Theoretical calculation of the power of wind turbine or tidal turbine

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Theoretical Calculation of Wind (Or Water) Turbine

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Abstract : The Betz limit sets a theoretical upper limit for the energy output of turbines, expressed as a maximum power coefficient of 16/27. Betz's theory is precise and is based on the calculation of kinetic energy. By taking into account the potential energy, the theoretical energy output of the turbines can be higher. For fast wind turbines, the kinetic energy of the wind is optimally recovered and at the same time a large amount of potential energy is created without additional energy input. The article presents a review of the consideration of this potential energy and it is possible for a wind turbine to transform potential energy into kinetic energy. The definition of the maximum power coefficient is the one established by Betz which remains valid for the horizontal axis turbine HAWT and no longer makes sense for vertical axis turbines VAWT. The results given are examined a conversion of the potential energy into kinetic energy through a mechanical system which is not applicable for horizontal axis turbines HAWT.

Keywords : Betz limit, Betz's law, Wind turbine, Tidal turbine, HAWT, VAWT.

1 Introduction

Lanchester, Betz, Joukowsky van Kuik (2007) have participated in the definition of the maximum power coefficient of a wind turbine. This limit is commonly called Betz limit. Huge research efforts have been made to optimize wind turbines to approach this limit, for example by optimizing the angle of incidence, the shape of the blade profile, etc. One can refer for example to the book "Wind Energy Handbook"Burton et al. for fast moving horizontal axis wind turbines (HAWT) or for example "Wind Turbine Design: With Emphasis on Darrieus Concept" Paraschivoiu (2002) for vertical axis wind turbines (VAWT).
Research has been done to improve this efficiency by, for example, ducting the turbine (Georgiou **(2016)** Georgiou and Theodoropoulos (2016)) or by placing a water turbine in a narrow channel, which allows the water level upstream to be increased by the resistance to the advancing fluid (Quaranta **(2018)** (Quaranta (2018))) or grouping turbines in farm in order to create an excess of power due to the proximity between machines (ducting effect) (Vennell **(2013)** Vennell (2013) and Broberg **(2018)** Broberg et al. (2018)).
Betz' theory is based on the calculation of kinetic energy. A wind turbine is defined according to the energy of the fluid for energy recovery and for the definition of its structure to resist. A carriage that can move freely, will be moved by the wind and will be subject to very little stress from the wind. If the carriage is stopped, the carriage will be subject to high stress on its structure. For a fast moving wind turbine, when the wind increases, the possible energy recovery increases and the stresses on its structure also increases, which is not the same situation as for the carriage.

Large wind turbines are stopped when the wind becomes too strong, not because they produce too much, but because they are under too much stress which may break their blades. Stresses in a structure are potential energy.
Betz's theory does not take into account the notion of stress (potential energy) for a wind turbine, it does not take into account only the notion of kinetic energy.

For a sailboat, a sail cannot recover all the energy of the wind. Betz' theory applies to the sail. At the kinetic energy level, the sail can only recover at most 60% (16/27) of the kinetic energy contained in the wind.
Many books present this limit, such as John Kimball's book "Physics of Sailing"Kimball (2009). The America Cup race is a good example that a sailing boat equipped with hydrofoils is very efficient.
The kinetic energy of the wind is used to move the boat forward. The additional kinetic energy to lift the boat and to optimize the boat's performance is also created by the kinetic energy of the wind.
The wind has not been doubled, but instead of creating stress on a keel, it is potential energy that is transformed into kinetic energy by a fluid flow around a wing profile.
The article presents the notion of kinetic energy and the notion of potential energy. It also presents that it is possible for a wind turbine to transform potential energy into kinetic energy.

2 Preliminary considerations of Betz's theory

The German mathematician A. Betz has shown that the power of turbine is (cf. Appendix 14, Betz (1920)):

$$P_k = F\ V\ = [\ \frac{C_k}{a}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^2]\ a\ V_{fluid} = C_k\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

$$where\quad V\ =\ a\ V_{fluid}\quad and\quad C_k = 4a^2(1-a)$$

P_k the kinetic turbine power, (W)
C_k the kinetic power coefficient,
ρ the fluid density,$(\frac{kg}{m^3})$
V_{fluid} the fluid velocity.$(\frac{m}{s})$
V the fluid velocity at the position of the turbine.
S the swept area.(m^2)

According to the work of Betz, a kinetic energy approach shows that the maximum power coefficient C_k can not exceed a maximum of $\frac{16}{27}$

$$C_{k\ Betz} = \frac{16}{27}\qquad C_k \leq C_{k\ Betz}\qquad P_{k\ maxi} = C_{k\ Betz}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3 \tag{1}$$

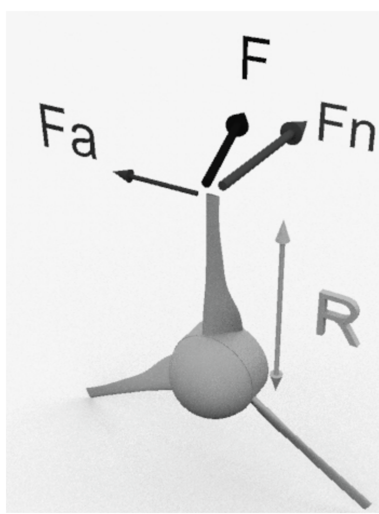


Figure 1. three-bladed VAWT wind turbine.;

The relative speed due to the rotation speed of the wind turbine and the fluid speed creates an induced force F on the profile. This induced force F can be broken down into an axial force Fa and a normal force Fn. The axial force Fa associated with the radius R creates a driving torque to produce energy. The normal force Fn associated with the same radius R creates a bending stress on the blade.

The power of the torque due to the axial forces and the angular speed of rotation of the wind turbine is limited to the 60% (16/27) of the kinetic power in the wind.

The force Fn is much greater than the force Fa. Both forces are associated with the same radius and have the same origin the fluid velocity. These constraints are the source of internal energy. Constraints are potential energy. Betz's theory does not take into account this potential energy which is as important as the kinetic energy. The theory behind Betz's limit is correct. However, it only takes into account the kinetic energy. In order to increase the efficiency of wind and hydrokinetic turbines, they should be designed to transform the potential energy into kinetic energy.
it is necessary to dissociate the slow speed turbines and the fast speed turbines.
$\lambda = \frac{\omega R}{V_{fluid}}$ is an important parameter which makes a difference in the behavior of the turbines in front of the wind.
As for the sail, it is necessary to dissociate the navigation in thrust (square sail, spinnaker) (unstuck flow) and the navigation in smoothness (wing profile sail) (laminar flow)

3 difference between a low speed turbine and a high speed turbine

3.1 The advantage of using an airplane wing

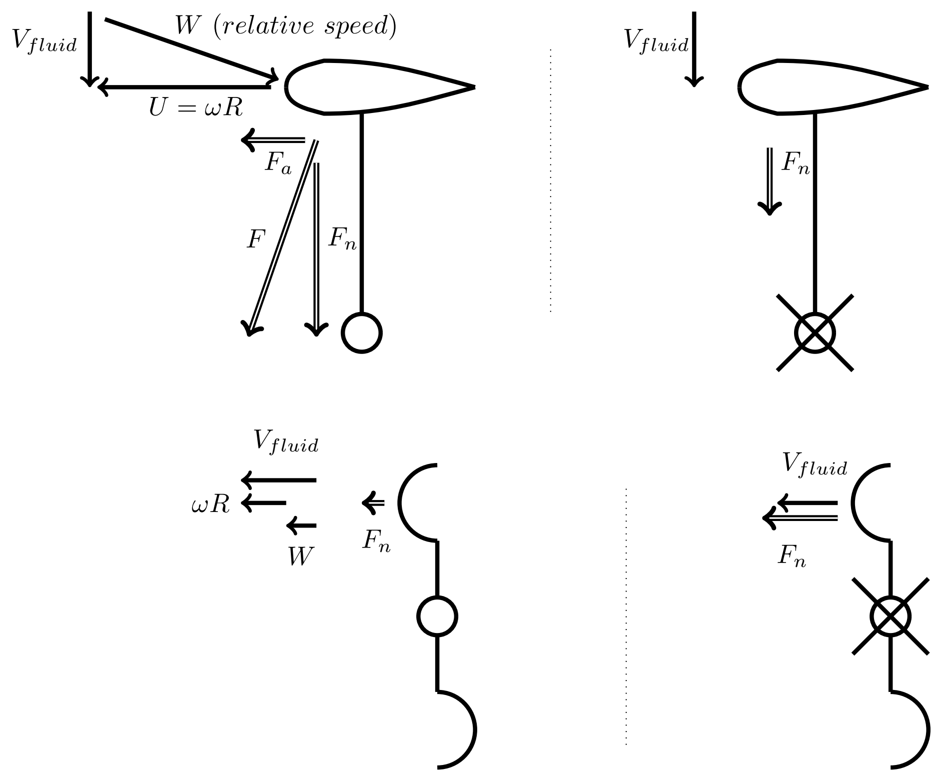


Figure 2. comparison of low and high speed turbine.;

The cups of an anemometer are almost free of stress, if the anemometer is free of charge. If the rotation of the anemometer is blocked, the cups are subjected to stress.
In the case of a wing profile, it's completely different. The tangential rotational speed U of the profile is transverse to the air flow of the fluid.
The relative speed W due to the rotational speed U and the fluid speed V_{fluid} creates an induced force F on the profile. The axial component F_a of this force F, combined with the radius, creates a driving torque. For *horizontal axis wind turbines* (HAWT) this torque allows to improve the efficiency closed to that the one defined by Betz. Adding a transverse speed U will optimize the efficiency defined by Betz, but will add significant stress on the wind turbine due to the normal component F_n of force F. That's why HAWT wind

turbines are stopped when the wind is too strong. They do not produce too much but they are subject to exceeding bending stress.

Anemometer ($\lambda < 1$)

case	$U = \omega R$	W	F_n	$F_n R$
blocked	0	$= V_{fluid}$	*max*	0
free	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

case	$U = \omega R$	W	*stress*	*torque*
blocked	0	$= V_{fluid}$	*max*	*null*
free	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

case	$U = \omega R$	W	*potential energy*	*kinetic energy*
blocked	0	$= V_{fluid}$	*max*	*null*
free	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

Horizontal axis wind turbine *($\lambda > 1$)* **(HAWT)**

case	$U = \omega R$	W	F_n	$F_a R$
blocked	0	$= V_{fluid}$	*min*	0
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

case	$U = \omega R$	W	*stress*	*torque*
blocked	0	$= V_{fluid}$	*min*	*null*
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

case	$U = \omega R$	W	*potential energy*	*kinetic energy*
blocked	0	$= V_{fluid}$	*min*	*null*
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

In the case of the anemometer, the kinetic energy varies in the opposite direction to the potential energy. This is not the case for HAWT turbines. In the case of HAWT turbines, we must not consider only the speed of the fluid. The kinetic energy and potential energy are related to the fluid speed and the to the tangential rotation speed U. The direction of U is perpendicular to the direction of the fluid speed. The potential energy cannot grow more than the kinetic energy. The tangential rotation speed U is related to the fluid speed by this relation $\lambda = \frac{U}{V_{fluid}} \quad U = \omega R$. For a low speed turbine, the kinetic energy of the wind is partially transformed into kinetic energy and potential energy.

3.1.1 Low speed turbine $\lambda < 1$:

Using the example anemometer with cups, when there is no resistant torque, the turbine rotates at maximum speed ($\omega R = V_{fluid} \quad W = 0$) and there is no energy production and the kinetic energy is maximum.
When the turbine is blocked, the stress on the turbine is maximum ($W = V_{fluid}$) and the speed of rotation is null. The potential energy is maximum.
For there is energy production, there must be kinetic energy and potential energy.
If the kinetic energy is greater than the potential energy, the energy production is limited by the potential energy.
Similarly, if the potential energy is greater than the kinetic energy, the energy production is limited by the kinetic energy.

To obtain kinetic energy of production, kinetic energy and potential energy are needed.

$$P_{k-max-productive} \le min(\ P_k \ , \ P_p \ )$$

Energy production is maximum when kinetic energy is equal to potential energy.
Total energy is equal to the sum of kinetic energy and potential energy.

$$E_T = E_k + E_p$$

The conservation of energy can be applied. The total energy remains constant (Prescott Joule (2011)). The variation of the total energy (kinetic energy + potential The variation of the total energy along the time is zero.

$$\frac{dE_{total}}{dt} = 0 \quad \rightarrow \quad \frac{dE_k}{dt} = -\frac{dE_p}{dt} \tag{2}$$

The potential energy is related to the relative speed W. The kinetic energy is related to the relative speed ωR (see Figure 2).
The tangential rotation speed must be equal to half of the fluid speed to have the optimal conditions.

$$|V_{fluid}| = |U| + |W| \qquad |U| = |W| \qquad U = \omega R = \frac{V_{fluid}}{2} \qquad \lambda = \frac{1}{2} \tag{3}$$

To have an energy production, the maximum power of the kinetic energy of a turbine at low speed ($\lambda < 1 \quad \lambda = \frac{\omega R}{V_{fluid}}$) is equal to

$$P_{k\ max(with\ \lambda<1)} = C_{k(with\ \lambda<1)}\frac{1}{2}\rho S V_{fluid}^3 \qquad C_{k\ max(with\ \lambda<1)} = \frac{1}{2} \tag{4}$$

For a low speed turbine, the kinetic energy of the wind is partially transformed into kinetic energy and potential energy.

$$E_{kwind} \Rightarrow E_k \ \&\ E_p$$

3.1.2 Fast speed turbine $\lambda > 1$:

For fast moving turbines, as the fluid speed increases, the potential energy and kinetic energy increase.
The maximum power of the energy is defined by the Betz limit

$$P_{k\ max(with\ \lambda>1)} = C_{k(Betz)}\frac{1}{2}\rho S V_{fluid}^3 \qquad C_{k(Betz)} = \frac{16}{27} \tag{5}$$

Energy production is related to kinetic energy and potential energy. Potential energy and kinetic energy cannot be separated to determine the maximum energy production.
The kinetic energy of the fluid is transformed partly into potential energy 8 and then into kinetic energy 9.

$$E_{kwind} \Rightarrow E_p \Rightarrow E_k$$

4 Summary calculation sheet for fast moving turbines

4.1 calculation of powers and forces $\lambda > 1$

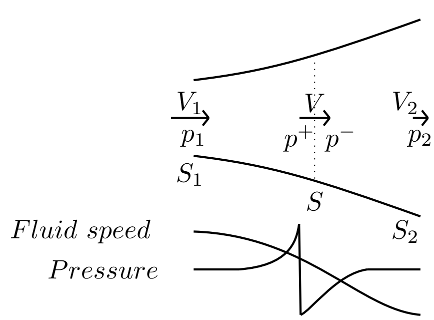


Figure3. current tube.;

4.1.1 Steady state and flow conservation

In steady state, the fluid velocity is constant in time: $\frac{dV}{dt} = 0$
By conservation of flow, the continuity equation is obtained

$$S_1 V_1 = SV = S_2 V_2 \qquad V_1 = V_{fluid}$$

4.1.2 Energy, power and kinetic force

The index k for "kinetic" is used

$$E_k = \frac{1}{2}mv^2 \qquad P_k = \frac{dE_k}{dt} = \frac{1}{2}\frac{dm}{dt}v^2 + \frac{1}{2}m\frac{dv^2}{dt} \qquad \frac{dv}{dt} = 0 \qquad m = \rho svdt \qquad P_k = \frac{1}{2}\rho sv^3 \qquad F_k = \frac{P_k}{v} = \frac{1}{2}\rho sv^2$$

4.1.3 Energy, power and potential force

The index p for "potential" is used

$$E_k = m\frac{p}{\rho} \qquad P_p = \frac{dE_p}{dt} = \frac{dm}{dt}\frac{p}{\rho} + m\frac{1}{\rho}\frac{dp}{dt} \qquad \frac{dp}{dt} = 0 \qquad m = \rho svdt \qquad P_p = svp \qquad F_p = \frac{P_p}{v} = ps$$

4.1.4 Calculation of the turbine power from the kinetic energy

The force applied to the rotor is $F_k = m\frac{dV}{dt} = \frac{dm}{dt}\Delta V = \rho SV(V_1 - V_2)$

$$P_k = F_k V = \rho SV^2(V_1 - V_2)$$

$$P_k = \frac{\Delta E_k}{dt} = \frac{1}{2}\rho SV^2(V_1^2 - V_2^2)$$

It is deducted $V = \frac{V_1+V_2}{2}$
By defining a as $a = \frac{V}{V_{fluid}}$ $\qquad V_{fluid} = V_1 \qquad 0 \le a \le 1 \qquad V_2 = V_1(2a-1) \qquad V_2 \ge 0 \qquad a \ge \frac{1}{2}$

$$P_k = 4a^2(1-a)\frac{1}{2}\rho SV_{fluid}^3 \qquad with \qquad C_k = 4a^2(1-a) \qquad P_k = C_k\frac{1}{2}\rho SV_{fluid}^3$$

Search for the maximum kinetic power (Betz limit) :

$$\frac{dP_k}{dt} = 0 \qquad a(2-3a) = 0 \qquad as \qquad a \ge \frac{1}{2} \qquad a = \frac{2}{3} \qquad C_{k-maxi} = \frac{16}{27} = C_{k-Betz}$$

4.2 Energy balance

4.2.1 Energy conservation in stationary state applied in the turbine area

$$Ek_+ \;+\; Ep_+ \;=\; Ek_- \;+\; Ep_- \;+\; E_{potential}$$

$$\rho\, m\, [\frac{V_+^2}{2} \;-\; \frac{V_-^2}{2} \;+\; \frac{p_+}{\rho} \;-\; \frac{p_-}{\rho}] \;=\; E_{potential} \qquad with \qquad V_+ \;=\; V_- \;=\; V$$

$$\rho\, m\, [\frac{p_+}{\rho} \;-\; \frac{p_-}{\rho}] \;=\; E_{potential} \tag{6}$$

4.2.2 Energy conservation in stationary state applied to the whole turbine

$$Ek_1 \;+\; Ep_1 \;=\; Ek_2 \;+\; Ep_2 \;+\; E_{Energy\ supplied\ to\ the\ turbine}$$

$$\rho\, m\, [\frac{V_1^2}{2} \;-\; \frac{V_2^2}{2} \;+\; \frac{p_1}{\rho} \;-\; \frac{p_2}{\rho}] \;=\; E_{Energy\ supplied\ to\ the\ turbine} \qquad with \qquad p_1 \;=\; p_2$$

$$\rho\, m\, [\frac{V_1^2}{2} \;-\; \frac{V_2^2}{2}] \;=\; E_{Energy\ supplied\ to\ the\ turbine} \tag{7}$$

4.2.3 remark

The Bernoulli equations

$$p_1 + \frac{1}{2}\rho s_1 V_1 = p^+ + \frac{1}{2}\rho s V \qquad p^- + \frac{1}{2}\rho s V = p_2 + \frac{1}{2}\rho s_2 V_2$$

$$V_1 = V_{fluid} \qquad P_1 = P_2 \qquad \rightarrow \qquad \frac{V_1^2}{2} - \frac{V_2^2}{2} = \frac{p_+}{\rho} - \frac{p_-}{\rho}$$

by using equations (8),(9)

$$E_{potential} = E_{Energy\ supplied\ to\ the\ turbine}$$

The kinetic energy of the fluid is partially transformed into potential energy and then into kinetic energy

4.3 Energy conservation for the fast speed turbine $\lambda > 1$

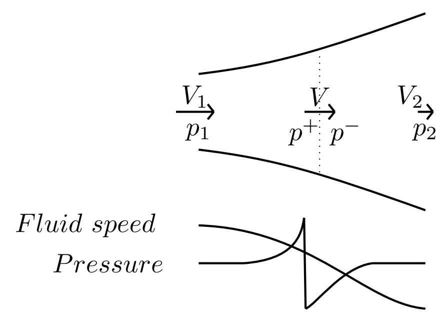


Figure3. current tube.;

Energy conservation in stationary state applied in the turbine area :

$$Ek_+ + Ep_+ = Ek_- + Ep_- + E_{potential}$$

$$\rho\ m\ [\frac{V_+^2}{2} - \frac{V_-^2}{2} + \frac{p_+}{\rho} - \frac{p_-}{\rho}] = E_{potential} \qquad with \qquad V_+ = V_- = V$$

$$\rho\ m\ [\frac{p_+}{\rho} - \frac{p_-}{\rho}] = E_{potential} \tag{8}$$

Energy conservation in stationary state applied to the whole turbine (HAWT or VAWT) :

$$Ek_1 + Ep_1 = Ek_2 + Ep_2 + E_{Energy\ supplied\ to\ the\ turbine}$$

$$\rho\ m\ [\frac{V_1^2}{2} - \frac{V_2^2}{2} + \frac{p_1}{\rho} - \frac{p_2}{\rho}] = E_{Energy\ supplied\ to\ the\ turbine} \qquad with \qquad p_1 = p_2$$

$$\rho\ m\ [\frac{V_1^2}{2} - \frac{V_2^2}{2}] = E_{Energy\ supplied\ to\ the\ turbine} \tag{9}$$

The kinetic energy of the wind is transformed initially into potential energy and then into kinetic energy.

$$E_{kwind} \Rightarrow E_p \Rightarrow E_k$$

5 Conversion of potential energy into additional kinetic energy

There can be energy production only if we have kinetic energy and potential energy.

For a sailboat, Betz' theory applies Kimball (2009) . The fluid flow due to the wind and the forward speed of the sailboat creates an induced force on the sail. A small part of this force is used to move the sailboat forward, the other part creates stress on the sail and causes the boat to heel. Constraints are created on the keel to resist the heel. Instead of having static stresses on the keel, the trick of using hydrofoils allows a kinetic flow. This flow lifts the sailboat, creates a reaction force against the tilt and greatly improves the performance of the sailboat. The principle of transforming potential energy into kinetic energy increases performance and the wind has not been doubled.

5.1 Conversion potential energy

Potential energy is the source of stress in the turbine. It may be possible to convert potential energy into kinetic energy
In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stresses in the blades for a defined wind speed, are constant.

$$\frac{d\sigma}{d\beta} = 0$$

σ Stress in turbine blade($\frac{N}{m^2}$)
β rotation angle of the blades (rad)

In fact, some variations of the stresses are existing due to gravitational forces and the differencial velocity within the boundary layer depending on the elevation.

In the case of vertical axis turbines (VAWT, Darrieus type) the blade and arm stresses are depending on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \qquad \frac{1}{2\ \pi} \int_0^{2\ \pi} \sigma d\beta \quad = \quad \epsilon \ \ (\epsilon \ small)$$

During a half-turn, the arms are submitted to compression stresses whereas extending stresses are dominant during the next half-turn.

In the case of a HAWT, the conversion is not possible thanks to a dynamic mechanical system. The additionnal stresses are constant during a rotating for a given wind speed. Alternative stresses, encountered in a vertical axis wind turbine VAWT can allow the extraction of additional energy.

5.2 Total recoverable power of the turbine

To determine the power, one selects the power defined from the kinetic energy and, in the case of conversion of stress into a mechanical movement, the power defined from the potential energy. It may be possible to convert potential energy into kinetic energy.
The total recoverable power coefficient is

$$C_T \ = \ C_k \ + \ C_p$$

$C_p = 0$ when constraints are not converted

$$P_T = \ C_T \ \frac{1}{2} \rho S V_{fluid}^3 \ = \ (C_k \ + \ C_p) \ \frac{1}{2} \rho S V_{fluid}^3$$

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the power coefficient $C_{T\ HAWT}$ is

$$C_{T\ HAWT} \ = \ C_k \ = \ 4\ a^2\ (1-a) \quad C_p = 0$$

In the case of vertical axis turbines (VAWT, Darrieus type)
the power coefficient $C_{T\ VAWT\ Darrieus}$ is

$$C_{T\ VAWT\ Darrieus} \ = \ C_k \ = \ 4\ a^2\ (1-a) \quad C_p = 0$$

In the case of vertical axis turbines (VAWT with conversion), these stresses convert into additional energy, and the power coefficient $C_{T\ VAWT\ with\ conversion}$ is

$$C_{T\ VAWT\ with\ conversion} \ = \ C_k \ + \ C_p \ = \ 8\ a^2\ (1-a)$$

Various powers will then be defined by

$$P = \ C_T \ \frac{1}{2} \rho S V_{fluid}^3 \tag{10}$$

$$P_{T\ HAWT} = \ 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3 \tag{11}$$

$$P_{T\ VAWT\ Darrieus} = \ 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3 \tag{12}$$

$$P_{T\ VAWT\ with\ conversion} = \ 8\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3 \tag{13}$$

6 HAWT-VAWT comparison and discusion

Following the work of Hau (2000), the power coefficient of different turbines is compared (a performance of 0.6 is applied for the supplementary energy recovery system).

$$With\ a = \frac{2}{3} \qquad C_k \approx 60\% \qquad C_p \approx 60\%$$

notation $C_p\ w.c\ =\ C_p\ with\ conversion$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
$perfect$	C_k	60%	60%	60%
$perfect$	$C_p\ w.c$	0%	0%	60%
$perfect$	C_T	= 60%	= 60%	= 120%
$in\ practice$	C_k	$0.8 \times 60\% \approx 48\%$	$0.7 \times 60\% \approx 42\%$	$0.7 \times 60\% \approx 42\%$
$in\ practice$	$C_p\ w.c$	0%	0%	$0.6 \times 0.7 \times 60\% \approx 25\%$
$in\ practice$	C_T	= 48%	= 42%	= 67%
....				
gain/HAWT		+ 0%	−12 %	+ 39%
gain/$C_k(Betz)$		− 20%	−30 %	+ 11%

$$With\ a \approx 0.8 \qquad C_k \approx 50\% \qquad C_p \approx 50\%$$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
$in\ practice$	C_k	$0.8 \times 50\% \approx 40\%$	$0.7 \times 50\% \approx 35\%$	$0.7 \times 50\% \approx 35\%$
$in\ practice$	$C_p\ w.c$	0%	0%	$0.6 \times 0.7 \times 50\% \approx 21\%$
$in\ practice$	C_T	= 40%	= 35%	= 56%
....				
gain / HAWT		+ 0%	−12 %	+ 27%
gain / C_k		− 20%	−30 %	+ 12%

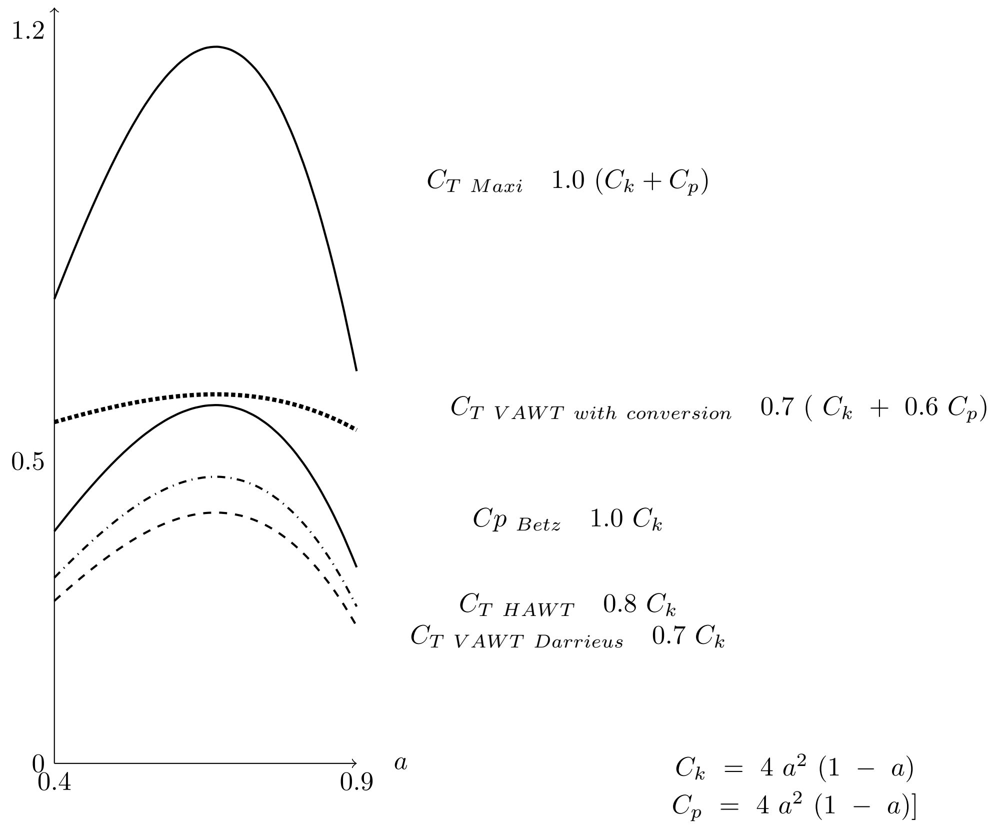


Figure4. Tip speed ratio performance curve;

6.1 Active lift turbine project

The project "Active lift turbine" is a transformation example of potential energy into kinetic energy (see preprint : Simplified theory of an active lift turbine with controlled displacement (Lecanu et al. (2016).

$$C_k = \frac{9\pi}{2^7}b^3 - \frac{2^3}{3}b^2 + \frac{\pi}{2}b$$

$$C_p = \frac{e}{R}\lambda(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

$$C_{active_lift_turbine} = (1 + \frac{e}{R}\lambda)(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

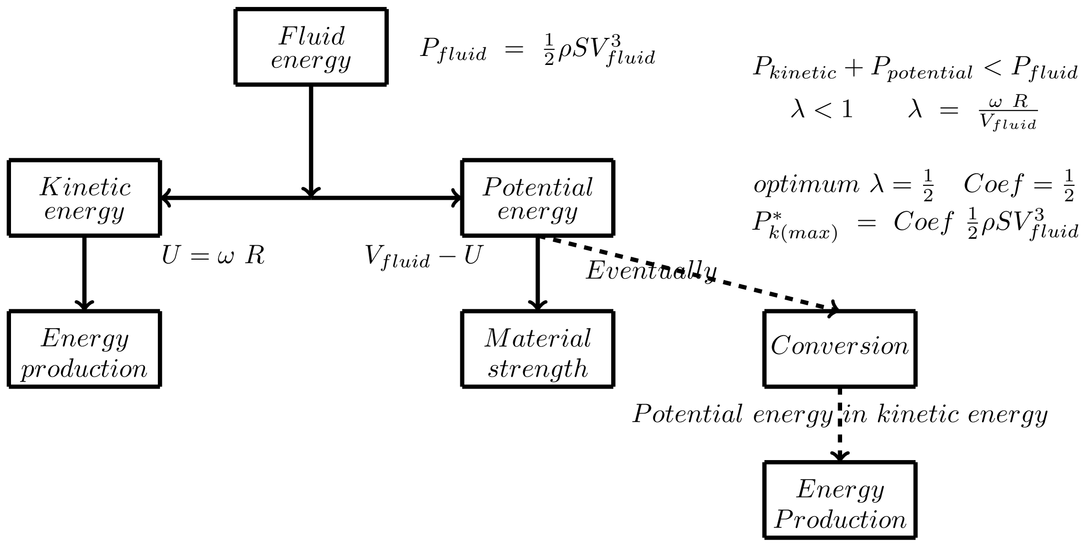
$$with \quad b = \sigma\lambda$$

σ stiffness coefficient
λ Tip speed ratio
e eccentric distance
R Turbine radius

6.2 Synthesis scheme:

6.2.1 Low speed turbine $\lambda < 1$:

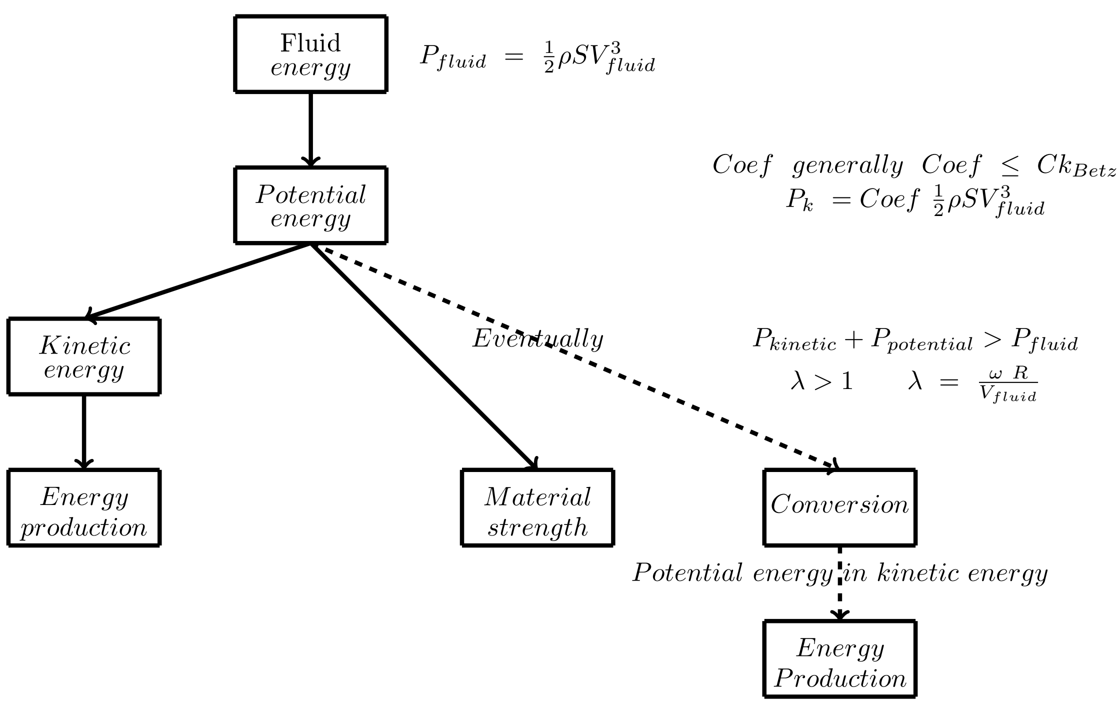
case : $\lambda < 1$ — turbine type Anemometer with cups



∗ to obtain an energy production

6.2.2 Fast speed turbine HAWT or VAWT $\lambda > 1$:

case : $\lambda > 1$ **— large wind turbines HAWT or VAWT**



In the case of a turbine with a lambda lower than 1, Of the recoverable energy , a part is transformed into kinetic energy and the other part into potential energy (stress in the turbine). Eventually a part of this potential energy could be transformed into kinetic energy.

In the case of a turbine with a lambda greater than 1, the recoverable energy defined by Betz, is transformed into potential energy. This energy creates kinetic energy and also creates potential energy (stress in the turbine). This is possible because we have created a speed of rotation higher than the speed of the fluid. A part of this potential energy (due to the stress) can be transformed into kinetic energy.

6.3 Synthesis on the extracted power:

With a mechanical conversion system, the powers for the different turbines are

$$P_{T\ HAWT} = \ 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

$$P_{T\ VAWT\ Darrieus} = \ 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

$$P_{T\ VAWT\ with\ conversion} = \ 8\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system is in practice from 20% to 50%.

Concerning a vertical axis turbine with a conversion system, the power factor is higher than the one defined by betz.

In the comparative table, a yield of 0.6 was chosen for the mechanical conversion system of the potential energy into mechanical energy. By choosing a powerful technology, this yield can be greatly increased, which will increase the performance of the turbine.

The definition of the maximum power coefficient is the one established by Betz which remains valid for the horizontal axis turbine HAWT and no longer makes sense for vertical axis turbines VAWT. The results given are examined a conversion of the potential energy into kinetic energy through a mechanical system which is not applicable for horizontal axis turbines HAWT. The calculation of the powers are the sum of the powers which one wants to consider.

The maximum power coefficient for a wind turbine or tidal turbine is

$$C_{T\ maxi} = \ \frac{32}{27}\ (\approx\ 118.5\%)$$

$C_{T\ maxi}$ is a limit value with $C_p\ \leq\ C_k$.

The coefficient is greater than 1 because the Betz limit only takes into account the kinetic power of the fluid.

Using piezo-electric materials would make it possible to transform potential energy into electrical energy. However, in order to achieve higher efficiency, the potential energy should be transformed into kinetic energy.

Thus, the power coefficient will be $C_T\ =\ C_k\ +\ C_p$ whatever the type of turbine.

Note : In the case of a turbine in a channel (see appendix 15), the maximum kinetic power coefficient is to 1. The maximum total power coefficient is then 200%.

7 Conclusion

Betz defined the maximum power coefficient $C_{k\ Betz}(=\frac{16}{27})$ of a wind turbine or tidal turbine from the calculation of kinetic energy.
To obtain an energy production, one cannot take into account only the kinetic energy. Energy production is linked to potential energy and kinetic energy.

In order to obtain productive kinetic energy, kinetic energy and potential energy are needed. The maximum productive kinetic power is equal to

$$\boldsymbol{P_{k-max-productive}\ \leq\ min(\ P_k\ ,\ P_p\ )}$$

The limit of this depends on whether it is a low or high speed turbine

$$P_{k(max)}\ =\ Coef\ \frac{1}{2}\rho S V_{fluid}^3$$

$$
\begin{array}{lcll}
Coef & = & \frac{1}{2} & \text{for low speed turbines } (\lambda < 1) \\
Coef & = & \frac{16}{21} & \text{for fast moving turbines } (\lambda > 1)
\end{array}
$$

For fast turbine, the stresses on the blades are defined from this force.

$$F_{maxi} = Ck_{Betz}\ \frac{1}{2}\ \rho\ S_{swept\ area}\ V_{fluid}^2$$

Instead of creating stress, using this force to create additional torque significantly increases turbine efficiency.

Taking into account the kinetic energy and the potential energy, the coefficient of maximum power becomes $C_{T\ maxi}(= \frac{32}{27})$:

Transforming potential energy into kinetic energy greatly increases turbine performance.

The maximum power becomes

$$P_{T\ maxi} = C_{T\ maxi}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3 \quad with \quad C_{T\ maxi} = \frac{32}{27}$$

It is impossible to recover more than 100% kinetic energy from a fluid. However, $C_{T\ maxi} > 1$ due to the conversion of the fluids potential energy into kinetic energy.

The proposed evolution of VAWT can be compared to the evolution of sailboats. Square sails evolved to sails with a wing profile and now there are sailboats with hydrofoils. With the same source of wind, by using relative speed, the performance of the sailboats was improved and by transforming the stresses on the keel by flowing around the profile, we were able to improve again the performance of the sailboats.

The use of a fast speed wind turbine rather than a Savonius turbine means improved efficiency due to the use of relative speed.

Using an active lift turbine instead of a fast speed turbine improves efficiency due to the fact that stresses are transformed into extra torque.

A : Appendix

Maximum wind power recovered from kinetic energy

$$E_k = \frac{1}{2}\, m\, V^2$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\frac{dm}{dt}\, V^2 + \frac{1}{2}\, m\, \frac{dV^2}{dt} \qquad \frac{dV}{dt} = 0$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\dot{m}\, V^2 = \frac{1}{2}\rho\, S\, V^3$$

V Wind speed at the turbine level
Force applied by the wind on the rotor

$$F = m\frac{dV}{dt} = \dot{m}\Delta V \; = \; \rho SV\,(V_{fluid} \; - \; V_{wake})$$

V_{wake} streamwise velocity in the far wake

$$P = FV \; = \; \rho SV^2\,(V_{fluid} \; - \; V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2}mV_{fluid}^2 - \frac{1}{2}mV_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} \; = \; \frac{1}{2}\dot{m}(V_{fluid}^2 \; - \; V_{wake}^2) = \; \frac{1}{2}\rho SV(V_{fluid}^2 \; - \; V_{wake}^2)$$

From theses equalities

$$V = \bar{V} = \frac{V_{fluid} + V_{wake}}{2}$$

$$F = \; \rho SV\,(V_{fluid} \; - \; V_{wake}) \; = \; \frac{1}{2}\rho S(V_{fluid}^2 \; - \; V_{wake}^2)$$

$$P = FV \; = \; \rho SV^2\,(V_{fluid} \; - \; V_{wake})$$

defining $a = \frac{V}{V_{fluid}}$

$$V_{wake} \;\; = \;\; V_{fluid}\,(2\;a \;\; - \;\; 1) \quad as \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P = \;\; 4\;a^2\;(1-a)\;\frac{1}{2}\;\rho SV_{fluid}^3$$

defining power coefficient $C_k = \frac{P}{\frac{1}{2}\rho SV_{fluid}^3} \;\; = \;\; 4\;a^2\;(1-a)$
Search of maximum power coefficient

$$\frac{dC_k}{da} = 0 \quad a\;(2 \;\; - \;\; 3\;a) \;\; = \;\; 0 \qquad a = 0 \qquad or \qquad a = \frac{2}{3}$$

$$a = \frac{2}{3} \qquad C_k = \frac{16}{27} = 0.593$$

The maximum power coefficient $C_{k\,maxi}$ is defined by Betz

$$C_{k\,maxi} \;\; = \;\; C_{k\,Betz} = \frac{16}{27} \approx 60\%$$

The maximum power of the fluid is

$$P_{fluid} = \frac{1}{2}\rho S_{fluid} V_{fluid}^3$$

$S_{fluid} \; = \; \frac{S \; V}{V_{fluid}} \; = \; a \; S$
The power of the turbine is

$$P = \frac{C_k}{a} \; P_{fluid} \; = \; C_k \; \frac{1}{2}\rho \frac{S_{fluid}}{a} V_{fluid}^3 \; = \; C_k \; \frac{1}{2}\rho S V_{fluid}^3$$

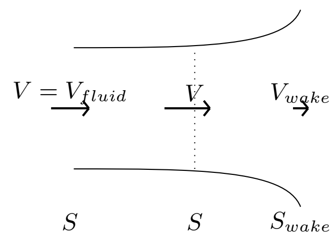
The maximum power of the turbine is

$$P_{max} = \frac{C_{k \; Betz}}{\frac{2}{3}} \; P_{fluid} \; = \; \frac{8}{9} \; P_{fluid} \; = \; C_{k \; Betz} \; \frac{1}{2}\rho S V_{fluid}^3 \; = \; \frac{16}{27} \; (\frac{1}{2}\rho S V_{fluid}^3)$$

$$P_{max} \; = \; C_{k \; Betz} \; \frac{1}{2}\rho S V_{fluid}^3 \tag{14}$$

B : Appendix

Maximum wind power for a turbine in a channel (from kinetic energy)



$$E_k = \frac{1}{2}\, m\, V^2$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\frac{dm}{dt}\, V^2 + \frac{1}{2}\, m\, \frac{dV^2}{dt} \qquad \frac{dV}{dt} = 0$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\dot{m}\, V^2 = \frac{1}{2}\rho\, S\, V^3$$

V Wind speed at the turbine level

Force applied by the wind on the rotor

$$F = m\frac{dV}{dt} = \dot{m}\Delta V \;=\; \rho S V\,(V \;-\; V_{wake})$$

V_{wake} streamwise velocity in the far wake

$$P = FV \;=\; \rho S V^2\,(V \;-\; V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2}mV^2 - \frac{1}{2}mV_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} \;=\; \frac{1}{2}\dot{m}(V^2 \;-\; V_{wake}^2) = \frac{1}{2}\rho S V(V^2 \;-\; V_{wake}^2)$$

From theses equalities

$$V_{wake} \;=\; \frac{1}{2}\; V \qquad S_{wake} = \frac{S\; V}{V_{wake}} \;=\; 2\; S$$

$$F \;=\; \rho S V\;(V_{fluid} \;-\; V_{wake}) \;=\; \frac{1}{2}\rho S(V_{fluid}^2 \;-\; V_{wake}^2)$$

$$P = FV \;=\; \rho S V^2\,(V_{fluid} \;-\; V_{wake})$$

$$P = \;\frac{1}{2}\;\rho S V_{fluid}^3$$

defining power coefficient

$$C_k = \frac{P}{\frac{1}{2}\rho S V_{fluid}^3} \;=\; 1 \tag{15}$$

C : Appendix

Additional recovery power (from potential energy)

The fluid creates stresses in the blade. They are due to thrust force. The energy of this force is

$$E_p \ = \ m \ \frac{f_s}{\rho}$$

f_s thrust force

For HAWT horizontal wind turbines (fast wind turbine type), the thrust force F_s are constant.

$$\frac{dE_p}{dt} \ = \ 0$$

For a VAWT, the thrust force depends on the time or the rotation angle $F_s(t)$ or $F_s(\beta)$ $\quad \beta = \omega \ t$

$\omega \ = \ \frac{d\dot{\beta}}{dt}$ angular frequency

$$\frac{dE_p}{dt} \neq 0 \quad and \quad \frac{1}{2\ \pi} \int_0^{2\ \pi} E_p(\beta) d\beta \ = \ \epsilon \ \ (\epsilon \ small)$$

As

$$F_s \ = \ f_s S \ = \ C_x \frac{1}{2} \ \rho \ S \ V_{fluid}^2$$

$$E_p \ = \ m \ C_x \frac{1}{2} \ V_{fluid}^2$$

V fluid speed at the level turbine

The power is

$$P_p \ = \ \frac{dE_p}{dt} \quad dm = \rho S V dt$$

$$P_p \ = \ \frac{dE_p}{dt} \ = \ \frac{dm}{dt} \ C_x \ \frac{1}{2} \ S \ V_{fluid}^2 \ + \ m \ C_x \frac{1}{2} \ S \ \frac{dV_{fluid}^2}{dt} \qquad \frac{dV_{fluid}}{dt} \ = \ 0$$

$$P_p \ = \ a \ C_x \ \frac{1}{2} \ \rho \ S \ V_{fluid}^3 \quad with \ a = \frac{V}{V_{fluid}} \tag{16}$$

$$\frac{1}{2\ \pi} \int_0^{2\ \pi} E_p(\beta) d\beta \ = \ \epsilon \ \ (\epsilon \ small) \quad E_{p-max} \ \approx \ -E_{p-min}$$

the power depends on a potential energy difference

$$P_p \ = \ \frac{\Delta E_p}{\Delta t} \quad T \ = \ \frac{2\ \pi}{\omega} \quad P_p \ \leq \ \frac{E_{p-max} - E_{p-min}}{T} \quad P_p \ \leq \ \frac{E_{p-max}}{\pi} \ \omega$$

for a half-turn

$$E_p(\beta) \ R \ d\beta \ = \ dE_p \ \pi \ R$$

in particular

$$E_{p-max} \ = \ \frac{dE_p}{d\beta} \pi \ = \ \frac{dE_p}{dt} \ \frac{\pi}{\omega}$$

As

$$\frac{dE_p}{dt} \ = \ a \ C_x \ \frac{1}{2} \ \rho \ S \ V_{fluid}^3 \quad and \quad P_p \ \leq \ \frac{E_{p-max}}{\pi} \ \omega$$

$$P_p \ \leq \ a \ C_x \ \frac{1}{2} \ \rho \ S \ V_{fluid}^3$$

D : Appendix

Variation of energy in opposite sense

Along a streamline, the Bernoulli's equation is see Bernoulli (1738)

$$\frac{p}{\rho} + \frac{v^2}{2} = constant \quad with \;\; z = 0$$

By multiplying by m

$$m\,\frac{p}{\rho} + m\,\frac{v^2}{2} = constant$$

The differential of this equation is

$$d(\frac{1}{2}\,m\,v^2) = -\,d(m\,\frac{p}{\rho}) \tag{17}$$

the variations of energy vary simultaneously and in opposite sense.

$$As \quad \frac{dV}{dt} = 0 \quad and \quad \frac{dp}{dt} = 0 \quad p = -\,\frac{1}{2}\,\rho\,v^2$$

References

D. Bernoulli. *Hydrodynamica*. doi:10.3931, eth-zurich e-rara-3911 edition, 1738.

A. Betz. Das maximum der theoretisch moglichen ausnutzung des windesdurch windmotoren. *Zeitschrift fur das gesamte Turbinenwesen*, 0(26):307–309, 1920.

D. Broberg, D. Shah, S. Drapcho, and A. Brockway. Getting more out of the wind: Extending betz's law to multiple turbines. *arXiv preprint arXiv:1805.01938*, 2018.

T. Burton, N. Jenkins, D. Sharpe, and E. Bossanyi. *Wind Energy Handbook, 2nd Edition*, volume isbn= 978-0-470-69975-1. Wiley.

D. Georgiou and N. Theodoropoulos. The loading of water current turbines: The betz limit and ducted turbines. *Alternative Energy and Shale Gas Encyclopedia*, pages 601–605, 2016.

E. Hau. *Wind Turbines*. Springer Germany, 2000.

J. Kimball. *Physics of Sailing*. CRC Press, 2009. ISBN 9781420073775. URL `https://books.google.fr/books?id=Xe_i23UL4sAC`.

P. Lecanu, J. Bréard, and D. Mouazé. Simplified theory of an active lift turbine with controlled displacement. Apr. 2016. URL `https://hal.inria.fr/hal-01300531`.

I. Paraschivoiu. Wind turbine design : with emphasis on darrieus concept, Jul 2002.

J. Prescott Joule. *The Scientific Papers of James Prescott Joule*, volume volume 1 : isbn= 978-0-470-69975-1. Cambridge Library Collection - Physical Sciences, 2011.

E. Quaranta. Stream water wheels as renewable energy supply in flowing water: Theoretical considerations, performance assessment and design recommendations. *Energy for Sustainable Development*, 45:96–109, 2018.

G. A. van Kuik. The lanchester–betz–joukowsky limit. *Wind Energy*, 10(3):289–291, 2007. doi: https://doi.org/10.1002/we.218. URL `https://onlinelibrary.wiley.com/doi/abs/10.1002/we.218`.

R. Vennell. Exceeding the betz limit with tidal turbines. *Renewable Energy*, 55:277–285, 2013.