WKB eigenmode construction for analytic Toeplitz operators

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Abstract

We provide almost eigenfunctions for Toeplitz operators with real-analytic symbols, at the bottom of non-degenerate wells. These quasimodes follow the WKB ansatz; the error is $O(e^{-cN})$, where $c > 0$ and $N \to +\infty$ is the inverse semiclassical parameter.

1 Introduction

This article is concerned with Berezin-Toeplitz quantization. We associate, to a real-valued function f on a compact Kähler manifold M, a sequence of self-adjoint operators $(T_N(f))_{N\geq 1}$ acting on spaces of sections over M. These operators are called *Toeplitz operators*. Examples of Toeplitz operators are spin systems (where M is a product of two-spheres), which are indexed by the total spin $S = \frac{N}{2}$. Motivated by questions arising in the physics literature about the behaviour of spin systems at low temperature, we wish to study the lowest-lying eigenvalues and associated eigenvectors of Toeplitz operators in the limit $N \to +\infty$. In this article we specifically study exponential estimates, that is, approximate expressions with $O(e^{-cN})$ remainder for some $c > 0$.

Given $f : M \to \mathbb{R}$, we say that $P_0 \in M$ is an elliptic point when $\nabla f(P_0) = 0$ and all eigenvalues of the Hessian of f at P_0 are nonzero and have the same sign. Elliptic points are always local extrema, while local extrema generically are elliptic points.

We provide, in the special case where f is real-analytic and has an elliptic point at $P_0 \in M$, a construction of quasimodes for $T_N(f)$: we build (Theorem A) a sequence of normalised sections $(v(N))_{N\geq 1}$ and a real sequence $(\lambda(N))_{N\geq 1}$, with asymptotic expansions in decreasing powers of N, such that

$$T_N(f)v(N) = \lambda(N)v(N) + O(e^{-cN}).$$

The sequence $v(N)$ takes the form of a Wentzel-Kramers-Brillouin (WKB) ansatz: it is written as

$$v(N) : x \mapsto CN^{\frac{\dim(M)}{2}}\psi^{\otimes N}(x)(v_0(x) + N^{-1}v_1(x) + \ldots), \tag{1}$$

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MSC 2010 Subject classification: 32A25 32W50 35A20 35P10 35Q40 58J40 58J50 81Q20 where the *symbol* $(v_k)_{k\geq 0}$ is a sequence of functions on M that are holomorphic in a neighbourhood V of P_0, and the *phase* ψ is a section over M, holomorphic on V, and decaying away from P_0:

$$\exists \epsilon > 0, \forall x \in M, |\psi(x)| \leq e^{-\epsilon \operatorname{dist}(x,P_0)^2}.$$

The multiplicative factor $CN^{\frac{\dim(M)}{2}}$ then ensures that $v(N)$ is normalised.

Since $T_N(f)$ is self-adjoint, the existence of a quasimode implies that $\lambda(N)$ is exponentially close to the spectrum of $T_N(f)$, but not necessarily that $v(N)$ is exponentially close to an eigenfunction. In Theorem A, we also prove that, if f is Morse (all critical points have non-degenerate Hessian), the eigenvectors associated with the lowest eigenvalue of $T_N(f)$ are exponentially close to a finite sum of quasimodes of the form (1), attached to the elliptic points corresponding to global minima of f.

1.1 Bergman kernels and Toeplitz operators

Let us rapidly present the basic definitions associated with semiclassical Berezin-Toeplitz quantization as introduced in full generality in [3]; see the in-depth introductions [5, 22].

Let (M,ω) be a compact boundaryless symplectic manifold. Berezin-Toeplitz quantization associates, to a function $f : M \to \mathbb{R}$, a sequence of Toeplitz operators $(T_N(f))_{N\geq 1}$. To perform this quantization, we have to provide a supplementary geometrical information: a complex structure J, which encodes a notion of holomorphic objects on M, and which is compatible with ω : (M,ω,J) is a Kähler manifold.

Definition 1.1. Let (M,ω,J) be a compact Kähler manifold. Let L be a complex line bundle over M, and let h be a Hermitian metric on L such that $\operatorname{curv} h = 2i\pi\omega$. The couple (L,h) exists if and only if the integral of ω over each closed surface in M is an integer multiple of 2π. We then say that M is *quantizable.*

Let $N \in \mathbb{N}$. The *Bergman projector* S_N is the orthogonal projector, from the space of square-integrable sections $L^2(M,L^{\otimes N})$ to the finite-dimensional subspace of holomorphic sections $H_0(M,L^{\otimes N})$.

Let also $f : M \to \mathbb{R}$. The Toeplitz operator $T_N(f)$ associated with f is the following operator:

$$\begin{array}{rcl} T_N(f) : H_0(M,L^{\otimes N}) & \to & H_0(M,L^{\otimes N}) \\ u & \mapsto & S_N(fu). \end{array}$$

It is convenient to extend $T_N(f)$ into an operator on $L^2(M,L^{\otimes N})$ by the formula

$$T_N(f) = S_N f S_N;$$

in this way, $(T_N(f))_{N\in\mathbb{N}}$ is a family of finite rank self-adjoint operators.

Given a Hilbert basis $(s_1,\ldots,s_{d_N})$ of $H_0(M,L^{\otimes N})$, the Bergman projector S_N admits the following integral kernel:

$$S_N(x,y) = \sum_{i=1}^{d_N} s_i(x) \otimes \overline{s_i(y)}.$$

The study of the Bergman kernel as $N \to +\infty$ lies at the core of the semiclassics of Toeplitz quantization. In a previous article [10], we developed a semiclassical machinery in real-analytic regularity, in order to give asymptotic formulas up to an exponential remainder for S_N, and Toeplitz operators, in the case where the symplectic form ω and the function f are real-analytic on the complex manifold (M,J). The analysis of the Bergman kernel in real-analytic geometry is a recent and active topic [1, 20, 26, 6, 21, 11].

Definition 1.2. Let (M,ω,J) be a compact quantizable Kähler manifold and $(S_N)_{N\geq 1}$ be the associated sequence of Bergman projectors. Let $x \in M$ and $N \in \mathbb{N}$. The *coherent state* ψ_x^N at x is the element of $H_0(M,L^{\otimes N}) \otimes \overline{L}_x^{\otimes N}$ given by freezing the second variable of the Bergman kernel: for every $y \in M$, one has

$$\psi_x^N(y) = S_N(y,x).$$

Theorem A. *Let M be a quantizable compact real-analytic Kähler manifold. Let f be a real-analytic, real-valued function on M.*

1. *Let $P_0 \in M$ be an elliptic point of f which is a local minimum. Then there exist*
   - *positive constants C, c, c', R, ϵ,*
   - *a neighbourhood V of P_0,*
   - *a holomorphic function φ on V such that*

   $$\exists \epsilon \in (0,1), \forall x \in V,\ |\varphi(x)| \leq (1-\epsilon)\frac{d(x,P_0)^2}{2},$$

   - *a sequence of holomorphic functions $(u_k)_{k\geq 0}$ on V, with $u_0(P_0) = 1$ and $u_k(P_0) = 0$ for $k \neq 0$, satisfying*

   $$\forall k \geq 0,\ \sup_V |u_k| \leq CR^k k!, \tag{2}$$

   - *a real sequence $(\lambda_k)_{k\geq 0}$, where $\lambda_0 = f(P_0)$ and where λ_1 is the ground state energy of the quantization of the Hessian of f at P_0 (see [8]), satisfying*

   $$\forall k \geq 0,\ |\lambda_k| \leq CR^k k!, \tag{3}$$

   such that, for every $N \geq 1$, if $\psi^N_{P_0}$ denotes the coherent state at P_0, then with

   $$u(N) = \mathbb{1}_V \psi^N_{P_0} e^{N\varphi}\left(\sum_{k=0}^{cN} N^{-k} u_k\right)$$
   $$\lambda(N) = \sum_{k=0}^{cN} \lambda_k N^{-k},$$

   one has

   $$\|T_N(f)u(N) - \lambda(N)u(N)\|_{L^2(M,L^{\otimes N})} \leq Ce^{-c'N}.$$

2. *If the minimal set of f consists in a finite number of non-degenerate minimal points, then any normalised eigenfunction of $T_N(f)$ with minimal eigenvalue is at distance $Ce^{-c'N}$ from a linear combination of the functions constructed in item 1 at each minimal point.*

The coherent state $\psi^N_{P_0}$ has a WKB-type expansion (see Proposition 2.4), and one can recover the section ψ in (1) from there and φ. The analytic symbol $(v_k)_{k\geq 0}$ is then obtained by normalising $u(N)$, an operation which preserves the growth property (2). Thus the expression of $u(N)$ above implies (1).

Our method of proof consists in constructing φ, satisfying a Hamilton-Jacobi type equation, then solve by induction a transport equation on the coefficients u_k, and finally to prove the analytic growth controls (2) and (3).

The pseudodifferential equivalent of Theorem A is claimed in [23], however all details are not given: the growth property (eqs. (2) and (3)), which is crucial to the ability to sum until $k = cN$ terms in (1), is stated without proof. The verification of estimates of this nature is often non trivial, and in this case, it is the subject of Propositions 3.4 and 4.2. The purpose of this article is not merely to fix the gap in the strategy proposed in [23], but to extend it to the more general setting of Berezin-Toeplitz operators.

Indeed, a pseudodifferential operator on $\mathbb{R}^d$ with real-analytic symbol can be written *exactly* as a Toeplitz operator on $M = \mathbb{C}^d$ if the symbol can be extended to a constant width strip in $\mathbb{C}^d$, since the exact formula for "Toeplitz to pseudodifferential" which can be found for instance in [32], formula (13.4.12), is a heat-type evolution at time $N^{-1} = \hbar$ which can be reversed if the pseudo-differential symbol is analytic. Hence, up to a careful check of the behaviour at infinity which we do not carry out here, Theorem A should be enough to provide a complete proof of the result stated in [23].

The Toeplitz point of view on pseudodifferential operators is relevant for WKB eigenmode construction and exponential estimates, both from the perspective of physics [31] and mathematics (at the core of analytic microlocal analysis is the Fourier-Bros-Iagolnitzer transformation, which relates pseudodifferential operators to Toeplitz operators). In addition, the Toeplitz setting contains other semiclassical quantum operators such as spin systems, on which tunnelling estimates are widely studied in the physics community [25], although not always in a rigorous way.

WKB estimates for low-energy eigenfunctions in a Morse energy landscape are well-known for purely electric Schrödinger operators, of the form $-\hbar^2\Delta + V(x)$. Without analyticity assumptions on V, one can construct a *formal* WKB ansatz [16] : a sequence of functions of the form

$$u_\hbar(x) \sim e^{-\frac{\varphi(x)}{\hbar}} \sum_{k=0}^{+\infty} \hbar^k u_k(x),$$

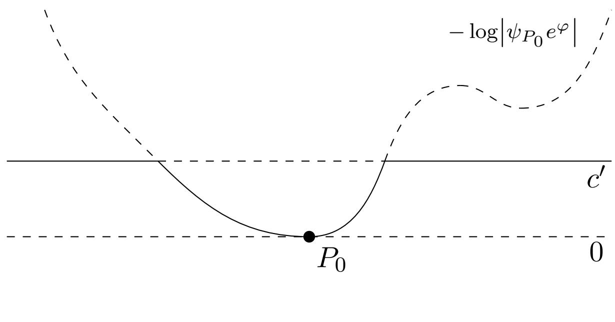
(where $\sim$ denotes formal summation of classical symbols), which is a $O(\hbar^\infty)$-quasimode for the first eigenvalue $\lambda_\hbar$ of the Schrödinger operator, in the weighted norm associated with φ:

$$\left\| \left(-\hbar^2\Delta + V - \lambda_\hbar \right) u_\hbar \right\|_{L^2\left(e^{-\frac{\varphi}{\hbar}}\right)} = O(\hbar^\infty).$$

WKB expansions of quasimodes have been the subject of recent activity in the context of purely magnetic Schrödinger operators, of the form $(i\hbar\partial + A(x))^2$, in an increasing order or generality [2, 13, 14, 12]. In this context, the symbol f reaches a non-degenerate minimum on a symplectic submanifold of $(\mathbb{R}^{2d}, \omega_{st})$, but subprincipal effects force the ground state to be microlocalised at one "miniwell" (as in [18, 9]). Contrary to the electric case, real-analyticity of the magnetic potential A is, most of the time, necessary to obtain exponential decay of the ground state away from the miniwell : in [2, 13, 14, 12], one assumes real-analyticity of A and concludes in a formal WKB expansion. We conjecture that, as in Theorem A, the coefficients of these formal WKB expansions can be in fact summed into an analytic symbol.

Remark 1.3. If the minimal set of f consists in several non-degenerate wells, then applying the Part 1 of Theorem A at every well yields that the actual ground state, which is exponentially close to an orthogonal linear combination of quasimodes as above, has Agmon-type exponential decay in a neighbourhood of the minimal set, as in [16].

Even if the function φ can be extended to all of M and yields, formally, exponential decay everywhere except at the minimal point, this rate of decay is blurred, not only by the error terms in the expression of the Bergman kernel (Proposition 2.4) but also by the fact that we can only sum up to cN with c small when summing analytic symbols (see Proposition 2.2), which yields a fixed error of order $e^{-c'N}$ with $c' > 0$ small. This yields a lower bound to the decay rate for the actual ground state, as a function of the position, which follows the blue, continuous line in the following picture:



The solid line above is our best estimate for a function $g : M \to [0, +\infty)$ such that the ground state v_N of $T_N(f)$ satisfies

$$\forall \epsilon > 0, \exists C > 0, \forall x \in M, \forall N \in \mathbb{N}, |v_N(x)| \leq Ce^{-(g(x)-\epsilon)N}.$$

Near P_0, the rate of decay is sharp, but we have no explicit control on the constant c'.

Theorem A has applications to tunnelling between (locally) symmetric wells, in the spirit of [17]. In Proposition 5.1 we prove that, if f has two symmetrical wells, and λ_0, λ_1 denote the two first eigenvalues of $T_N(f)$ (with multiplicity), then

$$\lambda_1 - \lambda_0 \leq Ce^{-c'N}, \tag{4}$$

where c' and C are as in Theorem A. In the physics community, the tunnelling rate $-N^{-1}\log(\lambda_1 - \lambda_0)$ is often estimated using the degree zero approximation φ in the WKB ansatz, which solves a Hamilton-Jacobi equation (see Proposition 3.3). However, in Proposition 5.2, we provide a series of examples which tend to illustrate that the tunnelling rate is not given by φ, and is not bounded from above by the best possible constant c' in Theorem A. This contrasts with the case of an electric Schrödinger operator, where it is well-known that the tunnelling rate corresponds to the behaviour of the Hamilton-Jacobi equation, as detailed in [17]. The difference between the two cases is the ability to extend the problem "far away" into the complex, and in particular, to prove sharp exponential decay, as we explained in Remark 1.3.

Let us now discuss possible alternative strategies for the proof of Theorem A. The method we follow is the most direct one, inspired from the C^∞ case, and proceeds by a sequence of perturbations of the Toeplitz operator with a quadratic symbol corresponding to the Hessian (for which the ground state is explicit). The necessary verification that the terms of the perturbation sum into an analytic symbol, i.e. controls (2) and (3), occupies most of the proof.

In some situations, it might be easier to prove that one can conjugate $T_N(f)$ (microlocally and up to an exponentially small error) into an operator for which the eigenfunctions are explicit, such as a quadratic operator. One can hope to do so when M has complex dimension 1, or more generally for integrable systems near elliptic points. This fact is used, for instance, in appendix B of [19] concerned with pseudo-differential operators on $\mathbb{R}$, and leads to a result similar to Theorem A, with a shorter and simpler proof. In this integrable case, if one can build a quantum action-angle theorem near an elliptic point in the analytic category (which remains to be done), one could describe all eigenfunctions and eigenvectors modulo an exponentially small error, not just the ground state.

Apart from the complete integrability assumption, there is hope that KAM-like theorems can be of use, and more precisely, that under a suitable genericity assumption on the symplectic diagonalisation of the Hessian, a Birkhoff normal form near the elliptic point is enough to describe the spectrum, but it is not clear whether it would provide a simpler proof of Theorem A.

1.2 Outline

In Section 2 we briefly present the tools which we developed in [10] to tackle problems from semiclassical analysis in real-analytic regularity in the context of Berezin-Toeplitz quantization. We then proceed to the proof of Theorem A.

Section 3 recalls the geometrical ingredients required in order to build a *formal* WKB ansatz, that is, for every $K \in \mathbb{N}$, a quasimode of the form (1), where there are K terms inside the parenthesis and which satisfies the eigenvalue equation up to $O(N^{-K-2})$. Each of the coefficients u_k solves a transport equation, with a source term depending on $u_0, \ldots, u_{k-1}$. The main novel result of Section 3 is a control the solution of this transport equation, in an analytic norm, by the source term.

In Section 4, we prove that the sequences $(u_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$ belong to an analytic class. In particular, they satisfy the growth condition (eqs. (2) and (3)). This allows us to perform an analytic summation and produce a sequence of sections (indexed by N) which satisfy the eigenvalue equation for $T_N(f)$ up to

$O(e^{-c'N})$, for some $c' > 0$. A standard analysis of the distribution of low-lying eigenvalues of $T_N(f)$ allows us to conclude the proof in Section 5, where we also discuss the constant c' in the statement of Theorem A.

2 Calculus of analytic Toeplitz operators

To be able to prove the growth condition (eqs. (2) and (3)), we use the framework developed in a previous article [10], which allowed us to study Toeplitz operators with real-analytic regularity.

Given two real parameters $r > 0, m$, we say that a function $f : U \to \mathbb{C}$ on a smooth open set U of $\mathbb{R}^d$ belongs to the space $H(m, r, U)$ when there exists $C > 0$ such that, for every $j \geq 0$, one has

$$\|f\|_{C^j(U)} := \sup_{x\in U} \sum_{|\mu|=j} |\partial^\mu f(x)| \leq C \frac{r^j j!}{(j+1)^m}.$$

The minimal C such that the control above is true is a Banach norm for the space $H(m, r, U)$. Such functions are real-analytic, and can be extended as holomorphic functions in an tube of radius proportional to r^{-1} around U. Reciprocally, by the Cauchy integral formula, for all $V \subset\subset U$, every real-analytic function on U belongs to $H(m, r, V)$, for all $m \in \mathbb{R}$ and for some $r > 0$ depending on $\mathrm{dist}(V, \mathbb{R}^d \setminus U)$ and the radius of analyticity of the function near V (see [10], Proposition 2.15).

We will often use, in this article, the pointwise version of the C^j seminorm above:

$$\|\nabla^j f(x)\|_{\ell^1} := \sum_{|\mu|=j} |\partial^\mu f(x)|.$$

Generalising the definition of $H(m, r, U)$, we obtain *analytic (formal) symbols.*

Definition 2.1. Let X be a compact real-analytic manifold, with real-analytic boundary. We fix a finite set $(\rho_V)_{V\in\mathcal{V}}$ of local real-analytic charts on open sets V which cover X.

- Let $j \geq 0$. The C^j seminorm of a function $f : X \to \mathbb{C}$ which is continuously differentiable j times is defined as
$$\|f\|_{C^j(X)} = \max_{V\in\mathcal{V}} \|f \circ \rho_V\|_{C^j(V)} = \max_{V\in\mathcal{V}} \sup_{x\in V} \sum_{|\mu|=j} |\partial^\mu (f \circ \rho_V)(x)|.$$
- Let r, R, m be positive real numbers. The space of analytic symbols $S^{r,R}_m(X)$ consists of sequences $(a_k)_{k\geq 0}$ of real-analytic functions on X, such that there exists $C \geq 0$ such that, for every $j \geq 0, k \geq 0$, one has
$$\|a_k\|_{C^j(X)} \leq C \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$
The norm of an element $a \in S^{r,R}_m(X)$ is defined as the smallest C as above; then $S^{r,R}_m(X)$ is a Banach space.

The definition of $S^{r,R}_m(X)$ depends on the chosen atlas, but not in an essential way: elements of $S^{r,R}_m(X)$ for a given atlas belong to $S^{r',R'}_{m'}(X)$ for another atlas, with r', R', m' suitably chosen as a function of r, R, m and the two atlases.

These analytic classes, which we defined and studied in [10], are well-behaved with respect to standard manipulations of functions (multiplication, change of variables, ...) and, most importantly, with respect to the stationary phase lemma. Another important property is the summation of such symbols: if $\hbar$ is a semiclassical parameter (here $\hbar = N^{-1}$), then for $c > 0$ small depending on R, the sum

$$\sum_{k=0}^{c\hbar^{-1}} \hbar^k u_k$$

is uniformly bounded as $\hbar \to 0$; in this sum, terms of order $k = \hbar^{-1}$ are exponentially small, so that the choice of c has an exponentially small influence on the sum.

Proposition 2.2. *[See [10], Propositions 3.6 and 3.8] Let X be a compact real-analytic manifold with boundary and fix a real-analytic atlas on X.*

Summation *Let $f \in S_m^{r,R}(X)$. Let $c_R = \frac{e}{3R}$. Then*

1. *The function*
$$f(N) : x \mapsto \sum_{k=0}^{c_R N} N^{-k} f_k(x)$$
is bounded on X uniformly for $N \in \mathbb{N}$.
2. *For every $0 < c_1 < c_R$, there exists $c_2 > 0$ such that*
$$\sup_{x \in X} \left| \sum_{k=c_1 N}^{c_R N} N^{-k} f_k(x) \right| = O(e^{-c_2 N}).$$

Cauchy product *There exists $C_0 \in \mathbb{R}$ such that the following is true.*

Let $r, R \geq 0$ and $m \geq 4$. For $a, b \in S_m^{r,R}(X)$, let us define the Cauchy product of a and b as
$$(a * b)_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

1. *The space $S_m^{r,R}(X)$ is an algebra for this Cauchy product, that is,*
$$\|a * b\|_{S_m^{r,R}} \leq C_0 \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}},$$
Moreover, there exists $c > 0$ depending only on R such that as $N \to +\infty$, one has
$$(a * b)(N) = a(N)b(N) + O(e^{-cN}).$$
2. *Let r_0, R_0, m_0 positive and $a \in S_{m_0}^{r_0,R_0}(X)$ with a_0 nonvanishing. Then, for every m large enough depending on a, for every $r \geq r_0 2^{m-m_0}, R \geq R_0 2^{m-m_0}$, a is invertible (for the Cauchy product) in $S_m^{r,R}(X)$, and its inverse a^{*-1} satisfies:*
$$\|a^{*-1}\|_{S_m^{r,R}(X)} \leq 2 \min(|a_0|)^{-4} \|a\|^3_{S_{m_0}^{r_0,R_0}(X)}.$$

Remark 2.3. A variant of Definition 2.1 reads
$$\|a_k\|_{C^j} \leq C \frac{r^j R^k j! k!}{(j+k+1)^m};$$
ultimately, the controls on the symbol $(u_k)_{k \geq 0}$ in Theorem A will take a mixed form between this and $S_m^{r,R}$, see (29) and (30). Other definitions can be found in the literature, as in [4], equation (1.2), or [28], chapter 1. These alternative definitions of analytic symbol spaces are all morally equivalent (they can be embedded into each other by changing the values of r,R,m). In practice, one has to choose the convention which suits the particular combinatorial arguments.

The summation property in Proposition 2.2, together with the stationary phase lemma, allows us to study Toeplitz operators up to an exponentially small error. One of the main results of [10], proved independently [26], then simplified in [6, 11, 21], is an expansion of the Bergman kernel on a real-analytic Kähler manifold, with error $O(e^{-c'N})$, in terms of an analytic symbol.

Proposition 2.4. *(See [10], Theorem A, and [26], Theorem 3.1) Let M be a quantizable compact real-analytic Kähler manifold of complex dimension d. There exists positive constants r, R, m, c, c', C, a neighbourhood U of the diagonal in $M \times M$, a section Ψ of $L \boxtimes \overline{L}$ over U, and an analytic symbol $a \in S_m^{r,R}(U)$, holomorphic in the first variable, anti-holomorphic in the second variable, such that the Bergman kernel S_N on M satisfies, for each $x, y \in M \times M$ and $N \geq 1$:*

$$\left| S_N(x,y) - \mathbb{1}_{(x,y)\in U} \Psi^{\otimes N}(x,y) \sum_{k=0}^{cN} N^{d-k} a_k(x,\overline{y}) \right| \leq Ce^{-c'N}.$$

Note that the constants c, c', C here are different from that of Theorem A.

Similar ideas appear in the literature, and have been successfully applied to the theory of pseudodifferential operators with real-analytic symbols. Early results [4] use a special case of our analytic classes, when $m = 0$; from there, a more geometrical theory of analytic Fourier Integral operators was developed [28], allowing one to gradually forget about the parameters r and R when applying the analytic stationary phase lemma. It is surprising that the introduction of the parameter m, which mimics the definition of the Hardy spaces on the unit ball, was never considered, although it simplifies the manipulation of analytic functions (for instance, the space $H(m, r, V)$ is stable by product if and only if $m \geq 3$). In [10] and in the present article, it is crucial that we are able to choose m arbitrarily large.

Along with the definition of symbol classes in Definition 2.1, we will use another analytic symbol class, which is a mixture of Definition 2.1 and Remark 2.3. The basic remark is that, by the Stirling formula,

$$\frac{(2k)!}{4^k k! k!}\sqrt{2k+1} \in \left[\tfrac{1}{2}, 1\right],$$

and in particular, the following classes of symbols are well-behaved:

$$\|a_k\|_{C^j} \leq C_a \begin{cases} \dfrac{r^j R^k j! k!}{(j+k+1)^m} & \text{if } j < k \\[2ex] \dfrac{(r/4)^j R^k (j+k)!}{(j+k+1)^{m-\frac{1}{2}}} & \text{if } j \geq k. \end{cases}$$

We end this section with a technical lemma, which is a refinement of Lemma 4.6 appearing in [10], adapted to the symbol class above.

Lemma 2.5. *Let U, V, Λ be domains in $\mathbb{C}^d$ containing 0. Let κ_λ be a biholomorphism from V with image contained in U, with real-analytic dependence on $\lambda \in \Lambda$ and suppose that $\kappa_0(0) = 0$.*

Let $\kappa : (\lambda, v) \mapsto \kappa_\lambda(v)$ and suppose that there exists constants C_κ, r_0, m_0 such that, for all $j \in \mathbb{N}_0$, one has

$$\|\kappa\|_{C^j(V\times\Lambda)} \leq C_\kappa \frac{r_0^j j!}{(j+1)^{m_0}}.$$

Then the following is true for all $m \geq m_0$ and all $r \geq r_0 2^{m-m_0+5}$. Let f be a real-analytic function on $U \times \Lambda$ and suppose that there exists C_f and $k \geq 0$ such that, for all $j \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$,

$$\|\nabla^j f(0,0)\|_{\ell^1} \leq C_f \frac{r^j j! k!}{(j+k+1)^m}$$

and furthermore, for all $j \geq k$,

$$\|\nabla^j f(0,0)\|_{\ell^1} \leq C_f \frac{(r/4)^j (j+k)!}{(j+k+1)^{m-\frac{1}{2}}}.$$

Let $n \leq k$ and $i \leq 2n$. Let ∇_v^i denote the i-th gradient over the first set of variables, acting on $V \times \Lambda$; then

$$g \mapsto (\lambda \mapsto \nabla_v^i g(\kappa_\lambda(v), \lambda)_{v=0})$$

is a differential operator of degree i acting on functions on $U \times \Lambda$. Let $(\nabla_\kappa^i)^{[\leq n]}$ denote the truncation of this differential operator to a differential operator of degree less or equal to n. Then, with

$$\gamma = 16 C_\kappa r$$

one has, for every $j \geq 0$,

$$\|\nabla^j (\nabla_\kappa^i)^{[\leq n]} f(0)\|_{\ell^1} \leq i^{d+1} j^{d+1} \gamma^i C_f \frac{r^{j+i} k!}{(i+j+k+1)^{m-\frac{1}{2}}} \begin{cases} (i+j)! & \text{if } i \leq n \\ \max((n+j)!(i-n)!, j!i!) & \text{otherwise,} \end{cases}$$

and, for every $j \geq k - \min(i,n)$,

$$\|\nabla^j (\nabla_\kappa^i)^{[\leq n]} f(0)\|_{\ell^1} \leq i^{d+1} j^{d+1} \gamma^i C_f \frac{(r/4)^{j+i} k!}{(i+j+k+1)^m} \times \begin{cases} (i+j+k)! & \text{if } i \leq n \\ \max((n+j+k)!(i-n)!, (j+k)!i!) & \text{otherwise.} \end{cases}$$

Proof. We proceed as in Lemma 4.6 in [10]. By the Faà di Bruno formula, one has

$$\|\nabla^j ((\nabla_\kappa^i)^{[\leq n]} f)(0)\|_{\ell^1} \leq i^d j^d \times \sum_{|P|=1}^{\min(n,i)} \sum_{\substack{e_0+\cdots+e_{|P|}=j \\ s_1+\cdots+s_{|P|}=|P|}} \frac{j!}{e_0! e_1! \cdots e_{|P|}!} \frac{i!}{(|P|)! s_1! \cdots s_{|P|}!} \|\nabla^{|P|+e_0} f(0)\|_{\ell^1} \prod_{i=1}^{|P|} \|\kappa\|_{C^{s_i+e_i}}. \quad (5)$$

We now inject the controls on f and κ. First of all, for all $j_1 \in \mathbb{N}_0$,

$$\|\kappa\|_{C^{j_1}} \leq C \frac{(r/32)^{j_1} j_1!}{(j_1+1)^m},$$

and in particular, if $j_1 \geq 1$,

$$\|\kappa\|_{C^{j_1}} \leq C \frac{(r/16)^{j_1} (j_1-1)!}{j_1^m}$$

since $2^{j_1} \geq j_1$.

Injecting this along with the control on f, the general term in the sum (5) is bounded by

$$\frac{j! i! r^{|P|+e_0} (r/16)^{i+j-e_0} (|P|+e_0)! k! (s_1+e_1-1)! \cdots (s_{|P|}+e_{|P|}-1)!}{(|P|)! e_0! \cdots e_{|P|}! s_1! \cdots s_{|P|}!} \times \frac{1}{(|P|+e_0+k+1)^m (s_1+e_1)^m \cdots (s_{|P|}+e_{|P|})^m}$$

and, if $|P|+e_0 \geq k$, there holds the more precise bound

$$\frac{j!i!(r/4)^{|P|+e_0}(r/4)^{i+j-e_0}(|P|+e_0+k)!(s_1+e_1-1)!\cdots(s_{|P|}+e_{|P|}-1)!}{(|P|)!e_0!\cdots e_{|P|}!s_1!\cdots s_{|P|}!} \times \frac{1}{(|P|+e_0+k+1)^{m-\frac{1}{2}}(s_1+e_1)^m\cdots(s_{|P|}+e_{|P|})^m}.$$

The constraints on (s_j) and (e_j) are such that one can simplify the second factors:

$$\frac{1}{(|P|+e_0+k+1)^m(s_1+e_1)^m\cdots(s_{|P|}+e_{|P|})^m} \leq \frac{1}{(i+j+k+1)^m}$$

$$\frac{1}{(|P|+e_0+k+1)^{m-\frac{1}{2}}(s_1+e_1)^m\cdots(s_{|P|}+e_{|P|})^m} \leq \frac{1}{(i+j+k+1)^{m-\frac{1}{2}}}.$$

Moreover, by Lemma 2.5 in [10],

$$\frac{(s_1+e_1-1)!(s_{|P|}+e_{|P|}-1)!}{s_1!\cdots s_{|P|}!e_1!\cdots e_{|P|}!} \leq \frac{(i-|P|+j-e_0)!}{(i-|P|+1)!(j-e_0)!}.$$

Thus, one has the following general bound on the general term of the sum in (5):

$$C_f(C_\kappa)^{|P|}\frac{j!i!r^{|P|+e_0}(r/16)^{i+j-e_0}(|P|+e_0)!k!(i-|P|+j-e_0)!}{(|P|)!e_0!(i-|P|+1)!(j-e_0)!}\frac{1}{(i+j+k+1)^m},$$

and, provided $|P|+e_0 \geq k$, the more precise bound

$$C_f(C_\kappa)^{|P|}\frac{j!i!(r/4)^{|P|+e_0}(r/16)^{i+j-e_0}(|P|+e_0+k)!(i-|P|+j-e_0)!}{(|P|)!e_0!(i-|P|+1)!(j-e_0)!}\frac{1}{(i+j+k+1)^{m-\frac{1}{2}}}.$$

In both cases, one can isolate

$$\frac{i!}{(|P|)!(i-|P|+1)!} \leq 2^i$$

and

$$\frac{(i-|P|+j-e_0)!}{(j-e_0)!} \leq 2^{i+j-e_0}(i-|P|)!;$$

thus, the general bound simplifies into

$$C_f\frac{(C_\kappa)^{|P|}r^{|P|}}{8^{j-e_0}4^i}\frac{r^{i+j}(|P|+e_0)!k!(i-|P|)!j!}{e_0!}\frac{1}{(i+j+k+1)^m},$$

and the specific bound into

$$C_f\frac{(C_\kappa)^{|P|}r^{|P|}}{2^{j-e_0}}\frac{(r/4)^{i+j}(|P|+e_0+k)!(i-|P|)!j!}{e_0!}\frac{1}{(i+j+k+1)^{m-\frac{1}{2}}}.$$

Let us now count the number of terms. For fixed $|P|-e_0$ and j, there are $\binom{i}{|P|} \leq 2^i$ choices for $s_1,\ldots,s_{|P|}$ (since each of them must be positive) and $\binom{j-e_0+|P|}{|P|} \leq 2^{j-e_0+|P|}$ choices for $e_1,\ldots,e_{|P|}$, which are non-negative. Thus, fixing $|P|$ and e_0 and summing over $s_1,\ldots,s_{|P|},e_1,\ldots,e_{|P|}$, the resulting sum is bounded by

$$C_f(C_\kappa)^{|P|}r^{|P|}\frac{r^{i+j}(|P|+e_0)!k!(i-|P|)!j!}{4^{j-e_0}e_0!}\frac{1}{(i+j+k+1)^m}$$

and, provided $|P|+e_0\geq k$,

$$C_f(C_\kappa)^{|P|}r^{|P|}2^{i+|P|}\frac{(r/4)^{i+j}(|P|+e_0+k)!(i-|P|)!j!}{e_0!}\frac{1}{(i+j+k+1)^{m-\frac{1}{2}}}.$$

Both formulas above are increasing with respect to e_0. If $e_0=k-|P|$, moreover, the second formula is larger than the first one up to losing a power of $|P|$: indeed, the ratio between the two is

$$\frac{2^{i+|P|}}{4^{i+k-|P|}}\frac{(2k)!}{k!k!}\sqrt{i+j+k+1}\geq\frac{8^{|P|}}{2^{i-1}}.$$

To conclude, if $j+|P|\leq k$, then the sum over e_0 is bounded by

$$jC_f(16C_\kappa)^{|P|}r^{|P|}r^{i+j}(|P|+j)!k!(i-|P|)!\frac{1}{(i+j+k+1)^m}$$

and if $j+|P|\geq k$, then this sum is bounded by

$$jC_f(16C_\kappa)^{|P|}r^{|P|}2^i(r/4)^{i+j}(|P|+j+k)!(i-|P|)!\frac{1}{(i+j+k+1)^{m-\frac{1}{2}}}.$$

We artificially added the factor $16^{|P|}$ in the first bound so that, if $j+|P|\geq k$, then the second bound implies the first one.

We can now conclude: if $j+\min(i,n)\leq k$ (that is to say, if $j+|P|$ is always less than k), we sum the first bound over $|P|$, remarking that it is log-convex with respect to $|P|$. We obtain that the sum appearing in (5) is bounded by

$$ijC_f(16C_\kappa r)^i\frac{r^{i+j}k!}{(i+j+k+1)^m}\max_{|P|\in\{0,\min(i,n)\}}(|P|+j)!(i-|P|)!,$$

If $j+\min(i,n)\geq k$, then we can apply the second bound for all $|P|$, so that we similarly obtain

$$ijC_f(16C_\kappa r)^i\frac{(r/4)^{i+j}}{(i+j+k+1)^m}\max_{|P|\in\{0,\min(i,n)\}}(|P|+j+k)!(i-|P|)!.$$

This concludes the proof. □

3 Geometry of the WKB Ansatz

In this section we provide the geometric ingredients for the proof of Theorem A. We formally proceed as in the case of a Schrödinger operator [15]. If a real-analytic, real-valued function f has a non-degenerate local minimum at $P_0\in M$, we seek a sequence of eigenfunctions of $T_N(f)$ of the form

$$\psi^N_{P_0}e^{N\varphi}(u_0+N^{-1}u_1+\ldots),$$

where $\psi^N_{P_0}$ denotes the coherent state at P_0. If $f(P_0)=0$, then the associated sequence of eigenvalues should be of order $O(N^{-1})$, that is to say, follow the asymptotic expansion:

$$N^{-1}\lambda_0+N^{-2}\lambda_1+\ldots.$$

When solving the eigenvalue problem, the terms of order 0 in

$$e^{-N\varphi}T_N(f)\psi^N_{P_0}e^{N\varphi}(u_0+N^{-1}u_1+\ldots)$$

yield an equation on φ. In the case of a Schrödinger operator this is the eikonal equation $|\nabla\varphi|^2 = V$, which is solved using the Agmon metric. In our more general case, we are in presence of a form of the Hamilton-Jacobi equation (see (11) below), which we solve in Proposition 3.3 using a geometric argument based on the existence of a stable manifold, in the spirit of [29]. Associated with f and φ are transport equations which we must solve in order to recover the sequence of functions $(u_k)_{k\geq 0}$. In Proposition 3.4 we study this transport equation under the point of view of symbol spaces of Definition 2.1. This will allow us, in Proposition 4.2, to perform an analytic summation of the u_k's in order to find an exponentially accurate eigenfunction for $T_N(f)$, with exponential decay away from P_0.

The plan of this section is as follows: we begin in Subsection 3.1 with the study of an analytic phase which will be a deformation of the phase Φ_1 considered above. We then define and study the Hamilton-Jacobi equation associated with a real-analytic function near a non-degenerate minimal point, and the associated transport equations, in Subsections 3.2 and 3.3 respectively. This geometric insight on the construction of a quasimode attached to an elliptic point is not new, but the purpose of this section is to fix notations, to present these ideas in a self-sustained way and in the geometric context of Berezin-Toeplitz situation, and to prove an analytic estimate for the solution of the transport equation (Proposition 3.4).

In the rest of this article,

- (M,ω,J) is a quantizable real-analytic compact Kähler manifold (which means that ω is real-analytic on the complex manifold (M,J)); $L, (S_N)_{N\in\mathbb{N}}$ and $(T_N)_{N\in\mathbb{N}}$ are the prequantum line bundle, the Bergman projectors and the Toeplitz quantizations of Definition 1.1;

- f is a real-valued function on M with real-analytic regularity.

- U_0 is a small neighbourhood of an elliptic point P_0 of f which is a local minimum (such that the objects below exist on U_0); without loss of generality $f(P_0) = 0$;

- ϕ is a Kähler potential near U_0 such that, in a chart where P_0 is mapped to 0,
$$\phi(y) = \frac{|y|^2}{2} + O(|y|^3) :$$
that is, $\phi : U_0 \to \mathbb{R}$ satisfies
$$\partial\overline{\partial}\phi = i\omega;$$

- $\widetilde{\phi}$ is the holomorphic function on $U_0 \times \overline{U_0}$ such that $\widetilde{\phi}(x,\overline{x}) = \phi(x)$ (holomorphic extension or polarisation of ϕ);

- More generally, $\widetilde{\ }$ represents holomorphic extension of real-analytic functions: for instance, $\widetilde{f}$ is the extension of f and is defined on $U_0 \times \overline{U_0}$;

- $\Phi_1 : U_0^2 \times \overline{U_0}^2$ is defined by
$$\Phi_1 : (x,y,\overline{w},\overline{z}) \mapsto 2\widetilde{\phi}(x,\overline{w}) - 2\widetilde{\phi}(y,\overline{w}) + 2\widetilde{\phi}(y,\overline{z}) - 2\widetilde{\phi}(x,\overline{z}).$$

The function Φ_1 is associated with the Bergman kernel S_N in the following way: the section Ψ of Proposition 2.4 satisfies, for all $(x,y,z) \in U^3$:
$$\langle \Psi^{\otimes N}(x,y), \Psi^{\otimes N}(y,z)\rangle_{L_y^{\otimes N}} = \Psi^{\otimes N}(x,z)\exp(N\Phi_1(x,y,\overline{y},\overline{z})).$$

3.0 Formal identification of the WKB ansatz

We search for an eigenfunction of $T_N(f)$ of the form

$$x \mapsto \psi_0^N(x)e^{N\varphi(x)}(u_0(x)+N^{-1}u_1(x)+\ldots),$$

where ψ_0^N is the coherent state at 0 (see Definition 1.2), and $\varphi, u_0, u_1, \ldots$ are holomorphic functions on a fixed neighbourhood of 0.

This construction is local. Indeed, the only situation where the holomorphic functions $\varphi, u_0, u_1, \ldots$ can be extended to the whole of M is when they are constant. However, if φ does not grow too fast (see Definition 3.1), then the trial function above is exponentially small outside any fixed neighbourhood of zero. In particular, applying $T_N(f)$ yields, by Proposition 2.4,

$$\begin{aligned}&T_N(f)(e^{N\varphi}(u_0+N^{-1}u_1+\ldots)\psi_0^N):\\ &\quad x \mapsto \psi_0^N(x)e^{N\varphi(x)}\int_U e^{N\Phi_1(x,y,\overline{y},0)+N\varphi(y)-N\varphi(x)}f(y)\left(\sum_{k=0}^{cN}N^{d-k}a_k(x,y)\right)\left(u_0(y)+N^{-1}u_1(y)+\ldots\right)\mathrm{d}y\\ &\qquad\qquad + O(e^{-cN}).\end{aligned}$$

If the function appearing in the exponential

$$\Phi_2 : (x,y) \mapsto \Phi_1(x,y,\overline{y},0)+\varphi(x)-\varphi(y)$$

is a positive phase function in the sense of [24] (which is guaranteed if φ does not grow too fast, see Proposition 3.2), one can apply the stationary phase lemma ([28], Theorem 2.8). If $y_*(x)$ is the critical point of this phase (which belongs to the complexification $\widetilde{U_0} = U_0 \times \overline{U_0}$), at dominant order, one has

$$T_N(f)(e^{N\varphi}u_0\psi_0^N)(x) = \psi_0^N(x)e^{N\varphi(x)}\widetilde{f}(y_*(x))\widetilde{a_0}(x,y_*(x))\widetilde{u_0}(y_*(x))J(x)+O(N^{-1}).$$

where J is a non-vanishing Jacobian.

Since we search for an eigenfunction with eigenvalue close to zero, we want this principal term to vanish. As J and a_0 do not vanish, this yields

$$\widetilde{f}(y_*(x)) = 0,$$

which boils down to a particular PDE on φ, the *Hamilton-Jacobi equation*. We provide a geometric solution to this equation in Proposition 3.3.

To study the higher orders of the stationary phase lemma we introduce, as in [28], Lemma 2.7, a x-dependent, holomorphic change of variables κ_x, from a neighbourhood of $y_*(x)$ in $\widetilde{U_0}$ to a neighbourhood of 0 in $\mathbb{C}^{2d}$, such that

$$\widetilde{\Phi_2}\circ\kappa_x^{-1}(v_1,\overline{v_2}) = v_1\cdot\overline{v_2}, \tag{6}$$

as well as the associated gradient and Laplacian, acting as follows on holomorphic functions on $U_0 \times \widetilde{U_0}$:

$$(\nabla_{\kappa_x}b):(x,y,\overline{w}) \mapsto \left(\frac{\partial b(x,\kappa_x^{-1}(v_1,\overline{v_2}))}{\partial v_{1,j}}(x,\kappa_x(y,\overline{w})), \frac{\partial b(x,\kappa_x^{-1}(v_1,\overline{v_2}))}{\partial \overline{v_{2,j}}}(x,\kappa_x(y,\overline{w}))\right)_{1\leq j\leq d} \tag{7}$$

$$(\Delta_{\kappa_x}b):(x,y,\overline{w}) \mapsto \sum_{j=1}^{d}\frac{\partial^2 b(x,\kappa_x^{-1}(v_1,\overline{v_2}))}{\partial v_{1,j}\partial\overline{v_{2,j}}}. \tag{8}$$

At next order, the eigenvalue equation reads, for all $x \in U_0$,

$$\begin{aligned}N^{-1}\lambda_0u_0(x) &= T_N(f)(e^{N\varphi}(u_0+N^{-1}u_1)\psi_0^N)(x) \bmod N^{-2}\\ &= N^{-1}\psi_0^N(x)e^{N\varphi(x)}\left(\widetilde{f}J(\widetilde{a_0}\widetilde{u_1}+\widetilde{a_1}\widetilde{u_0})(x,y_*(x))+\Delta_{\kappa_x}(\widetilde{f}\widetilde{a_0}\widetilde{u_0}J)(x,y_*(x))\right).\end{aligned}$$

Since $\widetilde{f}(y_*(x)) = 0$, there is no contribution from u_1 at this order. Moreover, one can distribute

$$\Delta_{\kappa_x}(\widetilde{f}\widetilde{a_0}\widetilde{u_0}J) = \widetilde{f}\widetilde{a_0}J\Delta_{\kappa_x}\widetilde{u_0} + \widetilde{u_0}\Delta_{\kappa_x}(\widetilde{f}\widetilde{a_0}J) + \nabla_{\kappa_x}(\widetilde{f}\widetilde{a_0}J)\cdot\nabla_{\kappa_x}(\widetilde{u_0}).$$

The first term of the right-hand side is zero when evaluated at $y_*(x)$ since $\widetilde{f}(y_*(x)) = 0$. We obtain

$$\left(\nabla_{\kappa_x}(\widetilde{f}\widetilde{a_0}J)\right)(y_*(x))\cdot(\nabla_{\kappa_x}\widetilde{u_0})(y_*(x)) = u_0(x)\left(\lambda_0 - \Delta_{\kappa_x}(\widetilde{f}\widetilde{a_0}J)(y_*(x))\right).$$

Observe that $\widetilde{f}$, as the complex extension of f, has a critical point at $x = 0$, so that, as long as $y_*(0) = 0$ (which is proved in Proposition 3.2), there holds $\nabla_{\kappa_0}(\widetilde{f}\widetilde{a_0}J)(y_*(0)) = 0$. Hence, the equation above implies

$$\lambda_0 = \Delta_{\kappa_0}(\widetilde{f}\widetilde{a_0}J)(0).$$

We will see in Proposition 4.1 that this λ_0 indeed corresponds to the ground state energy of the Hessian of f at zero. It remains to solve an equation of the form

$$\left(\nabla_{\kappa_x}(\widetilde{f}\widetilde{a_0}J)\right)(y_*(x))\cdot(\nabla_{\kappa_x}\widetilde{u_0})(y_*(x)) = u_0(x)h(x), \tag{9}$$

where $h(x) = \lambda_0 - \Delta_{\kappa_x}(\widetilde{f}\widetilde{a_0}J)(y_*(x))$ vanishes at $x = 0$. Similar equations are satisfied by the successive terms u_k. This family of equations is solved (with a convenient control on the size of the solution) in Proposition 3.4. Then, in Section 4 we will prove by induction that the sequence $(u_k)_{k\geq 0}$ indeed forms an analytic symbol and that the eigenvalue equation admits a solution up to an $O(e^{-c'N})$ error.

3.1 A family of phase functions

In this subsection we study a family of analytic phases (in the sense of Definition 3.11 in [10]) given by a WKB ansatz at the bottom of a well. To begin with, we describe the conditions on a holomorphic function φ at a neighbourhood of zero, such that $e^{N\varphi}\psi_0^N$ is a convenient first-order candidate for the ground state of $T_N(f)$.

Definition 3.1. A holomorphic function φ on U_0 is said to be *admissible* under the following conditions:

$$\varphi(0) = 0$$
$$\nabla\varphi(0) = 0$$
$$\exists t < 1,\ \forall x \in U_0,\ |\varphi(x)| \leq \frac{t}{2}|x|^2.$$

Proposition 3.2. *Let φ be an admissible function. The function from $U_0 \times U_0$ to $\mathbb{C}$ defined by*

$$\Phi_2 : (x, y) \mapsto \Phi_1(x, y, \overline{y}, 0) + \varphi(y) - \varphi(x) \tag{10}$$

is, for all x in a small neighbourhood of zero, a positive phase function of y in the sense of [24].

The complex critical point of Φ_2 is $y_*(x) = (x, \overline{y}_c(x))$, where the holomorphic function $x \mapsto \overline{y}_c(x)$ satisfies

$$-2\partial_1\widetilde{\phi}(x, \overline{y}_c(x)) + 2\partial_1\widetilde{\phi}(x, 0) = -\partial\varphi(x).$$

In particular, $\overline{y}_c(0) = 0$.

Proof. Near $y = \overline{w} = 0$, there holds

$$\Phi_1(0, y, \overline{w}, 0) = -y\cdot\overline{w} + O(|y, \overline{w}|^3).$$

In particular, for $x = 0$, the function $(y, \overline{w}) \mapsto \widetilde{\Phi_2}(0, y, \overline{w})$ has a critical point at $(0, 0)$ whose Hessian has a non-degenerate, negative real part (because $|\varphi(y)| \leq \frac{t|y|^2}{2}$). In particular, for x small enough, $\widetilde{\Phi_2}$ has exactly one critical point near 0, with non-degenerate, negative Hessian real part. The critical point $(y, \overline{w})$ satisfies the two equations

$$\overline{\partial}_{\overline{w}}\widetilde{\phi}(x, \overline{w}) - \overline{\partial}_{\overline{w}}\widetilde{\phi}(y, \overline{w}) = 0$$
$$-2\partial_y\widetilde{\phi}(y, \overline{w}) + 2\partial_y\widetilde{\phi}(y, 0) = -\partial\varphi(y).$$

The first equation yields $y = x$, then the second equation has only one solution $\overline{w} = \overline{y}_c(x)$, so that the phase at this critical point is equal to

$$2\widetilde{\phi}(x, \overline{y}_c(x)) - 2\widetilde{\phi}(x, \overline{y}_c(x)) + 2\widetilde{\phi}(x, 0) - 2\widetilde{\phi}(x, 0) + \varphi(x) - \varphi(x) = 0.$$

This concludes the proof. □

3.2 Hamilton-Jacobi equation

Let φ be an admissible function. For every $x \in M$ close to 0, there exists one $\overline{y_c}(x)$ in $\overline{U_0}$ such that $(x, \overline{y_c}(x))$ is a critical point for the phase of Proposition 3.2.

In order to find the phase of the WKB ansatz, we want to solve, in a neighbourhood of 0, the following system of equations on φ and $\overline{y_c}$, where φ is an admissible function:

$$\begin{cases} \widetilde{f}(x, \overline{y_c}(x)) = 0. \\ -2\partial_1\widetilde{\phi}(x, \overline{y}_c(x)) + 2\partial_1\widetilde{\phi}(x, 0) = -\partial\varphi(x). \end{cases} \tag{11}$$

This will be called the *Hamilton-Jacobi equation*. This equation is non-trivial already at the formal level: for fixed x the equation $\widetilde{f}(x, \overline{y}) = 0$ defines (a priori) a manifold of complex codimension 1, which has a singularity at $x = 0$. On the other hand, we need to ensure that $\partial_1\widetilde{\phi}(x, \overline{y}_c(x))$ is a closed holomorphic 1-form in order to solve for φ.

Proposition 3.3. *The Hamilton-Jacobi equation* (11) *admits a solution near* 0*, such that* φ *is analytic.*

Proof. We follow the usual method (see the appendix of [29]), which will consist in considering the stable manifold of the Hamiltonian flow of $\widetilde{f}$ for a certain symplectic form.

Since the Taylor expansion of ϕ at zero is

$$\phi(x) = \frac{1}{2}|x|^2 + O(|x|^3),$$

the map

$$\overline{w} \mapsto 2\partial_1\widetilde{\phi}(x, \overline{w}) = \overline{w} + O(|x, \overline{w}|^2)$$

is a biholomorphism in a neighbourhood of zero, for x small. Let γ_x denote its inverse, then γ_x is tangent to identity at $x = \overline{w} = 0$.

Letting

$$\widetilde{f_1} : (x, \overline{z}) \mapsto \widetilde{f}(x, \gamma_x(\overline{z})),$$

the Hamilton-Jacobi equation (11) is equivalent to the modified system:

$$\begin{cases} \widetilde{f_1}(x, \overline{z_c}(x)) = 0 \\ -\overline{z_c}(x) + 2\partial_1\widetilde{\phi}(x, 0) = -\partial\varphi(x). \end{cases}$$

The first step is to solve this equation at main order, that is, when $\widetilde{f}_1, \widetilde{\phi}, \varphi$ are quadratic. This can be done using a KAK decomposition, and for completeness and pedagogical purposes we detail how this is done. The construction of this decomposition will also play a role in the proof of Proposition 3.4.

Let Q be the Hessian of f at zero and $\widetilde{Q}$ its holomorphic extension (as a quadratic form). Then $\widetilde{f}_1(x,\overline{z}) = \widetilde{Q}(x,\overline{z}) + O(|x,\overline{z}|^3)$ since γ_x is tangent to identity at $x = \overline{w} = 0$.

In the modified system, there holds $\overline{z_c}(x) = \partial(2\widetilde{\phi}(x,0) + \varphi(x))$, so that finding $x \mapsto \overline{z_c}(x)$ amounts to finding a holomorphic Lagrange submanifold $L = \{x, \overline{z_c}(x)\}$ of $\mathbb{C}^d \times \overline{\mathbb{C}^d}$ near 0, for the standard symplectic form $\Im(\sum \mathrm{d}x_j \wedge \mathrm{d}\overline{z_j})$ (which extends the symplectic form $\sum \mathrm{d}\Re(x_j) \wedge \mathrm{d}\Im(x_j)$), such that L is contained in $\{\widetilde{f}_1 = 0\}$ and is transverse to the vertical $\{x = 0\}$. Then, near 0, one has $L = \{x, \partial F(x)\}$ for some holomorphic F, and it will only remain to check that $\varphi = F - 2\widetilde{\phi}(\cdot, 0)$ is admissible. As in [29], from f and the standard symplectic form, the Lagrangean L will be constructed as the stable manifold of the fixed point 0 for the symplectic flow of $\widetilde{f}_1$.

Suppose $\widetilde{f}_1$ is quadratic; that is, $\widetilde{f}_1 = \widetilde{Q}$. The quadratic form Q admits a symplectic diagonalisation with respect to the (real) symplectic form $\sum \mathrm{d}\Re(x_j) \wedge \mathrm{d}\Im(x_j)$: there exists a symplectic matrix S, and positive numbers $\omega_1, \ldots, \omega_d$, such that

$$Q = S^T \operatorname{diag}(\omega_1, \omega_1, \omega_2, \omega_2, \ldots, \omega_d, \omega_d) S.$$

Let us study how this symplectic change of variables S behaves under complexification. From the KAK decompostion of the semisimple Lie group $Sp(2d)$ (or, more practically, using a singular value decomposition), the matrix S can be written as $U_1 D U_2$, where U_1 and U_2 belong to $Sp(2d) \cap O(2d) \simeq U(d)$, and $D = \operatorname{diag}(\sigma_1, \sigma_1^{-1}, \ldots, \sigma_d, \sigma_d^{-1}) > 0$.

We now complexify U_1, U_2, D as $\mathbb{R}$-linear endomorphisms of $\mathbb{C}^d$ (in contrast with Q, which we complexified as a quadratic form). The complexified actions of U_1 and U_2 are straightforward: for $j = 1, 2$ one has $\widetilde{U}_j(x, \overline{z}) = (U_j x, U_j^{-1}\overline{z})$. The action of D is diagonal: $D = \operatorname{diag}(D_1, \ldots, D_d)$, with

$$D_j(\Re(x_j), \Im(x_j)) = \left(\sigma_j \Re(x_j), \sigma_j^{-1} \Im(x_j)\right).$$

Hence, the action of $\widetilde{D}$ is block-diagonal, with

$$\widetilde{D}_j(x_j, \overline{z_j}) = \left(\frac{\sigma_j + \sigma_j^{-1}}{2} x_j + \frac{\sigma_j - \sigma_j^{-1}}{2} \overline{z_j}, \frac{\sigma_j - \sigma_j^{-1}}{2} x_j + \frac{\sigma_j + \sigma_j^{-1}}{2} \overline{z_j} \right).$$

After applying successively the changes of variables $\widetilde{U}_1, \widetilde{D}, \widetilde{U}_2$, in the new variables, the quadratic form becomes

$$\widetilde{f}_1 \circ \widetilde{S} : (q, p) \mapsto \sum_{j=1}^{d} \omega_j q_j p_j.$$

Among the zero set of this form, a space of particular interest is $\{p = 0\}$. It is a holomorphic Lagrangean subspace, which is preserved by the symplectic gradient flow of $\widetilde{f}_1 \circ \widetilde{S}$, and such that every solution starting from this subspace tends to zero for positive time. This subspace $\{p = 0\}$ is the *stable manifold of zero* for the symplectic gradient of $\widetilde{f}_1 \circ \widetilde{S}$. Let us show that, in the starting coordinates $(x, \overline{z})$, the stable manifold of $\widetilde{f}_1$ leads to an admissible solution of the Hamilton-Jacobi equation.

- The inverse change of variables $\widetilde{U}_2^{-1}$ leaves $\{p = 0\}$ invariant.
- The inverse change of variables $\widetilde{D}^{-1}$ sends $\{p = 0\}$ to $\{\overline{z} = Ax\}$, with $\|Ax\|_{\ell^2} \leq t\|x\|_{\ell^2}$ for some $t < 1$. Indeed, the matrix A has diagonal entries $\frac{\sigma_j - \sigma_j^{-1}}{\sigma_j + \sigma_j^{-1}} \in (-1, 1)$.

- The inverse change of variables $\widetilde{U}_1^{-1}$ sends $\{\overline{z} = Ax\}$ to $\Lambda_0 = \{\overline{z} = U_1 A U_1^{-1} x\}$, with a similar property: for some $t < 1$, there holds $\|U_1 A U_1^{-1} x\|_{\ell^2} \le t\|x\|_{\ell^2}$.

Then Λ_0 is a linear space of the form $\{\overline{z} = \partial F_0(x)\}$, where F_0 is the holomorphic function

$$F_0 : x \mapsto \frac{1}{2}\langle x, U_1 A U_1^{-1} x\rangle.$$

Hence $\varphi : x \mapsto F_0(x) - 2\widetilde{\phi}(x, 0) = F_0(x) + O(|x|^3)$ is an admissible solution to the Hamilton-Jacobi equations.

If $\widetilde{f}_1$ is quadratic, we just identified a holomorphic Lagrange submanifold transverse to $\{x = 0\}$ and contained in $\{\widetilde{f}_1 = 0\}$, as the stable manifold of 0 for the Hamiltonian flow of $\widetilde{f}_1$. In the general case, $\widetilde{f}_1$ is a small perturbation of its quadratic part in a small neighbourhood of 0, so that, by the stable manifold Theorem ([27], Theorem 6.1), the stable subspace Λ_0 is deformed into a stable manifold L which has the same properties: L is Lagrangean (since it is a stable manifold of a symplectic flow, it must be isotropic, and L has maximal dimension), and it is transverse to x a small neighbourhood of zero since $T_0 L$ is the linear Lagrangean subspace Λ_0 described above. Moreover, the Hamiltonian flow of $\widetilde{f}_1$ preserves $\widetilde{f}_1$ so that L is contained in $\{\widetilde{f}_1 = 0\}$.

We finally let F be a holomorphic function such that $L = \{(x, \partial F(x))\}$. With $\varphi : x \mapsto F(x) - 2\widetilde{\phi}(x, 0)$, and $\overline{z_c}(x) = \partial F(x)$, we obtain a solution to the modified Hamilton-Jacobi equation

$$\begin{cases} \widetilde{f}_1(x, \overline{z_c}) = 0 \\ -\overline{z_c} + \partial_1 \widetilde{\phi}(x, 0) = -\partial\varphi(x). \end{cases}$$

Since $\widetilde{\phi}(x, 0) = O(|x|^3)$, one has $\varphi(x) = F(x) + O(|x|^3) = F_0(x) + O(|x|^3)$, so that

$$|\varphi(x)| = |F_0(x)| + O(|x|^3) \le \frac{t}{2}|x|^2$$

for some $t < 1$ on a neighbourhood of 0. This concludes the proof. $\square$

3.3 Transport equations

In the proof of Theorem A, one must solve recursively transport equations of the form (9), and prove that the solution is well-controlled. Let us prove that one can control the solution of this equation by the source term.

Proposition 3.4. *Let $f' : U_0 \times \widetilde{U_0} \mapsto \mathbb{C}$ be holomorphic and such that*

$$f'(x, y, \overline{w}) = \widetilde{f}(y, \overline{w}) + O(|x, y, \overline{w}|^3),$$

and let φ be an admissible solution of the Hamilton-Jacobi equation (11). Let $x \in U_0$ and let ∇_{κ_x} as defined in (7). Let also $\overline{y}_c$ be the holomorphic function of x such that $(x, \overline{y}_c(x))$ is the critical point of Φ_2 as defined in (10). Then there exists $U \subset U_0$ containing 0 such that the following is true.

For every $g : U \to \mathbb{C}$ holomorphic with $g(0) = 0$, and every $h : U \to \mathbb{C}$ holomorphic with $h(0) = 0$, there exists a unique holomorphic function $u : U \to \mathbb{C}$ with $u(0) = 0$ which solves the following transport equation:

$$(\nabla_{\kappa_x} f')(x, x, \overline{y}_c(x)) \cdot (\nabla_{\kappa_x}[(x, y, \overline{w}) \mapsto u(y)])(x, x, \overline{y}_c(x)) = h(x)u(x) + g(x).$$

Moreover, there exists a $\mathbb{C}$-linear change of variables $A(f', \varphi)$ on $\mathbb{C}^d$, and positive constants $r_0(h, f', \varphi)$, $m_0(h, f', \varphi)$, $C(h, f', \varphi)$ such that, for every

$$k \ge 0, \quad m \ge m_0(h, f', \varphi), \quad r \ge r_0(h, f', \varphi)2^{m - m_0(h, f', \varphi)}, \quad C_g > 0,$$

for every g as above which satisfies, for every $j \geq 0$,

$$\sum_{|\mu|=j} |\partial^\mu (g \circ A(f', \varphi))(0)| \leq C_g \frac{r^j (j+1)! k!}{(1+j+k+1)^m}, \tag{12}$$

one has, for every $j \geq 0$,

$$\sum_{|\mu|=j} |\partial^\mu (u \circ A(f', \varphi))(0)| \leq C(h, f', \varphi) C_g \frac{r^j j! k!}{(1+j+k)^m}. \tag{13}$$

If moreover g satisfies the sharper control

$$\sum_{|\mu|=j} |\partial^\mu (g \circ A(f', \varphi))(0)| \leq C_g \frac{(r/4)^j (j+1+k)!}{(1+j+k+1)^{m-\frac{1}{2}}} \qquad \forall j \geq k, \tag{14}$$

Then u satisfies

$$\sum_{|\mu|=j} |\partial^\mu (u \circ A(f', \varphi))(0)| \leq C(h, f', \varphi) C_g \frac{(r/4)^j (j+k)!}{(1+j+k)^{m-\frac{1}{2}}} \qquad \forall j \geq k. \tag{15}$$

Note that (14) is sharper than (12), and similarly (15) is sharper than (13), because for every $j \geq k$ one has

$$\frac{(j+k)!\sqrt{j+k+1}}{j!k!4^j} \leq 1;$$

in fact in the limit case $k = j$ one has

$$\frac{(2j)!\sqrt{2j+1}}{j!j!4^j} \in \left[\frac{1}{2}, 1\right].$$

Proof. We let X be the vector field on U such that

$$(\nabla_{\kappa_x} f')(x, x, \overline{y}_c(x)) \cdot (\nabla_{\kappa_x}[(x, y, \widetilde{w}) \mapsto u(y)])(x, x, \overline{y}_c(x)) = X \cdot u(x).$$

The proof consists in four steps. In the first step we prove that all trajectories of X converge towards 0 in negative time, so that there is no dynamical obstruction to the existence of u near 0 (if X had wandering or closed trajectories, solving $X \cdot u = fu + g$ would require specific conditions on f and g). In the second step, we identify the successive terms of a formal power expansion of u, which allows us to control successive derivatives of u at 0: that is, we prove inequality (13) using (12). In the third step, we prove that the solution u is well-defined on the whole of U for some small neighbourhood U of 0, using the Duhamel formula. In the fourth step, we finally prove that (14)⇒(15).

First step

We study the dynamics of the vector field X in a neighbourhood of zero. To this end, we relate κ_x to the linear change of variables which appeared in the proof of Proposition 3.3 in the case where f is quadratic.

We first note that, as the Taylor expansion of f' is

$$f' = \widetilde{f} + O(|x, y, \overline{w}|^3) = O(|x, y, \overline{w}|^2),$$

one has $X(0) = 0$. The Hessian of φ at zero is determined by the Hessian of f at zero; it then determines the linear part of κ_0 at 0, hence the linear part of X at 0. Up to a linear unitary change of variables, there exists a diagonal matrix A, a unitary matrix U_1, and positive $\omega_1, \ldots, \omega_d$, such that

$$f : x \mapsto \sum_{j=1}^{d} \omega_j |(U_1 A x)_j|^2 + O(|x|^3).$$

Then $\varphi(x) = \frac{1}{2}\langle x, U_1 A U_1^{-1} x\rangle + O(|x|^3)$, so that the phase reads

$$\Phi_1(x, y, \overline{w}, 0) + \varphi(y) - \varphi(x) = 2(x - y) \cdot \left(\overline{w} - \frac{1}{4} U_1 A U_1^{-1}(x + y)\right) + O(|x, y, \overline{w}|^3).$$

In particular, at first order, one can write

$$\kappa_x(y, \overline{w}) = \left(y - x, \overline{w} - \frac{1}{4} U_1^{-1} A U_1 (y + x)\right) + O(|x, y, \overline{w}|^2).$$

Hence, the inverse change of variables is of the form

$$\kappa_x^{-1}(v_1, \overline{v_2}) = \left(v_1 + x, \overline{v_2} + \frac{1}{4} U_1^{-1} A U_1 (v_1 + 2x)\right) + O(|x, v_1, \overline{v_2}|^2),$$

so that the restriction to the diagonal

$$u \circ \kappa_x^{-1}(v, \overline{v}) = u(v + x) + O(|x, v, \overline{v}|^2)$$

is holomorphic with respect to v, at first order.

We then wish to compute

$$\nabla_{\kappa_x} f' \cdot \nabla_{\kappa_x} u := [\overline{\partial}_v (f' \circ \kappa_x^{-1}) \cdot \partial_v (u \circ \kappa_x^{-1}) + \partial_v (f' \circ \kappa_x^{-1}) \cdot \overline{\partial}_v (u \circ \kappa_x^{-1})](v_1 = \overline{v_2} = 0)$$

which is equal, at first order, to the opposite symplectic flow (for the symplectic form $\Im(\mathrm{d}v \wedge \mathrm{d}\overline{v})$) of $\widetilde{f}$ applied to u:

$$\nabla_{\kappa_x} f' \cdot \nabla_{\kappa_x} u = [\overline{\partial}_v (\widetilde{f} \circ \kappa_x^{-1}) \cdot \partial_v (u \circ \kappa_x^{-1}) - \partial_v (\widetilde{f} \circ \kappa_x^{-1}) \cdot \overline{\partial}_v (u \circ \kappa_x^{-1})](v_1 = \overline{v_2} = 0) + O(|x|^2),$$

since $\partial_v (\tilde{f} \circ \kappa_x^{-1}) \cdot \overline{\partial}_v (u \circ \kappa_x^{-1}) = O(|x|^2)$.

As seen in the proof of Proposition 3.3, the critical manifold $\{v_1(y, \overline{w}) = \overline{v_2}(y, \overline{w}) = 0\}$ is the stable manifold for the Hamiltonian flow of $\widetilde{f}$, so that each trajectory of the vector field above is repulsed from zero in a non-degenerate way.

Second step.

Since X has 0 as non-degenerate repulsive point, it can be diagonalised: there exists a linear change of variables $A(f', \varphi)$ on $\mathbb{C}^d$ after which

$$X = \sum_{i=1}^{d} \omega_i x_i \partial_{x_i} + O(|x|^2),$$

for positive ω_i. From now on we apply this linear change of variables and we will control $\|\nabla^j u(0)\|_{\ell^1}$ in these coordinates, from $\|\nabla^j g(0)\|_{\ell^1}$ in the same coordinates.

Note that, by the Poincaré-Dulac theorem, after a non-linear change of variables, the non-linear $O(|x|^2)$ part in X commutes with the linear part; this additional simplification is not needed here. Note also that, generically, the ω_i's are independent over $\mathbb{Q}$. In this case, in principle, one could completely eliminate the non-linear part in X, and in particular, build WKB quasimodes corresponding to a higher eigenvalue, not only the microlocal ground state.

Let us expand

$$X \cdot u(x) = \sum_{i=1}^{d} \left(\omega_i x_i + \sum_{|\nu| \geq 2} \frac{a_{i,\nu}}{\nu!} x^\nu\right) \frac{\partial}{\partial x_i} u(x)$$

$$h(x) = \sum_{|\nu| \geq 1} \frac{h_\nu}{\nu!} x^\nu$$

$$g(x) = \sum_{|\nu| \geq 1} \frac{g_\nu}{\nu!} x^\nu.$$

Then, for some $V \subset\subset U_0$ which contains 0, for some positive r_0, m_0, one has $a_i \in H(m_0, r_0, V)$ and $h \in H(m_0, r_0, V)$, so that

$$|h_\nu| \leq C_h \frac{r_0^{|\nu|}\nu!}{(1+|\nu|)^{m_0}} \qquad \forall |\nu| \geq 1$$
$$|a_{i,\nu}| \leq C_a \frac{r_0^{|\nu|-1}\nu!}{|\nu|^{m_0}} \qquad \forall |\nu| \geq 2.$$

The index shift on the control of a_i will balance the one in (20) below.

Let $m \geq m_0$ and $r \geq r_0 2^{2+m-m_0}$, to be fixed later on. Then, one has also

$$|h_\nu| \leq C_h \frac{(r/4)^{|\nu|}\nu!}{(1+|\nu|)^{m}} \qquad \forall |\nu| \geq 1 \tag{16}$$
$$|a_{i,\nu}| \leq C_a \frac{(r/4)^{|\nu|-1}\nu!}{|\nu|^{m}} \qquad \forall |\nu| \geq 2. \tag{17}$$

Let us now suppose that (12) holds, that is, for some $k \geq 0$, for every $j \geq 0$, one has

$$\sum_{|\nu|=j} |g_\nu| \leq C_g \frac{r^j k!(j+1)!}{(1+k+j+1)^m}. \tag{18}$$

We will solve the transport equation with

$$u : x \mapsto \sum_{|\nu|\geq 1} \frac{u_\nu}{\nu!} x^\nu,$$

and prove by induction on $j \geq 0$ that (13) holds, i.e.

$$\sum_{|\mu|=j} |u_\mu| \leq C(h, f', \varphi) C_g \frac{r^j k! j!}{(1+k+j)^m}, \tag{19}$$

as long as m is large enough with respect to C_a and C_h, and r is large enough accordingly.

For $j = 0$, one has $u(0) = 0$ by hypothesis. The transport equation is equivalent to the following family of equations indexed by μ with $|\mu| \geq 1$:

$$u_\mu \frac{\sum_{i=1}^d \omega_i \mu_i}{\mu!} = \sum_{|\nu|\geq 1} \frac{h_\nu u_{\mu-\nu}}{\nu!(\mu-\nu)!} + \frac{g_\mu}{\mu!} - \sum_{i=1}^d \sum_{|\nu|\geq 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i}}{\nu!(\mu-\nu+\eta_i)!}. \tag{20}$$

Here, as in the rest of the proof, η_i denotes the base polyindex with coefficients $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is at the site i.

Observe that u_μ appears only on the left-hand side of the equation above, while the right-hand side contains coefficients u_ρ with $\rho < \mu$. As the eigenvalues ω_i are all positive, one can solve for u_μ by induction. Indeed, there exists $C_\omega > 0$ such that, for every $|\mu| \neq 0$ there holds

$$\sum_{i=1}^d \omega_i \mu_i \geq C_\omega^{-1}(|\mu|+1).$$

In particular,

$$|u_\mu| \leq \frac{C_\omega}{|\mu|+1} \left(|g_\mu| + \left| \sum_{|\nu|\geq 1} \frac{h_\nu u_{\mu-\nu}\mu!}{\nu!(\mu-\nu)!} \right| + \left| \sum_{i=1}^d \sum_{|\nu|\geq 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i}\mu!}{\nu!(\mu-\nu+\eta_i)!} \right| \right).$$

One has, directly, from (18),

$$\sum_{|\mu|=j} \frac{C_\omega |g_\mu|}{|\mu|+1} \leq C_g C_\omega \frac{r^j k! j!}{(1+k+j)^m}.$$

From (16), one has

$$\begin{aligned}\sum_{|\mu|=j} \left| \sum_{|\nu|\geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| &\leq \sum_{\ell=1}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu\geq\rho}} \frac{|h_{\mu-\rho}|}{(\mu-\rho)!} \frac{\mu!}{\rho!} \\ &\leq C_h \sum_{\ell=1}^{j-1} r^{j-\ell} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu\geq\rho}} \frac{\mu!}{\rho!} \frac{1}{(1+j-\ell)^m}.\end{aligned}$$

Note that, when applying (16), we have loosened $(r/4)^j$ into r^j ; the supplementary power 4^j will be used only in the fourth step.

For $|\rho| = \ell$ there holds

$$\sup_{\substack{|\mu|=j \\ \mu\geq\rho}} \frac{\mu!}{\rho!} \leq \frac{j!}{\ell!},$$

since if ρ_M denotes the largest index of ρ the supremum above is $(\rho_M+1)(\rho_M+2)\dots(\rho_M+j-\ell)$. Moreover, there are less than $(j-\ell+1)^d$ polyindices μ such that $|\mu| = j$ and $\mu \geq \rho$ with $|\rho| = \ell$.

Hence, by the induction hypothesis (19),

$$\begin{aligned}\sum_{|\mu|=j} \left| \sum_{|\nu|\geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| &\leq C_h \sum_{\ell=1}^{j-1} r^{j-\ell} \frac{j!}{\ell!} \frac{(1+j-\ell)^d}{(1+j-\ell)^m} \sum_{|\rho|=\ell} |u_\rho| \\ &\leq C_h C(h, f', \varphi) C_g \frac{r^j j! k!}{(1+k+j)^m} \sum_{\ell=1}^{j-1} \frac{(1+j-\ell)^d (1+k+j)^m}{(1+j-\ell)^m (1+k+\ell)^m}.\end{aligned}$$

From Lemma 2.13 in [10], if $m \geq \max(d+2, 2(d+1))$, there holds

$$\sum_{\ell=1}^{j-1} \frac{(1+j-\ell)^d (1+k+j)^m}{(1+j-\ell)^m (1+k+\ell)^m} \leq C(d) \frac{3^m}{4^m}.$$

In particular,

$$\sum_{|\mu|=j} \left| \sum_{|\nu|\geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| \leq C_h C(d) \frac{3^m}{4^m} C(h, f', \varphi) C_g \frac{r^j j! k!}{(1+k+j)^m}.$$

For m large enough with respect to $C_h C(d) C_\omega$, and $r \geq r_0 2^{2+m-m_0}$, one has

$$\sum_{|\mu|=j} \left| \sum_{|\nu|\geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| \leq \frac{1}{3C_\omega} C(h, f', \varphi) C_g \frac{r^j j! k!}{(1+k+j)^m}.$$

Similarly, from (17), one can control, for $1 \le i \le d$, the quantity

$$\left|\sum_{|\nu|\ge 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i}\mu!}{\nu!(\mu-\nu+\eta_i)!}\right| \le \sum_{\ell=1}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j\\ \mu\ge\rho-\eta_i}} \frac{|a_{i,\mu-\rho+\eta_i}|\mu!}{(\mu-\rho+\eta_i)!\rho!}$$
$$\le C_a \sum_{\ell=1}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j\\ \mu\ge\rho-\eta_i}} r^{j-\ell}\frac{\mu!}{\rho!}\frac{1}{(1+j-\ell)^m}.$$

Again we have loosened $(r/4)^j$ into r^j.

Letting ρ_M denote again the large index of ρ, and ρ_m its smallest non-zero index, then

$$\max_{\substack{|\mu|=j\\ \mu\ge\rho-\eta_i}} \frac{\mu!}{\rho!} = \frac{(\rho_M+j-\ell+1)!}{\rho_M!\rho_m} \le \frac{j!}{(\ell-1)!} \le \frac{(j+1)!}{\ell!}.$$

In particular, by the induction hypothesis (19),

$$\left|\sum_{|\nu|\ge 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i}\mu!}{\nu!(\mu-\nu+\eta_i)!}\right| \le C_a C(h,f',\varphi) C_g r^j (j+1)!k! \sum_{\ell=1}^{j-1} \frac{(1+j-\ell)^d}{(1+j-\ell)^m(1+k+\ell)^m}$$
$$\le C_a C(h,f',\varphi) C_g \frac{r^j(j+1)!k!}{(1+j+k)^m} \sum_{\ell=1}^{j-1} \frac{(1+j-\ell)^d(1+j+k)^m}{(1+j-\ell)^m(1+k+\ell)^m}$$
$$\le C_a C(d)\frac{3^m}{4^m} C(h,f',\varphi) C_g \frac{r^j(j+1)!k!}{(1+j+k)^m}.$$

Hence, for m large enough, and r large enough accordingly, one has, for every $1 \le i \le d$,

$$\left|\sum_{|\nu|\ge 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i}\mu!}{\nu!(\mu-\nu+\eta_i)!}\right| \le \frac{1}{3dC_\omega} C(h,f',\varphi) C_g \frac{r^j(j+1)!k!}{(1+k+j)^m}.$$

To conclude, if $C(h,f',\varphi) \ge 3C_\omega$, then

$$\sum_{|\mu|=j} |u_\mu| \le \frac{1}{j+1}\left(\frac{1}{3}C(h,f',\varphi) + \frac{1}{3}C(h,f',\varphi) + \frac{1}{3}C(h,f',\varphi)\right) C_g \frac{r^j(j+1)!k!}{(1+k+j)^m},$$

which concludes the induction.

Third step

Let U be a neighbourhood of 0 such that all trajectories of X, starting in U, converge to 0 (exponentially fast) in negative time. It remains to prove that u is well-defined and holomorphic on U. Since the sequence of derivatives of u at 0 enjoys an analytic-type growth control, the associated power series converges on some small neighbourhood W of 0. Then, from the knowledge of u on W one can build u on U using the geometric structure of the transport equation. Indeed, by definition 0 is the repulsive point of all trajectories of X on U. Letting $(\Phi_t)_{t\in\mathbb{R}}$ denote the flow of $-X$, there exists $T > 0$ such that $\Phi_T(U) \subset W$. Then the transport equation on u implies the Duhamel formula

$$u(x) = u(\Phi_T(x)) + \int_0^T g(\Phi_t(x))\mathrm{d}t + \int_0^T u(\Phi_t(x))h(\Phi_t(x))\mathrm{d}t.$$

By the analytic Picard-Lindelöf theorem, the unique solution of this degree 1 integral equation, where the initial data $u(\Phi_T(x))$ and the coefficients have real-analytic dependence on $\Phi_T(x) \in W$, is well-defined and real-analytic. Then u is well-defined on U, and holomorphic since the derived equation on $\overline{\partial} u$ is $\overline{\partial} u = 0$.

Fourth step

Now we impose the stronger control (14) on g and prove (15). Observe that, if $j \geq k$ and

$$\sum_{|\mu|=j} |g_\mu| \leq C_g \frac{(r/4)^j (j+k+1)!}{(1+j+k+1)^{m-\frac{1}{2}}},$$

and if $C(h,f,\varphi) \geq 6C_\omega$, then

$$\sum_{|\mu|=j} \frac{|g_\mu|}{\left|\sum_{i=1}^d \omega_i \mu_i\right|} \leq \frac{1}{6} C(h,f,\varphi) C_g \frac{(r/4)^j (j+k+1)!}{(1+k+j+1)^{m-\frac{1}{2}}(j+1)} \leq \frac{1}{3} C(h,f,\varphi) C_g \frac{(r/4)^j (j+k)!}{(1+k+j)^{m-\frac{1}{2}}}.$$

It then remains to study how the more precise condition on u propagates. Fix $j \geq k$; suppose that (13) is satisfied for all $\ell < j$, and that (15) is satisfied for all $k \leq \ell < j$. Then

$$\sum_{|\mu|=j} \sum_{|\nu|\geq 1} \left| \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| = \sum_{|\mu|=j} \sum_{1 \leq |\nu| \leq j-k-1} \left| \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| + \sum_{|\mu|=j} \sum_{|\nu|\geq j-k} \left| \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right|.$$

In the first sum, one has $|\mu - \nu| \geq k$. Hence

$$\sum_{|\mu|=j} \sum_{1 \leq |\nu| \leq j-k-1} \left| \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| = \sum_{\ell=k}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho}} \frac{|h_{\mu-\rho}| \mu!}{\nu!(\mu-\nu)!}.$$

From there and (16), one has, as previously,

$$\begin{aligned}
\sum_{|\mu|=j} \left| \sum_{1 \leq |\nu| \leq j-k-1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| &\leq \sum_{\ell=k}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho}} \frac{|h_{\mu-\rho}|}{(\mu-\rho)!} \frac{\mu!}{\rho!} \\
&\leq C_h \sum_{\ell=k}^{j-1} (r/4)^{j-\ell} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho}} \frac{\mu!}{\rho!} \frac{1}{(1+j-\ell)^m} \\
&\leq C_h \sum_{\ell=k}^{j-1} (r/4)^{j-\ell} \frac{j!}{\ell!} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho}} \frac{1}{(1+j-\ell)^{m-\frac{1}{2}}} \\
&\leq C_h C(h,f',\varphi) C_g \frac{(r/4)^j (j+k)!}{(1+j+k)^{m-\frac{1}{2}}} \sum_{\ell=1}^{k} \underbrace{\frac{j!(\ell+k)!}{\ell!(j+k)!}}_{\leq 1} \frac{(1+j-\ell)^d (1+j+k)^{m-\frac{1}{2}}}{(1+j-\ell)^{m-\frac{1}{2}}(1+\ell+k)^{m-\frac{1}{2}}}.
\end{aligned}$$

If m is large enough, and r is large accordingly, we obtain

$$\sum_{|\mu|=j} \left| \sum_{1 \leq |\nu| \leq j-k-1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu!(\mu-\nu)!} \right| \leq \frac{1}{6C_\omega} C(h,f',\varphi) C_g \frac{(r/4)^j (j+k)!}{(1+k+j)^{m+\frac{1}{2}}}.$$

In the second sum, we have

$$\sum_{\ell=1}^{k}\sum_{|\rho|=\ell}|u_\rho|\sum_{\substack{|\mu|=j\\ \mu\geq\rho}}\frac{|h_{\mu-\rho}|\mu!}{(\mu-\rho)!\rho!}\leq C_hC(h,f',\varphi)C_g\sum_{\ell=1}^{j}(r/4)^{j-\ell}\frac{j!(1+j-\ell)^d}{\ell!(1+j-\ell)^m}\sum_{|\rho|=\ell}|u_\rho|$$

$$\leq C_hC(h,f',\varphi)C_gr^jj!k!4^{k-j}\sum_{\ell=1}^{k}\frac{(1+j-\ell)^d}{(1+j-\ell)^m(1+k+\ell)^m}.$$

Let us prove that, since $k\leq j$, one has

$$\frac{4^kj!k!}{(j+k)!\sqrt{j+k+1}}\leq 2.$$

This is a log-convex function of k; at $k=0$ it is equal to $1/\sqrt{j+1}$. at $k=j$ we use the fact that

$$4^jj!j!\leq 4^j(2j)!\times\frac{j!j!}{(2j)!}\leq 2\sqrt{2j+1}(2j)!,$$

as remarked before the proof.

In particular, since $\sqrt{j+k+1}\leq\sqrt{(j-\ell+1)(k+\ell+1)}$, one has

$$\sum_{\ell=1}^{k}\sum_{|\rho|=\ell}|u_\rho|\sum_{\substack{|\mu|=j\\ \mu\geq\rho}}\frac{|h_{\mu-\rho}|\mu!}{(\mu-\rho)!\rho!}\leq 2C_hC(h,f',\varphi)C_g\frac{(r/4)^j(j+k)!}{(1+j+k)^{m-\frac{1}{2}}}\sum_{\ell=1}^{k}\frac{(1+j-\ell)^d(1+j+k)^{m-\frac{1}{2}}}{(1+j-\ell)^{m-\frac{1}{2}}(1+k+\ell)^{m-\frac{1}{2}}}.$$

We finally obtain, for m large enough, and $r\geq 2^{2+m-m_0}$,

$$\sum_{|\mu|=j}\left|\sum_{|\nu|\geq 1}\frac{h_\nu u_{\mu-\nu}\mu!}{\nu!(\mu-\nu)!}\right|\leq\frac{1}{3C_\omega}C(h,f',\varphi)C_g\frac{(r/4)^j(j+k)!}{(1+j+k)^{m-\frac{1}{2}}}.$$

The control on

$$\left|\sum_{i=1}^{d}\sum_{|\nu|\geq 2}\frac{a_{i,\nu}u_{\mu-\nu+\eta_i}\mu!}{\nu!(\mu-\nu+\eta_i)!}\right|$$

is very similar; the only notable difference is the combinatorial factor studied in Part 2,

$$\max_{\substack{|\mu|=j\\ \mu\geq\rho-\eta_i}}\frac{\mu!}{\rho!}\leq\frac{(j+1)!}{\ell!}=(j+1)\frac{j!}{\ell!},$$

which brings a supplementary factor $j+1$ in all cases. We obtain

$$\left|\sum_{|\mu|=j}\sum_{i=1}^{d}\sum_{|\nu|\geq 2}\frac{a_{i,\nu}u_{\mu-\nu+\eta_i}\mu!}{\nu!(\mu-\nu+\eta_i)!}\right|\leq\frac{1}{3C_\omega}C(h,f',\varphi)C_g\frac{(r/4)^j(j+k)!}{(1+j+k)^{m-\frac{1}{2}}}(j+1),$$

and finally,

$$\left|\sum_{|\mu|=j}\frac{1}{\sum_{i=1}^d\omega_i\mu_i}\left[\sum_{|\nu|\geq 1}\frac{h_\nu u_{\mu-\nu}\mu!}{\nu!(\mu-\nu)!}+g_\mu-\sum_{i=1}^{d}\sum_{|\nu|\geq 2}\frac{a_{i,\nu}u_{\mu-\nu+\eta_i}\mu!}{\nu!(\mu-\nu+\eta_i)!}\right]\right|\leq C(h,f',\varphi)C_g\frac{(r/4)^j(j+k)!}{(1+j+k)^{m-\frac{1}{2}}},$$

which concludes the proof. □

4 Construction of quasimodes

Solving the Hamilton-Jacobi equation then controlling successive transport equations allows us to prove the first part of Theorem A, which is the object of this section.

The strategy of proof is the following: we first exhibit sequences $(u_i)_{i\geq 0}$ and $(\lambda_i)_{i\geq 0}$ such that the eigenvalue equation (22) is valid up to $O(N^{-\infty})$, and we control these sequences in analytic spaces. Then we prove that one can perform an analytic summation in (22).

Before proceeding, we note that, if φ is admissible and $u(N)$ is the summation of an analytic symbol, both being defined on an open neighbourhood V of 0, then $\mathbb{1}_V e^{N\varphi}u(N)\psi_0^N$ concentrates at 0, in the sense that there exist $C>0, c>0$ such that for every open set $W\subset M$,

$$\|N^{-\frac{d}{2}}\mathbb{1}_V e^{N\varphi}u(N)\psi_0^N\|_{L^2(W)} \leq Ce^{-cN\,\mathrm{dist}(W,\{0\})^2}.$$

and moreover, by Proposition 2.4 and the stationary phase lemma, there exists $C>0$ such that, for every $N\in\mathbb{N}$, there holds

$$\frac{1}{C}N^{\frac{d}{2}} \leq \|\mathbb{1}_V e^{N\varphi}u(N)\psi_0^N\|_{L^2(M)} \leq CN^{\frac{d}{2}}.$$

In particular, if

$$\|(T_N(f)-\lambda(N))\mathbb{1}_V e^{N\varphi}u(N)\psi_0^N\|_{L^2(M)} \leq Ce^{-c'N}$$

then $\lambda(N)$ will be exponentially close to the spectrum of $T_N(f)$. Thus, through Proposition 4.4 we are indeed providing quasimodes of $T_N(f)$ which concentrate on 0.

Proposition 4.1. *Let φ denote an admissible solution to the Hamilton-Jacobi equations* (11)*, and let ψ_0^N denote the sequence of coherent states at* 0*. There exists $W\subset\subset V\subset\subset U\subset U_0$ containing zero, a sequence $(u_k)_{k\geq 0}$ of holomorphic functions on U, and a sequence $(\lambda_k)_{k\geq 0}$ of real numbers, such that for every $K\geq 0$ there holds*

$$\left\|\left(T_N(f)-\sum_{k=0}^{K}N^{-k-1}\lambda_k\right)\mathbb{1}_V\psi_0^N e^{N\varphi}\sum_{k=0}^{K}N^{-k}u_k\right\|_{L^2(W)} = O(N^{-\frac{d}{2}-K-2}).$$

One has

$$\lambda_0 = \min \mathrm{Sp}(T_1(\mathrm{Hess}(f)(0))).$$

Proof. Recall that, by Proposition 2.4, there exists an analytic symbol a and constants $c>0, c'>0$ such that

$$S_N(x,y) = N^d\Psi^{\otimes N}(x,\overline{y})\sum_{k=0}^{cN}N^{-k}a_k(x,\overline{y}) + O(e^{-c'N}).$$

In particular,

$$\psi_0^N(x) = N^d\Psi^{\otimes N}(x,0)\sum_{k=0}^{cN}N^{-k}a_k(x,0) + O(e^{-c'N}).$$

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of holomorphic functions on U and let

$$u(N) = \sum_{k=0}^{K}N^{-k}u_k.$$

With $a(N)=\sum_{k=0}^{cN}a_k$, by definition of Φ_1, one has, uniformly for $x\in W$,

$$\begin{aligned}&T_N(f)\left(\mathbb{1}_V\psi_0^N e^{N\varphi}u(N)\right)(x)\\ &\quad= \psi_0^N(x)e^{N\varphi(x)}\int_{y\in V}e^{N(\Phi_1(x,y,\overline{y},0)+\varphi(y)-\varphi(x))}\frac{a(N)(x,\overline{y})}{a(N)(x,0)}a(N)(y,0)\widetilde{f}(y,\overline{y})u(N)(y)\mathrm{d}y + O(e^{-c'N}).\end{aligned}$$

We are now able to apply the complex stationary phase Lemma (with analytic phase but, at this stage, smooth symbol, as in [24]). Let $*$ denote the Cauchy product of symbols, and let b be the analytic symbol such that

$$b(x,y,\overline{w}) = \widetilde{f}(y,\overline{w})a(x,\overline{w}) * a^{*-1}(x,0) * a(y,0)J(x,y,\overline{w}),$$

where J is the Jacobian of the change of variables κ_x defined in (6). One has

$$e^{-N\varphi(x)}T_N(f)\left(\psi_0^N e^{N\varphi}u\right)(x) = \psi_0^N(x)\sum_{k=0}^{+\infty} N^{-k}\sum_{n=0}^{k}\frac{\Delta_{\kappa_x}^n}{n!}(u(y)b_{k-n}(x,y,\overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y_c}(x))} + O(N^{-\infty}). \quad (21)$$

Using Proposition 3.4 with

$$f' : (x,y,\overline{w}) \mapsto b_0(x,y,\overline{w}),$$

which indeed coincides with f up to $O(|x,y,\overline{w}|^3)$, we will construct by induction a sequence of holomorphic functions u_i and a sequence of real numbers λ_i such that

$$T_N(f)\left(\psi_0^N e^{N\varphi}\sum_{k=0}^{+\infty} N^{-k}u_k\right)(x) = \psi_0^N(x)e^{N\varphi(x)}\left(\sum_{j=0}^{+\infty} N^{-j-1}\lambda_j\right)\left(\sum_{k=0}^{cN} N^{-k}u_k(x)\right) + O(N^{-\infty}). \quad (22)$$

We further require that

$$u_k(0) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{else.} \end{cases}$$

In the right-hand side of (22), there are no terms of order 0. In the left-hand side, the term of degree 0 is given by the term $k=0$ in (21), so that one needs to solve

$$\widetilde{f}(x,\overline{y}_c(x))u_0(x)\frac{a_0(x,0)}{a_0(y,0)}a_0(x,\overline{y})J(x,x,\overline{y}_c(x)) = b_0(x,x,\overline{y}_c(x))u_0(x) = 0.$$

Since $\widetilde{f}(x,\overline{y}_c(x)) = 0$, this equation is always satisfied.

By the stationary phase lemma (21), the order 1 in (22) reads

$$\lambda_0 u_0(x) - (\Delta_{\kappa_x} b_0)(x,x,\overline{y}_c(x))u_0(x) - (\nabla_{\kappa_x} b_0)(x,x,\overline{y}_c(x))\cdot\nabla_{\kappa_x}u_0(x) = 0. \quad (23)$$

Here, and until the end of this proof as well as that of Proposition 4.2, we (informally) denote

$$\nabla_{\kappa_x}u_k(x) = \nabla_{\kappa_x}[(x,y,\overline{w}) \mapsto u_k(y)]_{(y,\overline{w})=(x,\overline{y}_c(x))}.$$

The equation (23) allows us to solve for u_0 with the supplementary condition $u_0(0) = 1$. Indeed, as $\nabla_{\kappa_x}b_0(0) = 0$, at $x = 0$, the order 1 reads

$$\lambda_0 - (\Delta_{\kappa_x}b_0)(0,0,0) = 0,$$

so that we set

$$\lambda_0 = (\Delta_{\kappa_x}b_0)(0,0,0).$$

We now prove that λ_0 coincides with the ground state energy of the associated quadratic operator $T_N(\mathrm{Hess}(f)(0))$. Indeed, λ_0 depends only on the Hessian of f and ϕ at zero (which together determine the Hessian of φ at zero as seen in Proposition 3.3, thus they determine the linear part of the change of variables κ_0, which in turn determines Δ_{κ_0} and J at 0). If f and ϕ are quadratic, then the solution φ of the Hamilton-Jacobi equation is also quadratic as constructed in Proposition 3.3, so that $u_0 = 1$ satisfies (22) exactly. Thus, λ_0 is an eigenvalue of $T_N(\mathrm{Hess}(f)(0))$ which depends continuously on $\mathrm{Hess}(f)(0)$. Moreover, if $\mathrm{Hess}(f)(0) : y \mapsto |y|^2$, then $\mathrm{Hess}(\varphi) = 0$ so that the eigenvector of $T_N(\mathrm{Hess}(f)(0))$ associated with λ_0 is the coherent state (in $\mathbb{C}^d$) ψ_0^N, which is the ground state of $T_N(|y|^2)$; thus in this case λ_0 is the ground state energy. Since the set of positive definite quadratic forms in $\mathbb{R}^{2d}$ is connected, and since there is always a gap between the ground state energy and the first excited level, then λ_0 is always the ground state energy of $T_N(\mathrm{Hess}(f)(0))$.

We wish now to find u_0 such that $u_0(0) = 1$. Setting $v_0 = u_0 - 1$ yields

$$\nabla_{\kappa_x} v_0(x) \cdot (\nabla_{\kappa_x} b_0)(x, x, \overline{y}_c(x)) = v_0(x) \left[(\Delta_{\kappa_x} b_0)(x, x, \overline{y}_c(x)) - (\Delta_{\kappa_x} b_0)(0, 0, 0)\right].$$

We then solve for v_0 using Proposition 3.4 with $f' = b_0$, which indeed yields $v_0(0) = 0$.

Let us now find the remaining terms of the sequences $(u_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$ by induction. For $k \geq 1$, the term of order $k+1$ in (22) is given again by the stationary phase lemma (21): at this order, the equation is

$$\begin{aligned} \lambda_k u_0(x) + \lambda_0 u_k(x) - (\Delta_{\kappa_x} b_0)(x, x, \overline{y}_c(x)) u_k(x) - (\nabla_{\kappa_x} b_0)(x, x, \overline{y}_c(x)) \cdot \nabla_{\kappa_x} u_k(x) \\ = -\sum_{j=1}^{k-1} \lambda_j u_{k-j}(x) + \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_x}^n}{n!} (u_l(y) b_{k+1-n-l}(x, y, \overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y_c}(x))}. \end{aligned} \tag{24}$$

In this equation, we have put to the left-hand side all terms involving λ_k or u_k, and all terms involving λ_l and u_l with $l < k$ to the right-hand side. We can apply Proposition 3.4 to solve for u_k, λ_k once $(u_l, \lambda_l)_{0\leq l\leq k-1}$ are known. Indeed, (24) takes the form

$$(\nabla_{\kappa_x} b_0)(x, x, \overline{y}_c(x)) \cdot \nabla_{\kappa_x} u_k(x) = g_k(x) + h(x) u_k(x), \tag{25}$$

with $h(x) = \Delta_{\kappa_x} b_0(x, x, \overline{y}_c(x)) - \lambda_0$ and

$$g_k(x) = -\sum_{l=1}^{k-1} \lambda_l u_{k-l}(x) - \lambda_k u_0 + \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_x}^n}{n!} (u_l(y) b_{k+1-n-l}(x, y, \overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))}. \tag{26}$$

By construction of λ_0, one has $h(0) = 0$; moreover,

$$g_k(0) = \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_0}^n}{n!} (u_l(y) b_{k+1-n-l}(0, y, \overline{w}))\Big|_{(y,\overline{w})=(0,0)} - \lambda_k.$$

Thus, one can solve for λ_k by setting $g_k(0) = 0$, then solve for u_k using Proposition 3.4: the role of f' is played by b_0, which does not depend on k. Thus, letting U be as in Proposition 3.4, one can, by induction on k, define g_k as a holomorphic function on U using (26), then u_k as a holomorphic function on U using (25). □

It remains to prove that, because of Proposition 3.4, the coefficients $(u_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$ satisfy analytic growth controls.

Proposition 4.2. *Let $(u_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$ be the sequences constructed in the previous proposition. Then there exist $C > 0$, $R > 0$, $r > 0$, $m \in \mathbb{R}$ and an open set $V \subset\subset U$ containing 0 such that, for all $k \geq 0, j \geq 0$, one has*

$$\|u_k\|_{C^j(V)} \leq C \frac{r^j R^k j! k!}{(j+k+1)^m}$$

$$|\lambda_k| \leq C \frac{R^k k!}{(k+2)^m}.$$

Moreover, if $j \geq k$, then

$$\|u_k\|_{C^j(V)} \leq C \frac{(r/4)^j R^k (j+k)!}{(j+k+1)^{m-\frac{1}{2}}}.$$

Proof. The proof proceeds by induction on k and consists in three steps. In the first step, we show that in equation (24) (that is, in the definition of g_k), when expanding $\Delta^n_{\kappa_x}(u_l b_{k+1-n-l})$, no derivatives of u_l of order larger than n appear. This will allow us to apply Lemma 2.5. The second step is the core of the induction: we suppose some control on all derivatives of u_l at zero, for $0 \leq l \leq k-1$, and we apply Lemma 2.5 to deduce that the derivatives of g_k at zero are well-behaved. We then apply Proposition 3.4 to obtain a control on the derivatives of u_k at zero. In the last step, we deduce, from a control of the derivatives of u_k at zero, a control of the same nature on a small open neighbourhood.

First step.

Let f_0 be a holomorphic function near 0 in M. Then $T_N(f_0)$ is, locally, a multiplication operator, so that, for all holomorphic u,

$$e^{-N\varphi}T_N(f_0)(\psi_0^N e^{N\varphi}u) = \psi_0^N f_0 u + O(e^{-c'N}).$$

In this particular case, no derivative of u of order ≥ 1 appear in (21), hence in (24).

We then decompose the real-analytic function f as

$$\widetilde{f}(y,\overline{y}) = \widetilde{f}(y,\overline{y}_c(x)) + \left(\widetilde{f}(y,\overline{y}) - \widetilde{f}(y,\overline{y}_c(x))\right).$$

In the right-hand side, the second term vanishes when $\overline{y} = \overline{y}_c(x)$, so that, with

$$\Phi : (x,y,\overline{w}) \mapsto \Phi_1(x,y,\overline{w},0) + \varphi(y) - \varphi(x),$$

there exists a smooth vector-valued function f_1 such that

$$\widetilde{f}(y,\overline{y}) = \widetilde{f}(y,\overline{y}_c(x)) + \partial_y\Phi(x,y,\overline{y}) \cdot f_1(x,y,\overline{y}).$$

Now S_N acts as the identity on holomorphic functions and $\overline{y}_c$ is a holomorphic function of x so that, by integration by parts:

$$\begin{aligned}
&\int e^{-N\Phi(x,y,\overline{y})}a(N)(x,\overline{y})\widetilde{f}(y,\overline{y})u(y)\mathrm{d}y \\
&\quad = \psi_0^N(x)\widetilde{f}(x,\overline{y}_c(x))u(x) + \int e^{-N\Phi(x,y,\overline{y})}a(N)(x,\overline{y})\partial_y\Phi(x,y,\overline{y}) \cdot f_1(x,y,\overline{y})\, u(y)\mathrm{d}y + O(e^{-c'N}) \\
&\quad = \psi_0^N(x)\widetilde{f}(x,\overline{y}_c(x))u(x) + N^{-1}\int e^{-N\Phi(x,y,\overline{y})}a(N)(x,\overline{y})\partial_y\left[f_1(x,y,\overline{y})u(y)\right]\mathrm{d}y + O(e^{-c'N}).
\end{aligned}$$

In particular, in the term of order N^{-1} in (21), there only are derivatives of u of order 0 or 1.

One can in fact perform this decomposition iteratively: with

$$\partial_y\left[f_1(x,y,\overline{y})u(y)\right] = \partial_y \cdot f_1(x,y,\overline{y})u(y) + f_1(x,y,\overline{y})\partial u(y),$$

one can write

$$\begin{aligned}
f_1(x,y,\overline{y}) &= f_1(x,y,\overline{y}_c(x)) + \partial_y\Phi(x,y,\overline{y}) \cdot f_{2,0}(x,y,\overline{y}) \\
\partial_y f_1(x,y,\overline{y}) &= \partial_y f_1(x,y,\overline{y}_c(x)) + \partial_y\Phi(x,y,\overline{y}) \cdot f_{2,1}(x,y,\overline{y}),
\end{aligned}$$

so that the original integral is equal to

$$\begin{aligned}
&\psi_0^N(x)\widetilde{f}(x,\overline{y}_c(x))u(x) \\
&\qquad + N^{-1}\psi_0^N(x)\left[f_1(x,x,\overline{y}_c(x))\partial u(x) + \partial_y f_1(x,x,\overline{y}_c(x))u(x)\right] \\
&\qquad + N^{-2}\int e^{-N\Phi(x,y,\overline{y})}a(N)(x,\overline{y})\partial_y[f_{2,0}(x,y,\overline{y})\partial u(y) + f_{2,1}(x,y,\overline{y})u(y)]\mathrm{d}y \\
&\qquad + O(e^{-c'N}).
\end{aligned}$$

By induction, the terms of order N^{-k} in the expansion (21) only contain derivatives of u of order smaller than k. This means in particular that, in (24), in

$$\Delta^n_{\kappa_x}(u_l(y)b_{k+1-n-l}(x,y,\overline{w}))\big|_{(y,\overline{w})=(x,\overline{y}_c(x))},$$

there only appears derivatives of u_l of order less or equal to n.

Second step.

Let us prove by induction that the sequences $(u_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$ are analytic symbols. We will make use of the precise controls obtained in Proposition 3.4. Since $(b_k)_{k\geq 0}$ is an analytic symbol and u_0 is holomorphic, by Proposition 2.2 there exists a small open neighbourhood W of zero in $\mathbb{C}^d$, and a small open neighbourhood W_1 of 0 in $\mathbb{C}^{3d}$, and $r_0, R_0, m_0, C_b, C_0 > 0$ such that

$$\|(x,v_1,v_2)\mapsto b_k(x,\kappa_x^{-1}(v_1,v_2))\|_{C^j(W_1)} \leq C_b\frac{r_0^jR_0^k(j+k)!}{(j+k+1)^{m_0}}$$

$$\|u_0\|_{C^j(W)} \leq C_0\frac{r_0^jj!}{(j+1)^{m_0}}$$

$$\|\kappa^{-1}\|_{C^j(W_1)} \leq C_\kappa\frac{r_0^jj!}{(j+1)^{m_0}}.$$

Here, and the rest of this proof we again denote by κ^{-1} the map $(x,v_1,v_2)\mapsto(x,\kappa_x^{-1}(v_1,v_2))$.

Let us transform this into a control on b_k which is more suited to our needs. First, for all j and k, one has

$$\|b_k\circ\kappa^{-1}\|_{C^j(W_1)} \leq C_b\frac{(4r_0)^j(4R_0)^kj!k!}{(j+k+1)^{m_0+1}}.$$

Indeed $(j+k)! \leq 2^{j+k}j!k!$ and $2^{j+k}\geq j+k+1$. In particular,

$$\|b_0\circ\kappa^{-1}\|_{C^j(W_1)} \leq C_b\frac{(4r_0)^jj!}{(j+1)^{m_0+1}} \leq C_b\frac{(4r_0)^jj!}{(j+1)^{m_0}}.$$

On the other hand, for $k\geq 1$, one has

$$\|b_k\circ\kappa^{-1}\|_{C^j(W_1)} \leq C_b\frac{(4r_0)^j(4R_0)^kj!(k-1)!}{(j+k+1)^{m_0}},$$

since $\frac{k}{j+k+1}\leq 1$.

In particular, for any $m\geq m_0$, for any $r\geq 2^{m+5-m_0}r_0$ and $R\geq 2^{m+2-m_0}R_0$, one has

$$\|b_0\circ\kappa^{-1}\|_{C^j(W_1)} \leq C_b\frac{(r/8)^jj!}{(j+1)^m}$$

$$\|b_k\circ\kappa^{-1}\|_{C^j(W_1)} \leq C_b\frac{(r/8)^jR^kj!(k-1)!}{(j+k+1)^m} \qquad k\geq 1$$

$$\|u_0\|_{C^j(W)} \leq C_0\frac{(r/32)^jj!}{(j+1)^m} \leq C_0\frac{(r/4)^jj!}{(j+1)^{m-\frac{1}{2}}}$$

$$\|\kappa^{-1}\|_{C^j(W_1)} \leq C_\kappa\frac{(r/16)^j(j-1)!}{j^m} \qquad j\geq 1.$$

In equation (24), let us isolate the terms involving u_0. We obtain

$$\lambda_k u_0(x) + \lambda_0 u_k(x) - \Delta_{\kappa_x} b_0(x,x,\overline{y}_c(x))u_k(x) - \nabla_{\kappa_x} b_0(x,x,\overline{y}_c(x)) \cdot \nabla_{\kappa_x} u_k(x)$$
$$= \sum_{n=2}^{k+1} \frac{\Delta_{\kappa_x}^n}{n!}(u_0(y)b_{k+1-n}(x,y,\overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))}$$
$$-\sum_{j=1}^{k-1}\lambda_j u_{k-j}(x) + \sum_{n=2}^{k+1}\sum_{l=1}^{k+1-n} \frac{\Delta_{\kappa_x}^n}{n!}(u_l(y)b_{k+1-n-l}(x,y,\overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))}. \quad (27)$$

Let m, r, R, C_u, C_λ be large enough (they will be fixed in the course of the induction), and suppose that, for all $0 \leq l \leq k-1$ and all $j \geq 0$, one has

$$|\lambda_l| \leq C_\lambda \frac{R^l l!}{(l+2)^m} \quad (28)$$

$$\|\nabla^j u_l(0)\|_{\ell^1} \leq C_u \frac{r^j R^l j! l!}{(j+l+1)^m}. \quad (29)$$

Suppose further that for $j \geq l$ one has the more precise control

$$\|\nabla^j u_l(0)\|_{\ell^1} \leq C_u \frac{(r/4)^j R^l (j+l)!}{(j+l+1)^{m-\frac{1}{2}}}. \quad (30)$$

Our goal is now to prove the three inequalities (28), (29), and (30), in the case $l = k$.

To begin with, we estimate how the iterated modified Laplace operator $\Delta_{\kappa_x}^n$ acts on u_l using the fact that the former differentiates the latter at most n times (Part 1) and Lemma 2.5.

After a change of variables $\kappa_x : (y, \overline{w}) \mapsto v(x,y,\overline{w}) = (v_1(x,y,\overline{w}), \overline{v_2}(x,y,\overline{w}))$ for which the phase is the holomorphic extension of the standard quadratic form $-|v|^2$, one has, by definition,

$$\Delta_{\kappa_x} = \Delta_v = \sum_{i=1}^d \frac{\partial^2}{\partial v_{1,i}\partial\overline{v_{2,i}}}.$$

Hence, denoting the inverse change of variables by $(x,v) \mapsto (x, y(x,v), \overline{w}(x,v))$, we obtain

$$\frac{\Delta_{\kappa_x}^n}{n!}(u_0(y)b_{k+1-n}(x,y,\overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))}$$
$$= \sum_{|\mu|=n}\sum_{\nu\leq 2\mu} \frac{n!(2\mu)!}{\mu!\nu!(2\mu-\nu)!}\partial_v^\nu u_l(y(x,v))|_{v=0}\partial_v^{2\mu-\nu} b_{k+1-n-l}(x,y(x,v),\overline{w}(x,v))|_{v=0}.$$

Since at most n derivatives on u_l appear in (27) by the first step, in the expression above, the differential operator

$$\partial_v^\nu u_l(y(x,v,\overline{v}))_{v=0}$$

can be replaced with its truncation into a differential operator of degree less or equal to n, which we denote by $(\nabla_\kappa^\nu)^{[\leq n]}u_l(x)$ as in [10], Lemma 4.6. In particular, for every $\rho \in \mathbb{N}^d$,

$$\nabla_x^\rho \Delta_v^n[u_l(y(v,\overline{v}))b_{k+1-n-l}(x,y(v,\overline{v}),\overline{w}(v,\overline{v}))]_{v=0} =$$
$$\sum_{|\mu|=n}\sum_{\nu\leq 2\mu}\sum_{\rho_1\leq\rho} \frac{n!(2\mu)!\rho!}{\mu!\nu!(2\mu-\nu)!\rho_1!(\rho-\rho_1)!}\nabla_x^{\rho_1}(\nabla_\kappa^\nu)^{[\leq n]}u_l(x)\nabla_x^{\rho-\rho_1}\nabla_v^{2\mu-\nu}b_{k+1-n-l}(x,y(x,v,\overline{v}),\overline{w}(x,v,\overline{v}))_{v=0}.$$

Moreover, if $|\mu| = n$ then

$$\frac{n!}{\mu!} \leq (2d)^n,$$

and if $\nu \leq 2\mu$ then, by Lemma 2.4 in [10],

$$\frac{(2\mu)!\rho!}{\nu!(2\mu-\nu)!\rho_1!(\rho-\rho_1)!} = \binom{2\mu}{\nu}\binom{\rho}{\rho_1} \leq \binom{2n}{|\nu|}\binom{|\rho|}{|\rho_1|}.$$

Hence,

$$\begin{aligned}&\|\nabla_x^j \Delta_v^n [u_l(y(v,\overline{v}))b_{k+1-n-l}(x, y(v,\overline{v}), \overline{y}(v,\overline{v}))]_{v=x=0}\|_{\ell^1} \\ &\qquad \leq (2d)^n \sum_{i_1=0}^{2n}\sum_{j_1=0}^{j} \binom{2n}{i_1}\binom{j}{j_1} \|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\leq n]} u_l|_{x=0}\|_{\ell^1} \|b_{k+1-n-l}\circ\kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)}.\end{aligned}$$

By the induction hypothesis, one has

$$\|\nabla^j u_l(0)\|_{\ell^1} \leq C_u \frac{r^j R^l j! l!}{(j+l+1)^m},$$

then, by Lemma 2.5, there exists a fixed $C_\kappa > 0$ such that

$$\begin{aligned}&\|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\leq n]} u_l(y(v,\overline{v}))_{x=v=0}\|_{\ell^1} \\ &\qquad \leq i_1^{d+1} j_1^{d+1} C_u \frac{r^{j_1+i_1} R^l l!}{(i_1+j_1+l+1)^m}(C_\kappa)^{i_1} \times \begin{cases}\max((n+j_1)!(i_1-n)!\ ,\ j_1! i_1!) & \text{if } i_1 \geq n \\ (i_1+j_1)! & \text{else.}\end{cases}\end{aligned} \tag{31}$$

If $j_1 + \min(i_1, n) \geq l$, one has the more precise control

$$\begin{aligned}&\|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\leq n]} u_l(y(v,\overline{v}))_{x=v=0}\|_{\ell^1} \\ &\qquad \leq i_1^{d+1} j_1^{d+1} C_u \frac{(r/4)^{j_1+i_1} R^l}{(i_1+j_1+l+1)^{m-\frac{1}{2}}}(C_\kappa)^{i_1} \times \begin{cases}\max((n+j_1+l)!(i_1-n)!\ ,\ (j_1+l)! i_1!) & \text{if } i_1 \geq n \\ (i_1+j_1+l)! & \text{else.}\end{cases}\end{aligned} \tag{32}$$

In the case $l = 0$, the constant C_u can be replaced with the smaller constant C_0.

Let us now control λ_k using equation (24) at $x = 0$:

$$\lambda_k = \sum_{n=2}^{k+1}\sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_0}^n}{n!}(u_l(y) b_{k+1-n-l}(0, y, \overline{w}))\Big|_{(y,\overline{w})=(0,0)}.$$

Then, by the induction hypothesis, (31), and the fact that

$$\|\nabla^{j_1}(b_l \circ \kappa^{-1})\|_{\ell^1} \leq C_b \frac{r^{j_1} R^l j_1! (l-1+\mathbb{1}_{l=0})!}{(1+j_1+l)^m},$$

we obtain

$$\begin{aligned}|\lambda_k| \leq{}& C_u C_b \sum_{n=2}^{k+1} \frac{R^k k!}{(k+2)^m}(2d)^n R \left(\frac{C_\kappa^2 r^2}{R}\right)^n \times \\ & \sum_{i_1=0}^{2n} \frac{(2n)! A(i_1, 0, n)}{i_1! n! k!}\left(\sum_{l=0}^{k-n} \frac{(k-n-l)! l! (k+2)^m}{(i_1+l+1)^m (k+2+n-l-i_1)^m} + \frac{(k+1-n)!(k+2)^m}{(i_1+k-n+2)^m(1+2n-i_1)^m}\right),\end{aligned}$$

with

$$A(i_1, j_1, n) = \begin{cases} \max((n+j_1)!(i_1-n)!\ ,\ j_1!i_1!) & \text{if } i_1 \geq n \\ (i_1+j_1)! & \text{else;} \end{cases}$$

in the sum above, we separated the case $l = k+1-n$, corresponding to the specific control on b_0.

For $l \leq k-n$, one has

$$\frac{(2n)!l!A(i_1,0,n)(k-n-l)!}{i_1!n!k!} \leq 4^n.$$

Indeed, in this case where $j_1 = 0$, one has always $n!(i_1-n)! \leq i_1!$ if $i_1 \geq n$, so that $A(i_1,0,n) = i_1!$ in all cases. We obtain

$$\frac{(2n)!l!(k-n-l)!}{n!k!} \leq \binom{2n}{n}\frac{n!l!(k-n-l)!}{k!} \leq 4^n.$$

In the specific case $l = k-n+1$, one has similarly

$$\frac{(2n)!(k+1-n)!}{n!k!} \leq 4^n\frac{n!(k+1-n)!}{k!}$$

and the right-hand side is a log-convex function of n. At $n=2$ we obtain

$$32\frac{(k-1)!}{k!} \leq 6^2,$$

and at $n = k+1$,

$$4^{k+1}(k+1) \leq 6^{k+1},$$

so that one has always

$$\frac{(2n)!(k+1-n)!}{n!k!} \leq 6^n.$$

Getting back to the control on λ_k, we obtain

$$|\lambda_k| \leq C_u C_b \sum_{n=2}^{k+1}\frac{R^k k!}{(k+2)^m}(2d)^n R\left(\frac{6C_\kappa^2 r^2}{R}\right)^n \sum_{i_1=0}^{2n}\sum_{l=0}^{k+1-n}\frac{(k+2)^m}{(i_1+l+1)^m(k+2+n-l-i_1)^m},$$

Since $(k+2)^m \leq (k+2+n)^m$, one has

$$|\lambda_k| \leq C_u C_b \sum_{n=2}^{k+1}\frac{R^k k!}{(k+2)^m}(2d)^n R\left(\frac{6C_\kappa^2 r^2}{R}\right)^n \sum_{i_1=0}^{2n}\sum_{l=0}^{k+1-n}\frac{(k+n+2)^m}{(i_1+l+1)^m(k+2+n-l-i_1)^m},$$

Then, by Lemma 2.13 in [10], there holds

$$|\lambda_k| \leq C_u C_b \frac{R^k k!}{(k+2)^m} R \sum_{n=2}^{k+1}\left(\frac{12dC_\kappa^2 r^2}{R}\right)^n.$$

If R is large enough (once r, m, C_u, C_λ are fixed), then one can conclude:

$$|\lambda_k| \leq C_\lambda \frac{R^k k!}{(k+2)^m}.$$

We now pass to the control on u_k. We recall that u_k solves an equation of the form

$$X \cdot u_k = hu_k + g_k,$$

with X and h independent on k and

$$g_k : x \mapsto -\sum_{l=1}^{k-1} \lambda_l u_{k-l}(x) - \lambda_k u_0(x) + \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_x}}{n!} (u_l(y) b_{k+1-n-l}(x, y, \overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))}.$$

We want to prove

$$\|\nabla^j g_k(0)\|_{\ell^1} \leq \epsilon C_u \frac{r^j R^k (j+1)! k!}{(j+k+2)^m} \tag{33}$$

and, if $j \geq k$, the more precise control

$$\|\nabla^j g_k(0)\|_{\ell^1} \leq \epsilon C_u \frac{(r/4)^j R^k (j+k+1)!}{(j+k+2)^{m-\frac{1}{2}}}, \tag{34}$$

in order to apply Proposition 3.4. Here $\epsilon > 0$ must be smaller than $\frac{1}{C(h,b_0,\varphi)}$ in Proposition 3.4, in order to conclude the induction and prove the claimed controls on u_k.

One has first

$$\|\lambda_k \nabla^j u_0(0)\|_{\ell^1} \leq C_\lambda C_0 \frac{(r/4)^j R^k j! k!}{(j+1)^m (k+2)^m}$$

Once C_λ and ϵ are fixed, one has $C_\lambda C_0 \leq \epsilon C_u$ for C_u large enough. In particular, one has, for all j and k,

$$\|\lambda_k \nabla^j u_0(0)\|_{\ell^1} \leq \epsilon C_u \frac{(r/4)^j R^k j! k!}{(j+1)^{m-\frac{1}{2}} (k+2)^m} \leq \epsilon C_u \frac{r^j R^k j! k!}{(j+k+2)^m},$$

and for $j \geq k$,

$$\|\lambda_k \nabla^j u_0(0)\|_{\ell^1} \leq \epsilon C_u \frac{(r/4)^j R^k j! k!}{(j+1)^{m-\frac{1}{2}} (k+2)^m} \leq \epsilon C_u \frac{(r/4)^j R^k (j+k)!}{(j+k+2)^{m-\frac{1}{2}}}.$$

Moreover, for all j,

$$\left\| \sum_{l=1}^{k-1} \lambda_l \nabla^j u_{k-l}(0) \right\|_{\ell^1} \leq C_\lambda C_u \frac{r^j R^k j! k!}{(j+k+2)^m} \sum_{l=1}^{k-1} \underbrace{\frac{l!(k-l)!}{k!}}_{=\binom{k}{l}^{-1} \leq 1} \frac{(k+j+2)^m}{(l+2)^m (k-l+j+1)^m}.$$

Hence, by Lemma 2.13 in [10],

$$\left\| \sum_{l=1}^{k-1} \lambda_l \nabla^j u_{k-l}(0) \right\|_{\ell^1} \leq C C_\lambda C_u \frac{3^m}{4^m} \frac{r^j R^k k! j!}{(j+k+2)^m}.$$

Once C_λ and C_u are fixed, the constant $C C_\lambda C_u \frac{3^m}{4^m}$ is smaller than ϵC_u for m large enough (and r, R large enough accordingly), and we obtain

$$\left\| \sum_{l=1}^{k-1} \lambda_l \nabla^j u_{k-l}(0) \right\|_{\ell^1} \leq \epsilon C_u \frac{r^j R^k j! k!}{(j+k+2)^m}.$$

If in addition $j \geq k$, then in particular $j \geq k - l$ for all $1 \leq l \leq k-1$, so that one has the more precise control

$$\left\| \sum_{l=1}^{k-1} \lambda_l \nabla^j u_{k-l}(0) \right\|_{\ell^1} \leq C_\lambda C_u \frac{(r/4)^j R^k (j+k)!}{(j+k+2)^{m-\frac{1}{2}}} \sum_{l=1}^{k-1} \underbrace{\frac{l!(k-l+j)!}{(j+k)!}}_{=\binom{k+j}{l}^{-1} \leq 1} \frac{(k+j+2)^{m-\frac{1}{2}}}{(l+2)^{m-\frac{1}{2}} (k-l+j+1)^{m-\frac{1}{2}}}.$$

Again, by Lemma 2.13 in [10], we obtain, for m large enough,

$$\left\|\sum_{l=1}^{k-1} \lambda_l \nabla^j u_{k-l}(0)\right\|_{\ell^1} \leq C C_\lambda C_u \left(\frac{3}{4}\right)^{m-\frac{1}{2}} \frac{(r/4)^j R^k (k+j)!}{(j+k+2)^{m-\frac{1}{2}}} \leq \epsilon C_u \frac{(r/4)^j R^k (k+j)!}{(j+k+2)^{m-\frac{1}{2}}}.$$

It remains to estimate

$$\left\|\nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta^n_{\kappa_x}}{n!} (u_l(y) b_{k+1-n-l}(x,y,\overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))} \right]_{x=0}\right\|_{\ell^1}.$$

Let us first suppose $j \leq k$. By (31), and since

$$\|\nabla^{j_1}(b_l \circ \kappa^{-1})\|_{\ell^1} \leq \frac{(r/2)^{j_1} R^l j_1! l!}{(j+l+1)^m},$$

one has

$$\begin{aligned}
&\left\|\nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta^n_{\kappa_x}}{n!} (u_l(y) b_{k+1-n-l}(x,y,\overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))} \right]_{x=0}\right\|_{\ell^1} \\
&\qquad \leq C_u C_b \frac{r^j R^k (j+1)! k!}{(j+k+2)^m} \sum_{n=2}^{k+1} R \left(\frac{C_\kappa^2 r^2}{R}\right)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=0}^{j} \\
&\qquad \frac{(2n)! j! l! A(i_1,j_1,n)(k-n-l+1)!(2n-i_1+j-j_1)!}{2^{2n-i_1+j-j_1} i_1!(2n-i_1)! j_1!(j-j_1)! n!(j+1)! k!} \\
&\qquad\qquad \times \frac{(k+j+2)^m}{(i_1+l+j_1+1)^m (k+2+n-l-i_1+j-j_1)^m}.
\end{aligned}$$

Let us prove, similarly to the control on λ_k, that

$$\frac{(2n)! l! A(i_1,j_1,n)(k-n-l+1)!(2n-i_1+j-j_1)!}{2^{2n-i_1+j-j_1} i_1!(2n-i_1)! j_1!(j-j_1)! n! k!(j+1)} \leq 16^n.$$

First of all,

$$\frac{(2n-i_1+j-j_1)!}{(2n-i_1)!(j-j_1)!} \leq 2^{2n-i_1+j-j_1},$$

so we are left with

$$\frac{(2n)! l! A(i_1,j_1,n)(k-n-l+1)!}{i_1! j_1! n! k!(j+1)}.$$

Suppose first $i_1 \leq n$, so that $A(i_1,j_1,n) = (i_1+j_1)!$. We are left with trying to bound

$$\frac{(2n)! l!(i_1+j_1)!(k-n-l+1)!}{i_1! j_1! n! k!(j+1)}.$$

This is increasing with respect to i_1 and j_1, so that this is smaller than

$$\frac{(2n)! l!(n+j)!(k-n-l+1)!}{n! n! j! k!(j+1)} \leq 4^n \frac{l!(n+j)!(k-n-l+1)!}{k!(j+1)!}.$$

The right-hand side is log-convex with respect to l, and it is equal, at the boundaries $l=0$ and $l=k+1-n$, to

$$4^n\frac{(k+1-n)!(n+j)!}{k!(j+1)!}.$$

This is a log-convex function of n, which varies from 2 to $k+1$. At $n=2$ we obtain $4^2\frac{j+2}{k}\le 16^2$ (since $j\le k$). At $n=k+1$, we obtain instead

$$4^{k+1}\frac{(k+j+1)!}{(j+1)!k!}\le 4^{k+1}2^{k+1+j}\le 16^{k+1},$$

since $j\le k$. Hence, for all n it is smaller than 16^n.

If now $i_1\ge n$, and if $A(i_1,j_1,n)=j_1!i_1!$, then we must simply bound

$$\frac{(2n)!l!(k-n-l+1)!}{n!k!(j+1)}\le 4^n\frac{l!n!(k-n-l+1)!}{k!(j+1)}.$$

With respect to l, the right-hand side reaches a maximum at $l=0$ and $l=k-n+1$, yielding

$$4^n\frac{n!(k-n+1)!}{k!(j+1)}.$$

This log-convex function of n is equal to $4^2\frac{2}{k(j+1)}\le 16^2$ at $n=2$, and at $n=k+1$ we obtain

$$4^{k+1}\frac{k+1}{j+1}\le 16^{k+1};$$

thus, again, it is smaller than 16^n in all cases.

To conclude, if $i_1\ge n$ and $A(i_1,j_1,n)=(j_1+n)!(i_1-n)!$, then it remains to bound

$$\frac{(2n)!l!(i_1-n)!(j_1+n)!(k-n-l+1)!}{i_1!j_1!n!k!(j+1)}\le\frac{(2n)!l!(i_1-n)!(j_1+n)!(k-n-l+1)!}{i_1!j_1!n!k!(j+1)}.$$

This function is increasing with respect to j_1 and decreasing with respect to i_1, so that it is maximal at $i_1=n, j_1=j$, where we obtain

$$\frac{(2n)!l!(j+n)!(k-n-l+1)!}{n!(j+1)!n!k!}\le 4^n\frac{l!(j+n)!(k-n-l+1)!}{(j+1)!k!}$$

which we bounded a few lines above. In conclusion,

$$\begin{aligned}
&\left\|\nabla^j\left[x\mapsto\sum_{n=2}^{k+1}\sum_{l=0}^{k+1-n}\frac{\Delta^n_{\kappa_x}}{n!}(u_l(y)b_{k+1-n-l}(x,y,\overline{w}))\Big|_{(y,\overline{w})=(x,\overline{y}_c(x))}\right]_{x=0}\right\|_{\ell^1}\\
&\le C_uC_b\frac{r^jR^k(j+1)!k!}{(j+k+2)^m}\sum_{n=2}^{k+1}R\left(\frac{16C_\kappa^2r^2}{R}\right)^n\sum_{l=0}^{k+1-n}\sum_{i_1=0}^{2n}\sum_{j_1=0}^{j}\frac{(k+j+2)^m}{(i_1+l+j_1+1)^m(k+2+n-l-i_1+j-j_1)^m}\\
&\le C_uC_b\frac{r^jR^k(j+1)!k!}{(j+k+2)^m}\sum_{n=2}^{k+1}R\left(\frac{16C_\kappa^2r^2}{R}\right)^n\sum_{l=0}^{k+1-n}\sum_{i_1=0}^{2n}\sum_{j_1=0}^{j}\frac{(k+j+n+2)^m}{(i_1+l+j_1+1)^m(k+2+n-l-i_1+j-j_1)^m}.
\end{aligned}$$

By Lemma 2.13 in [10], there exists $C > 0$ such that, for m large enough, (and r, R large enough accordingly) one has

$$\left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_x}^n}{n!} (u_l(y) b_{k+1-n-l}(x, y, \overline{w})) \Big|_{(y,\overline{w})=(x,\overline{y}_c(x))} \right]_{x=0} \right\|_{\ell^1}$$

$$\leq C C_u C_b \frac{r^j R^k (j+1)! k!}{(j+k+2)^m} \sum_{n=2}^{k+1} R \left(\frac{16 C_\kappa^2 r^2}{R} \right)^n.$$

Thus, for R large enough (once C_u, C_λ, m, r are fixed),

$$\left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_x}^n}{n!} (u_l(y) b_{k+1-n-l}(x, y, \overline{w})) \Big|_{(y,\overline{w})=(x,\overline{y}_c(x))} \right]_{x=0} \right\|_{\ell^1} \leq \epsilon C_u \frac{r^j R^k (j+1)! k!}{(j+k+2)^m}.$$

This concludes the proof of the control (33).

Suppose now that $j \geq k$. We start again from

$$\left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\Delta_{\kappa_x}^n}{n!} (u_l(y) b_{k+1-n-l}(x, y, \overline{w})) \Big|_{(y,\overline{w})=(x,\overline{y}_c(x))} \right]_{x=0} \right\|_{\ell^1}$$
$$\leq \sum_{n=2}^{k+1} (2d)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=0}^{j} \frac{(2n)! j!}{i_1!(2n-i_1)! j_1!(j-j_1)! n!} \|\nabla_x^{j_1} (\nabla_\kappa^{i_1})^{[\leq n]} u_l|_{x=0}\|_{\ell^1} \|b_{k+1-n-l} \circ \kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)}.$$

We decompose the sum into two parts, corresponding to $j_1 + \min(i_1, n) < l$ and $j_1 + \min(i_1, n) \geq l$.

In the first part, the control on u_l is the same as previously: one has

$$\|\nabla_x^{j_1} (\nabla_\kappa^{i_1})^{[\leq n]} u_l|_{x=0}\| \leq C_u (C_\kappa)^{i_1} \frac{r^{j_1+i_1} R^l l! A(i_1, j_1, n)}{(i_1 + j_1 + l + 1)^m},$$

and

$$\|b_l \circ \kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)} \leq C_b \frac{(r/8)^{j_1} R^l j_1! l!}{(j_1 + l + 1)^m},$$

so that

$$\sum_{n=2}^{k+1} (2d)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=0}^{l-\min(i_1,n)} \frac{(2n)! j!}{i_1!(2n-i_1)! j_1!(j-j_1)! n!} \|\nabla_x^{j_1} (\nabla_\kappa^{i_1})^{[\leq n]} u_l|_{x=0}\|_{\ell^1} \|b_{k+1-n-l} \circ \kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)}$$

$$\leq C_u C_b \frac{(r/4)^j R^k (j+k+1)!}{(j+k+2)^{m-\frac{1}{2}}} \sum_{n=2}^{k+1} \left(\frac{2d C_\kappa^2 r^2}{R} \right)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=0}^{l-\min(i_1,n)}$$
$$\frac{4^{i_1+j_1} (2n)! j! l! A(i_1, j_1, n)! (j - j_1 + 2n - i_1)! (k+1-n-l)!}{2^{j-j_1+2n-i_1} n! i_1! (2n-i_1)! j_1! (j-j_1)! (j+k+1)!}$$
$$\times \frac{(j+k+2)^{m-\frac{1}{2}}}{(j+k+2-n-l-j_1-i_1)^m (i_1+j_1+l+1)^m}.$$

Let us now prove that

$$4^{j_1+i_1}\frac{j!(2n)!l!A(i_1,j_1,n)(k-n-l+1)!(2n-i_1+j-j_1)!}{2^{2n-i_1+j-j_1}n!i_1!(2n-i_1)!j_1!(j-j_1)!(j+k+1)!}\leq 256^n\sqrt{j}.$$

First, as before

$$\frac{(2n-i_1+j-j_1)!}{(2n-i_1)!(j-j_1)!}\leq 2^{2n-i_1+j-j_1},$$

and we obtain

$$4^{j_1+i_1}\frac{j!(2n)!l!A(i_1,j_1,n)(k-n-l+1)!}{n!i_1!j_1!(j+k+1)!}.$$

If $i_1\leq n$, then $A(i_1,j_1,n)=(i_1+j_1)!$, and we obtain

$$4^{i_1+j_1}\frac{j!(2n)!l!(i_1+j_1)!(k-n-l+1)!}{i_1!j_1!n!(j+k+1)!}$$

This quantity is increasing with respect to j_1, so that it is maximal at $j_1=l-i_1$, yielding

$$4^l\frac{j!(2n)!l!l!(k-n-l+1)!}{i_1!(l-i_1)!n!(j+k+1)!}$$

Suppose first $n\geq\frac{l}{2}$. Then, with respect to i_1, this quantity reaches a maximum at $i_1=\frac{l}{2}$, and we obtain

$$4^l\binom{l}{\frac{l}{2}}\binom{2n}{n}\frac{j!n!l!(k-n-l+1)!}{(j+k+1)!}\leq 4^n8^l\frac{j!n!l!(k-n-l+1)!}{(j+k+1)!}\leq 256^n.$$

Suppose next $n\leq\frac{l}{2}$. Then, with respect to i_1, the maximum of

$$4^l\frac{j!(2n)!l!l!(k-n-l+1)!}{i_1!(l-i_1)!n!(j+k+1)!}$$

is reached at $i_1=n$, yielding

$$4^l\binom{2n}{n}\frac{j!l!l!(k-n-l+1)!}{(l-n)!(j+k+1)!}\leq 4^{l+n}\frac{j!l!l!(k-n-l+1)!}{(l-n)!(j+k+1)!}.$$

This decreasing function of k reaches its maximum at $k=n+l-1$ (the minimal value for k for n,l,j fixed). We obtain

$$4^{n+l}\frac{j!l!l!}{(l-n)!(j+l+n)!}\leq 4^n\left(4^l\frac{j!l!l!}{(l-n)!(j+l+n)!}\right).$$

To conclude, the quantity inside parentheses is a decreasing function of n ; at $n=0$, we obtain

$$4^l\frac{j!l!}{(j+l)!}\leq 2\sqrt{j},$$

since $l\leq j$. Thus, we can bound the original quantity by $4^{n+\frac{1}{2}}\sqrt{j}\leq 256^n\sqrt{j}$.

If $i_1\geq n$ and $A(i_1,j_1,n)=i_1!j_1!$, it remains to bound

$$4^{i_1+j_1}\frac{j!(2n)!l!(k-n-l+1)!}{n!(j+k+1)!}.$$

Again, this decreasing function of k is maximal at $k = l + n - 1$, yielding

$$4^{i_1+j_1}\frac{j!(2n)!l!}{n!(j+n+l)!} \le 4^{2n}4^{j_1}\frac{j!(2n)!l!}{n!(j+n+l)!} \le 4^{n+l}\frac{j!(2n)!l!}{n!(j+n+l)!} \le 16^n 4^l \frac{j!n!l!}{(j+n+l)!}.$$

Now

$$4^l \frac{j!n!l!}{(j+n+l)!}$$

is a decreasing function of n, and at $n = 0$ it is equal to

$$4^l \frac{j!l!}{(j+l)!} \le \sqrt{j}.$$

Hence, in this case the original quantity is bounded by $16^n\sqrt{j}$.

If $i_1 \ge n$ and $A(i_1, j_1, n) = (i_1 - n)!(j_1 + n)!$, we have to bound

$$4^{i_1+j_1}\frac{j!(2n)!l!(i_1-n)!(j_1+n)!(k-n-l+1)!}{i_1!j_1!n!(j+k+1)!}.$$

This quantity is decreasing with respect to k, and at the minimal value $k = l + n - 1$ it is equal to

$$4^{i_1+j_1}\frac{j!(2n)!l!(i_1-n)!(j_1+n)!}{i_1!j_1!n!(j+n+l)!}.$$

This is now increasing with respect to j_1, and at the maximal value $j_1 = l - n$, it is equal to

$$\begin{aligned} 4^{i_1+l-n}\frac{j!(2n)!l!(i_1-n)!l!}{i_1!(l-n)!n!(j+n+l)!} &\le 4^{i_1-n}\frac{\binom{2n}{n}}{\binom{i_1}{n}}4^l\frac{j!l!l!}{(l-n)!(j+n+l)!} \\ &\le 16^n 4^l \frac{j!l!l!}{(l-n)!(j+l+n)!}. \end{aligned}$$

We proved above that

$$4^l \frac{j!l!l!}{(l-n)!(j+l+n)!} \le \sqrt{j},$$

and we obtain that the original quantity is bounded by $16^n\sqrt{j}$.

We thus obtain

$$\begin{aligned} &\sum_{n=2}^{k+1}(2d)^n \sum_{l=0}^{k+1-n}\sum_{i_1=0}^{2n}\sum_{j_1=0}^{l-\min(i_1,n)} \frac{(2n)!j!}{i_1!(2n-i_1)!j_1!(j-j_1)!n!}\|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\le n]}u_l|_{x=0}\|_{\ell^1}\|b_{k+1-n-l}\circ\kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)} \\ &\le C_u C_b \frac{(r/4)^j R^k (j+k+1)!}{(j+k+2)^{m-\frac{1}{2}}} \sum_{n=2}^{k+1}\left(\frac{512 d C_\kappa^2 r^2}{R}\right)^n \sum_{l=0}^{k+1-n}\sum_{i_1=0}^{2n}\sum_{j_1=0}^{l-\min(i_1,n)} \frac{\sqrt{j}(j+k+2)^{m-\frac{1}{2}}}{(j+k+2-n-l-j_1-i_1)^m(i_1+j_1+l+1)^m}. \end{aligned}$$

Since $\sqrt{j} \le \sqrt{j+k+2}$, one can apply Lemma 2.13 in [10] and obtain, for R large enough,

$$\begin{aligned} &\sum_{n=2}^{k+1}(2d)^n \sum_{l=0}^{k+1-n}\sum_{i_1=0}^{2n}\sum_{j_1=0}^{l-\min(i_1,n)} \frac{(2n)!j!}{i_1!(2n-i_1)!j_1!(j-j_1)!n!}\|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\le n]}u_l|_{x=0}\|_{\ell^1}\|b_{k+1-n-l}\circ\kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)} \\ &\qquad \le \epsilon C_u \frac{(r/4)^j R^l (j+k+1)!}{(j+k+2)^{m-\frac{1}{2}}}. \end{aligned}$$

If $j_1 + \min(i_1, n) \geq l$, then the control on u_l takes the form

$$\|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\leq n]} u_l|_{x=0}\|_{\ell^1} \leq (C_\kappa)^{i_1} C_u \frac{(r/4)^j R^l B(i_1, j_1, l, n)}{(1 + j_1 + i_1 + l)^{m-\frac{1}{2}}}$$

with

$$B(i_1, j_1, l, n) = \begin{cases} \max((n + j_1 + l)!(i_1 - n)!,\ (j_1 + l)!i_1!) & \text{if } i_1 \geq n \\ (i_1 + j_1 + l)! & \text{else.} \end{cases}$$

Together with

$$\|b_l \circ \kappa^{-1}\|_{C^{j_1}(W_1)} \leq C_b \frac{(r/8)^{j_1} R^l j_1! l!}{(j_1 + l + 1)^m},$$

we obtain

$$\sum_{n=2}^{k+1} (2d)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=l-\min(i_1,n)}^{j} \frac{(2n)!j!}{i_1!(2n-i_1)!j_1!(j-j_1)!n!} \|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\leq n]} u_l|_{x=0}\|_{\ell^1} \|b_{k+1-n-l} \circ \kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)}$$

$$\leq C_u C_b \frac{(r/4)^j R^k (j+k+1)!}{(j+k+2)^{m-\frac{1}{2}}} \sum_{n=2}^{k+1} \left(\frac{2dC_\kappa^2 r^2}{R}\right)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=l-\min(i_1,n)}^{j}$$
$$\frac{(2n)!j!B(i_1, j_1, l, n)(k+1-n-l)!(j-j_1+2n-i_1)!}{2^{2n-i_1+j-j_1} i_1!(2n-i_1)!j_1!(j-j_1)!n!(j+k+1)!}$$
$$\times \frac{(j+k+2)^{m-\frac{1}{2}}}{(1+j_1+i_1+l)^{m-\frac{1}{2}}(2+k+n+j-l-j_1-i_1)^m}.$$

Let us prove that, in this sum, one has always

$$\frac{(2n)!j!B(i_1, j_1, l, n)(k+1-n-l)!(j-j_1+2n-i_1)!}{2^{2n-i_1+j-j_1} i_1!(2n-i_1)!j_1!(j-j_1)!n!(j+k+1)!} \leq 4^n.$$

First,

$$\frac{(j-j_1+2n-i_1)!}{(j-j_1)!(2n-i_1)!} \leq 2^{2n-i_1+j-j_1},$$

and it remains to bound

$$\frac{(2n)!j!B(i_1, j_1, l, n)(k+1-n-l)!}{i_1!j_1!n!(j+k+1)!}.$$

If $i_1 \leq n$, we obtain

$$\frac{(2n)!j!(i_1+j_1+l)!(k+1-n-l)!}{i_1!j_1!n!(j+k+1)!}.$$

This quantity is increasing with respect to i_1 and j_1, so that it is maximal at $i_1 = n$ and $j_1 = j$, where we obtain

$$\frac{(2n)!(n+j+l)!(k+1-n-l)!}{n!n!(j+k+1)!} = \frac{\binom{2n}{n}}{\binom{j+k+1}{j+n+l}} \leq 4^n.$$

If $i_1 \geq n$ and $B(i_1, j_1, l, n) = i_1!(j_1 + l)!$, we obtain

$$\frac{(2n)!j!(j_1+l)!(k+1-n-l)!}{j_1!n!(j+k+1)!}.$$

This increasing function of j_1 reaches a maximum at $j_1 = j$, where we obtain

$$\frac{(2n)!(j+l)!(k+1-n-l)!}{n!(j+k+1)!} = \frac{\binom{2n}{n}}{\binom{j+k+1}{n,j+l}} \leq 4^n.$$

If $i_1 \geq n$ and $B(i_1, j_1, l, n) = (i_1 - n)!(j_1 + l + n)!$, then we obtain

$$\frac{(2n)!j!(i_1-n)!(j_1+l+n)!(k+1-n-l)!}{j_1!n!i_1!(j+k+1)!}.$$

This is an increasing function of j_1, as well as a decreasing function of i_1, so that it is maximal at $i_1 = n, j_1 = j$, where we obtain again

$$\frac{(2n)!(j+l+n)!(k+1-n-l)!}{n!n!(j+k+1)!} = \frac{\binom{2n}{n}}{\binom{j+k+1}{j+l+n}} \leq 4^n.$$

As before, we conclude using Lemma 2.13 in [10]; if m, r, R are large enough, then we obtain

$$\sum_{n=2}^{k+1}(2d)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=l-\min(i_1,n)}^{j} \frac{(2n)!j!}{i_1!(2n-i_1)!j_1!(j-j_1)!n!} \|\nabla_x^{j_1}(\nabla_\kappa^{i_1})^{[\leq n]}u_l|_{x=0}\|_{\ell^1} \|b_{k+1-n-l} \circ \kappa^{-1}\|_{C^{j-j_1+2n-i_1}(W_1)}$$
$$\leq \epsilon C_u \frac{(r/4)^j R^l (j+k+1)!}{(j+k+2)^{m-\frac{1}{2}}}.$$

This concludes the proof of (34). Now, we can apply Lemma 3.4: there exists $C(b_0, \varphi)$ such that

$$\|\nabla^j u_k(0)\|_{\ell^1} \leq \epsilon C(b_0, \varphi) C_u \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

If ϵ is chosen such that $\epsilon < C(b_0, \varphi)^{-1}$, one can conclude the induction.

Third step.

We successfully constructed and controlled the sequences $(\lambda_k)_{k \geq 0}$ and $(u_k)_{k \geq 0}$ that satisfy (22) at every order. Let us now prove that u_k is controlled on a small neighbourhood of 0.

In the second step, we controlled the functions u_k as follows, *at zero*:

$$\|\nabla^j u_k(0)\|_{\ell^1} \leq C_u \frac{r^j R^k j! k!}{(j+k+1)^m}.$$

Since u_k is real-analytic, in a small neighbourhood of zero, it is given by the power series

$$u_k(y) = \sum_\nu \frac{\nabla^\nu u_k(0)}{\nu!} y^\nu.$$

Since

$$\frac{\nabla^\nu u_k(0)}{\nu!} \leq C_u R^k k! \frac{|\nu|!}{\nu!} r^{|\nu|} \leq C_u R^k k! (rd)^{|\nu|},$$

the power series above converges for $y \in P(0, (rd)^{-1})$, the polydisk centred at zero with radius $(rd)^{-1}$. Moreover, for every $a < 1$, there exists $C(a)$ such that

$$\sup_{P(0, a(rd)^{-1})} |u_k| \leq C(a) C_u R^k k!.$$

In particular, by Proposition 2.14 in [10], for every $a < \frac{1}{2}$, there exists $C(a)$ such that

$$\|u_k\|_{H\left(-d,\frac{d^2 r}{a},P(0,\frac{a}{rd})\right)} \leq C(a)C_u R^k k!.$$

In other terms, letting $V = P(0, a(2rd)^{-1})$, for every $j \geq 0$, one has

$$\|u_k\|_{C^j(V)} \leq C(a)C_u \frac{R^k(\frac{d^2}{a}r)^j j!k!}{(j+1)^{-d}}.$$

In particular, u is an analytic symbol on V. $\square$

We are now in position to perform an analytic summation.

Lemma 4.3. *Let f, V, φ, $(u_k)_{k\geq 0}$, be as in Proposition 4.2. There exists $c' > 0$, $c_0 > 0$ and $C > 0$ such that, for all $0 \leq c < c_0$, for all $N \in \mathbb{N}$, with*

$$u(N) = \mathbb{1}_V \psi_0^N e^{N\varphi} \sum_{k=0}^{cN} N^{-k} u_k,$$

one has

$$\|(1 - S_N)u(N)\|_{L^2(M)} \leq Ce^{-c'N}.$$

Proof. Let $R > 0$ be as in Proposition 4.2. There exists $C_u > 0$ such that, for all $k \in \mathbb{N}$,

$$\sup_V \|u_k\| \leq C_u R^k k!.$$

In particular, by Proposition 2.2, for all $c \leq \frac{e}{3R}$, the sum

$$\sum_{k=0}^{cN} N^{-k} u_k$$

is bounded uniformly with respect to N.

Let now $W \subset\subset V$ be such that $0 \in W$, and let $\chi : M \to [0,1]$ be a smooth function such that $\mathbb{1}_W \leq \chi \leq \mathbb{1}_V$.

Since there exists $\epsilon > 0$ and $C > 0$ such that, for all $x \in V$, for all $N \in \mathbb{N}$,

$$|\psi_0^N(x)e^{N\varphi(x)}| \leq CN^d e^{-\epsilon\, \mathrm{dist}(x,0)^2 N},$$

and since $1 - \chi$ is supported outside W, then there exists $C > 0$ and $c' > 0$ such that, for all $N \in \mathbb{N}$,

$$\|(1 - \chi)u(N)\|_{L^2(M)} \leq Ce^{-c'N}.$$

In particular, since S_N is an orthogonal projection,

$$\|S_N[(1 - \chi)u(N)]\|_{L^2(M)} \leq Ce^{-c'N}.$$

Now

$$\overline{\partial}(\chi u(N)) = (\overline{\partial}\chi)u(N)$$

satisfies

$$\|(\overline{\partial}\chi)u(N)\|_{L^2(M)} \leq Ce^{-c'N}$$

because $\overline{\partial}\chi$ is supported outside W as well.

We conclude using the Hörmander $\overline{\partial}$ inequality (see for instance [30], Proposition 1.1, or [7], Proposition 2.3.3)

$$\|(1-S_N)v\|_{L^2(M)} \leq N^{-\frac{1}{2}}\|\overline{\partial}v\|_{L^2(M)}.$$

Hence

$$\|(1-S_N)\chi u(N)\|_{L^2(M)} \leq Ce^{-c'N},$$

and we can conclude:

$$\|(1-S_N)u(N)\|_{L^2(M)} \leq \|(1-\chi)u(N)\|_L^2(M) + \|(1-S_N)\chi u(N)\|_{L^2(M)} + \|S_N(1-\chi)u(N)\|_{L^2(M)} \leq 3Ce^{-c'N}.$$

□

Proposition 4.4. *Let f, V, $(u_k)_{k\geq 0}$, $(\lambda_k)_{k\geq 0}$ be as above. There exists $c > 0$, $c' > 0$ and $C > 0$ such that, for every $N \in \mathbb{N}$,*

$$\left\|\left(T_N(f) - \sum_{j=0}^{cN} N^{-j-1}\lambda_j\right)\left(\mathbb{1}_V \psi_0^N e^{N\varphi}\sum_{k=0}^{cN} N^{-k}u_k\right)\right\|_{L^2(M)} \leq Ce^{-c'N}.$$

Proof. Let $c > 0$ be small enough, so that one can apply Proposition 2.2: $\sum_{k=0}^{cN} N^{-k}u_k$, $\sum_{j=0}^{cN} N^{-j-1}\lambda_j$ are bounded independently on N. Let

$$u(N) = \mathbb{1}_V \psi_0^N e^{N\varphi}\sum_{k=0}^{cN} N^{-k}u_k$$

$$\lambda(N) = \sum_{j=0}^{cN} N^{-j-1}\lambda_j.$$

Outside of V, our presumed quasimode $u(N)$ is 0, and one has also

$$\|T_N(f)u_N\|_{L^2(M\setminus V)} = O(e^{-c'N}):$$

indeed, since φ is admissible, outside any open set $W \subset\subset V$ such that $0 \in W$, one has $|\psi_0^N e^{-N\varphi}| \leq Ce^{-c'N}$ for some $c' > 0$, and the Szegő projector S_N decays away from the diagonal, so that $\|\mathbb{1}_{M\setminus V}S_N\mathbb{1}_W\| \leq Ce^{-c'N}$ as well.

Since $S_N u(N) = u(N) + O(e^{-c'N})$ by Lemma 4.3, we now replace $S_N f S_N u(N) - \lambda(N)u(N)$ with $S_N f u(N) - \lambda(N)u(N)$, and estimate the L^2 norm of the latter on V. By construction, on V, there holds

$$\begin{aligned}&[(S_N f - \lambda(N))\, u(N)](x)\\ &\quad = -\sum_{j=0}^{cN}\sum_{k=cN-j}^{cN} N^{-1-j-k}\psi_0^N(x)e^{N\varphi(x)}\lambda_j u_k(x) + \sum_{j+k\leq cN} N^{-1-j-k}\psi_0^N(x)e^{N\varphi(x)}R(j,k,N)(x),\end{aligned}$$

where $R(j,k,N)$ is the remainder at order $cN-k-j$ in the stationary phase Lemma applied to

$$N^{2d}\lambda_j e^{-N\varphi(x)}\int_{y\in M} e^{-N\Phi_1(x,y,\overline{y},0)+N\varphi(y)}(u*b)_k(x,y,\overline{y})\mathrm{d}y.$$

Since $\lambda * u$ is an analytic symbol by Proposition 2.2, we have, for $c > 0$ and $c' > 0$ small enough,

$$\left\|\sum_{j=0}^{cN}\sum_{k=cN-j}^{cN} N^{-1-j-k}\lambda_j u_k\right\|_{L^\infty(V)} \leq Ce^{-c'N},$$

so that

$$\left\| \left(\sum_{j=0}^{cN} \sum_{k=cN-j}^{cN} N^{-1-j-k} \lambda_j u_k(x) \right) \psi_0^N(x) e^{N\varphi(x)} \right\|_{L^2(V)} \leq Ce^{-c'N}.$$

The remainder $R(j,k,N)$ can be estimated using Proposition 3.13 in [10]. Indeed, let $r > 0$ and $R > 0$ be such that $u \in S_4^{r,R}(V)$ and $b \in S_4^{r,R}(V)$. By Proposition 2.2, $u * b$ is an analytic symbol of the same class, so that

$$\|(u*b)_k\|_{C^j(V)} \leq CC_uC_bR^kr^j(j+k)! \leq (CC_uC_b(2R)^kk!)(2r)^jj!.$$

In particular, $(u*b)_k$ admits a holomorphic extension to a k-independent complex neighbourhood $\widetilde{V}$ of V, with

$$\sup_{\widetilde{V}} |(u*b)_k| \leq CC_uC_b(2R)^kk!.$$

In particular, by Proposition 3.13 in [10], one has, for some $c_1 > 0$, that the remainder at order c_1N in the stationary phase Lemma applied to

$$N^{2d}\lambda_j e^{-N\varphi(x)} \int_{y\in M} e^{-N\Phi_1(x,y,\overline{y},0)+N\varphi(y)}(u*b)_k(x,y,\overline{y})\mathrm{d}y$$

is smaller than $CC_uC_b(2R)^k(2R)^jj!k!e^{-c'N}$. In particular,

$$\left(\frac{1}{n!}\Delta^n_{\kappa_x}((u*b)_kJ)(y_c)\right)_n$$

is an analytic symbol in a fixed class, with norm smaller than $C(2R)^kk!$.

If $j + k < \frac{1}{2}cN$, we will compare $R(j,k,N)$ to the remainder at order c_1N. If $j + k \geq \frac{1}{2}cN$, we will compare $R(j,k,N)$ to the remainder at order 0.

Without loss of generality, $c < c_1$. Then, for all j,k such that $j + k < \frac{1}{2}cN$, since the expansion in the stationary phase

$$\sum_{n=cN-j-k}^{c_1N} (n!N^{d+n})^{-1}\Delta^n_{\kappa_x}((u*b)_kJ)(y_c)$$

corresponds to an analytic symbol, then by Lemma 2.2 this sum is $O(e^{-c'N})$; thus if $j + k < c/2$ one has

$$R(j,k,N) \leq Ce^{-c'N}.$$

If $\frac{1}{2}cN < j + k < cN$, then, on one hand

$$N^{-1-j-k}\left|N^{2d}\lambda_j e^{-N\varphi(x)} \int_{y\in M} e^{-N\Phi_1(x,y,\overline{y},0)+N\varphi(y)}(u*b)_k(x,y,\overline{y})\mathrm{d}y\right| \leq C\left(\frac{2R}{N}\right)^{j+k}(j+k)!$$

is smaller than $Ce^{-c'N}$ if c is small enough; on the other hand, again

$$\left(\frac{1}{n!}\Delta^n_{\kappa_x}((u*b)_kJ)(y_c)\right)_n$$

is an analytic symbol in a fixed class (with norm smaller than $C(2R)^kk!$), so that, by Proposition 2.2, if c is small enough,

$$N^{d-1-j-k}\lambda_j \sum_{n=0}^{cN-j-k} \frac{1}{n!N^n}\Delta^n_{\kappa_x}((u*b)_kJ)(y_c) < C\left(\frac{2R}{N}\right)^{j+k}(j+k)! \leq Ce^{-c'N}.$$

This concludes the proof. $\square$

5 Spectral estimates at the bottom of a well

5.1 End of the proof of Theorem A

We now prove part 2 of Theorem A. Suppose that $\min(f) = 0$ and that the minimal set of f consists in a finite-number of non-degenerate minimal points $P_1, \ldots, P_j$. At each of these points P_i with $1 \leq i \leq j$, one can construct (see Proposition 4.4) a sequence $v_i(N)$ of $O(e^{-c'N})$-eigenfunctions of $T_N(f)$. From Proposition 4.1, if μ denotes the Melin value (see Section 3.3 of [8]), then, for every $1 \leq i \leq j$ one has

$$T_N(f)v_i(N) = N^{-1}\mu(P_i)v_i(N) + O(N^{-2}).$$

Moreover, from Theorem B in [8], for $\epsilon > 0$ small, the number of eigenvalues of $T_N(f)$ in the interval $[0, \min_{1\leq i\leq j} \mu(P_i) + N^{-1}\epsilon]$ is exactly the number of i's such that P_i minimises μ.

Hence, any normalised sequence of ground states of $T_N(f)$ is $O(Ne^{-c'N}) = O(e^{-(c'-\epsilon)N})$-close to a linear combination of those $v_i(N)$ whose associated well P_i minimises μ (as the spectral gap is of order N^{-1} and the the $v_i(N)$'s are $O(e^{-c'N})$-eigenvectors). This concludes the proof.

5.2 Tunnelling

The main physical application of Theorem A is the study of the spectral gap for Toeplitz operators that enjoy a local symmetry. Let us formulate a simple version of this result.

Proposition 5.1. *Suppose that* $\min(f) = 0$ *and that the minimal set of f consists of two non-degenerate critical points P_0 and P_1. Suppose further that these wells are* symmetrical*: there exist neighbourhoods U_0 of P_0 and U_1 of P_1, and a ω-preserving biholomorphism $\sigma : U_0 \mapsto U_1$, such that $\sigma \circ f = f$.*

Then there exists $c > 0$ and $C > 0$ such that, for every $N \geq 1$, the gap between the two first eigenvalues of $T_N(f)$ is smaller than $Ce^{-c'N}$.

Proof. Near P_0, one can build a sequence of $O(e^{-c'N})$-eigenvectors as in Proposition 4.4, with $c > 0$; near P_1 one can build another sequence of $O(e^{-c'N})$-eigenvectors. Since M and f are equivalent near P_0 and near P_1, the associated sequences of eigenvalues are identical up to $O(e^{-c'N})$, and the approximate eigenvectors are orthogonal with each other since they have disjoint support, so that there are at least two eigenvalues in an exponentially small window near the approximate eigenvalue. As above (see Theorem B in [8]), there are no more than two eigenvalues in the window $[\min Sp(T_N(f)), \min Sp(T_N(f)) + \epsilon N^{-1}]$, for ϵ small; hence the claim. □

Unfortunately, the actual spectral gap between two symmetrical wells cannot be recovered from Proposition 4.2 or the solution φ of the Hamilton-Jacobi equation, apart from the upper bound (4).

Proposition 5.2. *Suppose that* $\min(f) = 0$ *and that the minimal set of f consists of two symmetrical wells. Let λ_0 and λ_1 denote the two first eigenvalues of $T_N(f)$ (with multiplicity), and let*

$$\alpha = \liminf_{N\to+\infty} \left(-N^{-1}\log(\lambda_1 - \lambda_0)\right).$$

Then α cannot be bounded from above in terms of the best possible constant c' in Proposition 4.4, and moreover α is unrelated to the solution φ of the Hamilton-Jacobi equation.

Proof. We first let $\chi : [-1, 1] \mapsto \mathbb{R}$ be an even smooth function; we suppose that χ reaches its minimum only at -1 and 1, with $\chi(-1) = 0$ and $\chi'(-1) > 0$. We consider the function $f = \chi \circ z$ on $\mathbb{S}^2$, where $z : \mathbb{S}^2 \to [-1, 1]$ is the height function. Then f is invariant under a rotation around the vertical axis, so that $T_N(f)$ is diagonal in the natural spin basis (which consists of the eigenfunctions for $T_N(z)$). Since $\chi'(-1) > 0$,

f has two global minima (the North and South pole) and they are elliptic points. Among the spin basis, the states that minimise the energy are the coherent states at the North and South poles, respectively; they have the same energy since f is invariant under the symmetry $z \to -z$. In this setting the first eigenvalue is degenerate, and shared between two states which localise at either of the two non-degenerate wells; one has $\alpha = +\infty$.

Let us give a formal solution to the Hamilton-Jacobi equation. In stereographic coordinates near one of the poles, the symbol reads $g(|r|^2) = g(r\overline{r})$ for some $g \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$. The expression $g(rs)$ does not make sense if rs is not a real number, but taking $s = 0$ yields $g(r \times 0) = 0$. A formal solution of $\widetilde{g}(x, \partial\varphi) = 0$ is thus given by $\varphi = 0$. This corresponds indeed to the exponential decay of the exact ground states: $\varphi = 0$ means that the ground state decays as fast as the coherent state (they actually coincide).

In the system above, the formal solution of the Hamilton-Jacobi equation yields the correct decay rate. However, from the point of view of Proposition 4.4, one has $c' = 0$: if χ is not real-analytic near 1 we cannot hope to perform an analytic summation for the sequence λ_i as in Proposition 4.4. To be more precise, the ground state is

$$\frac{N+1}{\pi} \int_{\mathbb{S}^2} \chi(z(x)) \left(\frac{1-z(x)}{2}\right)^{2N} \mathrm{dVol}(x),$$

so that, if χ is not real-analytic near -1, one cannot approximate λ_0 by an analytic symbol up to $O(e^{-c'N})$ for some $c' > 0$.

We consider now a smooth perturbation of the function χ above: let $\chi_1 : \mathbb{R} \mapsto [0, 1]$ be a smooth, non-zero function supported on a compact subset of $[0, 1)$. If we replace χ with $\chi + \chi_1$ in the previous discussion, we still get a symbol invariant under vertical rotation, so that it is diagonal in the spin basis. Since $(\chi+\chi_1)\circ z = \chi \circ z$ where the latter is smallest (near the poles), the two candidates for the ground state are still the coherent states associated with the North and South pole, for N large enough (all other states have an energy gap of order at least N^{-1}). The Hamilton-Jacobi equation has the same formal solution. However, the two candidates for the ground state now have different energies, with an exponentially small but non-zero gap, of order $e^{-\alpha N}$. In fact, from

$$\lambda_1 - \lambda_0 = \frac{N+1}{\pi} \int_{\mathbb{S}^2} [\chi_1(z(x)) - \chi_1(-z(x))] \left(\frac{1+z(x)}{2}\right)^{2N} \mathrm{dVol}(x),$$

one obtains

$$\alpha = -2\log\left(\frac{1 + \max(\mathrm{supp}\chi_1)}{2}\right).$$

Here, α can be made arbitrarily small by choosing χ_1 with support arbitrarily close to 1. In this case, we identified a family of Toeplitz operators with symmetrical wells, with identical (formal) admissible solution of the Hamilton-Jacobi equation, and identically $c' = 0$, but such that one has possibly $\alpha = +\infty$ (if $\chi_1 = 0$) or α arbitrarily small. □

The counterexample proposed in the proof is not entirely satisfactory, because it is not real-analytic on the whole manifold. In fact, in the situation of Proposition 5.1, if f is real-analytic everywhere, then there is a global symmetry $\sigma : M \to M$, whose square is the identity, and such that $\sigma \circ f = f$. However, what Proposition 5.2 illustrates is that even if the solution of the Hamilton-Jacobi equation can be globally defined (as a section), the fact that one can perform analytic extensions only in a fixed, not necessarily large neighbourhood of the real set means that large errors may occur.

Another possible obstruction comes from the fact that, contrary to the case of two symmetric wells for Schrödinger operators [17], the symmetry $\sigma : M \to M$ may not be quantizable. For instance, on the unit torus $M = \mathbb{C}/\mathbb{Z}^2$, consider f invariant under horizontal translation by $\frac{1}{2}$, and having two non-degenerate wells. Then one cannot quantize σ and decompose $H_0(M, L)$ into odd and even sections (for the action of σ) if N is an odd integer. The tunnelling rate α may actually be different in the odd and even case.

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