Theoretical calculation of the power of wind turbine or tidal turbine

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Theoretical Calculation of Wind (Or Water) Turbine

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Abstract :
The Betz limit sets a theoretical upper limit for the energy output of turbines, expressed as a maximum power coefficient of 16/27. Betz's theory is precise and based on the calculation of kinetic energy. However, taking into account the potential energy, the theoretical energy output of a turbine can be higher. Fast wind turbines optimally recover the wind's kinetic energy while a large amount of potential energy is created without being recovered. The present article considers this potential energy and the possibility for a wind turbine to transform it into kinetic energy. The Betz limit remains valid for horizontal axis turbine HAWT but can be extended for vertical axis turbines VAWT.
The following article considers converting the potential energy into kinetic energy through a mechanical system which can only be fitted to vertical axis turbines VAWT.

Keywords : Betz limit, Betz's law, Wind turbine, Tidal turbine, HAWT, VAWT.

1 Introduction

Lanchester, Betz, Joukowsky van Kuik (2007) van Kuik et al. (2015) Lanchester (1915) have defined the maximum power coefficient of a wind turbine. This limit is commonly called the Betz limit. Considerable research efforts have been deployed to optimize wind turbines in order to reach this limit, for instance by optimizing the angle of incidence, the shape of the blade profile etc. One may for example refer to "Wind Energy Handbook"Burton et al. for fast moving horizontal axis wind turbines (HAWT), to "Hydrodynamic modelling of marine renewable energy devices: A state of the art review" Day et al. (2015) or to "Wind Turbine Design: With Emphasis on Darrieus Concept" Paraschivoiu (2002) for vertical axis wind turbines (VAWT).

Research has been made to reach the Betz limit: - by ducting the turbine (Georgiou (2016) Georgiou and Theodoropoulos (2016)) - by placing a water turbine in a narrow channel, which allows the water level upstream to be increased by the resistance to the advancing fluid (Quaranta (2018) (Quaranta (2018))) - by grouping turbines in wind farms in order to create an excess of power due to the proximity between machines (ducting effect) (Vennell (2013) Vennell (2013) and Broberg (2018) Broberg et al. (2018)).

Betz' limit is based on the calculation of kinetic energy. Designing a wind turbine, the energy of the fluid is taken into account in order to calculate the recoverable energy and to design a structure withstanding the stress.

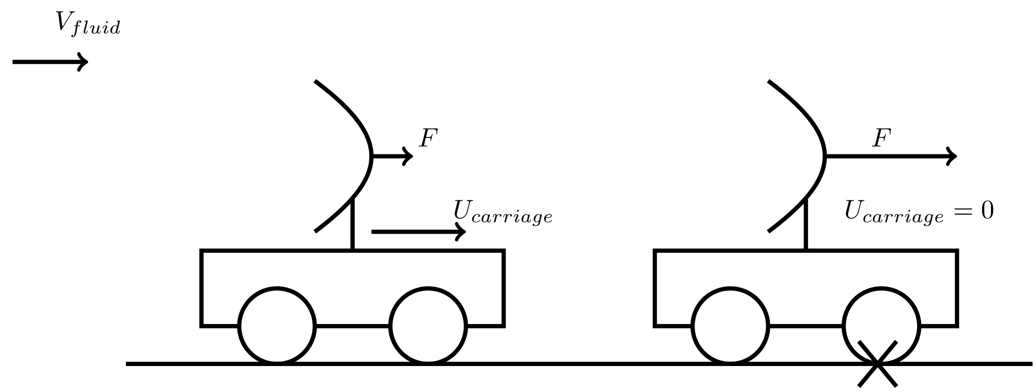


fig. 0

A carriage with a square sail that can move freely, will be set in motion by the wind and thus be subject to very low wind stress. If the carriage is blocked, its structure will be subject to high stress.

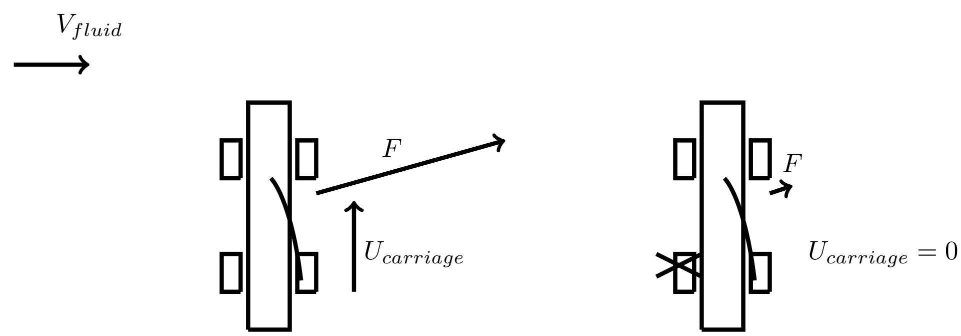


fig. 1

A carriage with a modern sail that can move freely will be set in motion by the wind and subject to high stress from the wind. However, if the carriage is blocked, it will be subject to low stress on its structure.

In the case of fast moving wind turbines, when the wind increases, the turbine structure is subject to high stress.

Large wind turbines are stopped when the wind becomes too strong, not because they produce too much energy but because excessive stress could damage or destroy their structure. The stress induced in a structure is potential energy. Betz's theory does not take into account this potential energy. It exclusively takes into consideration kinetic energy.

In the case of a sailboat, the sole sail cannot recover all the energy from the wind. Betz' theory applies to the sail that cannot retrieve more than 16/27 of the wind's kinetic energy. Many books explore the Betz limit, for instance John Kimball's "Physics of Sailing"Kimball (2009).

However, the America's Cup is a good example showing that the Betz limit can be exceeded. Indeed, a sailboat equipped with hydrofoils transforms potential energy into kinetic energy. The kinetic energy of the wind sets the boat in motion. The fluid flowing around the hydrofoils lifts the boat out of the water, thus reducing the drag and improving efficiency. The wind power has not increased but the potential energy that applied constraints on the keel has been converted into kinetic energy by the flow of fluid around the profiles of the foils.

This article introduces the notions of kinetic and potential energy. It suggests that a vertical axis wind turbine (VAWT) with an appropriate design can transform potential energy into kinetic energy.

2 Preliminary considerations of Betz's theory

The German mathematician A. Betz has demonstrated that the power of a turbine is (cf. Appendix 14, Betz (1920)):

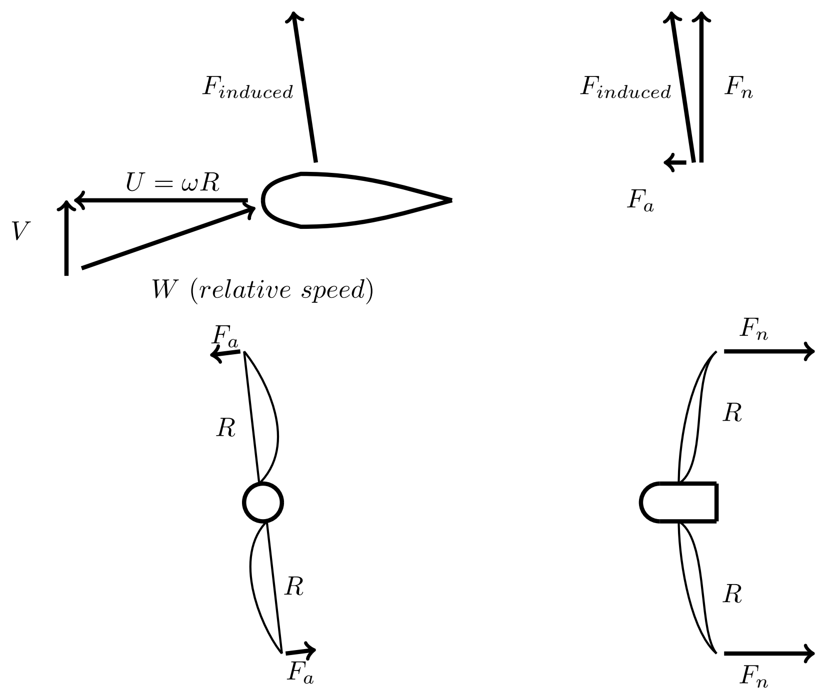
$$P_{kin} = F\ V\ = [\ \frac{C_{kin}}{a}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^2]\ a\ V_{fluid} = C_{kin}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

$$where \quad V \ = \ a\, V_{fluid} \quad and \quad C_{kin} = 4a^2(1-a)$$

P_{kin} the kinetic turbine power (W),
C_{kin} the kinetic power coefficient,
ρ the fluid density $(\frac{kg}{m^3})$,
V_{fluid} the fluid velocity$(\frac{m}{s})$,
V the fluid velocity at the position of the turbine,
S the swept area.(m^2).

According to Betz 's work (preliminarily by Lanchester), a kinetic energy approach shows that the maximum power coefficient C_k can not exceed a maximum of $\frac{16}{27}$

$$C_{kin\ Betz} = \frac{16}{27} \qquad C_{kin} \leq C_{kin\ Betz} \qquad P_{kin\ maxi} = C_{kin\ Betz}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3 \tag{1}$$



tow-bladed HAWT wind turbine.;

fig. 2

The relative speed due to the rotation speed of the wind turbine and the fluid speed creates an induced force F on the profile. This induced force F is composed of an axial force Fa and of a normal force Fn. The axial force Fa associated with the radius R creates a driving torque to produce energy. The normal force Fn associated with the same radius R creates a bending stress on the blade.

The power of the torque due to the axial forces and the angular rotation speed of the wind turbine is limited to 16/27 of the wind's kinetic power.

The forces Fn and Fa are associated with the same radius and have the same origin: the fluid velocity. Fn creates constraints that are the source of internal energy. These constraints are potential energy. Betz's theory does not take into account this potential energy which is as important as the kinetic energy. Betz limit is correct. However, it only takes into account the kinetic energy. In order to increase the efficiency of wind and hydrokinetic turbines, they could be designed to transform the potential energy into kinetic energy. It is necessary to dissociate slow speed turbines and fast speed turbines.
$\lambda = \frac{\omega R}{V_{fluid}}$ is an important parameter which makes a difference in the behaviour of the turbines when exposed to the wind.
As for the sail, it is necessary to dissociate the navigation in thrust (square sail, spinnaker) (unstuck flow) and the navigation in smoothness (wing profile sail) (laminar flow).

2.1 Steady state, flow conservation and energy conservation

2.1.1 Flow conservation

In steady state, the fluid velocity is constant in time: $\frac{dV}{dt} \; = \; 0$
With a constant flow, the continuity equation is obtained

$$S_1V_1 \; = \; SV \; = \; S_2V_2 \qquad\qquad V_1 \; = \; V_{fluid}$$

2.1.2 Energy, power and kinetic force

The index *kin* for "kinetic" is used

$$E_{kin} = \frac{1}{2}mv^2 \qquad P_{kin} = \frac{dE_{kin}}{dt} = \frac{1}{2}\frac{dm}{dt}v^2 + \frac{1}{2}m\frac{dv^2}{dt} \qquad \frac{dv}{dt} = 0 \qquad m = \rho svdt \qquad P_{kin} = \frac{1}{2}\rho sv^3 \qquad F_{kin} = \frac{P_{kin}}{v} = \frac{1}{2}\rho sv^2$$

2.1.3 Energy, power and potential force

The index *pot* for "potential" is used

$$E_{pot} = m\frac{p}{\rho} \qquad P_{pot} = \frac{dE_{pot}}{dt} = \frac{dm}{dt}\frac{p}{\rho} + m\frac{1}{\rho}\frac{dp}{dt} \qquad \frac{dp}{dt} = 0 \qquad m = \rho svdt \qquad P_{pot} = svp \qquad F_{pot} = \frac{P_{pot}}{v} = ps$$

2.1.4 Energy conservation (Prescott Joule (2011))

$$E_{kin} \; + \; E_{pot} \; = \; E_{kin-fluid}$$

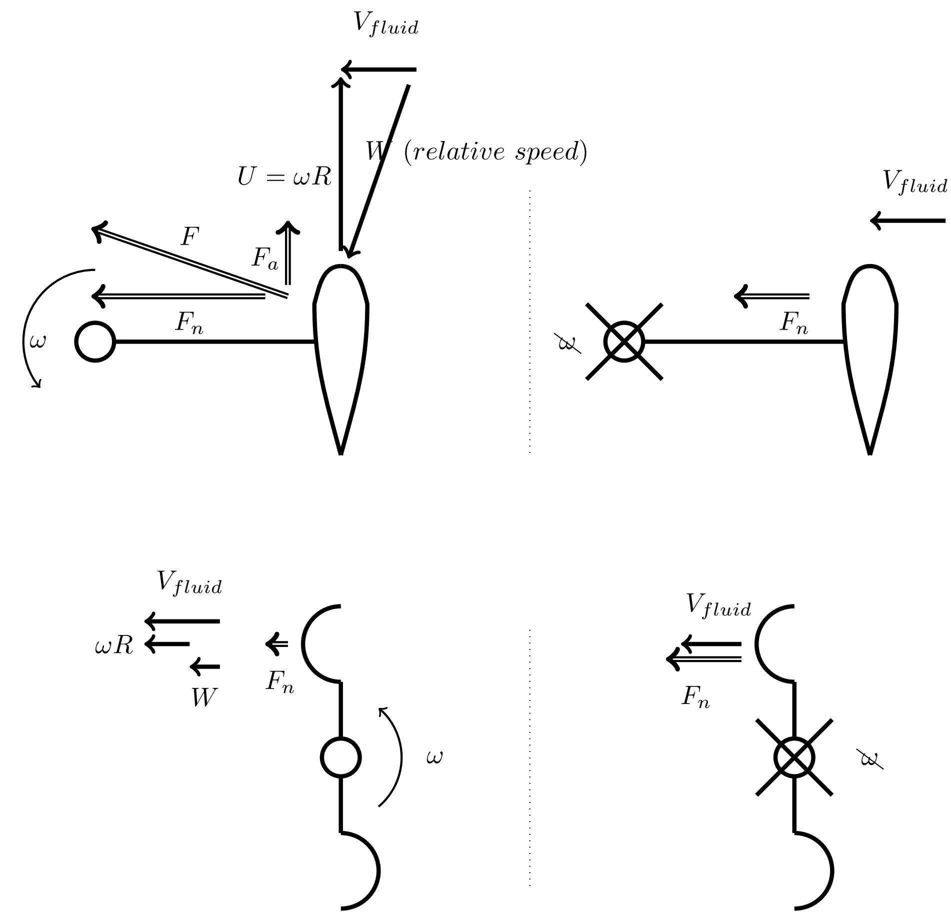
In steady state, $\frac{dE_{kin-fluid}}{dt} \; = \; 0$

$$\Rightarrow \qquad \frac{dE_{kin}}{dt} \; = \; - \, \frac{dE_{pot}}{dt}$$

3 Difference between a low speed turbine and a high speed turbine

3.1 The advantage of using an airplane wing

Fast moving turbines can not be described as slow moving turbines such as Savionus. The comparison is made between the airfoil of a fast moving turbine and the cup of an anemometer.



comparison of low and high speed turbine.;
fig. 3

If the movement of an anemometer is not impeeded, then its cups are almost free of stress. If the rotation of an anemometer is hampered, then its cups are subject to stress.
In the case of a wing profile, this effect is totaled different. Indeed, the tangential rotational speed U of the profile is transverse to the flow of the fluid.
The relative speed W due to the rotational speed U and the fluid speed V_{fluid} creates an induced force F on the wind profile. The axial component F_a of this force F, combined with the radius, creates a driving torque.

For *horizontal axis wind turbines* (HAWT), this torque improves the efficiency which can then approach the Betz limit. Adding a transverse speed U will optimize the efficiency defined by Betz, but will add significant stress on the wind turbine due to the normal component F_n of force F. That's why HAWT wind turbines have to be stopped when the speed of the fluid is too high. They do not produce too much but they are subject to excessive bending stress.

Anemometer ($\lambda < 1$)

case	$U = \omega R$	W	F_n	$F_n R$
blocked	0	$= V_{fluid}$	*max*	0
free	↗	↘	↘	↗

case	$U = \omega R$	W	*stress*	*torque*
blocked	0	$= V_{fluid}$	*max*	*null*
free	↗	↘	↘	↗

case	$U = \omega R$	W	*potential energy*	*kinetic energy*
blocked	0	$= V_{fluid}$	*max*	*null*
free	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

Horizontal axis wind turbines ***($\lambda > 1$)*** **(HAWT)**

case	$U = \omega R$	W	F_n	$F_a R$
blocked	0	$= V_{fluid}$	*min*	0
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

case	$U = \omega R$	W	*stress*	*torque*
blocked	0	$= V_{fluid}$	*min*	*null*
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

case	$U = \omega R$	W	*potential energy*	*kinetic energy*
blocked	0	$= V_{fluid}$	*min*	*null*
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

In the case of an anemometer, the kinetic energy varies in a direction opposite to the potential energy. This is not the case for HAWT turbines.
In the case of HAWT turbines, we must not only consider the speed of the fluid. The kinetic energy and potential energy are related to the fluid speed and to the tangential rotation speed U. The direction of U is perpendicular to the direction of the fluid.
The tangential rotation speed U is related to the fluid speed by this relation $\lambda = \frac{U}{V_{fluid}} \quad U = \omega\ R.$

In the case of an anemometer and in the case of a wing profile, we have this equation $E_{kin} + E_{pot} = E_{kin-fluid}$.
The conservation of energy is not verified in the case of a wing profile.
The variations of potential energy and energy increase when it is not blocked.

3.1.1 Low speed turbines $\lambda < 1$:

Using the example as an anemometer with cups, when there is no resistant torque, the turbine rotates at maximum speed ($\omega R = V_{fluid} \quad W = 0$), there is no energy production and the kinetic energy is maximum.
When the turbine is blocked, the stress on the turbine is maximum ($W = V_{fluid}$) and the speed of rotation is null. The potential energy is maximum.
To obtain energy production, both kinetic and potential energy are required.
If the kinetic energy is higher than the potential energy, the energy production is limited by the potential energy.
Similarly, if the potential energy is higher than the kinetic energy, the energy production is limited by the kinetic energy.

There can be no production of kinetic energy without potential energy.

$$P_{kin-max-productive} \leq min(\ P_{kin}\ ,\ P_{pot}\ )$$

Energy production is at its maximum when kinetic energy is equal to the potential energy.
Total energy is equal to the sum of the kinetic energy and the potential energy.

$$E_{total} = E_{kin} + E_{pot}$$

Conservation of energy can be applied and verified.

$$\frac{dE_{total}}{dt} = 0 \qquad \rightarrow \qquad \frac{dE_{kin}}{dt} = -\frac{dE_{pot}}{dt} \tag{2}$$

The potential energy is related to the relative speed W. The kinetic energy is related to the relative speed ωR (see Figure 2).

The tangential rotation speed must be equal to half of the fluid speed to reach optimal conditions.

$$|V_{fluid}| = |U| + |W| \qquad |U| = |W| \qquad U = \omega R = \frac{V_{fluid}}{2} \qquad \lambda = \frac{1}{2} \tag{3}$$

To produce energy, the maximum power of the kinetic energy of a turbine at low speed ($\lambda < 1 \quad \lambda = \frac{\omega R}{V_{fluid}}$) is equal to

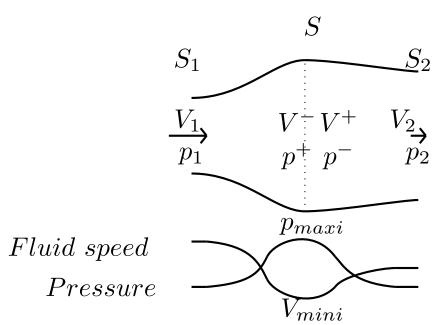
$$P_{kin\ max(with\ \lambda<1)} \ = \ C_{kin(with\ \lambda<1)} \frac{1}{2}\rho S V_{fluid}^3 \qquad C_{kin\ max(with\ \lambda<1)} = \frac{1}{2} \qquad E_{kin} = \frac{E_{kin-wind}}{2} \tag{4}$$

For a low speed turbine, the wind 's kinetic energy is partially transformed into kinetic and potential energy.

$$E_{kinwind} \ \Rightarrow \ E_{kin} \ \& \ E_{pot} \tag{5}$$

4 Calculation of energy

4.1 Calculation of energy $\lambda \ < \ 1$



Current tube.;

fig. 4

4.1.1 Conservation of steady state energy applied in the area upstream of the turbine

$$\frac{1}{2}\rho S(V^-)^3 \ + \ SV^- p^+ \ = \ \frac{1}{2}\rho S_1 {V_1}^3 \qquad => \qquad \frac{1}{2}\rho (V^-)^2 \ + \ p^+ \ = \ \frac{1}{2}\rho \frac{S_1}{S} {V_1}^2$$

$$\frac{1}{2}\rho S {V_{mini}}^2 \ + \ p_{maxi} \ = \ \frac{1}{2}\rho S_1 {V_1}^2$$

$$\frac{d}{dt}(\frac{1}{2}\rho S(V^-)^2 \ + \ p^+) \ = \ 0 \qquad => \qquad {V_{mini}}^2 \ = \ \frac{1}{2}\frac{S_1}{S}{V_1}^2$$

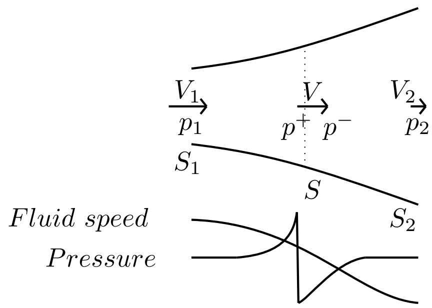
$$if \quad S_1 \ = \ \frac{1}{2} \ S \quad then \quad V_{mini} \ = \ \frac{1}{2} \ V_1$$

the kinetic energy of the fluid is transformed into kinetic and potential energy.

$$E_{kwind} \ \Rightarrow \ E_{pot} \ \& \ E_{kin}$$

see 3.1.1

4.2 Calculation of energy $\lambda > 1$



current tube.;

fig. 5

4.2.1 Calculation of the turbine power from the kinetic energy

The force applied to the rotor is $F_{kin} = m\frac{dV}{dt} = \frac{dm}{dt}\Delta V = \rho SV(V_1 - V_2)$

$$P_{kin} = F_{kin}V = \rho SV^2(V_1 - V_2)$$

$$P_{kin} = \frac{\Delta E_{kin}}{dt} = \frac{1}{2}\rho SV^2(V_1^2 - V_2^2)$$

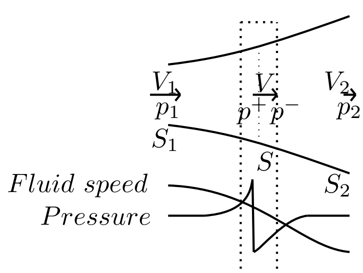
It is deducted $V = \frac{V_1+V_2}{2}$
By defining a as $a = \frac{V}{V_{fluid}}$ $\quad V_{fluid} = V_1 \quad 0 \le a \le 1 \quad V_2 = V_1(2a-1) \quad V_2 \ge 0 \quad a \ge \frac{1}{2}$

$$P_k = 4a^2(1-a)\frac{1}{2}\rho SV_{fluid}^3 \qquad with \qquad C_{kin} = 4a^2(1-a) \qquad P_{kin} = C_{kin}\frac{1}{2}\rho SV_{fluid}^3$$

Searching for the maximum kinetic power (Betz limit) :

$$\frac{dP_{kin}}{dt} = 0 \qquad a(2-3a) = 0 \qquad as \qquad a \ge \frac{1}{2} \qquad a = \frac{2}{3} \qquad C_{kin-maxi} = \frac{16}{27} = C_{kin-Betz}$$

4.2.2 Energy in stationary state applied in the turbine area



turbine area

fig. 6

$$Ekin_+ \ + \ Epot_+ \ = \ Ekin_- \ + \ Epot_- = \ E_{pot}$$

$$\rho\ m\ [\frac{V_+^2}{2} \ - \ \frac{V_-^2}{2} \ + \ \frac{p_+}{\rho} \ - \ \frac{p_-}{\rho}] \ = \ E_{pot} \qquad with \qquad V_+ \ = \ V_- \ = \ V$$

$$\rho\ m\ [\frac{p_+}{\rho} \ - \ \frac{p_-}{\rho}] \ = \ E_{pot} \tag{6}$$

4.2.3 Energy in stationary state applied to the whole turbine

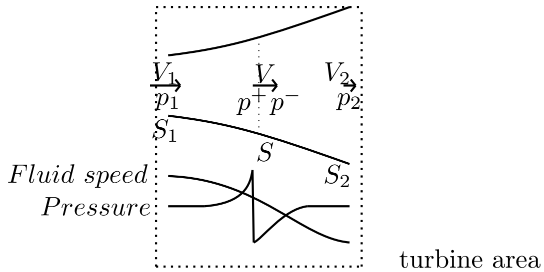


fig. 7

$$Ekin_1 \ + \ Epot_1 \ = \ Ekin_2 \ + \ Epot_2 \ + \ E_{kin}$$

$$\rho \ m \ [\frac{V_1^2}{2} \ - \ \frac{V_2^2}{2} \ + \ \frac{p_1}{\rho} \ - \ \frac{p_2}{\rho}] \ = \ E_{kin} \qquad with \qquad p_1 \ = \ p_2$$

$$\rho \ m \ [\frac{V_1^2}{2} \ - \ \frac{V_2^2}{2}] \ = \ E_{kin} \tag{7}$$

4.2.4 Remark

The Bernoulli equations

$$p_1 + \frac{1}{2}\rho s_1 V_1 = p^+ + \frac{1}{2}\rho s V \qquad p^- + \frac{1}{2}\rho s V = p_2 + \frac{1}{2}\rho s_2 V_2$$

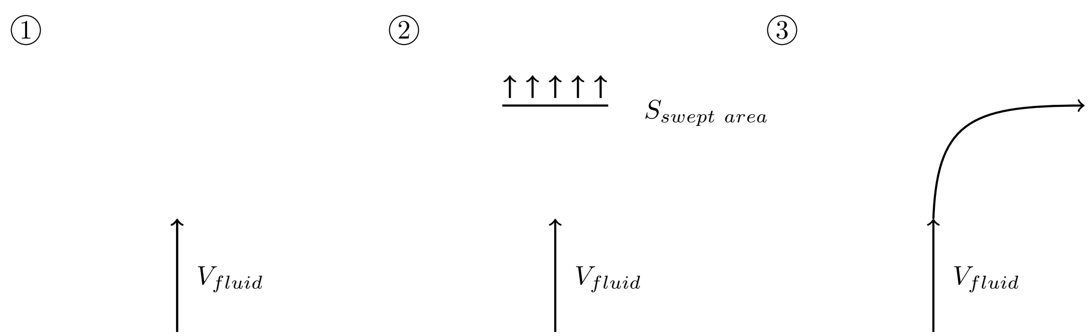
$$V_1 = V_{fluid} \qquad P_1 = P_2 \qquad \rightarrow \qquad \frac{V_1^2}{2} \ - \ \frac{V_2^2}{2} = \frac{p_+}{\rho} \ - \ \frac{p_-}{\rho}$$

by using the equation (6),

$$E_{pot} = E_{kin}$$

The kinetic energy of the fluid is partially transformed into potential energy and then into kinetic energy

$$E_{kwind} \ \Rightarrow \ E_{pot} \ \Rightarrow \ E_{kin} \tag{8}$$



energy transfer stage;
fig. 8

① Kinetic energy of the fluid
② Stresses are created on the turbine
③ The change of direction of the fluid creates a driving torque to generate energy

$$E_{kwind} \ \Rightarrow \ E_{pot} \ \Rightarrow \ E_{kin} \tag{9}$$

$$① \Rightarrow ② \Rightarrow ③$$

The fluid ① creates stress on the surface swept ② by the turbine, then ther is a change of direction ③ of the fluid .

Successively, there is a transformation of energy from the kinetic energy of the fluid into potential energy, then into kinetic energy.

$$E_{kwind} \Rightarrow E_{pot} \Rightarrow E_{kin}$$

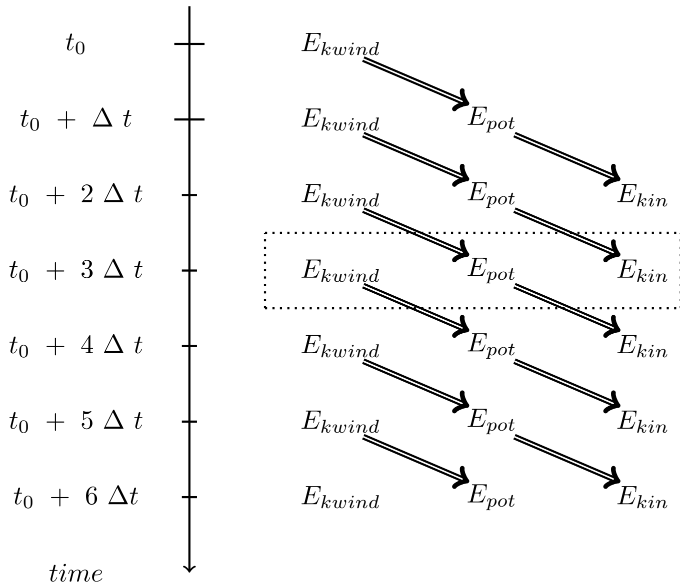


fig. 9

Successively, there is a transformation of energy from the kinetic energy of the fluid into potential energy, and from this into kinetic energy.
As the regime is stationary, although it is a transfer of energies, we have a summation of energies.

As there is a transfer of temporal energy $E_{kwind}(t - 2\Delta\ t) \Rightarrow E_{pot}(t - \Delta\ t) \Rightarrow E_{kin}(t)$, in the case of an increase in the speed of the fluid, we have successively these energy variations

$$\frac{dE_{kwind}(t - 2\Delta\ t)}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{pot}(t - \Delta\ t)}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{kin}(t)}{dt} > 0$$

This formulation allows to be consistent with the conservation of energy.

remark : The kinetic energy $E_{kin}(v)$ and the potential energy $E_{pot}(p)$ are not one energy and one co-energy Borel (1965) Max Marty (2005) . The pressure varies according to the square of the speed.

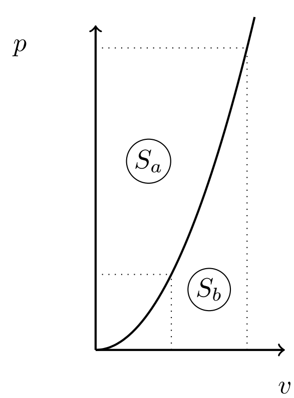


fig. 10

When the fluid velocity increases, the energy variations are both positive, but the area of S_a is different from S_b.

5 Conversion of potential energy into additional kinetic energy

There can be energy production only if we have kinetic energy and potential energy.
For a sailboat, Betz' theory applies Kimball (2009). The fluid flow due to the wind and the forward speed of the sailboat creates an induced force on the sail. A small part of this force is used to move the sailboat forward, the other part creates stress on the sail and causes the boat to heel. Constraints are created on the keel to resist the heel. Instead of having static stresses on the keel, using hydrofoils allows a kinetic flow. This flow lifts the sailboat, creates a reaction force against the tilt and greatly improves the performance of the sailboat. The principle of transforming potential energy into kinetic energy increases performance and the wind has not been doubled.

5.1 Conversion potential energy

Potential energy is the source of stress in the turbine. It is possible to convert potential energy into kinetic energy
In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stresses in the blades for a defined wind speed, are constant.

$$\frac{d\sigma}{d\beta} = 0$$

σ Stress in turbine blade$(\frac{N}{m^2})$
β rotation angle of the blades (rad)

In fact, some variations of the stress existe due to gravitational forces and the differencial velocity within the boundary layer depending on the elevation.

In the case of vertical axis turbines (VAWT, Darrieus type) the stress of blade and arm are depending on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \qquad \frac{1}{2\ \pi}\int_0^{2\ \pi} \sigma d\beta \ = \ \epsilon \ \ (\epsilon \ small)$$

During a half-turn, the arms are submitted to compression stress whereas extending stress is dominant during the next half-turn.

In the case of a HAWT, the conversion is not possible with a dynamic mechanical system. The additionnal stresses are constant during a rotating for a given wind speed. Alternative stresses, encountered in a vertical axis wind turbine VAWT can allow the extraction of additional energy.

5.2 Total recoverable power of the turbine

To determine the power, one selects the power defined from the kinetic energy and, in the case of conversion of stress into a mechanical movement, the power defined from the potential energy. It is possible to convert potential energy into kinetic energy.
The total recoverable power coefficient is

$$C_{total} \ = \ C_{kin} \ + \ C_{pot}$$

$C_p = 0$ when constraints are not converted

$$P_{total} = \ C_{total} \ \frac{1}{2}\rho S V_{fluid}^3 \ = \ (C_{kin} \ + \ C_{pot}) \ \frac{1}{2}\rho S V_{fluid}^3$$

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the power coefficient $C_{T\ HAWT}$ is

$$C_{Total\ HAWT} = C_{kin} = 4\ a^2\ (1-a) \quad C_{pot} = 0$$

In the case of vertical axis turbines (VAWT, Darrieus type)
the power coefficient $C_{T\ VAWT\ Darrieus}$ is

$$C_{Total\ VAWT\ Darrieus} = C_{kin} = 4\ a^2\ (1-a) \quad C_{pot} = 0$$

In the case of vertical axis turbines (VAWT with conversion), these stresses is converted into additional energy, and the power coefficient $C_{T\ VAWT\ with\ conversion}$ is

$$C_{Total\ VAWT\ with\ conversion} = C_{kin} + C_{pot} = 8\ a^2\ (1-a)$$

Depending on the turbine type, the power will be different by

$$P = C_{total}\ \frac{1}{2}\rho S V_{fluid}^3 \tag{10}$$

$$P_{Total\ HAWT} = 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3 \tag{11}$$

$$P_{Total\ VAWT\ Darrieus} = 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3 \tag{12}$$

$$P_{Total\ VAWT\ with\ conversion} = 8\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3 \tag{13}$$

6 HAWT-VAWT comparison and discusion

Following the work of Hau (2000), the power coefficient of different turbines is compared (a performance of 0.6 is applied for the supplementary energy recovery system).

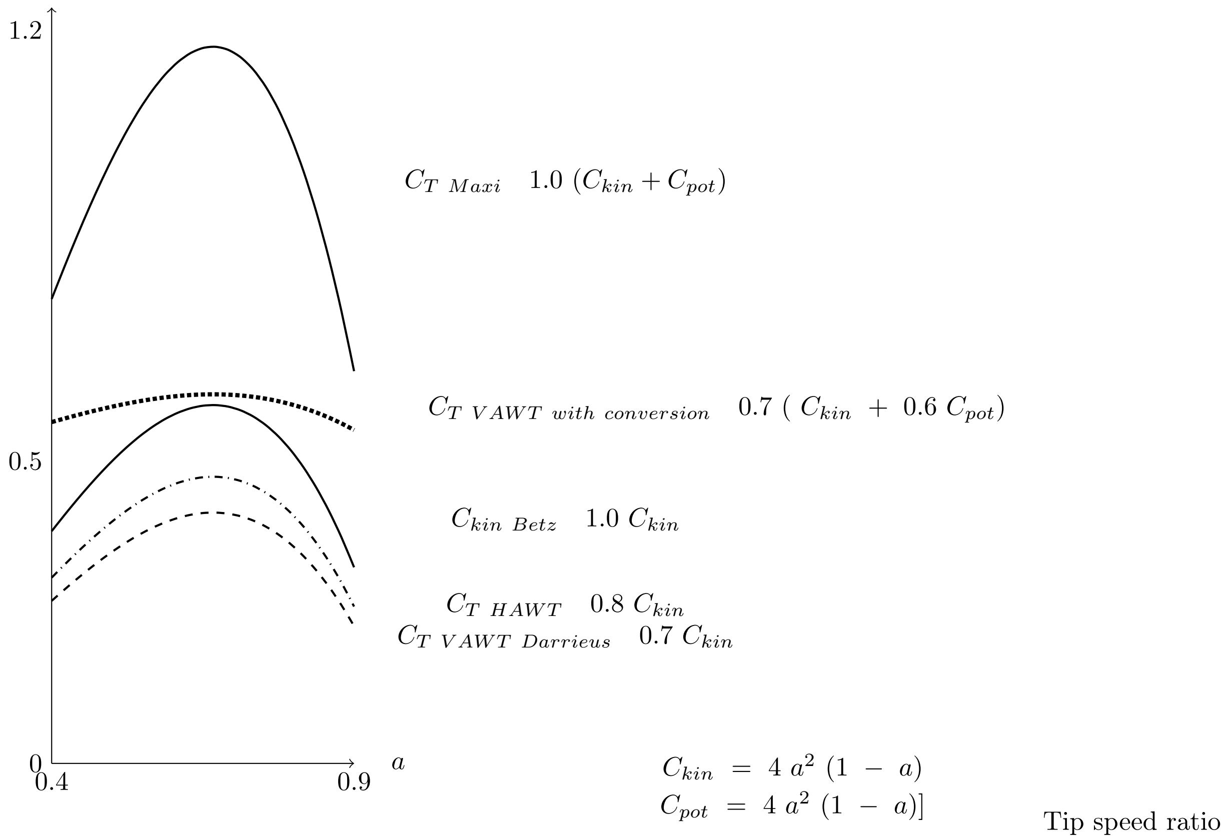
$$With\ a = \frac{2}{3} \qquad C_{kin} \approx 60\% \qquad C_{pot} \approx 60\%$$

notation $C_{pot}\ w.c = C_{pot}\ with\ conversion$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
perfect	C_{kin}	60%	60%	60%
perfect	$C_{pot}\ w.c$	0%	0%	60%
perfect	C_{total}	= 60%	= 60%	= 120%
in practice	C_{kin}	$0.8 \times 60\% \approx 48\%$	$0.7 \times 60\% \approx 42\%$	$0.7 \times 60\% \approx 42\%$
in practice	$C_{pot}\ w.c$	0%	0%	$0.6 \times 0.7 \times 60\% \approx 25\%$
in practice	C_{total}	= 48%	= 42%	= 67%
....				
gain/HAWT		+ 0%	−12 %	+ 39%
gain/$C_{kin}(Betz)$		− 20%	−30 %	+ 11%

$$With\ a \approx 0.8 \qquad C_{kin} \approx 50\% \qquad C_{pot} \approx 50\%$$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
in practice	C_{kin}	$0.8 \times 50\% \approx 40\%$	$0.7 \times 50\% \approx 35\%$	$0.7 \times 50\% \approx 35\%$
in practice	$C_{pot}\ w.c$	0%	0%	$0.6 \times 0.7 \times 50\% \approx 21\%$
in practice	C_{total}	= 40%	= 35%	= 56%
....				
gain / HAWT		+ 0%	−12 %	+ 27%
gain / C_k		− 20%	−30 %	+ 12%



performance curve;
fig. 11

6.1 The Active lift turbine project

The "Active lift turbine" project" is an example of transformation of potential energy into kinetic energy (see preprint : Simplified theory of an active lift turbine with controlled displacement (Lecanu et al. (2016).

$$C_{kin} = \frac{9\pi}{2^7}b^3 - \frac{2^3}{3}b^2 + \frac{\pi}{2}b$$

$$C_{pot} = \frac{e}{R}\lambda(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

$$C_{active.lift.turbine} = (1 + \frac{e}{R}\lambda)(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

$$with \quad b = \sigma\lambda \qquad \sigma = \frac{Nc}{R} \qquad \lambda = \frac{R\dot{\theta}}{V_{fluid}} \qquad b \leq \frac{4}{3}$$

σ stiffness coefficient
N Number of blades
c Chord of profil
λ Tip speed ratio
e eccentric distance
R Turbine radius
θ rotation angle of the turbine

numerical application

if $e = 0.2$ and $\sigma = 0.25$

$$\frac{dC_{active.lift.turbine}}{db} = \frac{d[(1+\frac{e}{R}\frac{b}{\sigma})(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)]}{db} = 0 \qquad b_1 \approx 0.84 \qquad C_{active.lift.turbine} \approx .85$$

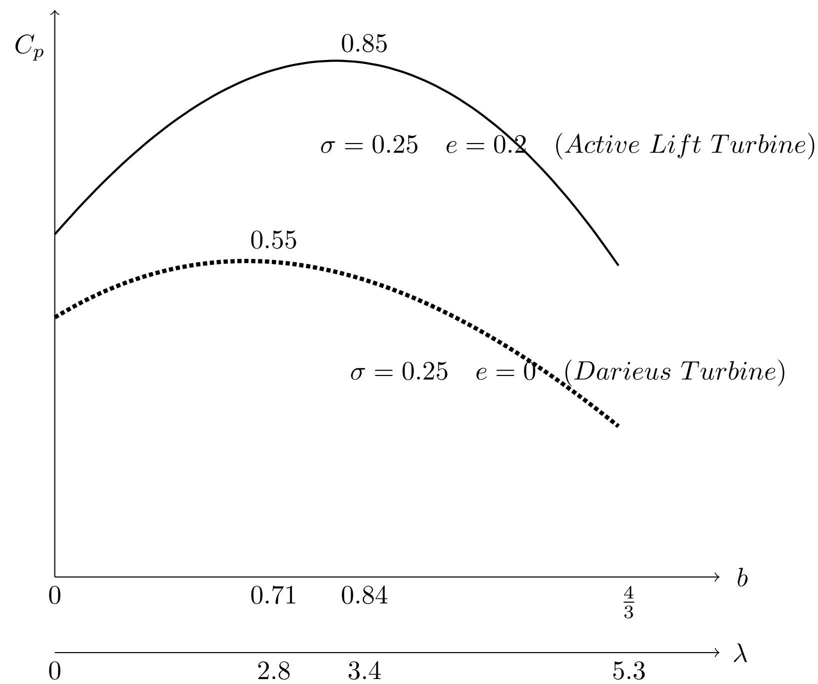


fig. 12

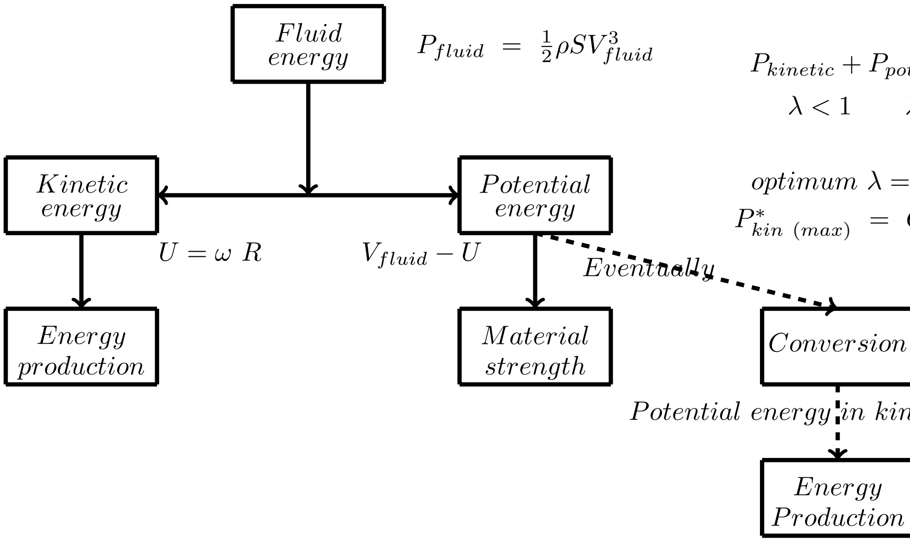
6.2 Synthesis scheme:

6.2.1 Low speed turbine $\lambda < 1$:

case : $\lambda < 1$ — **turbine type Anemometer with cups** $\mathbf{E_{kin\ wind}} \Rightarrow \mathbf{E_{pot}}$ & $\mathbf{E_{kin}}$

For a constant wind speed, if the load of the electric generator is decreased, then we have **simultaneously** these energy variations :

$$\mathbf{\frac{dE_{pot}}{dt} < 0} \quad \& \quad \mathbf{\frac{dE_{kin}}{dt} > 0}$$



$P_{kinetic} + P_{potential} < P_{fluid}$

$\lambda < 1 \qquad \lambda \ = \ \frac{\omega \ R}{V_{fluid}}$

$optimum \ \lambda = \frac{1}{2} \quad Coef = \frac{1}{2}$

$P^*_{kin \ (max)} \ = \ Coef \ \frac{1}{2}\rho S V_{fluid}^3$

fig. 13

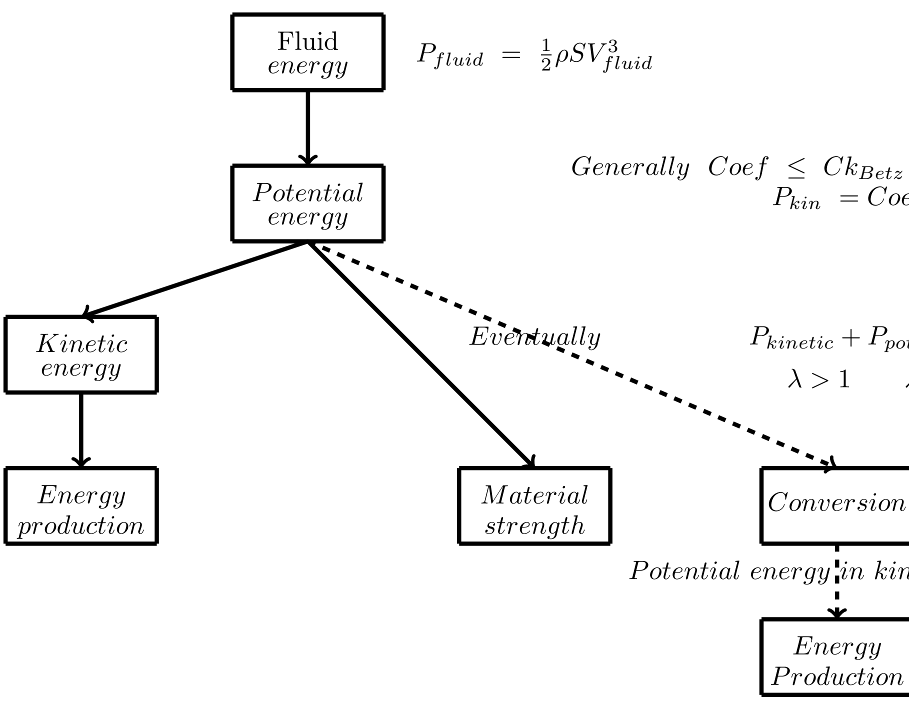
$*$ to obtain an energy production

6.2.2 Fast speed turbines HAWT or VAWT $\lambda \ > \ 1$:

case : $\lambda \ > \ 1$ **— large wind turbines HAWT or VAWT**
Temporal transfer of energy $\quad \mathbf{E_{kwind}} \ \Rightarrow \ \mathbf{E_{pot}} \ \Rightarrow \ \mathbf{E_{kin}}$

For a constant wind speed, if the load of the electric generator is decreased, then we have **successively** these energy variations :

$$\frac{\mathbf{dE_{pot}}}{\mathbf{dt}} > \mathbf{0} \quad \Rightarrow \quad \frac{\mathbf{dE_{kin}}}{\mathbf{dt}} > \mathbf{0}$$



$Generally \ \ Coef \ \leq \ Ck_{Betz} \qquad Ck_{Betz} = \frac{16}{27}$

$P_{kin} \ = Coef \ \frac{1}{2}\rho S V_{fluid}^3$

$P_{kinetic} + P_{potential} > P_{fluid}$

$\lambda > 1 \qquad \lambda \ = \ \frac{\omega \ R}{V_{fluid}}$

fig. 14

In the case of a turbine with a lambda lower than 1, a part of the recoverable energy, is transformed into kinetic energy and the other part into potential energy (stress in the turbine). Eventually a part of this potential energy could be transformed into kinetic energy.

In the case of a turbine with a lambda higher than 1, the recoverable energy defined by Betz, is transformed into potential energy. This energy creates kinetic but also potential energy (stress in the turbine). This is possible because we have created a rotation speed that is higher than the speed of the fluid. A part of this potential energy (due to the stress) can be transformed into kinetic energy.

6.3 Synthesis on the extracted power:

With a mechanical conversion system, the power is, for each turbine type

$$P_{T\ HAWT} = \ 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V^3_{fluid}$$

$$P_{T\ VAWT\ Darrieus} = \ 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V^3_{fluid}$$

$$P_{T\ VAWT\ with\ conversion} = \ 8\ a^2\ (1-a)\ \frac{1}{2}\rho S V^3_{fluid}$$

Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system is in practice from 20% to 50%.

Concerning a vertical axis turbine with a conversion system, the power factor is higher than the one defined by Betz.

In the comparative table, a yield of 0.6 was chosen for the mechanical conversion system of the potential energy into mechanical energy. By choosing a powerful technology, this yield can be greatly increased, which will increase the performance of the turbine.

The definition of the maximum power coefficient is the one established by Betz which remains valid for horizontal axis turbines HAWT but not for vertical axis turbines VAWT. The given results examined a conversion of the potential energy into kinetic energy through a mechanical system which is not applicable for horizontal axis turbines HAWT. The calculation of the powers are the sum of the powers which one wants to consider.

In order to obtain productive kinetic energy, kinetic energy and potential energy are needed. The maximum productive kinetic power is equal to

$$\boldsymbol{P_{k-max-productive}\ \leq\ min(\ P_k\ ,\ P_p\ )}$$

The limit of this depends on whether it is a low or high speed turbine

$$P_{k(max)}\ =\ Coef\ \frac{1}{2}\rho S V^3_{fluid}$$

$Coef\ =\ \frac{1}{2}$ for low speed turbines ($\lambda < 1$)
$Coef\ =\ \frac{16}{21}$ for high speed turbines ($\lambda > 1$)

The maximum power coefficient for a wind or tidal turbine is

$$C_{Total\ maxi\ \lambda<1} = \ \frac{100}{100}\ (\approx\ 100\%)$$

$$C_{Total\ maxi\ \lambda>1} = \ \frac{32}{27}\ (\approx\ 118.5\%)$$

$C_{Total\ maxi}$ is a limit value with $C_{pot}\ \leq\ C_{kin}$.

Using piezo-electric materials would make it possible to transform potential energy into electrical energy. However, in order to achieve higher efficiency, the potential energy should be transformed into kinetic energy.

Thus, the power coefficient will be $C_{total} = C_{kin} + C_{pot}$ whatever the type of turbine.

Note : In the case of a turbine in a channel (see appendix 15), the maximum kinetic power coefficient is to 1. The maximum total power coefficient is then 200%.

7 Conclusion

The article presented a new formulation for power calculation of wind or water turbines.
Taking into account the potential energy allows to establish that the conservation of energy is well verified.
This was not the case in the formulation presented by Betz.
A new power coefficient has been defined for the turbines depending on their type (low or high speed).

A : Appendix

Maximum wind power recovered from kinetic energy

$$E_k = \frac{1}{2}\, m\, V^2$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\frac{dm}{dt}\, V^2 + \frac{1}{2}\, m\, \frac{dV^2}{dt} \qquad \frac{dV}{dt} = 0$$

$$\frac{dE_k}{dt} = \frac{1}{2}\, \dot{m}\, V^2 = \frac{1}{2} \rho\, S\, V^3$$

V Wind speed at the turbine level

Force applied by the wind on the rotor

$$F = m \frac{dV}{dt} = \dot{m} \Delta V \;=\; \rho S V \,(V_{fluid} \;-\; V_{wake})$$

V_{wake} streamwise velocity in the far wake

$$P = FV \;=\; \rho S V^2 \,(V_{fluid} \;-\; V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2} m V_{fluid}^2 - \frac{1}{2} m V_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} \;=\; \frac{1}{2} \dot{m} (V_{fluid}^2 \;-\; V_{wake}^2) = \; \frac{1}{2} \rho S V (V_{fluid}^2 \;-\; V_{wake}^2)$$

From theses equalities

$$V = \bar{V} = \frac{V_{fluid} + V_{wake}}{2}$$

$$F = \;\rho S V \,(V_{fluid} \;-\; V_{wake}) \;=\; \frac{1}{2} \rho S (V_{fluid}^2 \;-\; V_{wake}^2)$$

$$P = FV \;=\; \rho S V^2 \,(V_{fluid} \;-\; V_{wake})$$

defining $a = \frac{V}{V_{fluid}}$

$$V_{wake} \;\;=\;\; V_{fluid}\;(2\;a \;\;-\;\; 1) \quad as \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P = \;\; 4\; a^2\; (1 - a)\; \frac{1}{2}\; \rho S V_{fluid}^3$$

defining power coefficient $C_k = \frac{P}{\frac{1}{2} \rho S V_{fluid}^3} \;\;=\;\; 4\; a^2\; (1 - a)$

Search of maximum power coefficient

$$\frac{dC_k}{da} = 0 \quad a\;(2 \;\;-\;\; 3\;a) \;\;=\;\; 0 \qquad a = 0 \qquad or \qquad a = \frac{2}{3}$$

$$a = \frac{2}{3} \qquad C_k = \frac{16}{27} = 0.593$$

The maximum power coefficient $C_{k\,maxi}$ is defined by Betz

$$C_{k\,maxi} \;\;=\;\; C_{k\,Betz} = \frac{16}{27} \approx 60\%$$

The maximum power of the fluid is

$$P_{fluid} = \frac{1}{2}\rho S_{fluid} V_{fluid}^3$$

$S_{fluid} \ = \ \frac{S\ V}{V_{fluid}} \ = \ a\ S$
The power of the turbine is

$$P = \frac{C_k}{a}\ P_{fluid} \ = \ C_k\ \frac{1}{2}\rho \frac{S_{fluid}}{a} V_{fluid}^3 \ = \ C_k\ \frac{1}{2}\rho S V_{fluid}^3$$

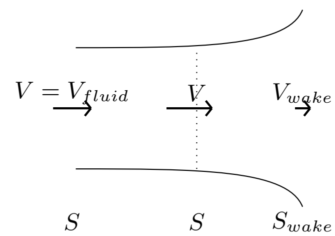
The maximum power of the turbine is

$$P_{max} = \frac{C_{k\ Betz}}{\frac{2}{3}}\ P_{fluid} \ = \ \frac{8}{9}\ \ P_{fluid} \ = \ \ C_{k\ Betz}\ \frac{1}{2}\rho S V_{fluid}^3 \ = \ \ \frac{16}{27}\ (\frac{1}{2}\rho S V_{fluid}^3)$$

$$P_{max} \ = \ \ C_{k\ Betz}\ \frac{1}{2}\rho S V_{fluid}^3 \tag{14}$$

B : Appendix

Maximum wind power for a turbine in a channel (from kinetic energy)



$$E_k = \frac{1}{2}\, m\, V^2$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\frac{dm}{dt}\,V^2 + \frac{1}{2}\,m\,\frac{dV^2}{dt} \qquad \frac{dV}{dt} = 0$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\dot{m}\,V^2 = \frac{1}{2}\rho\,S\,V^3$$

V Wind speed at the turbine level
Force applied by the wind on the rotor

$$F = m\frac{dV}{dt} = \dot{m}\Delta V \ = \ \rho SV\,(V \ - \ V_{wake})$$

V_{wake} streamwise velocity in the far wake

$$P = FV \ = \ \rho SV^2\,(V \ - \ V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2}mV^2 - \frac{1}{2}mV_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} \ = \ \frac{1}{2}\dot{m}(V^2 \ - \ V_{wake}^2) = \ \frac{1}{2}\rho SV(V^2 \ - \ V_{wake}^2)$$

From theses equalities

$$V_{wake} \ = \ \frac{1}{2}\ V \qquad S_{wake} = \frac{S\ V}{V_{wake}} \ = \ 2\ S$$

$$F \ = \ \rho SV\,(V_{fluid} \ - \ V_{wake}) \ = \ \frac{1}{2}\rho S(V_{fluid}^2 \ - \ V_{wake}^2)$$

$$P = FV \ = \ \rho SV^2\,(V_{fluid} \ - \ V_{wake})$$

$$P = \ \frac{1}{2}\ \rho SV_{fluid}^3$$

defining power coefficient

$$C_k = \frac{P}{\frac{1}{2}\rho SV_{fluid}^3} \ = \ 1 \tag{15}$$

C : Appendix

Additional recovery power (from potential energy)

The fluid creates stresses in the blade. They are due to thrust force. The energy of this force is

$$E_p \;=\; m\,\frac{f_s}{\rho}$$

f_s thrust force

For HAWT horizontal wind turbines (fast wind turbine type), the thrust force F_s are constant.

$$\frac{dE_p}{dt} \;=\; 0$$

For a VAWT, the thrust force depends on the time or the rotation angle $F_s(t)$ or $F_s(\beta)$ $\quad \beta = \omega\, t$

$\omega \;=\; \frac{d\dot{\beta}}{dt}$ angular frequency

$$\frac{dE_p}{dt} \;\neq\; 0 \quad and \quad \frac{1}{2\,\pi}\int_0^{2\,\pi} E_p(\beta)d\beta \;=\; \epsilon \;\;(\epsilon\ small)$$

As

$$F_s \;=\; f_s S \;=\; C_x \frac{1}{2}\,\rho\, S\, V_{fluid}^2$$

$$E_p \;=\; m\, C_x \frac{1}{2}\, V_{fluid}^2$$

V fluid speed at the level turbine

The power is

$$P_p \;=\; \frac{dE_p}{dt} \qquad dm = \rho S V dt$$

$$P_p \;=\; \frac{dE_p}{dt} \;=\; \frac{dm}{dt}\, C_x\, \frac{1}{2}\, S\, V_{fluid}^2 \;+\; m\, C_x \frac{1}{2}\, S\, \frac{dV_{fluid}^2}{dt} \qquad \frac{dV_{fluid}}{dt} \;=\; 0$$

$$P_p \;=\; a\, C_x\, \frac{1}{2}\,\rho\, S\, V_{fluid}^3 \quad with\ a = \frac{V}{V_{fluid}} \tag{16}$$

$$\frac{1}{2\,\pi}\int_0^{2\,\pi} E_p(\beta)d\beta \;=\; \epsilon \;\;(\epsilon\ small) \quad E_{p-max} \;\approx\; -\,E_{p-min}$$

the power depends on a potential energy difference

$$P_p \;=\; \frac{\Delta E_p}{\Delta t} \quad T \;=\; \frac{2\,\pi}{\omega} \quad P_p \;\leq\; \frac{E_{p-max} - E_{p-min}}{T} \quad P_p \;\leq\; \frac{E_{p-max}}{\pi}\,\omega$$

for a half-turn

$$E_p(\beta)\; R\; d\beta \;=\; dE_p\, \pi\, R$$

in particular

$$E_{p-max} \;=\; \frac{dE_p}{d\beta}\pi \;=\; \frac{dE_p}{dt}\,\frac{\pi}{\omega}$$

As

$$\frac{dE_p}{dt} \;=\; a\, C_x\, \frac{1}{2}\,\rho\, S\, V_{fluid}^3 \quad and \quad P_p \;\leq\; \frac{E_{p-max}}{\pi}\,\omega$$

$$P_p \;\leq\; a\, C_x\, \frac{1}{2}\,\rho\, S\, V_{fluid}^3$$

D : Appendix

Variation of energy in opposite sense

Along a streamline, the Bernoulli's equation is see Bernoulli (1738)

$$\frac{p}{\rho} + \frac{v^2}{2} = constant \quad with \quad z = 0$$

By multiplying by m

$$m\,\frac{p}{\rho} + m\,\frac{v^2}{2} = constant$$

The differential of this equation is

$$d(\frac{1}{2}\,m\,v^2) = -\,d(m\,\frac{p}{\rho}) \tag{17}$$

the variations of energy vary simultaneously and in opposite sense.

$$As \quad \frac{dV}{dt} = 0 \quad and \quad \frac{dp}{dt} = 0 \quad p = -\,\frac{1}{2}\,\rho\,v^2$$

References

D. Bernoulli. *Hydrodynamica*. doi:10.3931, eth-zurich e-rara-3911 edition, 1738.

A. Betz. Das maximum der theoretisch moglichen ausnutzung des windesdurch windmotoren. *Zeitschrift fur das gesamte Turbinenwesen*, 0(26):307–309, 1920.

L. Borel. Fonctions d'état, bilan de travail, pertes et rendement thermodynamique, 1965. URL `https://dx.doi.org/10.5169/seals-67674`.

E. Hau. *Wind Turbines*. Springer Germany, 2000.

J. Kimball. *Physics of Sailing*. CRC Press, 2009. ISBN 9781420073775. URL `https://books.google.fr/books?id=Xe_i23UL4sAC`.

P. Lecanu, J. Bréard, and D. Mouazé. Simplified theory of an active lift turbine with controlled displacement. Apr. 2016. URL `https://hal.inria.fr/hal-01300531`.

D. G. G. Max Marty, Daniel Dixneuf. *Principes d'electrotechnique*. Dunod, 2005. ISBN 2-10-048550-4. EAN 9782100485505.

J. Prescott Joule. *The Scientific Papers of James Prescott Joule*, volume volume 1 : isbn= 978-0-470-69975-1. Cambridge Library Collection - Physical Sciences, 2011.