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The weak-core of a game in normal form with a continuum of players*

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Abstract

This paper deals with the weak-core of normal form games with a continuum set of players and without side payments. This concept is an approximation of the core introduced by Weber, Shapley and Shubik. The weak-core is slightly larger than Aumann's α -core when adapted to large anonymous games. A non emptiness result is obtained based on the well known Scarf's non vacuity theorem for finite games.

keyword: Weak-Core; α -Core; Game with a Continuum of Players; Large Anonymous Games; Normal Form Games

JEL Classification codes : C02; C71.

1 Introduction

In this paper, we formulate the weak-core concept for strategic (normal form) games without side payments. The starting point of this work is the α -core introduced by Aumann (1961) and investigated, for finite games, by Scarf (1967) and Kajii (1992). Precisely, we focus on an approximation of the α -core introduced, as a quasi-core, by Shapley and Shubik (1966) for finite games and adapted to games with a continuum of players by Weber (1981). The studied games belong to the class of large anonymous games. The weak-core is slightly larger than the α -core of Aumann (1961) and seems, in our context, more appropriate for mathematical treatment.

For finite (large) games in characteristic function form, the approximate solutions of the core are introduced in order to relax the convexity assumption generally assumed in studies of cores and competitive equilibria of exchange economies. It has been shown that the convexity (of preference sets) is not much important when the set of agents become large enough.

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In order to illustrate the intuitive idea of the different approximations of the core (known by the author) let us consider a game in characteristic function form $F = (N, \mathcal{V})$. Here N is a finite set of players. In side payments case, $\mathcal{V}$ is a function which associate to each subset S of N , a real number $\mathcal{V}(S)$. A coalition S blocks an “allocation” $x \in \mathbb{R}^N$, if $\mathcal{V}(S) > \sum_{i \in S} x_i$.

Then, the core of F is the set of elements $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = \mathcal{V}(N)$ and x cannot

be blocked by any coalition $S \subset N$. Shapley and Shubik (1966) defined two approximate concepts : a weak ε -core and a strong ε -core. This was done by weakening the concept of blocking. For the weak ε -core (resp. strong ε -core), a coalition S blocks $x \in \mathbb{R}^N$, if $\mathcal{V}(S) > \sum_{i \in S} x_i + |S|\varepsilon$ (resp. $\mathcal{V}(S) > \sum_{i \in S} x_i + \varepsilon$). The notation $|S|$ stands for the cardinal of S .

In non transferable utility case, the function $\mathcal{V}$ is a correspondence from the subsets of N to that of $\mathbb{R}^N$. For this category of games, let us speak only about one approximation type (weak ε -core). Before defining it, an allocation x is said to be blocked by a coalition S , if there exists $y \in \mathcal{V}(S)$ such that $y_i > x_i$ for every $i \in S$. Then, the core of F is defined to be the elements of $\mathcal{V}(N)$ not blocked by any coalition. Let $\varepsilon > 0$ be fixed. As defined in (Wooders, 2008), the blocking concept corresponding to the weak ε -core can be stated as follows : x is blocked by a coalition $S \subset N$, if there exists $y \in \mathcal{V}(S)$ such that $y_i > x_i + \varepsilon$, for all $i \in S$.

The comparison between these approximations with the exact core is obvious. One can note that all these approximations contain the exact core and the latter corresponds to the particular case $\varepsilon = 0$.

In transferable utility case, Shapley and Shubik obtained interesting (non vacuity of approximate cores) results for exchange economies with sufficiently many participants. The concept of weak ε -core is successfully applied for (finite large) games induced from *pergames* (a configuration in which the payoff achievable by a coalition is a function of the number of its players and their characteristics or attributes) in (Wooders, 1983, 1994) and (Wooders and Zame, 1984). In case of a continuum set of players, Weber (1979) studied the concept of weak ε -core and proved its existence for balanced games in characteristic function form and without side payments.

A slightly different concept of weak ε -core is introduced by Kannai (1970, 1972) and applied for finite markets. The approach of Kannai consists in a representation of a finite market by a continuum one as defined in (Aumann, 1964). The non vacuity results are based on this representation. The approximation concept of Kannai is applied by Hildenbrand et al. (1973) to generalize the Shapley-Shubik results for large economies without side payments.

In a more recent paper, Wooders (2008) investigated the weak ε -core for (finite large) games (induced from *pergames*) without side payments. In (Wooders, 2008), the approximation is enlarged and deals, in addition, with the set of players. Roughly speaking, the approximation related to players was done by limiting the size of the possible blocking coalitions. More precisely, an element $x \in \mathcal{V}^N$ is in the weak ε -core if there exists $N_0 \subset N$ such that $|N \setminus N_0|/|N| < \varepsilon$ and no coalition in N_0 can block x (in the weak sense defined above). The approximation obtained by relaxing this limitation, is defined as the uniform

ε -core by Wooders and its existence requires, as expected, additional conditions. Other studies about approximations of the core can be found in (Anderson, 1985; Starr, 1969).

It is obvious that these weak solutions can produce loss in stability, but as raised by the different authors, this can be offset, for example, by cooperation costs, which can be proportional to the number of cooperating players.

In this paper, we deal with Weber's approximation (Weber, 1981) as stated in Section 2 below. It is different from the above, its purpose is to "make easier" the passage from the finite to the continuum case rather than overcoming the nonexistence of the exact core caused by the non convexity assumption. Indeed, for finite games our weak core is exactly the α -core of Aumann.

Following the idea of Scarf (1971), using a utility characteristic function game and another result of Scarf (1967), we show a non-emptiness result. However, we make some restrictions. We assume that the payoffs depend only on the distribution of the actions. Then, it is assumed that a payoff of an agent does not depend on its own action, but only on the resulting distribution of all (taken) actions.

For utility characteristic function games, there are very interesting results dealing with the continuum set of players case. We refer the reader, for instance, to Weber (1981, 1979), Ichiishi and Weber (1978), Rosenmüller (1975) and Kaneko and Wooders (1996) (for particular games and by assuming that the admissible coalitions are finite). For more general frameworks concerning economies (exchange) and markets, we refer the reader to the famous works of Aumann (1964, 1966), Hildenbrand (1974, 1968), Mas-Colell (1975), Hart et al. (1974) and Khan (1981).

In a non cooperative setting, Schmeidler (1973) initiates the study of equilibria for games with a continuum of players. Mas-Colell (1984) developed this approach by defining the games by means of distributions on the set of agents' characteristics. The Mas-Colell equilibrium existence result has been generalized later by Khan (1989). By restraining the dependency of the agents' payoffs to the average of the actions, Rath (1992) obtained an existence result of a pure strategy equilibrium. We refer to Khan et al. (1997) for a complete review on Nash equilibria for games with a continuum set of agents.

2 Main result

By a normal form (strategic) game with a continuum of players, we mean a triple

$$G = ((T, \lambda), \mathcal{U}_A, A)$$

$T = [0, 1]$ is the set of players endowed with its usual topology. λ stands for the Lebesgue measure on T . A is the set of actions. Assume that A is a compact metric space. Let us denote by $\mathcal{M}_+^1(A)$ the set of Radon probability measures on A endowed with the weak (star) topology. Since A is a Suslin space, all Borel measures on A are Radon (Bogachev, 2007). Then, $\mathcal{M}_+^1(A)$ is the set of all Borel probability measures on A . In the sequel, other sets of positive Radon measures on A are used. They are always considered as subspaces

of the space $\mathcal{M}_+(A)$ of all Radon positive measures on A endowed with the weak topology. Note that the convexity of A is not needed in this paper. We use, instead, the convexity of $\mathcal{M}_+^1(A)$ which is more advantageous.

The players' utility functions are described by a function $u : T \times \mathcal{M}_+^1(A) \rightarrow \mathbb{R}$. We assume that u is continuous with respect to the resulting product topology. To each player t corresponds its utility function $u(t, \cdot) : \mathcal{M}_+^1(A) \rightarrow \mathbb{R}$. In the sequel, this function is denoted by $u(t)$. When each player t chooses its strategy $f(t)$, we obtain a function f , $f(t) \in A, t \in T$. Assuming that this function is measurable, each player t receives the gain $u(t)(\lambda f^{-1})$. Here, λf^{-1} stands for the image measure of λ under the function f . Then, $u(t)$ depends only on the distribution λf^{-1} . The set of utility functions $u(t)$, $t \in T$, is denoted by $\mathcal{U}_A$ above and is endowed with the sup-norm topology. Thus, the continuity of u yields that of the function $t \mapsto u(t)$ from T to $\mathcal{U}_A$.

Let $\mathfrak{B}(T)$ be the set of all Borel subsets of T . The set of permissible coalitions is the set of Borel subsets of T of strictly positive λ -measure, denote it by $\mathfrak{B}_+(T)$. For a Borel subset X of T , we denote by $\mathfrak{B}(X, A)$ the set of Borel functions defined from X into A . If X_1 and X_2 are two disjoint Borel subsets of T , $h_{X_1} \in \mathfrak{B}(X_1, A)$ and $h_{X_2} \in \mathfrak{B}(X_2, A)$, then $h = (h_{X_1}, h_{X_2})$ stands for the Borel function defined as h_{X_1} on X_1 and h_{X_2} on X_2 . Clearly, $h \in \mathfrak{B}(X_1 \cup X_2, A)$. The complement of a subset X of a given space Y is denoted by $Y \setminus X$ or simply $\mathbb{C}X$ (if no confusion may occur). For instance, all complements of subsets of T stand for complements to T . In the whole of this paper, the measurability is taken with respect to the Borel σ -algebra. The closure of a subset X in a topological space is denoted by $\bar{X}$. If c is a real number and X a subset of a linear space, then $c.X$ stands for the set $\{cx : x \in X\}$.

Definition 1. *We say that a coalition $E \in \mathfrak{B}_+(T)$ blocks the strategy $f \in \mathfrak{B}(T, A)$ (or the distribution $p = \lambda f^{-1}$), if there is $\varepsilon > 0$ and a Borel function $h_E \in \mathfrak{B}(E, A)$, such that for every $h_{\mathbb{C}E} \in \mathfrak{B}(\mathbb{C}E, A)$, letting $h = (h_E, h_{\mathbb{C}E})$, we have :*

$$u(t)(\lambda h^{-1}) > u(t)(\lambda f^{-1}) + \varepsilon \text{ a.e. on } E$$

The weak-core of the game G is the set of distributions that are not blocked by any coalition $E \in \mathfrak{B}_+(T)$.

Following the representation result of Skorokhod (1956) (see also (Blackwell and Dubins, 1983; Bogachev, 2007)), for each element $p \in \mathcal{M}_+^1(A)$, there corresponds a function $f \in \mathfrak{B}(T, A)$, such that $p = \lambda f^{-1}$. Then the weak-core can be defined either by means of measures $p \in \mathcal{M}_+^1(A)$ or by functions in $\mathfrak{B}(T, A)$. Note that in this last case, there may be many functions corresponding to a same probability $p \in \mathcal{M}_+^1(A)$. Our definition is more precise with measures. When speaking about the weak-core in terms of functions (strategies), we mean a class of functions inducing the same probability. Furthermore, considering the blocking concept from the previous definition, any two coalitions having a null set as a symmetric difference can be confused.

The weak-core is slightly larger than the α -core of Aumann (1961). More precisely, if we remove ε in the previous definition, we obtain naturally the α -core in case of a continuum

set of players. Moreover, one can remark easily, as mentioned in (Weber, 1981), that for finite games (finite set of players) these two concepts coincide. Our proof of existence works only for the weak-core. In our context, the proof of Lemma 1 below fails if we remove ε in Definition 1.

Theorem 1. *Under the hypotheses described above and the concavity of the functions $p \mapsto u(t)(p)$ for all $t \in T$, the weak-core of G is non empty.*

Associate to each $E \in \mathfrak{B}_+(T)$, the set

$$\mathcal{H}(E) = \{p \in \mathcal{M}_+^1(A) : p \text{ is not blocked by } E\} \quad (1)$$

In order to prove that the weak-core of G is non empty, we need to prove that the intersection $\bigcap_{E \in \mathfrak{B}_+(T)} \mathcal{H}(E)$ is non empty. Since $\mathcal{M}_+^1(A)$ is compact for the weak topology, it suffices to prove that the sets $\mathcal{H}(E)$ are closed and every intersection of a finite subfamily $\mathcal{H}(E_i), i \in I$, is non empty. This is the objective of the following two lemmas :

Lemma 1. *For every $E \in \mathfrak{B}_+(T)$, $\mathcal{H}(E)$ is closed in $\mathcal{M}_+^1(A)$.*

Proof. Let $E \in \mathfrak{B}_+(T)$ and $p \in \mathcal{H}(E)$. Then, there exist $\varepsilon > 0$ and $h_E \in \mathfrak{B}(E, A)$ such that,

$$\begin{aligned} &\text{for all } h_{\mathbb{C}E} \in \mathfrak{B}(\mathbb{C}E, A), \text{ for } h = (h_E, h_{\mathbb{C}E}), \\ &u(t)(\lambda h^{-1}) > u(t)(p) + \varepsilon, \text{ for a.e. } t \in E \end{aligned} \quad (2)$$

By the Skorokhod representation theorem (Skorokhod, 1956; Blackwell and Dubins, 1983; Bogachev, 2007), the set of measures on A obtained as images of λ by the Borel functions $h_{\mathbb{C}E} \in \mathfrak{B}(\mathbb{C}E, A)$ is

$$\lambda(\mathbb{C}E). \mathcal{M}_+^1(A)$$

Indeed, if $\lambda(E) = \lambda(T)$, then the unique possible measure obtained by the functions $h_{\mathbb{C}E} \in \mathfrak{B}(\mathbb{C}E, A)$ is the null measure. Otherwise, $\lambda(\mathbb{C}E) > 0$. From Proposition 9.1.11 in Bogachev (2007), there is a Borel function $\varphi : \mathbb{C}E \rightarrow T$, such that $\lambda|_{\mathbb{C}E} \varphi^{-1} = \lambda(\mathbb{C}E). \lambda$. Then by Skorokhod's representation theorem, the set of measures $\lambda|_{\mathbb{C}E} (f \circ \varphi)^{-1}$ obtained when f ranges $\mathfrak{B}(T, A)$ is $\lambda(\mathbb{C}E). \mathcal{M}_+^1(A)$. Thus, the set of measures, λh^{-1} , appearing in Formula (2) is exactly the set

$$R(E, h_E) = \mu_E + \lambda(\mathbb{C}E). \mathcal{M}_+^1(A)$$

where $\mu_E = \lambda h_E^{-1}$. Clearly $R(E, h_E)$ is a compact set.

In terms of measures, the formula (2) is equivalent to the following :

$$\forall q \in R(E, h_E), u(t)(q) > u(t)(p) + \varepsilon, \text{ for a.e. } t \in E \quad (3)$$

For every $q \in R(E, h_E)$, let

$$E_q = \{t \in E : u(t)(q) > u(t)(p) + \varepsilon\}$$

Note that, for all $q \in R(E, h_E)$, E_q is measurable as intersection of an open set and E and $\lambda(E_q) = \lambda(E)$.

Using the continuity of u , we can state that for all $t \in \overline{E_q}$, $u(t)(q) \geq u(t)(p) + \varepsilon$. Choose $\varepsilon_1 \in]0, \varepsilon[$. Using the continuity of u again, we can find, for every $q \in R(E, h_E)$, two neighborhoods $\theta_q, \theta_p(q)$ of q and p respectively such that,

$$\forall t \in \overline{E_q}, \forall q' \in \theta_q, \forall p' \in \theta_p(q), u(t)(q') > u(t)(p') + \varepsilon_1$$

Let us consider all the sets $E_q, \theta_q, \theta_p(q)$ constructed as above. Since $R(E, h_E)$ is compact, there exists a finite subfamily $\{\theta_{q_i}, i \in I\}$ of the family $\{\theta_q, q \in R(E, h_E)\}$ covering $R(E, h_E)$. Let $E' = \bigcap_{i \in I} \overline{E_{q_i}} \cap E$, $\theta_{R(E, h_E)} = \bigcup_{i \in I} \theta_{q_i}$ and $\theta_p = \bigcap_{i \in I} \theta_p(q_i)$.

We obtained

$$\forall p' \in \theta_p, \forall t \in E', \forall q \in \theta_{R(E, h_E)}, u(t)(q) > u(t)(p') + \varepsilon_1$$

Moreover, it is obvious that $\lambda(E') = \lambda(E)$. Hence, we can replace in the previous formula $t \in E'$ by the expression “for a.e. $t \in E$ ” together with other simplifications, we obtain : for all $p' \in \theta_p$,

$$\forall q \in R(E, h_E), u(t)(q) > u(t)(p') + \varepsilon_1 \text{ for a.e. } t \in E$$

It results that E blocks all the measures in θ_p . This proves that the complement of $\mathcal{H}(E)$ is open and ends the proof. $\square$

Lemma 2. *Let I be a finite set of indices and for each $i \in I$, a coalition $E_i \in \mathfrak{B}_+(T)$. Consider the sets $\mathcal{H}(E_i), i \in I$, defined in (1). Assume that the functions $p \mapsto u(t)(p)$, $t \in T$, are concave. Then, $\bigcap_{i \in I} \mathcal{H}(E_i) \neq \emptyset$.*

Proof. Let $K_j, j \in J$, be a finite family of piecewise disjoint measurable subsets of T of strictly positive measure, such that every E_i is an union (up to a λ -null set) of some sets K_j . For a subset $B \subset J$, we denote $K_B = \bigcup_{j \in B} K_j$. Furthermore, we can assume that

$\bigcup_{j \in J} K_j = T$ up to a λ -null set. Indeed, if it is not the case, one can add the set $\mathcal{H}(T)$ to the family $\mathcal{H}(E_i), i \in I$.

For each $j \in J$, denote for every $\rho \in \mathcal{M}_+^1(A)$,

$$g_j(\rho) = \int_{K_j} u(t)(\rho) d\lambda(t)$$

Note that the functions g_j are concave on $\mathcal{M}_+^1(A)$. This follows from the concavity of the functions $u(t)$. Also, the functions g_j are continuous. This fact can be verified by applying, for example, Lebesgue's dominated convergence theorem.

In the sequel, we follow the idea of Scarf (1971) using a utility characteristic function game and another result of Scarf (1967).

Associate the utility characteristic function game $G_1 = (J, V)$, where, for every $B \subseteq J$, $V(B)$ is the subset of $\mathbb{R}^J$ defined by :

$$v \in V(B) \Leftrightarrow \left| \begin{array}{l} \exists h_{K_B} \in \mathfrak{B}(K_B, A), \forall h_{\mathbb{C}K_B} \in \mathfrak{B}(\mathbb{C}K_B, A), \text{ for } h = (h_{K_B}, h_{\mathbb{C}K_B}), \\ g_j(\lambda h^{-1}) \geq v_j, \forall j \in B \end{array} \right.$$

A vector $v \in \mathbb{R}^J$ is in the core of the game G_1 if $v \in V(J)$ and v is not in the interior of $V(B)$ for every $B \subset J$. It is easy to verify that G_1 has the following properties :

- 1) For every $B \subset J$, $V(B)$ is closed and nonempty,
- 2) For every $B \subset J$, if $v \in V(B)$ and $v' \leq v$ then $v' \in V(B)$,
- 3) $V(J)$ is bounded from above.

Let us verify the closure of the sets $V(B)$, $B \subset J$. Their non emptiness and the other items are straightforward. Let $B \subset J$ be fixed. Similarly to Lemma 1, by applying the Skorokhod representation theorem, $v \in V(B)$ if there exists $\mu_B \in \lambda(K_B).\mathcal{M}_+^1(A)$ such that for all $\rho \in \lambda(\mathbb{C}K_B).\mathcal{M}_+^1(A)$, for all $j \in B$, $g_j(\mu_B + \rho) \geq v_j$. Thus, from the continuity of the functions g_j and the compactness of the sets $\lambda(K_B).\mathcal{M}_+^1(A)$ and $\lambda(\mathbb{C}K_B).\mathcal{M}_+^1(A)$, we can write $V(B)$ as :

$$V(B) = \left\{ v \in \mathbb{R}^J : \max_{\mu_B \in \lambda(K_B).\mathcal{M}_+^1(A)} \min_{\rho \in \lambda(\mathbb{C}K_B).\mathcal{M}_+^1(A)} \min_{j \in B} (g_j(\mu_B + \rho) - v_j) \geq 0 \right\}$$

It follows that the sets $V(B)$, $B \subset J$, are closed. Note further that if $v \in \mathbb{R}^J$ is not in the interior of $V(B)$, for a given $B \subset J$, then,

$$\begin{aligned} & \forall h_{K_B} \in \mathfrak{B}(K_B, A), \exists h_{\mathbb{C}K_B} \in \mathfrak{B}(\mathbb{C}K_B, A), \text{ for } h = (h_{K_B}, h_{\mathbb{C}K_B}), \\ & \exists j \in B, \text{ such that } g_j(\lambda h^{-1}) \leq v_j \end{aligned}$$

The game G_1 is said to be balanced (Scarf, 1971, 1967) if

$$\bigcap_{B \in \mathfrak{C}} V(B) \subset V(J)$$

for every balanced collection of coalitions $\mathfrak{C}$. Note that a collection of coalitions $\mathfrak{C}$ is said to be balanced provided there exist non negative weights $\delta_B, B \in \mathfrak{C}$, such that

$$\sum_{B \in \mathfrak{C}, B \ni j} \delta_B = 1, \forall j \in J$$

Remark that the definition of balancedness of a game does not change if we consider only balanced coalitions with strictly positives weights.

Following Scarf (1967), the core of G_1 is nonempty under the hypothesis of balancedness and the conditions 1)-3) above. Then, for the non vacuity of the core of G_1 , it remains only to prove its balancedness. Let, for this aim, $\mathfrak{C}$ be a balanced family of coalitions of G_1 , with strictly positive balancing coefficients $\delta_S, S \in \mathfrak{C}$, and $v \in \bigcap_{S \in \mathfrak{C}} V(S)$. Then, for every $S \in \mathfrak{C}$, there is a Borel function $h_{K_S} \in \mathfrak{B}(K_S, A)$ such that, for every $j \in S$, $g_j(\lambda(h_{K_S}, h_{\mathfrak{C}_{K_S}})^{-1}) \geq v_j$, for all $h_{\mathfrak{C}_{K_S}} \in \mathfrak{B}(\mathfrak{C}_{K_S}, A)$. Put $\mu_S = \lambda h_{K_S}^{-1}$. Observe that μ_S is a Radon measure on A . By the Skorokhod representation theorem, for every positive measure ρ , such that $\mu_S + \rho \in \mathcal{M}_+^1(A)$, there is a measurable function $h_{\mathfrak{C}_{K_S}} \in \mathfrak{B}(\mathfrak{C}_{K_S}, A)$ such that $\lambda h_{\mathfrak{C}_{K_S}}^{-1} = \rho$ (see also the proof of Lemma 1). Letting $h = (h_{K_S}, h_{\mathfrak{C}_{K_S}})$, we obtain $\lambda h^{-1} = \mu_S + \rho$.

Thus,

$$g_j(\mu_S + \rho) \geq v_j, \forall j \in S$$

Put

$$\mu = \sum_{S \in \mathfrak{C}} \delta_S \mu_S$$

In the sequel, we try to express, for a given $j \in J$, μ as a convex combination of probabilities with the form $\mu_S + \rho, S \ni j$. Before this, we have

$$\mu(A) = \sum_{S \in \mathfrak{C}} \delta_S \mu_S(A) = \sum_{S \in \mathfrak{C}} \delta_S \lambda[h_{K_S}^{-1}(A)] = \sum_{S \in \mathfrak{C}} \delta_S \lambda(K_S) = \sum_{S \in \mathfrak{C}} \delta_S \lambda\left(\bigcup_{j \in S} K_j\right)$$

This sum can be rewritten as :

$$\mu(A) = \sum_{S \in \mathfrak{C}} \delta_S \lambda\left(\bigcup_{j \in S} K_j\right) = \sum_{S \in \mathfrak{C}} \delta_S \sum_{j \in S} \lambda(K_j) = \sum_{j \in J} \sum_{S \ni j} \delta_S \lambda(K_j)$$

From the balancedness of $\mathfrak{C}$, the last equality reduces to $\mu(A) = \sum_{j \in J} \lambda(K_j)$. Since $T = \bigcup_{j \in J} K_j$ up to a null set, $\lambda(T) = \sum_{j \in J} \lambda(K_j)$. Hence, $\mu(A) = 1$. Then μ is a Radon probability on A . Fix an arbitrary index $j \in J$. We have

$$\mu = \sum_{S \in \mathfrak{C}, S \ni j} \delta_S \mu_S + \sum_{S \in \mathfrak{C}, j \notin S} \delta_S \mu_S$$

Let us denote

$$\mu_{-j} = \sum_{S \in \mathfrak{C}, j \notin S} \delta_S \mu_S$$

Since $\sum_{S \in \mathfrak{C}, S \ni j} \delta_S = 1$, we can write

$$\mu = \left(\sum_{S \in \mathfrak{C}, S \ni j} \delta_S \mu_S \right) + \mu_{-j} = \sum_{S \in \mathfrak{C}, S \ni j} \delta_S (\mu_S + \mu_{-j}) \quad (4)$$

If, for each $S \ni j$, $\mu_S + \mu_{-j}$ is a probability, from the foregoing, we can use the concavity of the function g_j to conclude that $g_j(\mu) \geq v_j$ and then the balancedness of the game. But, this is not necessarily true. In order to overcome this difficulty, let us transform another time the measure μ . Denote

$$\mathfrak{C}_j^+ = \{S \in \mathfrak{C} : S \ni j \text{ and } \mu_S(A) + \mu_{-j}(A) > 1\}$$

$$\mathfrak{C}_j^- = \{S \in \mathfrak{C} : S \ni j \text{ and } \mu_S(A) + \mu_{-j}(A) < 1\}$$

and

$$\mathfrak{C}_j^0 = \{S \in \mathfrak{C} : S \ni j \text{ and } \mu_S(A) + \mu_{-j}(A) = 1\}$$

Since $\mu(A) = 1$, if $\mathfrak{C}_j^+ = \emptyset$, necessarily $\mathfrak{C}_j^- = \emptyset$. Then, all the measures $\mu_S + \mu_{-j}$ are probability measures. Assume it is not the case. Since $\mu_S(A) \leq 1$, for all $S \in \mathfrak{C}$, necessarily $\mu_{-j}(A) > 0$. Let us prove also that in this circumstance μ can be written as a convex combination of the needed probability measures.

For all $S \in \mathfrak{C}_j^+$, let $\alpha_S \in [0, 1]$, such that $\mu_S(A) + \mu_{-j}(A) - \alpha_S \mu_{-j}(A) = 1$. The existence of α_S follows from the fact that $S \in \mathfrak{C}_j^+$, $\mu_S(A) \leq 1$ and $\mu_{-j}(A) > 0$.

For all $S \in \mathfrak{C}_j^-$, let $\beta_S > 0$, such that $\mu_S(A) + \mu_{-j}(A) + \beta_S \mu_{-j}(A) = 1$.

Then,

$$\begin{aligned} 1 &= \sum_{S \in \mathfrak{C}_j^+} \delta_S [\mu_S(A) + \mu_{-j}(A) - \alpha_S \mu_{-j}(A)] \\ &\quad + \sum_{S \in \mathfrak{C}_j^-} \delta_S [\mu_S(A) + \mu_{-j}(A) + \beta_S \mu_{-j}(A)] \\ &\quad + \sum_{S \in \mathfrak{C}_j^0} \delta_S [\mu_S(A) + \mu_{-j}(A)] \\ &= \sum_{S \ni j} \delta_S (\mu_S(A) + \mu_{-j}(A)) + \mu_{-j}(A) \left[- \sum_{S \in \mathfrak{C}_j^+} \delta_S \alpha_S + \sum_{S \in \mathfrak{C}_j^-} \delta_S \beta_S \right] \end{aligned}$$

Since $\mu(A) = 1$ and $\mu_{-j}(A) > 0$, we conclude from (4) that

$$- \sum_{S \in \mathfrak{C}_j^+} \delta_S \alpha_S + \sum_{S \in \mathfrak{C}_j^-} \delta_S \beta_S = 0 \tag{5}$$

This equation establishes that we can write μ as :

$$\mu = \sum_{S \in \mathfrak{C}_j^+} \delta_S [\mu_S + \mu_{-j} - \alpha_S \mu_{-j}] + \sum_{S \in \mathfrak{C}_j^-} \delta_S [\mu_S + \mu_{-j} + \beta_S \mu_{-j}] + \sum_{S \in \mathfrak{C}_j^0} \delta_S [\mu_S + \mu_{-j}]$$

That is, μ is a convex combination of probability measures and each one, associated to $S \ni j$, is in the form

$$\mu_S + \rho$$

where ρ is a positive measure. Then $g_j(\mu_S + \rho) \geq v_j$. From the concavity of g_j , we conclude that $g_j(\mu) \geq v_j$. Since j is arbitrarily chosen in J , we conclude that $g_{j'}(\mu) \geq v_{j'}$, for all $j' \in J$. To complete the proof of the balancedness of G_1 , it suffices to apply again the

Skorokhod representation theorem to establish that there is a function $h \in \mathfrak{B}(T, A)$, such that $\mu = \lambda h^{-1}$. Thus, we have proved that the core of G_1 is nonempty.

To end the proof of the lemma, remark that an element of the core of G_1 provides necessarily an element of the intersection $\bigcap_{i \in I} \mathcal{H}(E_i)$. Indeed, let v be an element of the core of G_1 and

$f \in \mathfrak{B}(T, A)$ such that $g_j(\lambda f^{-1}) \geq v_j$, for all $j \in J$. Put again $\mu = \lambda f^{-1} \in \mathcal{M}_+^1(A)$. Fix an arbitrary index $i \in I$. E_i is a finite union (up to a λ -null set) of some sets $K_j, j \in J_i$. Since v belongs to the core of G_1 , v cannot be in the interior of $V(J_i)$. Hence, for every $h_{E_i} \in \mathfrak{B}(E_i, A)$, there exist $h_{\mathbb{C}E_i} \in \mathfrak{B}(\mathbb{C}E_i, A)$ and an index $j \in J_i$ such that for $h = (h_{E_i}, h_{\mathbb{C}E_i})$,

$$\int_{K_j} u(t)(\lambda h^{-1}) d\lambda(t) \leq v_j \leq \int_{K_j} u(t)(\mu) d\lambda(t)$$

Thus, for every $h_{E_i} \in \mathfrak{B}(E_i, A)$, there exists $h_{\mathbb{C}E_i} \in \mathfrak{B}(\mathbb{C}E_i, A)$ and a measurable subset D_i of E_i of strictly positive measure, such that for $h = (h_{E_i}, h_{\mathbb{C}E_i})$,

$$u(t)(\lambda h^{-1}) \leq u(t)(\mu), \forall t \in D_i$$

This proves that $\mu \in \mathcal{H}(E_i)$, which completes the proof. □

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