_{\ell_2}^{(N)})^{b_2}\right]$ is the sum over $m_1 = 1$ to ℓ_1, $m_2 = 1$ to ℓ_2 and over families $(c_i^{(1)})_i \in \Sigma_{m_1,b_1}$ and $(c_i^{(2)})_i \in \Sigma_{m_2,b_2}$ of the terms

(5.14)
$$\frac{1}{\sqrt{N}^{\ell_1 b_1 + \ell_2 b_2}} \sum_{m_1=1}^{\ell_1} \frac{b_1! b_2!}{c_1^{(1)}! \cdots c_{m_1}^{(1)}! c_1^{(2)}! \cdots c_{m_2}^{(2)}}$$
$$\sum_{j_1^{(1)},\cdots,j_1^{m_1}=1}^{N} \sum_{i_1^{(1)},\cdots,i_1^{m_2}=1}^{N} \mathbb{E}\left[(J_{1,j_1^{(1)}} H_{\ell_1,j_1^{(1)}})^{c_1^{(1)}} \cdots (J_{1,j_1^{(m_1)}} H_{\ell_1,j_1^{(m_1)}})^{c_{m_1}^{(1)}} (J_{1,i_1^{(1)}} H_{\ell_2,i_1^{(1)}})^{c_1^{(2)}} \cdots (J_{1,i_1^{(m_2)}} H_{\ell_2,i_1^{(m_2)}})^{c_{m_2}^{(2)}}\right]$$

Using exactly the same arguments as in the proof of statement i), we see that if either $m_1 > \lfloor b_1/2 \rfloor$ or $m_2 > \lfloor b_2/2 \rfloor$ there is at least one of the $c_i^{(1)}$ or $c_i^{(2)}$ which is equal to one. Suppose that $c_1^{(1)} = 1$ then on the set where $j_1^{(1)} \neq i_1^{(i)}$ for all $i \in \{1, \cdots, \ell_2\}$ the expectation of the sum over the $j_1^{(i)}$, $i_1^{(j)}$ is null. Moreover, the sum over $m_1 < \lfloor b_1/2 \rfloor$ or over $m_2 < \lfloor b_2/2 \rfloor$ disappears in the limit. Indeed, as in the proof of statement i), there are not enough terms in these cases. Consequently, if at least one of the exponents b_1 or b_2 is odd, the limit is null. It is easy to see that it is exactly the same thing for $m \ge 2$ and statement ii) of the Lemma follows.

Furthermore, it yields that in the case $b_1 = 2a_1$ and $b_2 = 2a_2$, a_1, $a_2 \in \mathbb{N}$, the only surviving term in the limit $N \to +\infty$ is for $m_1 = a_1$ and $m_2 = a_2$:

$$\lim_{N\to+\infty} \mathbb{E}\left[\left(U_{\ell_1}^{(N)}\right)^{2a_1}\left(U_{\ell_2}^{(N)}\right)^{2a_2}\right] = \lim_{N\to+\infty} \frac{1}{N^{\ell_1 a_1+\ell_2 a_2}} \frac{(2a_1)!\,(2a_2)!}{2^p} \tag{5.15}$$

$$\sum_{1\le j_1^{(1)}<\cdots<j_1^{a_1}\le N} \sum_{1\le i_1^{(1)}<\cdots<i_1^{a_2}\le N} \mathbb{E}\left[J^2_{1,j_1^{(1)}}(H_{\ell_1,j_1^{(1)}})^2\cdots J^2_{1,j_1^{(a_1)}}(H_{\ell_1,j_1^{(a_1)}})^2 J^2_{1,i_1^{(1)}}(H_{\ell_2,i_1^{(1)}})^2\cdots J^2_{1,i_1^{(a_2)}}(H_{\ell_2,i_1^{(a_2)}})^2\right]$$

As before, only expectation of the sum over the set of indexes which are all different from each other contributes to the limit. Then it follows that, using (5.9),

$$\lim_{N\to+\infty} \mathbb{E}\left[\left(U_{\ell_1}^{(N)}\right)^{2a_1}\left(U_{\ell_2}^{(N)}\right)^{2a_2}\right] = \frac{(2a_1)!\,(2a_2)!}{2^p a_1! a_2!}(\sigma^2)^{\ell_1 a_1+\ell_2 a_2}\phi_{\ell_1,a_1}\phi_{\ell_2,a_2}. \tag{5.16}$$

The extension to the case $m \geq 3$ is similar. This conclude the proof of Lemma 3.4.

□

Acknowledgements

This research has received funding from the Europe an Union's Horizon 2020 Framework Programme for Research and Innovation under the Specific Grant Agreement No. 785907 (Human Brain Project SGA2).

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