Penalisation techniques for one-dimensional reflected rough differential equations

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Abstract

In this paper we solve real-valued rough differential equations (RDEs) reflected on a moving boundary. The solution is approached by a sequence of rough differential equations with an unbounded drift whose intensity increases with n (the penalisation). Hence we also provide an existence theorem for RDEs with a drift growing at most linearly. In addition, a speed of convergence of the sequence of penalised process to the reflected process is provided in the smooth case.

1 Introduction

Solving (stochastic) differential equations with a reflecting boundary condition is by now a classical problem. For a domain $D \subseteq \mathbb{R}^e$, a mapping $\sigma : \mathbb{R}^d \to \mathbb{R}^{e\times d}$, an initial value $y_0 \in \overline{D}$ and an $\mathbb{R}^d$-valued path $X = \{X_t\}_{t\in[0,T]}$ sometimes referred as the noise, this problem consists formally in finding $\mathbb{R}^e$-valued paths $\{Y_t\}_{t\in[0,T]}$ and $\{K_t\}_{t\in[0,T]}$ such that $\forall t \in [0, T]$,

$$\begin{aligned} &Y_t = y_0 + \int_0^t \sigma(Y_s)\mathrm{d}X_s + K_t,\\ &Y_t \in \overline{D},\ |K|_T < \infty,\\ &|K|_t = \int_0^t \mathbf{1}_{\{X_s\in\partial D\}} d|K|_s \ \text{ and }\ K_t = \int_0^t n(X_s) d|K|_s, \end{aligned}$$

where $|K|_t$ is the finite variation of K on $[0, t]$ and $n(x)$ is the unit inward normal of ∂D at x. If X is a Brownian motion and the integral is in the sense of Itô, this problem was first studied by Skorokhod [24], and then by McKean [23], El Karoui [10], Lions and Sznitman [21], to name but a few. For this reason, it is called the Skorokhod problem associated to X, σ and D (see Definition 2.8).

In the last few years, this problem has attracted a lot of attention when the driver X is a β-Hölder continuous path: in the "regular" case $\beta \in (\frac{1}{2}, 1)$, existence of a solution has been established in a multidimensional setting by Ferrante and Rovira [12] and uniqueness was then obtained by Falkowski and Słomiński [11]. In that case, the integral can be constructed by a Riemann sum approximation and is known as a Young integral [27]. Extensions of these results to the "irregular" case $\beta < \frac{1}{2}$ can be handled with rough paths. We recall that this theory was initiated by Lyons [22] and for a (multidimensional) β-Hölder continuous path X and σ a bounded vector field, it provides a way to solve the equation $\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}\mathbf{X}_t$, where $\mathbf{X} = (X, \mathbb{X})$ is the path X with a supplementary two-parameter path $\mathbb{X}$ (in fact higher order correction terms such as $\mathbb{X}$ are needed if $\beta \le \frac{1}{3}$, but we shall assume $\beta > \frac{1}{3}$ for simplicity). Solutions can be understood either as a limit of ODEs driven by a smooth driver X^k which converges to X ([16, Chapter 10]), or directly as an equality between Y_t and $\int_0^t \sigma(Y_s)\mathrm{d}\mathbf{X}_s$ when this integral is defined in the sense of controlled rough paths [14, 17] (see also the original definition of Lyons [22] and the one of Davie [6]). In this paper both notions will be useful and shown to coincide for the penalised RDEs. Existence of solutions of reflected RDEs with $\beta \in (\frac{1}{3}, \frac{1}{2})$ was proven

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by Aida [1], while Deya, Gubinelli, Hofmanová, and Tindel [8] provided uniqueness for a one-dimensional process reflected on the line. Note that existence was extended to processes reflected on a moving convex boundary by Castaing, Marie, and Raynaud de Fitte [5] and that except in this last work, the boundary is fixed.

We focus on one-dimensional solutions to rough differential equations which are reflected on a moving boundary $L : [0, T] \to \mathbb{R}$, where the driver is a d-dimensional rough path $\mathbf{X}$ with Hölder regularity $\beta \in (\frac{1}{3}, 1)$ (note that by a sligh abuse of notations, we may use $\mathbf{X}$ for X and the vocabulary of rough paths even in the smooth case). Following a classical method for reflected differential equations, we consider the following sequence of penalised RDEs with drift:

$$Y^n_t = y_0 + n \int_0^t (Y^n_s - L_s)_- \mathrm{d}s + \int_0^t \sigma(Y^n_s) \mathrm{d}\mathbf{X}_s. \tag{1.1}$$

For technical reasons, the drift function $n(\cdot)_-$ will be replaced by a smoother function ψ_n with at most linear growth, but the interpretation remains that of a stronger and stronger force pushing Y^n above L. However, solving RDEs with unbounded coefficients is known to be tricky [2, 19, 20] and to the best of our knowledge, there is no criterion in the literature allowing for the drift ψ_n (smooth and at most linearly growing) and any smooth and bounded σ. Hence as a slight extension of a result of Friz and Oberhauser [13], we prove that (1.1), with ψ_n as drift coefficient instead of $n(\cdot)_-$, has a unique global solution. Moreover, this solution has a Doss-Sussmann–like representation [9, 25]. This property turns to be extremely useful as it allows to transport the monotony of $\psi_n \le \psi_{n+1}$ to the penalised solution, leading to $Y^n \le Y^{n+1}$. We are then able to prove the uniform convergence of Y^n and $K^n := \int_0^\cdot \psi_n(Y^n_s - L_s)\mathrm{d}s$ to Y and K, which are identified as the solution to the Skorokhod problem described above, which reads in dimension 1:

$$Y_t = y_0 + \int_0^t \sigma(Y_s)\mathrm{d}\mathbf{X}_s + K_t \quad \text{ and } \quad Y_t \ge L_t \ , \ t \in [0, T], \tag{1.2}$$

and the non-decreasing path K increases only when Y hits L. Here, the reflection term also reads $K_t = \sup_{s \le t} \left((L_s - y_0 - \int_0^s \sigma(Y_u)\mathrm{d}\mathbf{X}_u) \vee 0 \right)$, and under appropriate conditions, the solution is unique ([8, Theorem 9]). Besides, when $\mathbf{X}$ is a Gaussian rough path, the convergence of the sequence of penalised processes also happens uniformly in $L^\gamma(\Omega)$, $\gamma \ge 1$. Finally, we are able to prove a rate of convergence of Y^n towards Y in the regular case.

In a forthcoming work, we will study the Euler scheme of the penalised processes and compute the distance between this approximation and the reflected process. Hence, computing the rate of convergence of the penalised solutions towards the reflected solution is the first step of this program, which may be extended to the rough case. Another natural extension of this work is to prove our results for $\mathbb{R}^e$-valued solutions, as in [5, 21, 26]. We believe this might be done with our monotony argument replaced by a tightness argument, as for instance in [26].

Organisation of the paper. In Section 2, a brief overview of rough paths definitions and techniques is presented, followed by a set of precise assumptions and the statement of our main results. Then the existence of a solution to the penalised equation is proven in Section 3, followed by some penalisation estimates. Most of the proofs that lead to the convergence of the penalised sequence to the reflected solution (Theorems 2.11 and 2.12) are contained in Section 4: first it is proven that Y^n and K^n converge uniformly (we show monotone convergence of Y^n towards a continuous limit), then that Y is controlled by X in the rough paths sense, which permits to use rough paths continuity theorems to show that Y and K solve the Skorokhod problem. In Section 5, we prove Theorem 2.13 which gives a rate of convergence of the sequence of penalised paths to the reflected solution. Eventually, Appendix A gathers the proof of existence of solutions for RDEs with unbounded drift (Proposition 2.10) and the proof of a Gronwall lemma for non-smooth paths.

Notations. C is a constant that may vary from line to line. For $k \in \mathbb{N}$ and $T > 0$, $\mathcal{C}^k_b([0, T]; F)$ (or simply $\mathcal{C}^k_b$) denotes the space of bounded functions which are k times continuously differentiable with bounded derivatives, with values in some linear space F. If E and F are two Banach spaces, $\mathcal{L}(E, F)$ denotes the space of continuous linear mappings from E to F. In the special case $E = \mathbb{R}^d$ and $F = \mathbb{R}$, we also write $(\mathbb{R}^d)'$ to denote the space of linear forms on $\mathbb{R}^d$. The tensor product of two finite-dimensional vector spaces E and

F is denoted by $E \otimes F$. In particular, $\mathbb{R}^d \otimes \mathbb{R}^e \simeq \mathbb{R}^{d\times e} \simeq \mathcal{M}^{d,e}(\mathbb{R})$, the space of real matrices of size $d \times e$. Let f be a function of one variable, we define

$$\delta f_{s,t} := f_t - f_s. \tag{1.3}$$

For $\beta \in (0,1)$ and a function $g : [0,T]^2 \to F$, the Hölder semi-norm of g on a sub-interval $I \subseteq [0,T]$, denoted by $\|g\|_{\beta,I}$ (or simply $\|g\|_\beta$ if $I = [0,T]$), is given by

$$\|g\|_{\beta,I} = \sup_{s\neq t\in I} \frac{|g_{s,t}|}{|t-s|^\beta}.$$

The β-Hölder space $\mathcal{C}_2^\beta([0,T];F)$ is the space of functions $g : [0,T]^2 \to F$ such that $\|g\|_\beta < \infty$. The β-Hölder space $\mathcal{C}^\beta([0,T];F)$ is the space of functions $f : [0,T] \to F$ such that $\|\delta f\|_\beta < \infty$ (hereafter $\|\delta f\|_\beta$ will simply be denoted by $\|f\|_\beta$). With a slight abuse of notations, we may write $g \in \mathcal{C}^\beta([0,T];F)$ even for a 2-parameter function, and if the context is clear, we may just write $g \in \mathcal{C}^\beta$.

Similarly, we also remind the definitions of the p-variation semi-norm and space. For $p \geq 1$, a sub-interval $I \subseteq [0,T]$ and $g : [0,T]^2 \to E$, denote by $\|g\|_{p,I}$ (or simply $\|g\|_p$ if $I = [0,T]$) the semi-norm defined by

$$\|g\|_{p,I}^p = \sup_\pi \sum_{i=0}^{n-1} |g_{t_i,t_{i+1}}|^p,$$

where the supremum is taken over all finite subdivisions $\pi = (t_0,\dots,t_n)$ of I with $t_0 < t_1 < \cdots < t_n \in I$, $\forall n \in \mathbb{N}$. With the same abuse of notations, we define $\mathcal{V}_2^p$ the set of *continuous* 2-parameter paths g with finite p-variation, and $\mathcal{V}^p$ the set of *continuous* paths $f : [0,T] \to F$ such that $\|\delta f\|_p \equiv \|f\|_p < \infty$.

Note that we shall use roman letters (p, q,...) for the variation seminorms and greek letters (α, β,...) for Hölder seminorms in order not to confuse $\|\cdot\|_p$ and $\|\cdot\|_\alpha$. In case there might be a confusion, we shall write $\|\cdot\|_{p\text{-var}}$ or $\|\cdot\|_{\alpha\text{-Höl}}$, for instance $\|f\|_{1\text{-var}}$.

Remark 1.1. *$\mathcal{C}^\beta$ (resp. $\mathcal{V}^p$) is a Banach space when equipped with the norm $f \mapsto |f_0| + \|f\|_\beta$. (resp. $|f_0| + \|f\|_p$). When this property will be needed, the paths will start from the same initial conditions, thus we may forget about the first term and consider $\|\cdot\|_\beta$ (resp. $\|\cdot\|_p$) as a norm.*

2 Preliminaries on rough paths and the Skorokhod problem

In this section, we briefly review the definitions of rough paths and rough differential equations, gathered mostly from Friz and Victoir [16] and Friz and Hairer [14]. We also give a sense to the Skorokhod problem written in Equation (1.2).

2.1 Geometric rough paths

Definition 2.1 (Rough path)**.** • *Let $\beta \in (\frac{1}{3}, \frac{1}{2}]$ (resp. $p \in [2,3)$). A β-Hölder rough path (resp. p-rough path) $\mathbf{X}$ is a couple $\mathbf{X} = \big((X_t)_{t\in[0,T]}, (\mathbb{X}_{s,t})_{s,t\in[0,T]}\big) \in \mathcal{C}^\beta([0,T];\mathbb{R}^d) \times \mathcal{C}^{2\beta}([0,T];\mathbb{R}^d \otimes \mathbb{R}^d)$ (resp. in $\mathcal{V}^p([0,T];\mathbb{R}^d) \times \mathcal{V}^{\frac{p}{2}}([0,T];\mathbb{R}^d \otimes \mathbb{R}^d)$) such that Chen's relation is satisfied:*

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = \delta X_{s,u} \otimes \delta X_{u,t},$$

for any $(s,t,u) \in [0,T]^3$. The space of such paths is denoted by $\mathscr{C}^\beta([0,T];\mathbb{R}^d)$, or simply $\mathscr{C}^\beta$ (resp. $\mathscr{V}^p([0,T];\mathbb{R}^d)$ and $\mathscr{V}^p$). For $\mathbf{X} \in \mathscr{C}^\beta$ (resp. in $\mathscr{V}^p$), we will need the following homogeneous rough path "norm":

$$|||\mathbf{X}|||_\beta = \|X\|_\beta + \sqrt{\|\mathbb{X}\|_{2\beta}} \quad (\textit{resp. } |||\mathbf{X}|||_p^p = \|X\|_p^p + \|\mathbb{X}\|_{\frac{p}{2}}^{\frac{p}{2}}).$$

• *$\mathbf{X} \in \mathscr{C}^\beta([0,T];\mathbb{R}^d)$ (resp. in $\mathscr{V}^p$) is a geometric rough path if the symmetric part of $\mathbb{X}$, $sym(\mathbb{X}) = \left(\mathbb{X}^{ij} + \mathbb{X}^{ji}\right)_{i,j=1\dots d}$, satisfies*

$$sym(\mathbb{X})_{s,t} = \frac{1}{2}\delta X_{s,t} \otimes \delta X_{s,t}.$$

Intuitively, this relation implies that geometric rough paths admit a first order chain rule, as for smooth paths or Stratonovich calculus. The space of geometric β-Hölder rough paths (resp. p-rough paths) is denoted by $\mathscr{C}_g^\beta$ (resp. $\mathscr{V}_g^p$).

Although our main results are expressed in Hölder spaces only, the p-variations play an important role in the proofs, due to the nature of the compensator process K (which is non-decreasing and thus in $\mathcal{V}^1$).

For the following definition, we follow [8].

Definition 2.2 (Control). *Let I be an interval and define the simplex $\mathcal{S}_I = \{(s,t) \in I^2 : \ s \le t\}$. A control is a map $w : \mathcal{S}_I \to \mathbb{R}_+$ which is super-additive, i.e. $w(s,t) + w(t,u) \le w(s,u)$ for all $s \le t \le u \in I$. A control is* regular *if $\lim_{|s-t|\to 0} w(s,t) = 0$.*

For instance, if $\|X\|_{p,I}^p < \infty$ for some interval $I \subseteq [0,1]$, then $w_X(s,t) = \|X\|_{p,[s,t]}^p$ is a control on $\mathcal{S}_I$. If $X \in \mathcal{V}^p(I)$, then w_X is a regular control.

2.2 Rough differential equations with drift

For a geometric rough path $\mathbf{X} \in \mathscr{C}_g^\beta\left([0,T], \mathbb{R}^d\right)$, we would like to give a meaning to the following formal equation:

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}\mathbf{X}_t. \tag{2.1}$$

We adopt the definition of solution given in [16, Definition 12.1] (see also [13, Definition 3]), which we recall for the reader's convenience. Note that this definition gives a meaning to (1.1), even though $\int \sigma(Y_s^n)\mathrm{d}\mathbf{X}_s$ has not been defined yet.

Definition 2.3 (RDE with drift). *Let $\mathbf{X} \in \mathscr{C}_g^\beta\left([0,T], \mathbb{R}^d\right)$, with $\beta \in (\frac{1}{3}, \frac{1}{2}]$. We call $Y \in \mathcal{C}^0([0,T], \mathbb{R}^e)$ a* solution to the RDE with drift *(2.1) started at $y_0 \in \mathbb{R}^e$ if there exists a sequence $(X^k)_{k\in\mathbb{N}}$ of $\mathbb{R}^d$-valued Lipschitz paths such that*

- *$\sup_{k\in\mathbb{N}} \left|\left|\left|\mathbf{X}^k\right|\right|\right|_\beta < \infty$, where $\mathbf{X}^k = (X^k, \mathbb{X}^k)$ and $\mathbb{X}_{s,t}^k = \int_s^t (X_u^k - X_s^k) dX_u^k$;*
- *$\mathbf{X}^k$ converges pointwise to $\mathbf{X}$;*
- *for all k, the ODE $dY_t^k = b(Y_t^k)dt + \sigma(Y_t^k)dX_t^k$ has a solution and $\|Y^k - Y\|_{\infty,[0,T]} \to 0$ as $k \to \infty$.*

The classical Doss-Sussmann representation (see Doss [9] and Sussmann [25]) provides a way to write the solution of a stochastic differential equation as the composition of the flow of σ with the solution of a random ODE. It works for one-dimensional noises, even in some rough cases. However its multidimensional generalization requires strong geometric assumptions on σ (see [9]). Instead we recall a less explicit formulation borrowed from Friz and Oberhauser [13], which requires no additional assumption on σ and shall be enough for our needs.

For some $\sigma : \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$, consider the RDE

$$\mathrm{d}\tilde{Y}_t = \sigma(\tilde{Y}_t)\ \mathrm{d}\mathbf{X}_t, \tag{2.2}$$

and if they exist, denote by $y_0 \mapsto U_{t\leftarrow 0}^{\mathbf{X};y_0}$ the flow of the solution (i.e. $\tilde{Y}_t = U_{t\leftarrow 0}^{\mathbf{X};y_0}$ when $\tilde{Y}_0 = y_0$), by $J_{t\leftarrow 0}^{\mathbf{X};y_0}$ its Jacobian and by $J_{0\leftarrow t}^{\mathbf{X};y_0}$ the inverse of the Jacobian.

Proposition 2.4 ([13]). *Assume that $b \in \mathcal{C}_b^1(\mathbb{R}^e;\mathbb{R}^e)$, $\sigma \in \mathcal{C}_b^4(\mathbb{R}^e;\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$ and that $\mathbf{X} \in \mathscr{C}_g^\beta\left([0,T];\mathbb{R}^d\right)$, with $\beta \in (\frac{1}{3}, \frac{1}{2})$. Then for any $y_0 \in \mathbb{R}^e$, there exists a unique solution Y to the RDE with drift (2.1) started from y_0. Moreover, this solution has the following Doss-Sussmann representation:*

$$\left\{ \begin{array}{ll} Y_t & = U_{t\leftarrow 0}^{\mathbf{X};Z_t} \\ Z_t & = y_0 + \int_0^t W(s, Z_s)\, ds \end{array} \right. ,\ t \in [0,T],$$

where

$$W(t,z) = J_{0\leftarrow t}^{\mathbf{X};z}\ b\left(U_{t\leftarrow 0}^{\mathbf{X};z}\right),\ (t,z) \in [0,T] \times \mathbb{R}^e. \tag{2.3}$$

2.3 Assumptions

We shall assume throughout the paper (except in the more general Proposition 2.10) that

$$\sigma \in \mathcal{C}_b^4(\mathbb{R}, (\mathbb{R}^d)'). \tag{2.4}$$

Since the penalisation term $n(\cdot)_-$ in (1.1) is not differentiable, we approximate it by a smooth non-increasing function ψ_n such that

$$\forall y \in \mathbb{R}, \quad \psi_n(y) = \begin{cases} 0 & \text{if } y > 0, \\ \text{smooth convex interpolation} & \text{if } -\frac{1}{n} < y \leq 0, \\ -\frac{1}{2} - ny & \text{if } y \leq -\frac{1}{n}. \end{cases} \tag{2.5}$$

In fact, for any $n \in \mathbb{N}$ it is possible to choose ψ_n as above and which also satisfies:

$$\begin{cases} \psi_n \in \mathcal{C}^\infty \text{ and } \psi_n' \in \mathcal{C}_b^\infty; \\ \forall y \in \mathbb{R}, \quad \psi_n(y) \leq \psi_{n+1}(y) \text{ and } -\frac{1}{2} + ny_- \leq \psi_n(y) \leq ny_-. \end{cases} \tag{2.6}$$

We assume that the driving signal is a geometric β-Hölder rough path, for some $\beta \in (\frac{1}{3}, \frac{1}{2}]$, i.e. $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}_g^\beta([0,T]; \mathbb{R}^d)$. The boundary process is assumed to have at least the same Hölder regularity as X, and further that

$$\widehat{X} := (X, L) \text{ can be enhanced into a geometric } \beta\text{-Hölder rough path } \widehat{\mathbf{X}} = (\widehat{X}, \widehat{\mathbb{X}}) \in \mathscr{C}_g^\beta([0,T]; \mathbb{R}^{d+1}). \tag{2.7}$$

In that case, we still denote by $\mathbb{X}$ the projection of $\widehat{\mathbb{X}}$ on the X component, and by $\mathbf{X} = (X, \mathbb{X})$ the associated rough path. Note that the previous assumption is not trivial in general because of the roughness $\beta \leq \frac{1}{2}$. In fact since we consider RDEs with drifts, we will also need (X, L, t) to be lifted into a geometric rough path. In that case, since the identity function of $\mathbb{R}$ is smooth, it is always possible to realise such a lift, in such a way that the projection on (X, L) coincides with $\widehat{\mathbf{X}}$ (see Young pairings [16, Section 9.4]). Observe that Young pairings can also be used to obtain (2.7), but then one has to assume more regularity on L, namely that $L \in \mathcal{V}^q$, with $q \geq 1$ such that $\frac{1}{q} + \frac{1}{p} > 1$ ($p = \beta^{-1}$).

With these notations and assumptions, we consider

$$Y_t^n = y_0 + \int_0^t \psi_n(Y_s^n - L_s)\mathrm{d}s + \int_0^t \sigma(Y_s^n)\mathrm{d}\mathbf{X}_s \ , \ t \in [0,T]. \tag{2.8}$$

For each $n \in \mathbb{N}$, we will see in Proposition 3.1 that there is a unique solution to (2.8).

2.4 Gaussian rough paths

In the case X is a Gaussian process, several papers give conditions (see in particular Cass, Hairer, Litterer, and Tindel [4]) for X to be enhanced into a geometric rough path. Cass, Litterer, and Lyons [3] also proved that such conditions yield that the Jacobian of the flow has moments of all order (see also [4] with a bounded drift).

Let $X = (X^1, \dots X^d)$ be a continuous, centred Gaussian process with independent and identically distributed components, then following Cass et al. [3]: Let $R(s,t) = \mathbb{E}\left(X_s^1 X_t^1\right)$ denote the covariance function of X^1, and

$$R\begin{pmatrix} s,t \\ u,v \end{pmatrix} = \mathbb{E}\left[(X_t^1 - X_s^1)(X_v^1 - X_u^1)\right]$$

be the rectangular increments of R. Then for $r \in [1, \frac{3}{2})$, we might assume that R has finite second-order r-variation in the sense

$$\|R\|_{r;[0,T]^2} := \left(\sup_{\substack{\pi=(t_i) \\ \pi'=(t_i')}} \sum_{\pi,\pi'} R\begin{pmatrix} t_i, t_{i+1} \\ t_i', t_{i+1}' \end{pmatrix}^r \right)^{\frac{1}{r}} < \infty \tag{H$_{\text{Cov}}$}$$

Under this assumption, X can almost surely be enhanced into a geometric rough path $\mathbf{X} = (X, \mathbb{X})$ and for any $\alpha \in (\frac{1}{3}, \frac{1}{2r})$, $\mathbf{X} \in \mathscr{C}_g^\alpha$. Moreover, this assumption permits to obtain upper bounds on the Jacobian of the flow of a Gaussian RDE, which shall help us to obtain better convergence results in Theorem 2.12.

2.5 Controlled rough paths

We choose to define controlled rough paths with respect to the p-variation topology. Roughly, this is because the compensator K and its approximations K^n are clearly in $\mathcal{V}^1$ while it seems much more difficult to prove that they have some Hölder regularity. Then, it becomes easier to use rough paths continuity results such as Theorem 2.6.

Definition 2.5 (Controlled rough path)**.** *Let $p \in [2,3)$ and $X \in \mathcal{V}^p([0,T];\mathbb{R}^d)$. A path $Y \in \mathcal{V}^p([0,T];E)$ is* controlled *by X if there exist a path $Y' \in \mathcal{V}^p([0,T];\mathcal{L}(\mathbb{R}^d,E))$ and a map $R^Y \in \mathcal{V}_2^{\frac{p}{2}}([0,T];E)$ such that*

$$\forall s \le t \in [0,T], \quad \delta Y_{s,t} = Y'_s \delta X_{s,t} + R^Y_{s,t}.$$

Y' is called the Gubinelli derivative (although not unique). The space of such couples of paths (Y,Y') controlled by X is denoted by $\mathcal{V}^p_X(E)$ ($\mathcal{C}^\beta_X(E)$ for a corresponding definition in β-Hölder norm, see [14, Definition 4.6]).

If now $\mathbf{X} \in \mathscr{V}^p([0,T];\mathbb{R}^d)$ and $(Y,Y') \in \mathcal{V}^p_X(\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$, then the rough integral of Y against $\mathbf{X}$ is defined by

$$\int_0^T Y_s \mathrm{d}\mathbf{X}_s = \lim_{n\to\infty} \sum_{\pi_n=(t_i)} Y_{t_i}\delta X_{t_i,t_{i+1}} + Y'_{t_i}\mathbb{X}_{t_i,t_{i+1}}, \tag{2.9}$$

where $(\pi_n)_{n\in\mathbb{N}}$ is an increasing sequence of subdivisions of $[0,T]$ with $t_0 = 0,\ t_n = T$, such that $\lim_{n\to\infty}\max(t_{i+1} - t_i) = 0$.

The existence of this integral has been established by Gubinelli [17] for the Hölder topology (see also [14, Proposition 4.10]). In the p-variation topology, we refer to Friz and Shekhar [15, Theorem 31]:

Theorem 2.6. *Let $p \in [2,3)$. If $\mathbf{X} \in \mathscr{V}^p([0,T];\mathbb{R}^d)$ and $(Y,Y') \in \mathcal{V}^p_X(\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$, then the rough integral of Y against $\mathbf{X}$ exists (and the limit in (2.9) does not depend on the choice of a subdivision). Moreover, for any $s,t \in [0,T]$,*

$$\left|\int_s^t Y_u\, d\mathbf{X}_u - Y_s\delta X_{s,t} - Y'_s\mathbb{X}_{s,t}\right| \le C_p\left(\|X\|_{p,[s,t]}\|R^Y\|_{\frac{p}{2},[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]}\|Y'\|_{p,[s,t]}\right). \tag{2.10}$$

Let us finally recall Proposition 2.12 of [5].

Proposition 2.7. *Let $p \in [2,3)$. Let $\mathbf{X} \in \mathscr{V}^p([0,T];\mathbb{R}^d)$ and assume that $(Y^n,(Y^n)')_{n\in\mathbb{N}} \subset \mathcal{V}^p_X(\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$ is a sequence such that:*

$$(Y^n)' \text{ and } R^{Y^n} \text{ converge in the uniform topology on } [0,T],$$

and

$$\sup_{n\in\mathbb{N}}\left(\|(Y^n)'\|_{p,[0,T]} + \|R^{Y^n}\|_{\frac{p}{2},[0,T]}\right) < \infty,$$

then $(Y^n,(Y^n)')$ converges uniformly to some $(Y,Y') \in \mathcal{V}^p_X$ and

$$\lim_{n\to\infty}\|\int_0^\cdot Y^n_s\, d\mathbf{X}_s - \int_0^\cdot Y_s\, d\mathbf{X}_s\|_{\infty,[0,T]} = 0.$$

2.6 The Skorokhod problem

Having at our disposal a rough integral in the sense of Equation (2.9), we can give a meaning to Equation (1.2), also referred to as Skorokhod problem associated to σ and L, denoted by $SP(\sigma,L)$.

Definition 2.8. *Let $\mathbf{X} \in \mathscr{V}^p([0,T];\mathbb{R}^d)$. We say that (Y,K) solves $SP(\sigma,L)$, or that it is a solution to the reflected RDE with diffusion coefficient σ started from $y_0 \ge L_0$ and reflected on the path L, if*

- *(i) $(Y,\sigma(Y)) \in \mathcal{V}^p_X$ and (Y,K) satisfies Equation (1.2), in the sense that both sides are equal, where the integral $\int_0^\cdot \sigma(Y_s)d\mathbf{X}_s$ is understood in the sense of (2.9);*

(ii) $\forall t \in [0,T],\ Y_t \geq L_t$;

(iii) K *is nondecreasing;*

(iv) $\forall t \in [0,T],\ \int_0^t (Y_s - L_s) dK_s = 0$, *or equivalently,* $\int_0^t \mathbf{1}_{\{Y_s \neq L_s\}} dK_s = 0$.

Remark 2.9. *In item (i), it is also possible to define solutions to reflected RDEs in the sense of Davie as in Deya et al. [8]. For RDEs with bounded coefficients (without reflection), Davie's solution and the solution in the sense of controlled rough paths coincide ([14, Proposition 8.8]).*

2.7 Main results

Hereafter, even if $\beta > \frac{1}{2}$, we use the notation $\mathbf{X} \in \mathscr{C}_g^\beta$ even though the iterated integral $\mathbb{X}$ is irrelevant in this case. This notation permits to present our results in a unified form.

Our first result states the global existence and uniqueness of solutions for RDEs with an unbounded drift which has at most linear growth. It is generally a difficult task to obtain global existence for RDEs when the vector fields are unbounded (which is the case of ψ_n), and known counter-examples show that global solutions may not exist in general. Nevertheless, for an RDE with coefficient $V = (V_1, \dots, V_d)$ on $\mathbb{R}^e$, where each V_i has components V_i^k, there are several results in this direction ([19, 20], [16, Exercise 10.56] and [2]) which ask very roughly for $V_i^k \nabla V_j^l$ to be bounded and Hölder continuous for all i, j, k, l. Observe that in our case (assuming $L \equiv 0$ for simplicity), the vector field V would be $V(y,t) = (\sigma(y),\ \psi_n(y))$ but that $\psi_n \sigma'$ is not bounded.

Proposition 2.10. *Let* $\sigma \in \mathcal{C}_b^4(\mathbb{R}^e, \mathcal{L}(\mathbb{R}^d;\mathbb{R}^e))$, $n \in \mathbb{N}$ *and* $b \in \mathcal{C}^1(\mathbb{R}^e, \mathbb{R}^e)$ *with* $\nabla b \in \mathcal{C}_b(\mathbb{R}^e, \mathbb{R}^{e\times e})$ *(i.e. b is not necessarily bounded). Let* $\beta \in (\frac{1}{3}, \frac{1}{2})$, *and let $\mathbf{X}$ be a d-dimensional β-Hölder geometric rough path. Then for any initial condition y_0, there exists a unique solution Y to the drifted RDE on $[0,T]$. Moreover, this solution is a path $Y \in \mathcal{C}^\beta$ which also solves:*

$$\begin{cases} Y_t &= U_{t\leftarrow 0}^{\mathbf{X};Z_t} \\ Z_t &= y_0 + \int_0^t W(s, Z_s)\ ds \end{cases},\ t \in [0,T],$$

where

$$W(t,z) = J_{0\leftarrow t}^{\mathbf{X};z}\ b\left(U_{t\leftarrow 0}^{\mathbf{X};z}\right),\ (t,z) \in [0,T] \times \mathbb{R}^e.$$

Since there is little difference between our proof and the original one of Friz and Oberhauser [13] (the difference is that b is bounded in [13]), it is postponed to the Appendix. The idea is to derive first the local existence and a Doss-Sussmann representation on a small time interval where the existence of the solution is known. Global existence is then achieved by stability of the ODE in the Doss-Sussmann representation.

Besides enabling us to prove the previous proposition, the Doss-Sussmann representation also has a monotonous property that will be very useful for the penalisation procedure. In particular, we will be able to deduce that there exists a path Y which is the non-decreasing limit of the sequence $(Y^n)_{n\in\mathbb{N}}$ and that this path is controlled by $\mathbf{X}$ (Proposition 4.10).
We are now in a position to state our first main theorem.

Theorem 2.11. *Let* $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}_g^\beta$ *be a geometric β-Hölder rough path,* $\beta \in (\frac{1}{3}, 1) \setminus \{\frac{1}{2}\}$. *Assume that* $\{\psi_n\}_{n\in\mathbb{N}}$, σ *and L satisfy conditions (2.4)-(2.7), and that* $y_0 \geq L_0$.

(i) Then the sequence $\left(Y^n, \int_0^\cdot \psi_n(Y_s^n - L_s) ds\right)_{n\in\mathbb{N}}$ *defined as the solution to (2.8) converges uniformly in* $C[0,T]$ *to some path* $(Y,K) \in \mathcal{C}_X^\beta \times \mathcal{V}^1$.

(ii) Besides, (Y,K) is the unique solution to the reflected RDE (1.2) (in the sense of Definition 2.8), i.e. it is the solution to the Skorokhod problem $SP(\sigma, L)$.

So far, the result only involved deterministic rough paths. Using some recent results on Gaussian rough paths leads to the following theorem.

Theorem 2.12. *Let σ and $\{\psi_n\}_{n\in\mathbb{N}}$ satisfy conditions (2.4)-(2.6) and let $y_0 \geq L_0$ almost surely. Let $X = (X^1, \dots, X^d)$ be a continuous, centred Gaussian process with independent and identically distributed components, and let R be its covariance function. Assume that either $X \in \mathcal{C}^\beta([0,T];\mathbb{R}^d)$ for some $\beta \in (\frac{1}{2}, 1)$, or that:*

- *R has finite second-order r-variations for some $r \in [1, \frac{3}{2})$, as in ($\mathrm{H_{Cov}}$);*
- *L satisfies almost surely condition (2.7) for any $\beta < \frac{1}{2r}$ and that $\mathbb{E}\left[\|L\|_\beta^\gamma\right] < \infty$, for any $\gamma \geq 1$.*

Then the conclusions of Theorem 2.11 hold in the almost sure sense and moreover, the convergence holds in the following sense: $\forall \gamma \geq 1$,

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t - Y_t^n|^\gamma\right] = 0. \tag{2.11}$$

In the "regular" case, we obtain a rate of convergence of the sequence of penalised processes to the reflected solution.

Theorem 2.13. *Let $X \in \mathcal{C}^\beta$ be a β-Hölder path, with $\beta \in (\frac{1}{2}, 1)$. Assume that $\{\psi_n\}_{n\in\mathbb{N}}$, σ and L satisfy conditions (2.4)-(2.7), and that $y_0 \geq L_0$. Then the solution Y^n converges to Y with the following rate: for any $\varepsilon \in (0, 2\beta - 1)$, there exists $C > 0$ (that depends only on ε, β and $\|X\|_p$ with $p = \beta^{-1}$) such that*

$$\sup_{t\in[0,T]} |Y_t^n - Y_t| \leq C\ n^{-(2\beta-1)+\varepsilon}.$$

Note that unless otherwise stated (for instance in the proof of Theorem 2.13), we will only consider the case $\beta \in (\frac{1}{3}, \frac{1}{2}]$. Indeed if $\beta \in (\frac{1}{2}, 1)$, Young integrals can be used which makes proofs easier.

3 Penalisation for RDEs

3.1 Flow of an RDE

In this paragraph, we gather several useful properties of the flow of the solution of an RDE, and of its Jacobian. Hereafter, $\left\{U_{t\leftarrow 0}^{\mathbf{X};y_0},\ t \in [0,T]\right\}$ denotes the solution to the RDE (2.2) with $\sigma \in \mathcal{C}_b^4(\mathbb{R}^e; \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$ (note that $\mathcal{C}_b^3$ is enough for existence and uniqueness in (2.2)).
First, we know that the smoothness of the flow depends on the smoothness of σ: for any t, $y_0 \mapsto U_{t\leftarrow 0}^{\mathbf{X};y_0}$ is Lipschitz continuous and twice differentiable (see for instance [13, Proposition 3]). Denote by $J_{t\leftarrow 0}^{\mathbf{X};y_0}$ its Jacobian matrix, which according to [3, Corollary 4.6] is uniformly (in $t \in [0,T]$ and $y_0 \in \mathbb{R}$) bounded by a quantity depending only on p, $\|\mathbf{X}\|_{p,[0,T]}$ and the so-called α-local p-variation of $\mathbf{X}$ (see [3, Definition 4.3]). We denote this upper bound by $C_J^{\mathbf{X}}$.
Denote also by $J_{0\leftarrow t}^{\mathbf{X};\cdot} := (J_{t\leftarrow 0}^{\mathbf{X};\cdot})^{-1}$ its inverse matrix, which can be seen as the Jacobian of the flow of the same RDE with X evolving backward. Hence as noticed in the proof of [4, Theorem 7.2], $J_{0\leftarrow t}^{\mathbf{X};\cdot}$ is also bounded by $C_J^{\mathbf{X}}$, so that altogether the following inequality is fulfilled:

$$\sup_{y_0\in\mathbb{R}} \max\left(\|J_{\cdot\leftarrow 0}^{\mathbf{X};y_0}\|_{\infty,[0,T]}, \|J_{0\leftarrow\cdot}^{\mathbf{X};y_0}\|_{\infty,[0,T]}\right) \leq C_J^{\mathbf{X}} < \infty. \tag{3.1}$$

Note also that with $\sigma \in \mathcal{C}_b^4$, $J_{0\leftarrow t}^{\mathbf{X};\cdot}$ and $J_{t\leftarrow 0}^{\mathbf{X};\cdot}$ are Lipschitz continuous, uniformly in t.
Besides, when X is Gaussian with iid components and satisfies ($\mathrm{H_{Cov}}$), $C_J^{\mathbf{X}}$ has moments of all orders ([3, Theorem 6.5] and [3, Theorem 7.2]). Note that b is not related to the definition of U and J, thus the above four inequalities depend only on σ.

As observed in [4, Section 7], $J_{0\leftarrow\cdot}^{\mathbf{X};z}$ satisfies the following linear RDE, for any fixed z:

$$dJ_{0\leftarrow t}^{\mathbf{X};z} = d\mathbf{M}_t J_{0\leftarrow t}^{\mathbf{X};z},$$

where $\mathbf{M}$ depends on the flow $U_{t\leftarrow 0}^{\mathbf{X};z}$. If $e = 1$ (e is the dimension of the space in which y lives), it is thus a consequence of the fact that $J_{0\leftarrow 0}^{\mathbf{X};z} = 1$ and of the uniqueness in the previous equation that $J_{0\leftarrow t}^{\mathbf{X};z} > 0$ for any $z \in \mathbb{R}$ and any $t \geq 0$. Hence it follows from Equation (3.1) that

$$\forall z \in \mathbb{R}, \quad J_{0\leftarrow t}^{\mathbf{X};z} \geq (C_J^{\mathbf{X}})^{-1} (> 0). \tag{3.2}$$

Finally, the mapping $W(t,z)$ defined in (2.3) is continuous in t, Lipschitz continuous in z uniformly in t if b is bounded (this is however not true anymore if b is unbounded). This ensures that there is a unique solution to

$$z'_t = W(t, z_t),\ z_0 = y_0.$$

3.2 Existence of a global solution to (2.8)

The result below states the global existence of a solution to the rough differential equation (2.8). Due to the boundary term in (2.8), we cannot apply directly Proposition 2.10. However, provided that (2.8) can be cast into a proper RDE with drift using Assumption (2.7), then the result will hold.

Proposition 3.1. *Let $\sigma \in \mathcal{C}_b^4(\mathbb{R}; (\mathbb{R}^d)')$, $n \in \mathbb{N}$ and ψ_n satisfying (2.5)-(2.6). Let $\beta \in (\frac{1}{3}, \frac{1}{2})$, let $\mathbf{X}$ be a β-Hölder geometric rough path and let $\{L_t\}_{t\in[0,T]}$ be a barrier process satisfying (2.7). Then for any initial condition y_0 such that $y_0 \geq L_0$, there exists a unique solution to (2.8). Moreover, this solution is a path $\{Y_s^n\}_{s\in[0,T]} \in \mathcal{C}^\beta$ which also solves:*

$$\begin{cases} Y_t^n &= U_{t\leftarrow 0}^{\mathbf{X};Z_t^n} \\ Z_t^n &= y_0 + \int_0^t W^n(s, Z_s^n)\ ds \end{cases},\ t \in [0,T], \tag{3.3}$$

where

$$W^n(t,z) = J_{0\leftarrow t}^{\mathbf{X};z}\ \psi_n\left(U_{t\leftarrow 0}^{\mathbf{X};z} - L_t\right),\ (t,z) \in [0,T] \times \mathbb{R}.$$

Remark 3.2.
- *For $\beta > \frac{1}{2}$, the previous Doss-Sussmann representation holds also true by a simple application of the usual chain rule. Moreover, our assumptions on the coefficients meet those from [18] and thus there exists a unique solution to (2.8).*
- *If the dimension of the noise d equals 1, then the usual Doss-Sussmann representation [9] can be used.*

Proof. For $y \in \mathbb{R}^2$, define $\widehat{b}(y) = (\psi_n(y^1 - y^2), 0)^T$, where we used the notation $y = (y^1, y^2)^T \in \mathbb{R}^2$. In the same way, define $\widehat{\sigma}(y) = \begin{pmatrix} \sigma(y^1) & 0 \\ 0 & 1 \end{pmatrix}$, so that $\widehat{\sigma} \in \mathcal{C}_b^4\left(\mathbb{R}^2; \mathcal{L}(\mathbb{R}^{d+1}, \mathbb{R}^2)\right)$. Finally, let $\widehat{\mathbf{X}} \in \mathscr{C}_g^\beta$ be the rough path above (X, L), as in (2.7). Proposition 2.10 ensures that there exists a unique solution $\widehat{Y}^n \in \mathcal{C}^\beta([0,T]; \mathbb{R}^2)$ to the following RDE with drift

$$\mathrm{d}\widehat{Y}_t^n = \widehat{b}(\widehat{Y}_t^n)\mathrm{d}t + \widehat{\sigma}(\widehat{Y}_t^n)\mathrm{d}\widehat{\mathbf{X}}_t.$$

Since Y^n corresponds to the first component of $\widehat{Y}^n$, the result follows. □

3.3 Penalisation estimates

We will rely on the previous Doss-Sussmann representation and the comparison theorem for ODEs to prove that $(Y^n)_{n\in\mathbb{N}}$ is a bounded nondecreasing sequence of continuous processes.

Lemma 3.3. *Let $\Psi > 0$, $\ell, \{g^n\}_{n\in\mathbb{N}}$ be continuous functions such that $g_0^n = 0$, and assume that for each $n \in \mathbb{N}$, f^n is a solution to:*

$$\begin{cases} f_t^n = f_0^n + g_t^n + \Psi \int_0^t \psi_n(f_u^n - \ell_u)du, \quad \forall t \in [0,T], \\ f_0^n = f_0 \geq \ell_0. \end{cases}$$

Then,

(i) For all $t \in [0,T]$,

$$\forall n \in \mathbb{N},\ |\delta f_{0,t}^n - \delta \ell_{0,t}| \leq \sqrt{26}\|g^n - \delta \ell_{0,\cdot}\|_{\infty,[0,t]}\ ;$$

(ii) Let $\beta \in (0,1)$. If $\ell, \{g^n\}_{n\in\mathbb{N}} \in \mathcal{C}^\beta([0,T], \mathbb{R})$ and $f_0^n \geq \ell_0$, then

$$\forall t \in [0,T],\ \forall n \in \mathbb{N}, \quad \psi_n(f_t^n - \ell_t) \leq \overline{\Psi}_n(\Psi^{-\beta} + \Psi^{1-\beta})n^{1-\beta},$$

where $\overline{\Psi}_n = C(\|\ell\|_\beta + \|g^n\|_\beta + \frac{1}{2}\Psi T^{1-\beta})$.

Proof. (i) Denoting $\Psi \int_0^t \psi_n(f_u^n - \ell_u)\mathrm{d}u$ by k_t^n, let $\overline{f}^n$ and $\overline{g}^n$ be defined as follows:

$$\begin{aligned}\overline{f}_t^n &:= \delta f_{0,t}^n - \delta \ell_{0,t} = -\delta \ell_{0,t} + g_t^n + \Psi \int_0^t \psi_n(f_u^n - \ell_u)\mathrm{d}u \\ &=: \overline{g}_t^n + k_t^n.\end{aligned}$$

Observe that

$$\begin{aligned}(\overline{f}_t^n)^2 = (\overline{g}_t^n)^2 + (k_t^n)^2 + 2\int_0^t \overline{g}_t^n \mathrm{d}k_u^n &= (\overline{g}_t^n)^2 + 2\int_0^t (k_u^n + \overline{g}_t^n)\, \mathrm{d}k_u^n \\ &\le (\overline{g}_t^n)^2 + 2\int_0^t (\overline{g}_t^n - \overline{g}_u^n)\, \mathrm{d}k_u^n\end{aligned}$$

where we used the inequality $\overline{f}_u^n \psi_n(f_u^n - \ell_u) \le (f_u^n - \ell_u)\psi_n(f_u^n - \ell_u) \le 0$. It follows that

$$\begin{aligned}(\overline{f}_t^n)^2 \le (\overline{g}_t^n)^2 + 2k_t^n \|\overline{g}_t^n - \overline{g}_\cdot^n\|_{\infty,[0,t]} &\le (\overline{g}_t^n)^2 + 2(|\overline{f}_t^n| + |\overline{g}_t^n|)\|\overline{g}_t^n - \overline{g}_\cdot^n\|_{\infty,[0,t]} \\ &\le 5\|\overline{g}_\cdot^n\|_{\infty,[0,t]}^2 + 4|\overline{f}_t^n|\|\overline{g}_\cdot^n\|_{\infty,[0,t]} \\ &\le 5\|\overline{g}_\cdot^n\|_{\infty,[0,t]}^2 + \tfrac{1}{2}\left(|\overline{f}_t^n|^2 + 16\|\overline{g}_\cdot^n\|_{\infty,[0,t]}^2\right),\end{aligned}$$

which implies the result.

(ii) The inequality $\psi_n(x) \ge -\frac{1}{2} - nx$ yields

$$f_t^n - \ell_t \ge f_0^n - \ell_0 + \overline{g}_t^n - \tfrac{1}{2}\Psi t - n\Psi \int_0^t (f_u^n - \ell_u)\mathrm{d}u.$$

Denote $\widetilde{g}_t^n := \overline{g}_t^n - \frac{1}{2}\Psi t$ and $\widetilde{f}^n$ the solution to

$$\widetilde{f}_t^n - \ell_t = f_0^n - \ell_0 + \widetilde{g}_t^n - n\Psi \int_0^t (\widetilde{f}_u^n - \ell_u)\mathrm{d}u. \tag{3.4}$$

It follows from the comparison principle of ODEs that for any $t \in [0,T]$, $f_t^n - \ell_t \ge \widetilde{f}_t^n - \ell_t$. Solving (3.4) yields

$$\begin{aligned}f_t^n - \ell_t &\ge (f_0^n - \ell_0)e^{-n\Psi t} - \int_0^t e^{-n\Psi(t-u)}\mathrm{d}\widetilde{g}_u^n \\ &\ge (f_0^n - \ell_0)e^{-n\Psi t} + \widetilde{g}_t^n e^{-n\Psi t} + n\Psi \int_0^t e^{-n\Psi(t-u)}(\widetilde{g}_u^n - \widetilde{g}_t^n)\mathrm{d}u\ , \quad t \in [0,T].\end{aligned} \tag{3.5}$$

Since $\psi_n(x) \le nx_-$, we now obtain from (3.5) that

$$\begin{aligned}\psi_n(f_t^n - \ell_t) &\le n\left(\widetilde{g}_t^n e^{-n\Psi t} + n\Psi \int_0^t e^{-n\Psi(t-u)}(\widetilde{g}_u^n - \widetilde{g}_t^n)\mathrm{d}u\right)_- \\ &\le n\|\widetilde{g}^n\|_\beta t^\beta e^{-n\Psi t} + \|\widetilde{g}^n\|_\beta n^2\Psi \int_0^t e^{-n\Psi(t-u)}(t-u)^\beta \mathrm{d}u\ , \quad t \in [0,T].\end{aligned}$$

It is clear that $n\|\widetilde{g}^n\|_\beta t^\beta e^{-n\Psi t} \le \|\widetilde{g}^n\|_\beta \Psi^{-\beta} n^{1-\beta}$. Thus one focuses now on the second term: an integration-by-parts and the change of variables $v = n\Psi u$ yield

$$\begin{aligned}n^2\Psi \int_0^t e^{-n\Psi(t-u)}(t-u)^\beta \mathrm{d}u &= -nt^\beta e^{-n\Psi t} + \beta n \int_0^t e^{-n\Psi u} u^{\beta-1}\mathrm{d}u \\ &= -nt^\beta e^{-n\Psi t} + \beta n^{1-\beta}\Psi^{1-\beta} \int_0^{n\Psi t} v^{\beta-1} e^{-v}\mathrm{d}v \\ &\le C n^{1-\beta}\Psi^{1-\beta}.\end{aligned} \tag{3.6}$$

□

4 Existence of a solution to the Skorokhod problem

4.1 Existence of the limit process

We use first comparison theorems and the Doss-Sussmann representation (3.3) to apply Lemma 3.3 and get the following result, which implies the existence of paths Z and Y as pointwise limits of (Z^n) and (Y^n).

Proposition 4.1. *(i) Let the notations and assumptions of Theorem 2.11 be in force. Then the sequences of paths $(Z^n)_{n\in\mathbb{N}}$ and $(Y^n)_{n\in\mathbb{N}}$ defined in (3.3) are nondecreasing with n. Besides, the following inequalities are satisfied:*

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}|Z_t^n|<+\infty \quad \textit{and} \quad \sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}|Y_t^n|<+\infty.$$

(ii) Now let the assumptions of Theorem 2.12 be in force. Then the previous conclusions hold in the almost sure sense and moreover, for any $\gamma\geq 1$,

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}|Z_t^n|^\gamma\right]<+\infty \quad \textit{and} \quad \mathbb{E}\left[\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}|Y_t^n|^\gamma\right]<+\infty.$$

Proof. (i) For each $n\in\mathbb{N}$, recall from (3.3) that Z^n is the solution of a (random) ODE with coefficient $W^n(t,z)=J^{\mathbf{X};z}_{0\leftarrow t}\psi_n(U^{\mathbf{X};z}_{t\leftarrow 0}-L_t)$. In view of (3.2) and the fact that $\psi_n\leq\psi_{n+1}$, it follows from the comparison theorem for ODEs that $Z^n\leq Z^{n+1}$. Besides, the mapping $z\mapsto U^{\mathbf{X};z}_{t\leftarrow 0}$ is increasing since its derivative is $J^{\mathbf{X};z}_{t\leftarrow 0}$ which, similarly to (3.2), is positive. Hence $Y^n\leq Y^{n+1}$ a.s.

To prove the boundedness of Z^n and Y^n, define $\widetilde{Z}^n$ as the solution of the following (random) ODE:

$$\widetilde{Z}^n_t=y_0+C^{\mathbf{X}}_J\int_0^t\psi_n\left(U^{\mathbf{X};\widetilde{Z}^n_s}_{s\leftarrow 0}-L_s\right)\mathrm{d}s,\ t\in[0,T],$$

which in view of the bound (3.1) and the comparison principle yields $\widetilde{Z}^n_t\geq Z^n_t$. Observing that

$$\begin{aligned}U^{\mathbf{X};\widetilde{Z}^n_s}_{s\leftarrow 0}&=U^{\mathbf{X};y_0}_{s\leftarrow 0}+U^{\mathbf{X};\widetilde{Z}^n_s}_{s\leftarrow 0}-U^{\mathbf{X};y_0}_{s\leftarrow 0}\\&=U^{\mathbf{X};y_0}_{s\leftarrow 0}+\int_{y_0}^{\widetilde{Z}^n_s}J^{\mathbf{X};z}_{s\leftarrow 0}\mathrm{d}s\\&\geq U^{\mathbf{X};y_0}_{s\leftarrow 0}+(C^{\mathbf{X}}_J)^{-1}(\widetilde{Z}^n_s-y_0),\end{aligned}$$

where the last inequality follows from (3.2), it comes that

$$\widetilde{Z}^n_t\leq y_0+C^{\mathbf{X}}_J\int_0^t\psi_n\left(U^{\mathbf{X};y_0}_{s\leftarrow 0}+(C^{\mathbf{X}}_J)^{-1}(\widetilde{Z}^n_s-y_0)-L_s\right)\mathrm{d}s,\ t\in[0,T].$$

Note that as the solution of an RDE, $U^{\mathbf{X};y_0}_{s\leftarrow 0}$ satisfies (see [14, Proposition 8.3]):

$$\|U^{\mathbf{X};y_0}_{\cdot\leftarrow 0}\|_{\beta,[0,T]}\leq C\left\{\left(\|\sigma\|_{\mathcal{C}^2_b}\|\mathbf{X}\|_{\beta,[0,T]}\right)\vee\left(\|\sigma\|_{\mathcal{C}^2_b}\|\mathbf{X}\|_{\beta,[0,T]}\right)^{\frac{1}{\beta}}\right\},\tag{4.1}$$

where C depends only on β. Hence, denoting temporarily by $C_{\sigma,\mathbf{X},\beta}$ the right-hand side of the previous inequality, and since $y_0\geq L_0$,

$$\widetilde{Z}^n_t\leq y_0+C^{\mathbf{X}}_J\int_0^t\psi_n\left(-C_{\sigma,\mathbf{X},\beta}s^\beta+(C^{\mathbf{X}}_J)^{-1}(\widetilde{Z}^n_s-y_0)-(L_s-L_0)\right)\mathrm{d}s,\ t\in[0,T],\tag{4.2}$$

so that the process $\overline{Z}^n$ which is the solution to the ODE

$$\overline{Z}^n_t=-C_{\sigma,\mathbf{X},\beta}t^\beta+\int_0^t\psi_n\left(\overline{Z}^n_s-(L_s-L_0)\right)\mathrm{d}s,\ t\in[0,T]$$

satisfies $-C_{\sigma,\mathbf{X},\beta}t^\beta + (C_J^{\mathbf{X}})^{-1}(\widetilde{Z}_t^n - y_0) \leq \overline{Z}_t^n$, $\forall t \in [0,T]$ (by the comparison principle of ODEs). By Lemma 3.3, $\overline{Z}^n$ satisfies:

$$|\overline{Z}_t^n - (L_t - L_0)| \leq \sqrt{26}\left(C_{\sigma,\mathbf{X},\beta}t^\beta + \sup_{s\in[0,t]}|L_s - L_0|\right),$$

which then leads to the following bound: there exists $C > 0$ which depends only on σ, β, T such that

$$Z_t^n \leq y_0 + C\ C_J^{\mathbf{X}}\ \left((\|\mathbf{X}\|_{\beta,[0,T]} \vee \|\mathbf{X}\|_{\beta,[0,T]}^{\frac{1}{\beta}})t^\beta + \sup_{s\in[0,t]}|L_s - L_0|\right). \tag{4.3}$$

Moreover, $Z^n \geq y_0$, hence $\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}|Z_t^n| < +\infty$. To prove the second part of claim *(i)*, observe that

$$\begin{aligned}|Y_t^n| &= |U_{t\leftarrow 0}^{\mathbf{X};Z_t^n}| = |U_{t\leftarrow 0}^{\mathbf{X};y_0} + U_{t\leftarrow 0}^{\mathbf{X};Z_t^n} - U_{t\leftarrow 0}^{\mathbf{X};y_0}| \\ &\leq |y_0| + \|U_{\cdot\leftarrow 0}^{\mathbf{X};y_0}\|_{\beta,[0,t]}t^\beta + C_J^{\mathbf{X}}|Z_t^n - y_0|. \end{aligned} \tag{4.4}$$

Claim *(i)* then follows from (4.1) and (4.3).

(ii) Now if X is a Gaussian process satisfying the assumptions of Theorem 2.12, it suffices to use the deterministic estimates (4.1), (4.3) and (4.4), as well as the following probabilistic estimates: for any $\gamma \geq 1$,

$$\mathbb{E}\left[\|\mathbf{X}\|_{\beta,[0,T]}^\gamma\right] < \infty, \quad \mathbb{E}\left[(C_J^{\mathbf{X}})^\gamma\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\|L\|_{\beta,[0,T]}^\gamma\right] < \infty \tag{4.5}$$

where the first bound is a classical consequence of Kolmogorov's continuity theorem (which follows from ($\mathrm{H}_{\mathrm{Cov}}$) for any $\beta < \frac{1}{2r}$), the second one is [3, Theorem 6.5] and the third one was an assumption in Theorem 2.12. Then Claim *(ii)* holds true. ☐

Remark 4.2. *Observe that in the previous proof, we carefully avoided to estimate directly the Hölder regularity of* $t \mapsto \int_0^t \sigma(U_{s\leftarrow 0}^{\mathbf{X};Z_s^n})d\mathbf{X}_s$, *since any basic* a priori *estimate would have depended on* n. *However, we will be able to treat such questions in the next section.*

4.2 Uniform continuity of the sequence of penalised processes

So far we only obtained pointwise convergence of the sequences of paths. Now, we obtain uniform convergence and derive Hölder continuity of the limiting paths. Thus this section is organised as follows: Lemmas 4.3 to 4.6 are technical results which permit to overcome the main difficulty here (Proposition 4.8), which is that the negative part of $Y^n - L$ converges to 0 as $n \to \infty$. Finally, we prove that this implies the desired uniform convergence of Y^n and Z^n (Proposition 4.9).

Define the mapping

$$\kappa_{\mathbf{X},Z}(s,t) := C(\|\mathbf{X}\|_{p,[s,t]} \vee \|\mathbf{X}\|_{p,[s,t]}^p) + C_J^{\mathbf{X}}(Z_t - Z_s)$$

for any $s < t \in [0,T]$, for some constant $C > 0$ which depends only on p. Define similarly $\kappa_{\mathbf{X},Z^n}$ with Z^n instead of Z in the previous expression.

Lemma 4.3. *(i) For any* $n \in \mathbb{N}$, $\kappa_{\mathbf{X},Z^n}$ *is a control in* p*-variation norm for* Y^n, *i.e. for any* $s < t \in [0,T]$,

$$\|Y^n\|_{p,[s,t]} \leq \kappa_{\mathbf{X},Z^n}(s,t).$$

Besides, $\kappa_{\mathbf{X},Z}$ *is a control in* p*-variation norm for* Y, *i.e.* $\forall s < t \in [0,T]$,

$$\|Y\|_{p,[s,t]} \leq \kappa_{\mathbf{X},Z}(s,t).$$

(ii) Moreover, if X *and* L *satisfy the assumptions of Theorem 2.12, then for any* $\gamma \geq 1$ *and for any* $n \in \mathbb{N}$, $\mathbb{E}\left[\kappa_{\mathbf{X},Z^n}(0,T)^\gamma\right] < \infty$ *and* $\mathbb{E}\left[\kappa_{\mathbf{X},Z}(0,T)^\gamma\right] < \infty$.

Proof. (*i*) For each $n \in \mathbb{N}$, Z^n is a non-decreasing path and invoking Proposition 4.1, it follows that $\sup_{n\in\mathbb{N}} \|Z^n\|_{1\text{-var},[0,T]} < \infty$ and thus $\sup_{n\in\mathbb{N}} \|Z^n\|_{p,[0,T]} < \infty$ (note that this bound depends only on $\mathbf{X}$ and L, see (4.3)). Moreover, since Z^n is a non-decreasing path,

$$\begin{aligned}\|Z^n\|^p_{p,[s,t]} &\le \sup_{\pi} \left\{ \sup_{t_i\in\pi} |\delta Z^n_{t_i,t_{i+1}}|^{p-1} \sum |\delta Z^n_{t_i,t_{i+1}}| \right\} \\ &= \sup_{\pi} \left\{ |\delta Z^n_{s,t}|^{p-1} \sum |\delta Z^n_{t_i,t_{i+1}}| \right\} = \delta Z^n_{s,t}.\end{aligned}$$

Hence it follows that

$$\limsup_{n\in\mathbb{N}} \|Z^n\|^p_{p,[s,t]} \le \limsup_{n\in\mathbb{N}} (Z^n_t - Z^n_s)^p = (Z_t - Z_s)^p. \tag{4.6}$$

As a consequence, using the Doss-Sussmann representation (3.3), one obtains

$$\begin{aligned}\|Y^n\|_{p,[s,t]} &= \sup_{\pi} \left(\sum_{\pi} |U^{\mathbf{X};Z^n_{t_{i+1}}}_{t_{i+1}\leftarrow 0} - U^{\mathbf{X};Z^n_{t_i}}_{t_i\leftarrow 0}|^p \right)^{\frac{1}{p}} \\ &\le \sup_{\pi} \left(\sum_{\pi} \|U^{\mathbf{X};Z^n_{t_{i+1}}}_{\cdot\leftarrow 0}\|^p_{p,[t_i,t_{i+1}]} \right)^{\frac{1}{p}} + \sup_{u\in[s,t]} \|U^{\mathbf{X};\cdot}_{u\leftarrow 0}\|_{\text{Lip}} \|Z^n\|_{p,[s,t]} \\ &\le \sup_{\pi} \left(\sum_{\pi} \|U^{\mathbf{X};Z^n_{t_{i+1}}}_{\cdot\leftarrow 0}\|^p_{p,[t_i,t_{i+1}]} \right)^{\frac{1}{p}} + C^{\mathbf{X}}_J (Z^n_t - Z^n_s).\end{aligned}$$

Since $(s,t) \mapsto \|U^{\mathbf{X};Z^n_{t_{i+1}}}_{\cdot\leftarrow 0}\|^p_{p,[s,t]}$ is super-additive, i.e. for any $s \le t \le u$, $\|U^{\mathbf{X};Z^n_{t_{i+1}}}_{\cdot\leftarrow 0}\|^p_{p,[s,t]} + \|U^{\mathbf{X};Z^n_{t_{i+1}}}_{\cdot\leftarrow 0}\|^p_{p,[t,u]} \le \|U^{\mathbf{X};Z^n_{t_{i+1}}}_{\cdot\leftarrow 0}\|^p_{p,[s,u]}$, we deduce that

$$\|Y^n\|_{p,[s,t]} \le \|U^{\mathbf{X};Z^n_{t_{i+1}}}_{\cdot\leftarrow 0}\|_{p,[s,t]} + C^{\mathbf{X}}_J (Z^n_t - Z^n_s).$$

Now the standard bound (4.1) also holds in p-variation norm (see [14, Exercise 10.20]), hence

$$\|Y^n\|_{p,[s,t]} \le C \left(\|\sigma\|_{\mathcal{C}^2_b} \|\mathbf{X}\|_{p,[s,t]} \vee (\|\sigma\|_{\mathcal{C}^2_b} \|\mathbf{X}\|_{p,[s,t]})^p \right) + C^{\mathbf{X}}_J (Z^n_t - Z^n_s).$$

Finally, noticing that for any finite subdivision π of $[s,t]$,

$$\begin{aligned}\sum_{\pi} (\delta Y_{t_i,t_{i+1}})^p &= \lim_{n\to\infty} \sum_{\pi} (\delta Y^n_{t_i,t_{i+1}})^p \\ &\le \lim_{n\to\infty} \|Y^n\|^p_{p,[s,t]},\end{aligned}$$

gives the desired result.

(*ii*) The result can be directly deduced from the definition of $\kappa_{\mathbf{X},Z}$ (resp. $\kappa_{\mathbf{X},Z^n}$) and from the combination of inequalities (4.3) and (4.5). □

Let us denote K^n the penalisation term in (2.8):

$$K^n_t := \int_0^t \psi_n(Y^n_s - L_s)\mathrm{d}s, \ t \in [0,T]. \tag{4.7}$$

The p-variations of K^n are controlled by those of Z^n:

Lemma 4.4. *Consider the continuous process K^n as defined in (4.7). Then for any $n \in \mathbb{N}$ and any $s < t \in [0,T]$,*

$$\delta K^n_{s,t} \le C^{\mathbf{X}}_J \delta Z^n_{s,t}.$$

This implies that for any $q \ge 1$, $\|K^n\|_{q,[s,t]} \le C^{\mathbf{X}}_J \|Z^n\|_{q,[s,t]}$.

Proof. Using the definition (3.3) of Z^n and the bound (3.1) on J (recall also that J is positive), one has:

$$\begin{aligned}K_t^n - K_s^n &= \int_s^t J_{u\leftarrow 0}^{\mathbf{X};Z_u^n} J_{0\leftarrow u}^{\mathbf{X};Z_u^n} \psi_n(Y_u^n - L_u)\mathrm{d}u \\ &\leq C_J^{\mathbf{X}} \int_s^t J_{0\leftarrow u}^{\mathbf{X};Z_u^n} \psi_n(Y_u^n - L_u)\mathrm{d}u = C_J^{\mathbf{X}} \left(Z_t^n - Z_s^n\right).\end{aligned}$$

Since K^n is non-decreasing, the result is a direct consequence of the previous inequality. □

Lemma 4.5. *The path $\sigma(Y^n)$ is controlled by X and its Gubinelli derivative is $\sigma'(Y^n)\sigma(Y^n)$ ($\sigma'(y)$ considered as an element of $\mathcal{L}(\mathcal{L}(\mathbb{R}^d, \mathbb{R}), \mathbb{R})$).*

Proof. Recall that Y^n is a solution obtained by approximation, i.e. in the sense of Definition 2.3. The first goal of this proof is to show that Y^n can also be understood as a solution in the sense of controlled rough paths.

We know from Proposition 3.1 that Y^n does not blow up in finite time. Let ω be fixed, denote $M(\omega) = \sup_{t\in[0,T]} |Y_t^n(\omega)|$ and consider a bounded smooth function $\psi_n^{(M)}$ equal to ψ_n on the ball $B(0, 2M(\omega))$. We denote by $Y^{n,M}$ the solution to (2.8) with $\psi_n^{(M)}$ instead of ψ_n. We have $Y_t^{n,M}(\omega) = Y_t^n(\omega)$ for all $t \in [0,T]$.

Now consider the following RDE in the augmented form (i.e. without drift) with bounded coefficient:

$$\mathrm{d}\widehat{Y}_t^{n,M} = \widehat{\sigma}(\widehat{Y}_t^{n,M})\mathrm{d}\widehat{\mathbf{X}}_t,$$

where $\widehat{Y}_t^{n,M} = (Y_t^{n,M}, L_t)^T$ is the solution in the sense of Definition 2.3, where for any $(y,l) \in \mathbb{R}^2$:

$$\widehat{\sigma}((y,l)) = \begin{pmatrix} \sigma(y) & \psi_n^{(M)}(y-l) & 0 \\ 0\dots 0 & 0 & 1 \end{pmatrix},$$

and $\widehat{\mathbf{X}}$ is the canonical rough path above $\widehat{X}_t = (X_t^T, t, L_t)^T$. It appears that this equation can now be solved in the sense of controlled rough paths (see [14, Theorem 8.4]), and that the solution that we denote by $\overline{Y}^{n,M}$ is controlled by $\widehat{X}$ (with Gubinelli derivative $\widehat{\sigma}(\overline{Y}^{n,M})$). Considering that $\widehat{\mathbf{X}}$ can be approximated by a sequence $((\widehat{X}^k, \widehat{\mathbb{X}}^k))_{k\in\mathbb{N}}$ of smooth paths in the α-Hölder rough path topology, with $\alpha < \beta$ and $\widehat{\mathbb{X}}^k$ the Riemann-Stieltjes iterated integral of $\widehat{X}^k$ (see [14, Proposition 2.5]), we can associate a unique solution $\widehat{Y}^{n,M,k}$ to the equation $\mathrm{d}\widehat{Y}_t^{n,M,k} = \widehat{\sigma}(\widehat{Y}_t^{n,M,k})\mathrm{d}\widehat{X}_t^k$, for each $k \in \mathbb{N}$. In view of the continuity of the Itô-Lyons map $\mathbf{X} \in \mathscr{C}^\beta \mapsto Y \in \mathcal{C}_X^\alpha$, where Y is the solution in the controlled rough paths sense ([14, Theorem 8.5]), we obtain that $\widehat{Y}^{n,M,k}$ converges in α-Hölder norm to $\overline{Y}^{n,M}$. Since $\widehat{Y}^{n,M,k}$ is in fact a solution in the usual sense of ODEs, it also converges in the uniform topology to $\widehat{Y}^{n,M}$. Hence $\widehat{Y}^{n,M} = \overline{Y}^{n,M}$, and as noticed in the first paragraph, $Y^{n,M} = Y^n$, so that the two notions of solution coincide.

In particular, $(\widehat{Y}^{n,M}, \widehat{\sigma}(\widehat{Y}^{n,M})) \in \mathcal{C}_{\widehat{X}}^\beta$. This immediately yields that $\left(Y^n, (\sigma(Y^n), \psi_n(Y^n - L))^T\right)$ is controlled by the paths $X_t^a := (X_t^T, t)^T$, which in other words states that the mapping $Q_{s,t}^n := \delta Y_{s,t}^n - (\sigma(Y_s^n), \psi_n(Y_s^n - L_s))\delta X_{s,t}^a$ satisfies $\|Q^n\|_{\frac{p}{2},[0,T]} < \infty$. Hence one deduces that $(Y^n, \sigma(Y^n)) \in \mathcal{C}_X^\beta$: Indeed, setting for any $s < t \in [0,T]$,

$$R_{s,t}^n := \delta Y_{s,t}^n - \sigma(Y_s^n)\delta X_{s,t}, \tag{4.8}$$

one gets that

$$\begin{aligned}|R_{s,t}^n| &= |Q_{s,t}^n + \psi_n(Y_s^n - L_s)(t-s)| \\ &\leq |Q_{s,t}^n| + \|\psi_n(Y^n - L)\|_{\infty,[0,T]}(t-s) \leq |Q_{s,t}^n| + n\|Y^n - L\|_{\infty,[0,T]}(t-s),\end{aligned}$$

Thus it follows from the above and Proposition 4.1 that

$$\|R^n\|_{\frac{p}{2},[0,T]} \leq C\left(\|Q^n\|_{\frac{p}{2},[0,T]} + n\|Y^n - L\|_{\infty,[0,T]}T\right) < \infty,$$

and in particular the Gubinelli derivative of Y^n is $\sigma(Y^n) \in \mathcal{V}^p([0,T]; (\mathbb{R}^d)')$.

Finally, $\sigma(Y^n)$ is controlled by X with Gubinelli derivative $\sigma'(Y^n)\sigma(Y^n)$ (see [14, Lemma 7.3]). □

That $\sigma(Y^n)$ is controlled by X means that the following quantity is bounded in $\frac{p}{2}$-var, for each $n \in \mathbb{N}$:

$$R^{\sigma,n}_{s,t} := \delta\sigma(Y^n)_{s,t} - \sigma'(Y^n_s)\sigma(Y^n_s)\delta X_{s,t}. \tag{4.9}$$

Lemma 4.6. (i) *Under the assumptions of Theorem 2.11, one has the following inequality:*

$$\Theta := \sup_{n\in\mathbb{N}} \left(\|\sigma(Y^n)^T\sigma'(Y^n)\|_{p,[0,T]} + \|R^{\sigma,n}\|_{\frac{p}{2},[0,T]} \right) < \infty. \tag{4.10}$$

(ii) *If in addition, the assumptions of Theorem 2.12 hold, then* $\mathbb{E}\left(\Theta^\gamma\right) < \infty$, *for any* $\gamma \geq 1$.

Proof. (i) First, observe that by the regularity assumption on σ and by Lemma 4.3, the first part of the inequality is fulfilled: $\sup_{n\in\mathbb{N}} \|\sigma(Y^n)^T\sigma'(Y^n)\|_{p,[0,T]} < \infty$.

Now consider $R^{\sigma,n}$. By a Taylor expansion,

$$\delta\sigma(Y^n)_{s,t} = \sigma'(Y^n_s)\delta Y^n_{s,t} + \int_{Y^n_s}^{Y^n_t} \sigma''(y)(Y^n_t - y)\mathrm{d}y. \tag{4.11}$$

The combination of (4.8) and (4.11) yields

$$R^{\sigma,n}_{s,t} = \sigma'(Y^n_s)R^n_{s,t} + \int_{Y^n_s}^{Y^n_t} \sigma''(y)(Y^n_t - y)\mathrm{d}y.$$

Hence

$$|R^{\sigma,n}_{s,t}| \leq \|\sigma'\|_\infty |R^n_{s,t}| + \|\sigma''\|_\infty (Y^n_t - Y^n_s)^2 \leq C\left(|R^n_{s,t}| + (Y^n_t - Y^n_s)^2\right).$$

From Lemma 4.5, the Definition (4.8) of R^n, and inequality (2.10) applied to $|\int_s^t \sigma(Y^n_u)\mathrm{d}\mathbf{X}_u - \sigma(Y^n_s)\delta X_{s,t}|$, one gets

$$|R^n_{s,t}| \leq \delta K^n_{s,t} + |\sigma'(Y^n_s)\sigma(Y^n_s)||\mathbb{X}_{s,t}| + C_p\left(\|X\|_{p,[s,t]}\|R^{\sigma,n}\|_{\frac{p}{2},[s,t]} + \|\sigma'(Y^n)\sigma(Y^n)\|_{p,[s,t]}\|\mathbb{X}\|_{\frac{p}{2},[s,t]}\right).$$

Hence

$$|R^{\sigma,n}_{s,t}| \leq M\left(\delta K^n_{s,t} + |\mathbb{X}_{s,t}| + \|X\|_{p,[s,t]}\|R^{\sigma,n}\|_{\frac{p}{2},[s,t]} + \|\sigma'(Y^n)\sigma(Y^n)\|_{p,[s,t]}\|\mathbb{X}\|_{\frac{p}{2},[s,t]} + (Y^n_t - Y^n_s)^2\right),$$

for M that depends only on p, the uniform norms of σ and its derivatives. Thus for any $s < t \in [0,T]$ such that $|s-t| \leq \delta_X$, $\delta_X := T \wedge \sup\left\{\delta > 0 :\ \|X\|_{p,[s,t]} \leq \frac{1}{2}M^{-1},\ \forall s,t \in [0,T] \text{ s.t. } |t-s| \leq \delta\right\}$, one gets

$$|R^{\sigma,n}_{s,t}| \leq 2M\left(\delta K^n_{s,t} + |\mathbb{X}_{s,t}| + \|\sigma'(Y^n)\sigma(Y^n)\|_{p,[s,t]}\|\mathbb{X}\|_{\frac{p}{2},[s,t]} + (Y^n_t - Y^n_s)^2\right).$$

Using the bound on δK^n from Lemma 4.4 and the bound on $\|Y^n\|_p$ from Lemma 4.3, one gets for any $|s-t| \leq \delta_X$ that

$$|R^{\sigma,n}_{s,t}| \leq C\left(\delta Z^n_{s,t} + |\mathbb{X}_{s,t}| + \kappa_{\mathbf{X},Z^n}(s,t)\|\mathbb{X}\|_{\frac{p}{2},[s,t]} + \kappa_{\mathbf{X},Z^n}(s,t)^2\right). \tag{4.12}$$

Then

$$\begin{aligned}
\|R^{\sigma,n}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} &= \sup_{\pi=(t_i)}\left(\sum_{t_{i+1}-t_i\leq\delta_X} |R^{\sigma,n}_{t_i,t_{i+1}}|^{\frac{p}{2}} + \sum_{t_{i+1}-t_i>\delta_X} |R^{\sigma,n}_{t_i,t_{i+1}}|^{\frac{p}{2}}\right) \\
&\leq C\left((\delta Z^n_{0,T})^{\frac{p}{2}} + \|\mathbb{X}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} + \kappa_{\mathbf{X},Z^n}(0,T)^{\frac{p}{2}}\|\mathbb{X}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} + \kappa_{\mathbf{X},Z^n}(0,T)^p\right) \\
&\quad + \sup_{\pi=(t_i)} \sum_{t_{i+1}-t_i>\delta_X} |R^{\sigma,n}_{t_i,t_{i+1}}|^{\frac{p}{2}}.
\end{aligned} \tag{4.13}$$

By a simple induction, one can verify that for any $s_0 < s_1 < \cdots < s_N$,

$$R^{\sigma,n}_{s_0,s_N} = \sum_{k=0}^{N-1} R^{\sigma,n}_{s_k,s_{k+1}} + \sum_{k=1}^{N-1} \delta R^{\sigma,n}_{s_0,s_k,s_{k+1}},$$

where $\delta R^{\sigma,n}_{s_0,s_k,s_{k+1}} = R^{\sigma,n}_{s_0,s_{k+1}} - R^{\sigma,n}_{s_0,s_k} - R^{\sigma,n}_{s_k,s_{k+1}}$. In view of (4.9), there is $\delta R^{\sigma,n}_{s_0,s_k,s_{k+1}} = -\delta\sigma'\sigma(Y^n)_{s_0,s_k}\delta X_{s_k,s_{k+1}}$. Hence

$$\begin{aligned} |\delta R^{\sigma,n}_{s_0,s_k,s_{k+1}}| &\le C|\delta Y^n_{s_0,s_k}||\delta X_{s_k,s_{k+1}}| \\ &\le C|\delta X_{s_k,s_{k+1}}| \sum_{j=0}^{k-1} |\delta Y^n_{s_j,s_{j+1}}| \end{aligned} \tag{4.14}$$

For any index i in (4.13) such that $t_{i+1} - t_i > \delta_X$, denote $K_i = \lfloor \frac{t_{i+1}-t_i}{\delta_X} \rfloor$ $(\le \frac{T}{\delta_X})$, $t_{i,k} = t_i + k\delta_X$ for any $k = 0 \dots K_i$ and $t_{i,K_i+1} = t_{i+1}$. We obtain

$$\begin{aligned} \sum_{t_{i+1}-t_i>\delta_X} |R^{\sigma,n}_{t_i,t_{i+1}}|^{\frac{p}{2}} &= \sum_{t_{i+1}-t_i>\delta_X} |\sum_{k=0}^{K_i} R^{\sigma,n}_{t_{i,k},t_{i,k+1}} + \delta R^{\sigma,n}_{t_i,t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} \\ &\le \sum_{t_{i+1}-t_i>\delta_X} \left(2\frac{T}{\delta_X}\right)^{\frac{p}{2}-1} \sum_{k=0}^{K_i} |R^{\sigma,n}_{t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} + |\delta R^{\sigma,n}_{t_i,t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} \\ &\le \left(2\frac{T}{\delta_X}\right)^{\frac{p}{2}-1} \sum_{t_{i+1}-t_i>\delta_X} \left\{ \sum_{k=0}^{K_i} |R^{\sigma,n}_{t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} + C\left(\sum_{k=0}^{K_i} |\delta Y^n_{t_{i,k},t_{i,k+1}}|\right)^{\frac{p}{2}} \sum_{k=0}^{K_i} |\delta X_{t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} \right\} \\ &\le C\left(\frac{T}{\delta_X}\right)^{\frac{p}{2}-1} \sum_{t_{i+1}-t_i>\delta_X} \left\{ \sum_{k=0}^{K_i} |R^{\sigma,n}_{t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} + \left(\frac{T}{\delta_X}\right)^{p-1} \sum_{k=0}^{K_i} |\delta Y^n_{t_{i,k},t_{i,k+1}}|^p + \frac{T}{\delta_X} \sum_{k=0}^{K_i} |\delta X_{t_{i,k},t_{i,k+1}}|^p \right\} \\ &\le C\left(\frac{T}{\delta_X}\right)^{\frac{p}{2}-1} \sum_{t_{i+1}-t_i>\delta_X} \sum_{k=0}^{K_i} |R^{\sigma,n}_{t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} + C\left(\frac{T}{\delta_X}\right)^{\frac{3p}{2}-2} \left(\|Y^n\|^p_{p,[0,T]} + \|X\|^p_{p,[0,T]} \right), \end{aligned}$$

where we used (4.14) in the third inequality. Now, since $t_{i,k+1} - t_{i,k} \le \delta_X$, we can use (4.12) to get that

$$\sum_{t_{i+1}-t_i>\delta_X} \sum_{k=0}^{K_i} |R^{\sigma,n}_{t_{i,k},t_{i,k+1}}|^{\frac{p}{2}} \le C\left((\delta Z^n_{0,T})^{\frac{p}{2}} + (1 + \kappa_{\mathbf{X},Z^n}(0,T)^{\frac{p}{2}}) \|\mathbb{X}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} + \kappa_{\mathbf{X},Z^n}(0,T)^p \right).$$

Eventually, one gets that

$$\sum_{t_{i+1}-t_i>\delta_X} |R^{\sigma,n}_{t_i,t_{i+1}}|^{\frac{p}{2}} \le C\left(\frac{T}{\delta_X}\right)^{\frac{3p}{2}-2} \left((\delta Z^n_{0,T})^{\frac{p}{2}} + (1 + \kappa_{\mathbf{X},Z^n}(0,T)^{\frac{p}{2}}) \|\mathbb{X}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} + \kappa_{\mathbf{X},Z^n}(0,T)^p + \|X\|^p_{p,[0,T]} \right),$$

so that

$$\|R^{\sigma,n}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} \le C(\delta_X)^{2-\frac{3p}{2}} \left((\delta Z^n_{0,T})^{\frac{p}{2}} + (1 + \kappa_{\mathbf{X},Z^n}(0,T)^{\frac{p}{2}}) \|\mathbb{X}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} + \kappa_{\mathbf{X},Z^n}(0,T)^p + \|X\|^p_{p,[0,T]} \right).$$

In view of the definition of $\kappa_{\mathbf{X},Z^n}$ and since $Z^n \le Z^{n+1}$,

$$\|R^{\sigma,n}\|^{\frac{p}{2}}_{\frac{p}{2},[0,T]} \le C(\delta_X)^{2-\frac{3p}{2}} \left(Z_T^{\frac{p}{2}} + Z_T^p + |||\mathbf{X}|||^{\frac{p}{2}}_{p,[0,T]} + |||\mathbf{X}|||^p_{p,[0,T]} + |||\mathbf{X}|||^{p^2}_{p,[0,T]} \right). \tag{4.15}$$

Finally, since we assumed that $\|X\|_\beta < \infty$, it follows that $\|X\|_{p,[s,t]} \le \|X\|_\beta |t-s|^\beta$ and then (since $\beta = p^{-1}$) that $\delta_X \ge \frac{1}{2}(\|X\|_\beta C)^{-p} > 0$. This proves (i).

(ii) Using Lemma 4.3 (ii), we deduce that $\mathbb{E}\left[\sup_{n\in\mathbb{N}} \|\sigma'(Y^n)\sigma(Y^n)\|^\gamma_{p,[0,T]}\right] < \infty$. In view of (4.15) and the fact that $\mathbb{E}[\delta_X^{(2-\frac{3p}{2})\gamma}] \lesssim \mathbb{E}[\|X\|_\beta^{p(\frac{3p}{2}-2)\gamma}] < \infty$ for any $\gamma \ge 1$, Lemma 4.3 (ii) also implies that $\mathbb{E}\left[\sup_{n\in\mathbb{N}} \|R^{\sigma,n}\|^\gamma_{p,[0,T]}\right] < \infty$. □

Corollary 4.7. *The rough integral in (2.8) is β-Hölder continuous on $[0,T]$, uniformly in n: there exists $C>0$ depending only on p, $\|\sigma\|_\infty$ and $\|\sigma'\|_\infty$ such that*

$$\forall n \in \mathbb{N},\ \forall s,t \in [0,T], \quad \Big| \int_s^t \sigma(Y^n_u) d\mathbf{X}_u \Big| \le C(1+\Theta) |||\mathbf{X}|||_\beta |s-t|^\beta,$$

where Θ was defined (independently of n) in (4.10).

Proof. In view of Lemmas 4.5 and 4.6, Theorem 2.6 implies that

$$w_n(s,t) := \| \int_0^\cdot \sigma(Y^n_u) \mathrm{d}\mathbf{X}_u \|^p_{p,[s,t]}$$

is bounded from above by a control, denoted by $w(s,t)$, which is independent of n. Namely:

$$\begin{aligned} w_n(s,t)^{\frac{1}{p}} &\le \|\sigma(Y^n)\|_{\infty,[s,t]} \|X\|_{p,[s,t]} + \|\sigma'(Y^n)\sigma(Y^n)\|_{\infty,[s,t]} \|\mathbb{X}\|_{p,[s,t]} \\ &\quad + C_p \left(\|X\|_{p,[s,t]} \|R^{\sigma,n}\|_{\frac{p}{2},[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \|\sigma'(Y^n)\sigma(Y^n)\|_{p,[s,t]} \right) \\ &\le C\,(1+\Theta) \left(\|X\|_{p,[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right) =: w(s,t)^{\frac{1}{p}}, \end{aligned}$$

where Θ was defined (independently of n) in (4.10). To conclude the proof, it remains to notice that since $(X,\mathbb{X}) \in \mathscr{C}^\beta$, $\|X\|_{p,[s,t]} \le \|X\|_\beta |s-t|^\beta$ and $\|\mathbb{X}\|_{\frac{p}{2},[s,t]} \le \|\mathbb{X}\|_{2\beta} |s-t|^{2\beta}$. □

Proposition 4.8. (i) *Under the assumptions of Theorem 2.11, there exists $C>0$ which depends only on p and T, such that for any $n \in \mathbb{N}^*$,*

$$\sup_{s\in[0,T]} (Y^n_s - L_s)_- \le C \left(1 + (1+\Theta) |||\mathbf{X}|||_\beta + \|L\|_\beta \right) n^{-\beta},$$

where Θ was defined in (4.10). In particular, $\lim_{n\to\infty} \sup_{s\in[0,T]} (Y^n_s - L_s)_- = 0$.

(ii) *If in addition, the assumptions of Theorem 2.12 hold, then $\lim_{n\to\infty} \mathbb{E}\big[\sup_{s\in[0,T]} |(Y^n_s - L_s)_-|^\gamma \big] = 0$, for any $\gamma \ge 1$.*

Proof. (i) Applying Lemma 3.3(ii) and Corollary 4.7, one gets that

$$\begin{aligned} \forall n, \quad \sup_{s\in[0,T]} (Y^n_s - L_s)_- &\le C \left(1 + \|L\|_\beta + \| \int_0^\cdot \sigma(Y^n_u) \mathrm{d}\mathbf{X}_u \|_\beta \right) n^{-\beta} \\ &\le C \left(1 + \|L\|_\beta + (1+\Theta) |||\mathbf{X}|||_\beta \right) n^{-\beta}. \end{aligned} \tag{4.16}$$

which is the desired result.

(ii) Using (4.5) and Lemma 4.6 (ii), one gets that $\lim_{|s-t|\to 0} \mathbb{E}[w(s,t)^\gamma] = 0$, $\forall \gamma \ge 1$, so the result follows from (4.16). □

Proposition 4.9. (i) *Under the assumptions of Theorem 2.11, $(Y^n)_{n\in\mathbb{N}}$ converges uniformly to some process $\{Y_t\}_{t\in[0,T]} \in \mathcal{C}^\beta$.*

(ii) *If in addition, the driving noise X is Gaussian and the assumptions of Theorem 2.12 are satisfied, then the convergence happens in $L^\gamma\left(\Omega; (\mathcal{C}^0, \|\cdot\|_{\infty,[0,T]})\right)$ (i.e. as in (2.11)), $\forall \gamma \ge 1$. Besides, Y has a β-Hölder continuous modification and $\mathbb{E}[\|Y\|^\gamma_{\beta,[0,T]}] < \infty$, $\forall \gamma \ge 1$.*

Proof. (i) This proof uses estimates which are similar to those in the proof of Proposition 4.1, but this time a bound on $Z^n_t - Z^n_s$ is needed.

For any $s \in [0,T)$, define $\{\widetilde{Z}^n_{t\leftarrow s}\}_{t\in[s,T]}$ as the solution of the following (random) ODE:

$$\widetilde{Z}^n_{t\leftarrow s} = Z^n_s + C^{\mathbf{X}}_J \int_s^t \psi_n \left(U^{\mathbf{X};\widetilde{Z}^n_u}_{u\leftarrow 0} - L_u \right) \mathrm{d}u,\ t \in [s,T],$$

Once again, there is $\widetilde{Z}^n_{t\leftarrow s} \geq Z^n_t$ and

$$\widetilde{Z}^n_{t\leftarrow s} \leq Z^n_s + C^{\mathbf{X}}_J \int_s^t \psi_n \left(Y^n_s - C_{\sigma,\mathbf{X},\beta}(u-s)^\beta + (C^{\mathbf{X}}_J)^{-1}(\widetilde{Z}^n_u - \widetilde{Z}^n_s) - L_u\right) \mathrm{d}u,\ t \in [s,T],$$

but unlike in (4.2), it is no longer true that the starting point Y^n_s is larger than L_s. Thus we only get that

$$\widetilde{Z}^n_{t\leftarrow s} \leq Z^n_s + C^{\mathbf{X}}_J \int_s^t \psi_n \left(-(Y^n_s - L_s)_- - C_{\sigma,\mathbf{X},\beta}(u-s)^\beta + (C^{\mathbf{X}}_J)^{-1}(\widetilde{Z}^n_u - \widetilde{Z}^n_s) - (L_u - L_s)\right) \mathrm{d}s,\ t \in [s,T].$$

As in (4.3), one can then verify that the previous bound leads to

$$\forall t \in [s,T], \quad Z^n_t \leq \widetilde{Z}^n_{t\leftarrow s} \leq Z^n_s + C\ C^{\mathbf{X}}_J \left((|||\mathbf{X}|||_{\beta,[0,T]} \vee |||\mathbf{X}|||^{\frac{1}{\beta}}_{\beta,[0,T]})(t-s)^\beta + \sup_{u\in[s,t]} |L_u - L_s| + (Y^n_s - L_s)_- \right),$$

where C depends only on σ, β, T (an in particular not in n or s). Now as in (4.4),

$$\begin{aligned} |Y^n_t - Y^n_s| &\leq \sup_{y\in\mathbb{R}} \|U^{\mathbf{X};y}_{\cdot\leftarrow 0}\|_{\beta,[0,T]}(t-s)^\beta + C^{\mathbf{X}}_J |Z^n_t - Z^n_s| \\ &\leq C\ (C^{\mathbf{X}}_J)^2 \left((|||\mathbf{X}|||_{\beta,[0,T]} \vee |||\mathbf{X}|||^{\frac{1}{\beta}}_{\beta,[0,T]})(t-s)^\beta + \sup_{u\in[s,t]} |L_u - L_s| + (Y^n_s - L_s)_- \right). \end{aligned} \tag{4.17}$$

Using Proposition 4.8, we can now take the (pointwise) limit as $n \to \infty$ in the two previous inequalities to get that for any $t \in [s,T]$,

$$\begin{aligned} Z_t &\leq Z_s + C\ C^{\mathbf{X}}_J \left((|||\mathbf{X}|||_{\beta,[0,T]} \vee |||\mathbf{X}|||^{\frac{1}{\beta}}_{\beta,[0,T]})(t-s)^\beta + \sup_{u\in[s,t]} |L_u - L_s| \right) \\ \text{and}\quad |Y_t - Y_s| &\leq C\ (C^{\mathbf{X}}_J)^2 \left((|||\mathbf{X}|||_{\beta,[0,T]} \vee |||\mathbf{X}|||^{\frac{1}{\beta}}_{\beta,[0,T]})(t-s)^\beta + \sup_{u\in[s,t]} |L_u - L_s| \right). \end{aligned}$$

Hence Z and Y are (Hölder-)continuous, so arguing with Dini's Theorem, we are now able to conclude that the convergences are uniform.

(ii) Under the assumptions of Theorem 2.12, one deduces from the previous point that $\lim_{n\to\infty} \|Y^n - Y\|_{\infty,[0,T]} = 0$ almost surely. Moreover, Proposition 4.1 states that $(Y^n)_{n\in\mathbb{N}}$ is a nondecreasing sequence. Thus $\|Y^n - Y\|_{\infty,[0,T]} \leq 2\|Y\|_{\infty,[0,T]}$, and since $\mathbb{E}[\|Y\|^\gamma_{\infty,[0,T]}] < \infty$ (by Proposition 4.1 (ii)), the convergence result is obtained by using Lebesgue's theorem.
The Hölder continuity of Y is a consequence of Inequality (4.17) and Proposition 4.8 (ii). □

4.3 Identification of the limit process X

To achieve the proof of Theorem 2.11, we will show that $(Y, \sigma(Y)) \in \mathcal{C}^\beta_X$ and deduce that $\int_0^\cdot \sigma(Y_s)\mathrm{d}\mathbf{X}_s$ is the uniform limit of the sequence $\{\int_0^\cdot \sigma(Y^n_s)\mathrm{d}\mathbf{X}_s\}_{n\in\mathbb{N}}$ and that K^n converges uniformly to a non-decreasing path K. Then, by checking the properties (i), (ii), (iii) and (iv) of Definition 2.8, we will be able to prove that the paths (Y,K) so constructed are indeed solutions to the Skorokhod problem $SP(\sigma, L)$.

4.3.1 Step 1: convergence of the rough integral.

Proposition 4.10. $(\sigma(Y), \sigma'(Y)\sigma(Y)) \in \mathcal{C}^\beta_X$ *and the following convergence happens in* $\mathcal{C}^0([0,T],\mathbb{R})$*:*

$$\lim_{n\to\infty} \left\| \int_0^\cdot \sigma(Y^n_s)\, d\mathbf{X}_s - \int_0^\cdot \sigma(Y_s)\, d\mathbf{X}_s \right\|_{\infty,[0,T]} = 0.$$

Proof. Our aim is to check that Proposition 2.7 can be applied. First recall that $\sigma(Y^n)$ is controlled by X and that its Gubinelli derivative is $\sigma'(Y^n)\sigma(Y^n)$ (Lemma 4.5). By Proposition 4.9, $\sigma'(Y^n)\sigma(Y^n)$ converges uniformly to $\sigma'(Y)\sigma(Y)$. Similarly, $R^{\sigma,n}_{s,t}$ converges uniformly a.s. to $R^\sigma_{s,t} := \delta\sigma(Y)_{s,t} - \sigma'(Y_s)\sigma(Y_s)\delta X_{s,t}$. Hence in view of Lemma 4.6, the assumptions of Proposition 2.7 are matched and the desired result follows. □

A direct consequence of the previous proposition and of Proposition 4.9 is that K^n converges (uniformly) to a limit path K so that

$$\forall t \in [0,T], \quad Y_t = y_0 + \int_0^t \sigma(Y_s)\mathrm{d}\mathbf{X}_s + K_t.$$

As a limit of non-decreasing paths, K is non-decreasing. Hence the properties (i) and (iii) of Definition 2.8 are verified.

4.3.2 Step 2: $Y \geq L$.

This is the result of Proposition 4.8. Thus property (ii) of Definition 2.8 is satisfied.

4.3.3 Step 3: points of increase of K.

By the uniform convergence of K^n and the non-decreasing property of K^n and K, it follows that $\mathrm{d}K^n$ weakly converges towards $\mathrm{d}K$ and since Y^n converges uniformly to Y,

$$0 \geq \int_0^t (Y_s^n - L_s)\psi_n(Y_s^n - L_s)\ \mathrm{d}s = \int_0^t (Y_s^n - L_s)\ \mathrm{d}K_s^n \to \int_0^t (Y_s - L_s)\ \mathrm{d}K_s,$$

where the last integral exists in the sense of Lebesgue-Stieltjes integrals, since K is a non-decreasing path. Since $Y_s - L_s \geq 0$ (by the previous step) and K is non-decreasing, it follows that $\int_0^t (Y_s - L_s)\ \mathrm{d}K_s \geq 0$. Hence for any $t \in [0,T]$, $\int_0^t (Y_s - L_s)\ \mathrm{d}K_s = 0$, which proves that the point (iv) is satisfied.

Remark 4.11. *In view of Subsections 4.3.1, 4.3.2 and 4.3.3, we conclude that (Y, K) is a solution to $SP(\sigma, L)$, which achieves the proof of Theorem 2.11. In addition, if X is a Gaussian process satisfying Assumption ($\mathrm{H}_{\mathrm{Cov}}$), we obtained all along Section 4 the probabilistic estimates to ensure that Theorem 2.12 holds.*

4.4 Uniqueness

Uniqueness is in general the most tricky part in reflection problems. Here we benefit from earlier results in the literature. In the case $\beta > \frac{1}{2}$, the uniqueness of the reflected is due to Falkowski and Słomiński [11].

In the case $\beta \leq \frac{1}{2}$, the uniqueness of the reflected RDE has been proven recently by Deya, Gubinelli, Hofmanová, and Tindel [8]. The difference between our work and [8] is that they have a fixed boundary process $L \equiv 0$. But their proof of uniqueness can adapt to a moving boundary.

5 Rate of convergence of the sequence of penalised processes

In this section, we prove Theorem 2.13, which gives a rate of convergence in Theorem 2.11 (in the regular case $\beta > \frac{1}{2}$).

The following result is similar to the rough Gronwall lemma of [7, 8], except that it only applies in the "regular" case of $q < 2$, however with a second member in the inequality which is not a control. The proof is very close to the proof of [7, Lemma 2.11], for this reason it is given in the Appendix.

Lemma 5.1. *Let $q \in [1, 2)$ and $\Delta \in \mathcal{V}^q([0,T])$. Let w_1 be a regular control with $M := w_1(0,T) < \infty$ and let w_2 be another control. Assume that for any $s < t \in [0,T]$,*

$$\|\Delta\|_{q,[s,t]} \leq \|\Delta\|_{\infty,[s,t]} w_1(s,t)^{\frac{1}{q}} + w_2(s,t)^{\frac{1}{q}}.$$

Then,

$$\|\Delta - \Delta_0\|_{\infty,[0,T]}^q \leq 2(2\alpha^{-1})^{q-1} e^{\frac{w_1(0,T)}{\alpha M}} \sup_{t\in[0,T]} w_2(0,t) e^{-\frac{w_1(0,t)}{\alpha M}},$$

where $\alpha = \min\left(1, \left(2^q M e^2\right)^{-\frac{1}{2-q}}\right)$.

Proposition 5.2. *Let $X \in \mathcal{C}^\beta$ with $\beta \in (\frac{1}{2}, 1)$ and assume that the assumptions of Theorem 2.11 are in force. Then for any $\varepsilon \in (0, 2\beta-1)$, there exists $C>0$ (that depends only on ε, β and $\|X\|_p, p=\beta^{-1}$) such that*

$$\sup_{t\in[0,T]} |Y_t^n - Y_t| \leq C\ n^{-(2\beta-1)+\varepsilon}.$$

Proof. In this proof, denote by $\Pi_t(y)$ the projection in the upper half "plane" delimited by $\{L_t\}_{t\in[0,T]}$, i.e. $\Pi_t(y) = y$ if $y \geq L_t$ and $\Pi_t(y) = L_t$ if $y < L_t$.
The proof relies on two important ingredients: the first one is the continuity of the Skorokhod mapping for any $p \geq 1$,

$$\begin{aligned} \mathcal{V}^p([0,T]) &\to \mathcal{V}^p([0,T]) \\ z &\mapsto y, \end{aligned}$$

where (z,k) is the solution of the Skorokhod problem driven by z: $y = z+k$ (see [11, Theorem 2.2]). The second one is the fact observed by Słomiński [26] that $\{\Pi_t(Y_t^n)\}_{t\in[0,T]}$ is the solution of $SP(\Upsilon^n, L)$, where $\Upsilon_t^n = \Pi_t(Y_t^n) - Y_t^n + y_0 + \int_0^t \sigma(Y_s^n)\mathrm{d}X_s$, and with compensator K^n.

Let $q = \frac{1}{1-\beta+\varepsilon}$. Note that since $\beta = p^{-1} \geq \frac{1+\varepsilon}{2}$, one has $q \geq p$. Moreover, $Y^n \in \mathcal{V}^q$ since $Y^n \in \mathcal{V}^p$. Thus for any $s<t \in [0,T]$,

$$\begin{aligned} \|Y - Y^n\|_{q,[s,t]} &\leq \|Y - \Pi(Y^n)\|_{q,[s,t]} + \|\Pi(Y^n) - Y^n\|_{q,[s,t]} \\ &\leq C\|y_0 + \int_0^\cdot \sigma(Y_u)\mathrm{d}X_u - \Upsilon^n\|_{q,[s,t]} + \|\Pi(Y^n) - Y^n\|_{q,[s,t]}, \end{aligned}$$

using the continuity of the Skorokhod map. Then

$$\begin{aligned} \|Y - Y^n\|_{q,[s,t]} &\leq C\|\int_0^\cdot (\sigma(Y_u) - \sigma(Y_u^n))\,\mathrm{d}X_u\|_{q,[s,t]} + (1+C)\|\Pi(Y^n) - Y^n\|_{q,[s,t]} \\ &\leq C\|\sigma'\|_\infty \|Y - Y^n\|_{\infty,[s,t]}\|X\|_{q,[s,t]} + C\ C_q\ \|X\|_{p,[s,t]}\|Y - Y^n\|_{q,[s,t]} \\ &\quad + (1+C)\ \|\Pi(Y^n) - Y^n\|_{q,[s,t]}, \end{aligned}$$

using Young's inequality at the second line (which is fine since $p^{-1} + q^{-1} > 1$). Now let

$$\tau := \inf\{\theta > 0 :\ CC_q\|X\|_{p,[s,t]} \leq \tfrac{1}{2},\ \forall s<t \in [0,T] \text{ such that } |t-s| \leq \theta\}.$$

Then for any $s<t\in[0,T]$ such that $t-s \leq \tau$,

$$\|Y - Y^n\|_{q,[s,t]} \leq 2C\|\sigma'\|_\infty\|Y - Y^n\|_{\infty,[s,t]}\|X\|_{q,[s,t]} + 2(1+C)\ \|\Pi(Y^n) - Y^n\|_{q,[s,t]}.$$

As in the proof of Lemma 4.6, it follows that for any $s<t\in[0,T]$,

$$\|Y - Y^n\|_{q,[s,t]} \leq 2C\tau^{1-q}\|\sigma'\|_\infty\|Y - Y^n\|_{\infty,[s,t]}\|X\|_{q,[s,t]} + 2(1+C)\tau^{1-q}\ \|\Pi(Y^n) - Y^n\|_{q,[s,t]}.$$

Hence, applying Lemma 5.1, we obtain

$$\|Y - Y^n\|_{\infty,[0,T]}^q \leq 2e^{\frac{w_1(0,T)}{\alpha M}} 2(1+C)\ \sup_{t\in[0,T]} \left\{\|\Pi(Y^n) - Y^n\|_{q,[0,t]}^q e^{-\frac{w_1(0,t)}{\alpha M}}\right\}, \tag{5.1}$$

where $w_1(s,t) = 2C\|\sigma'\|_\infty\|X\|_{q,[s,t]}^q$. Hence considering that $\Pi(Y^n) - Y^n = (Y^n - L)_-$ and

$$\|\Pi(Y^n) - Y^n\|_{q,[0,t]}^q \leq \|\Pi(Y^n) - Y^n\|_{\infty,[0,t]}^{q-p}\|\Pi(Y^n) - Y^n\|_{p,[0,t]}^p \lesssim \|\Pi(Y^n) - Y^n\|_{\infty,[0,t]}^{q-p}\|X\|_{p,[0,t]}^p,$$

it follows from Proposition 4.8(i) that

$$\|\Pi(Y^n) - Y^n\|_{q,[0,t]}^q \lesssim n^{-\beta(q-p)}\|X\|_{p,[0,t]}^p$$

which, when plugged in (5.1) leads to the desired result ($\frac{\beta(q-p)}{q} = \beta - (1-\beta+\varepsilon) = 2\beta - 1 - \varepsilon$). ☐

A Appendix

Proof of Proposition 2.10. The existence of a solution of (2.8) up to some time $\tau > 0$ comes from [19], Theorem 3 (see also [16, Theorem 10.21]). Uniqueness is also granted given the regularity of σ and b. In view of [19, Lemma 1] (see also [16, Theorem 10.21]), we know that either Y is a global solution on $[0,T]$, or that there is some time $\tau' > \tau$ such that for any $t \in [0,\tau')$, $\{Y_s\}_{s\in[0,t]}$ is a solution to (2.8) and that $\lim_{t\to\tau'} |Y_t| = \infty$. In the remaining of this proof, we shall prove that Y_t coincides on $[0,\tau')$ with the solution to (3.3). Since the latter does not explode, it will follow that Y is a global solution.

Let us turn to the Doss-Sussmann representation. Recall that according to (3.1), $J^{\mathbf{X};\cdot}_{0\leftarrow t}$ is Lipschitz uniformly in t but that due to the unboundedness of b, $W(t,\cdot)$ is only locally Lipschitz (uniformly in t). This suffices to prove existence and uniqueness of a solution to $\dot z_t = W(t, z_t)$ on a small enough time interval. In fact, $J^{\mathbf{X};\cdot}_{0\leftarrow t}$ is bounded (see (3.1)) and $C^{\mathbf{X}}_U := \sup_{t\in[0,T]} |U^{\mathbf{X};0}_{0\leftarrow t}| < \infty$. Denote by $B(C^{\mathbf{X}}_U)$ the ball of $\mathbb{R}^e$ centred in 0 and with radius $C^{\mathbf{X}}_U$. Thus

$$
\begin{aligned}
|W(t,z)| &\le |J^{\mathbf{X};z}_{0\leftarrow t} b(U^{\mathbf{X};0}_{t\leftarrow 0})| + |W(t,z) - J^{\mathbf{X};z}_{0\leftarrow t} b(U^{\mathbf{X};0}_{t\leftarrow 0})| \\
&\le C^{\mathbf{X}}_J \sup_{x\in B(C^{\mathbf{X}}_U)} |b(x)| + C^{\mathbf{X}}_J |b(U^{\mathbf{X};z}_{t\leftarrow 0}) - b(U^{\mathbf{X};0}_{t\leftarrow 0})| \\
&\le C^{\mathbf{X}}_J \left(\|b\|_{\infty, B(C^{\mathbf{X}}_U)} + \|\nabla b\|_\infty C^{\mathbf{X}}_J |z| \right),
\end{aligned}
$$

i.e. W has linear growth. This ensures the stability of the solution Z_t to the ODE $\dot Z_t = W(t, Z_t)$, and its global existence on any time interval (see e.g. [16, Theorem 3.7]). Thus the process $\{U^{\mathbf{X};Z_t}_{t\leftarrow 0}\}_{t\in[0,T]}$ is well-defined.

Now to prove that $\{Y_t\}_{t\in[0,\tau]}$ and $\{U^{\mathbf{X};Z_t}_{t\leftarrow 0}\}_{t\in[0,\tau]}$ coincide, one can follow closely the end of the proof of [13, Proposition 3]: let $(\mathbf{X}^k)_{k\in\mathbb{N}}$ a sequence of geometric rough paths such that $(X^k)_{k\in\mathbb{N}}$ is a sequence of Lipschitz paths with uniform β-Hölder bound, which converges pointwise to $\mathbf{X}$. Denote by Y^k the solution to (2.8) where $\mathbf{X}^k$ replaces $\mathbf{X}$. Then (Y^k, Z^k) is easily seen to solve (3.3) with $\mathbf{X}$ replaced by $\mathbf{X}^k$. As stated in [13], it suffices to prove the uniform convergence of Z^k to Z to get the result. Define

$$
M^k = \sup_{s\in[0,\tau], z\in\mathbb{R}^e} |J^{\mathbf{X};z}_{0\leftarrow s} - J^{\mathbf{X}^k;z}_{0\leftarrow s}| \vee |U^{\mathbf{X};z}_{s\leftarrow 0} - U^{\mathbf{X}^k;z}_{s\leftarrow 0}|
$$

and denote $J_{\mathrm{Lip}} = \sup_{s\in[0,T]} \|J^{\mathbf{X};\cdot}_{s\leftarrow 0}\|_{\mathrm{Lip}}$ which is finite (see the discussion of Section 3.1). Now the main difference with [13] lies again in the unboundedness of b: denote by $\overline{Z} = \sup_{t\in[0,T]} |Z_t| < \infty$ and $\overline{C}^{\mathbf{X}}_U = \sup_{t\in[0,T], \sup_{z\in B(0,\overline{Z})}} |U^{\mathbf{X};z}_{s\leftarrow 0}| < \infty$. Then for $t \le \tau$,

$$
\begin{aligned}
|Z^k_t - Z_t| &\le \int_0^t \Big\{ |J^{\mathbf{X};Z_s}_{0\leftarrow s} - J^{\mathbf{X};Z^k_s}_{0\leftarrow s}|\ |b(U^{\mathbf{X};Z_s}_{s\leftarrow 0})| + |J^{\mathbf{X};Z^k_s}_{0\leftarrow s} - J^{\mathbf{X}^k;Z^k_s}_{0\leftarrow s}|\ |b(U^{\mathbf{X}^k;Z^k_s}_{s\leftarrow 0})| \\
&\qquad + J^{\mathbf{X};Z^k_s}_{0\leftarrow s} |b(U^{\mathbf{X}^k;Z^k_s}_{s\leftarrow 0}) - b(U^{\mathbf{X};Z_s}_{s\leftarrow 0})| \Big\} \mathrm{d}s \\
&\le \int_0^t \Big\{ J_{\mathrm{Lip}} |Z^k_s - Z_s|\ \|b\|_{\infty, B(C^{\mathbf{X}}_U)} + M^k\ \left(\|b\|_{\infty, B(\overline{C}^{\mathbf{X}}_U)} + \|\nabla b\|_\infty |U^{\mathbf{X}^k;Z^k_s}_{s\leftarrow 0} - U^{\mathbf{X};Z_s}_{s\leftarrow 0}| \right) \\
&\qquad + C^{\mathbf{X}}_J \|\nabla b\|_\infty |U^{\mathbf{X}^k;Z^k_s}_{s\leftarrow 0} - U^{\mathbf{X};Z_s}_{s\leftarrow 0}| \Big\} \mathrm{d}s \\
&\le \int_0^t \Big\{ J_{\mathrm{Lip}} |Z^k_s - Z_s|\ \bar b + M^k\ \bar b + \|\nabla b\|_\infty \left(C^{\mathbf{X}}_J + M^k \right) |U^{\mathbf{X}^k;Z^k_s}_{s\leftarrow 0} - U^{\mathbf{X};Z_s}_{s\leftarrow 0}| \Big\} \mathrm{d}s \\
&\le \int_0^t \Big\{ J_{\mathrm{Lip}} |Z^k_s - Z_s|\ \bar b + M^k\ \bar b + \|\nabla b\|_\infty \left(C^{\mathbf{X}}_J + M^k \right) \left(M^k + C^{\mathbf{X}}_J |Z^k_s - Z_s| \right) \Big\} \mathrm{d}s,
\end{aligned}
$$

where $\bar b := \max\left(\|b\|_{\infty, B(C^{\mathbf{X}}_U)}, \|b\|_{\infty, B(\overline{C}^{\mathbf{X}}_U)} \right)$. Then, denoting $C^{Z,1} := \bar b + \|\nabla b\|_\infty C^{\mathbf{X}}_J$ and $C^{Z,2} := \bar b J_{\mathrm{Lip}} +$

$\|\nabla b\|_\infty (C_J^{\mathbf{X}})^2$, one gets that

$$\sup_{s\in[0,t]} |Z_s^k - Z_s| \leq tC^{Z,1}M^k + t\|\nabla b\|_\infty (M^k)^2 + C^{Z,2}(1+M^k)\int_0^t \sup_{u\in[0,s]} |Z_u^k - Z_u|\mathrm{d}s$$
$$\leq \left(C^{Z,1} + \|\nabla b\|_\infty M^k\right) TM^k \ \exp\left(C^{Z,2}(1+M^k)T\right) \tag{A.1}$$

applying Gronwall's lemma in the last inequality. By the continuity of the mapping $(y,\mathbf{X}) \mapsto U^{\mathbf{X};y}_{\cdot\leftarrow 0}$ (see [14, Theorem 8.5]), there is $M^k \to 0$ as $k\to\infty$, hence the inequality (A.1) implies that Z^k converges uniformly to Z. Since Y^k has the representation (3.3), then so has Y.

Hence $\{Y_t\}_{t\in[0,\tau]}$ and $\{U^{\mathbf{X};Z_t}_{t\leftarrow 0}\}_{t\in[0,\tau]}$ do coincide and since the latter does not explode in finite time, this implies that there cannot exist $\tau' > \tau$ such that $\lim_{t\to\tau'} |Y_t| = \infty$. Thus Y is defined on $[0,T]$. ☐

Proof of Lemma 5.1. This proof is very similar to the proof of [7, Lemma 2.11], but we reproduce most of it for the reader's convenience, emphasizing on the main differences.

Without loss of generality, assume that $\Delta_0 = 0$. Let $K = \lfloor \alpha^{-1} \rfloor$ be the integer part of α^{-1}. For $k = 0,\dots,K$, define $t_k \in [0,T]$ such that $w_1(0,t_k) = \alpha M k$. Hence $0 = t_0 < t_1 < \cdots < t_K \leq T$ and $w_1(t_k,t_{k+1}) \leq \alpha M$. For $t\in[0,T]$, let k such that $t\in[t_{k-1},t_k)$, then

$$\sum_{j=0}^{k-2} |\delta\Delta_{t_j,t_{j+1}}|^q + |\delta\Delta_{t_{k-1},t}|^q \leq \sum_{j=0}^{k-2} \|\Delta\|^q_{q,[t_j,t_{j+1}]} + \|\Delta\|^q_{q,[t_{k-1},t]}$$
$$\leq 2^{q-1}\alpha M \sum_{j=0}^{k-2} \|\Delta\|^q_{\infty,[t_j,t_{j+1}]} + 2^{q-1} w_2(0,t_{k-1}) + 2^{q-1}\alpha M \|\Delta\|^q_{\infty,[t_{k-1},t]} + 2^{q-1} w_2(t_{k-1},t)$$
$$\leq 2^{q-1}\alpha M \sum_{j=0}^{k-1} \|\Delta\|^q_{\infty,[0,t_{j+1}]} + 2^{q-1} w_2(0,t),$$

using the super-additivity property of w_2.
Now set $H_t = \|\Delta\|^q_{\infty,[0,t]} \exp\left(-\frac{w_1(0,t)}{\alpha M}\right)$. We get

$$\sum_{j=0}^{k-2} |\delta\Delta_{t_j,t_{j+1}}|^q + |\delta\Delta_{t_{k-1},t}|^q \leq 2^{q-1}\alpha M \sum_{j=0}^{k-1} H_{t_{j+1}} \exp\left(\frac{w_1(0,t_{j+1})}{\alpha M}\right) + 2^{q-1} w_2(0,t)$$
$$\leq 2^{q-1}\alpha M \|H\|_{\infty,[0,T]} e^{k+1} + 2^{q-1} w_2(0,t).$$

Since $|\Delta_t|^q \leq K^{q-1}\sum_{j=0}^{k-2} |\delta\Delta_{t_j,t_{j+1}}|^q + K^{q-1}|\delta\Delta_{t_{k-1},t}|^q$, we obtain from the previous equation that for any $t < t_k$,

$$\|\Delta\|^q_{\infty,[0,t]} \leq (2K)^{q-1}\alpha M \|H\|_{\infty,[0,T]} e^{k+1} + (2K)^{q-1} w_2(0,t).$$

Hence, using that $t\in[t_{k-1},t_k)$, so that $w_1(0,t) \geq \alpha M(k-1)$,

$$H_t = \|\Delta\|^q_{\infty,[0,t]} \exp\left(-\frac{w_1(0,t)}{\alpha M}\right) \leq (2K)^{q-1}\alpha M e^2 \|H\|_{\infty,[0,T]} + (2K)^{q-1} w_2(0,t)\exp\left(-\frac{w_1(0,t)}{\alpha M}\right),$$

thus, using the fact that $K \leq \alpha^{-1}$,

$$\|H\|_{\infty,[0,T]} \leq (2K)^{q-1}\alpha M e^2 \|H\|_{\infty,[0,T]} + (2K)^{q-1} \sup_{t\in[0,T]} \left(w_2(0,t)\exp\left(-\frac{w_1(0,t)}{\alpha M}\right)\right)$$
$$\leq \alpha^{2-q} 2^{q-1} M e^2 \|H\|_{\infty,[0,T]} + (2\alpha^{-1})^{q-1} \sup_{t\in[0,T]} \left(w_2(0,t)\exp\left(-\frac{w_1(0,t)}{\alpha M}\right)\right).$$

In view of the definition of α and the assumption $q < 2$,

$$\|H\|_{\infty,[0,T]} \leq 2(2\alpha^{-1})^{q-1} \sup_{t\in[0,T]} \left(w_2(0,t)\exp\left(-\frac{w_1(0,t)}{\alpha M}\right)\right),$$

and the conclusion follows from the definition of H. ☐

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