Interpolation, extrapolation, Morrey spaces and local energy control for the Navier–Stokes equations.

Pierre Gilles Lemarié-Rieusset

Interpolation, extrapolation, Morrey spaces and local energy control for the Navier–Stokes equations

Pierre Gilles Lemarié–Rieusset*

Abstract

Barker recently proved new weak-strong uniqueness results for the Navier–Stokes equations based on a criterion involving Besov spaces and a proof through interpolation between Besov-Hölder spaces and L^2. We improve slightly his results by considering Besov-Morrey spaces and interpolation between Besov-Morrey spaces and $L^2_{\rm uloc}$.

Keywords : Navier–Stokes equations, Morrey spaces, Besov spaces, uniformly locally square integrable functions, weak-strong uniqueness

AMS classification : 35K55, 35Q30, 76D05.

1 The Navier–Stokes equations

Let $\vec u_0$ a divergence-free vector field on $\mathbb{R}^3$. We shall consider weak solutions to the Cauchy initial value problem for the Navier–Stokes equations which satisfy energy estimates.

The differential Navier–Stokes equations read as

$$\partial_t \vec u + \vec u.\vec\nabla \vec u = \Delta \vec u - \vec\nabla p$$

$$\mathrm{div}\ \vec u = 0$$

$$\vec u(0,.) = \vec u_0$$

*LaMME, Univ Evry, CNRS, Université Paris-Saclay, 91025, Evry, France; e-mail : pierregilles.lemarierieusset@univ-evry.fr

Under reasonable assumptions, the problem is equivalent to the following integro-differential problem :

$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u})(t,x)$$

where

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\ \mathrm{div}\ (\vec{u}\otimes\vec{v})\, ds \tag{1}$$

and $\mathbb{P}$ is the Leray projection operator. (See [LR 2, LR 6] for details).

Weak Leray solutions for the Navier–Stokes equations

When $\vec{u}_0 \in L^2$, Leray proved existence of solutions $\vec{u}$ on $(0,+\infty)\times\mathbb{R}^3$ such that :

- $\vec{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$
- $\lim_{t\to 0^+} \|\vec{u}(t,.) - \vec{u}_0\|_2 = 0$
- we have the Leray energy inequality

$$\|\vec{u}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{u}\|_2^2\, ds \le \|\vec{u}_0\|_2^2 \tag{2}$$

Such solutions are weak solutions : the derivatives in the Navier–Stokes solutions are taken in the sense of distributions. Those solutions (that satisfy the energy inequality (2)) are called *Leray weak solutions.*

When $\vec{u}_0 \in L^2$, Leray's proof of existence of solutions [Le] is based on mollification, energy estimates and compactness arguments :

- he solves
$$\partial_t\vec{u}_\epsilon + (\varphi_\epsilon * \vec{u}_\epsilon).\vec{\nabla}\vec{u}_\epsilon = \Delta\vec{u}_\epsilon - \vec{\nabla}p_\epsilon$$
with div $\vec{u}_\epsilon = 0$ and $\vec{u}_\epsilon(0,.) = \vec{u}_0$. Here, $\varphi \in \mathcal{D}$, $\int \varphi\, dx = 1$ and $\varphi_\epsilon(x) = \frac{1}{\epsilon^3}\varphi(\frac{x}{\epsilon})$.
- the solution holds on an interval $(0,T_\epsilon)$ where T_ϵ depends on ϵ and on $\|\vec{u}_0\|_2$ and we have the equality
$$\|\vec{u}_\epsilon(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{u}_\epsilon\|_2^2\, ds = \|\vec{u}_0\|_2^2$$
- the solution is then global; moreover by Rellich theorem, we find a subsequence that converges strongly in $(L^2_t L^2_x)_{loc}$ to a Leray solution $\vec{u}$

Such solutions (i. e. obtained by this mollification/extraction process) will be called in the following *restricted Leray weak solutions.*

Restricted Leray solutions satisfy the Leray energy inequality which takes into account the energy on the whole space. But they enjoy as well a pointwise inequality property : for a non-negative locally finite measure μ we have

$$\partial_t(|\vec{u}|^2) + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta(|\vec{u}|^2) - \operatorname{div}\, ((2p + |\vec{u}|^2)\vec{u}) - \mu \tag{3}$$

Leray solutions that enjoy the pointwise energy inequality are called *suitable Leray solutions* [CKN].

Local weak Leray solutions

The pointwise energy inequality allows one [LR 1, LR 2] to develop a theory of weak solutions with infinite energy. Consider $\vec{u}_0$ a divergence-free vector field that is uniformly locally square integrable :

$$\sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0|<1} |\vec{u}_0(x)|^2 \, dx < +\infty.$$

A *local Leray solution* on $(0,T) \times \mathbb{R}^3$ is a solution such that

- $\vec{u} \in L^\infty_t(L^2_{\text{uloc}}) \cap (L^2_t \dot{H}^1_x)_{\text{uloc}}$
- for all compact subset K of $\mathbb{R}^3$, $\lim_{t\to 0^+} \int_K |\vec{u}(t,.) - \vec{u}_0|^2 \, dx = 0$
- we have the pointwise energy inequality (3).

Local in time existence of restricted local Leray solutions has been proved for a positive T that depends only on $\|\vec{u}_0\|_{L^2_{\text{uloc}}}$ (see section 8).

2 The Prodi–Serrin criterion for weak-strong uniqueness

Based on a compactness criterion, the proof of existence of Leray solutions does not provide any clue on the would-be uniqueness of the solution to the Cauchy initial value problem.

A classical case of uniqueness of Leray weak solutions is Serrin's criterion for weak-strong uniqueness [Pr, Se]. If $\vec{u}_0 \in L^2$ and if the Navier-Stokes equations has a solution $\vec{u}$ on $(0,T)$ such that

$$\vec{u} \in X_T = L^p_t L^q_x \text{ with } \frac{2}{p} + \frac{3}{q} = 1 \text{ and } 2 \le p < +\infty$$

then if $\vec{v}$ is a Leray solution we have $\vec{u} = \vec{v}$ on $(0,T)$.

The proof of the criterion is based on the fact that if $\vec{v}$ is a Leray solution and if $\vec{u}$ is the mild solution with $\|\vec{u}\|_{X_T} < +\infty$, then the difference $\vec{w} = \vec{u} - \vec{v}$ satisfies a Gronwall estimate :

$$\|\vec{w}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2\, ds \le 2\int_0^T |\int \vec{u}.(\vec{w}.\vec{\nabla}\vec{w})\, dx|\, ds.$$

We have (for $\frac{2}{p} + \frac{3}{q} = 1$)

$$\|\vec{u} \otimes \vec{w}\|_2 \le C\|\vec{u}\|_q \|\vec{w}\|_2^{\frac{2}{p}} \|\vec{\nabla} \otimes \vec{w}\|_2^{\frac{3}{q}}$$

so that

$$|\int \vec{u}.(\vec{w}.\vec{\nabla}\vec{w})\ dx| \le \|\vec{u}\|_q \|\vec{w}\|_2^{\frac{2}{p}} \|\vec{\nabla} \otimes \vec{w}\|_2^{1+\frac{3}{q}} = \|\vec{u}\|_q (\|\vec{w}\|^2)^{\frac{1}{p}} (\|\vec{\nabla} \otimes w\|^2)^{1-\frac{1}{p}}.$$

Thus, we find

$$2\int_0^T |\int \vec{u}.(\vec{w}.\vec{\nabla}\vec{w})\, dx|\, ds \le C\int_0^t \|\vec{u}\|_q^p \|\vec{w}\|_2^2\, ds + \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2\, ds.$$

The facts that $\|\vec{w}(0,.)\|_2^2 = 0$ and $\|\vec{w}(t,.)\|_2^2 \le C\int_0^t \|\vec{u}\|_q^p \|\vec{w}\|_2^2\, ds$ then gives $\vec{w} = 0$ on $(0,T)$.

We may comment a little further in the case $2 < p < +\infty$. In that case, the bilinear operator B (given by (1)) is bounded on $X_T = L^p_t L^q_x$. Thus, we find that the existence of $T > 0$ and of a solution in $L^p L^q$ with $\frac{2}{p} + \frac{3}{q}$ is equivalent to the existence of T' such that $e^{t\Delta}\vec{u}_0 \in L^p L^q$ on $(0,T')$ (and on $(0,+\infty)$, since $\vec{u}_0 \in L^2$). Using the thermic characterization of Besov spaces, we can see that this is equivalent with

$$\vec{u}_0 \in \dot{B}_{q,p}^{-1+\frac{3}{q}}.$$

Thus, the initial value is not only in L^2, but it must belong as well to a Besov space with a better regularity than provided by the embedding

$$L^2 \subset \dot{B}_{q,2}^{-\frac{3}{2}+\frac{3}{q}} \subset \dot{B}_{q,p}^{-\frac{3}{2}+\frac{3}{q}}.$$

3 The Koch and Tataru theorem and T. Barker's question

We may now wonder how to generalize the Prodi–Serrin criterion. It means : given $\vec{u}_0 \in L^2$ and weak Leray solutions associated to $\vec{u}_0$, find a space $\mathbb{X}$

(as large as possible) such that if moreover $\vec{u}_0 \in \mathbb{X}$ then we have a solution $\vec{u} \in X_T$ for some space X_T of functions on $(0,T) \times \mathbb{R}^3$ and such that the existence of a solution in X_T implies that any other weak Leray solution is equal to this solution $\vec{u}$ for $0 < t < T$.

The space BMO^{-1}

First of all, we precise which kind of space $\mathbb{X}$ we are going to study. The idea is to look at an initial value which generates a solution in some uniqueness class (where uniqueness holds for small solutions). The setting where to construct such solutions is the setting of mild solutions, as introduced by Kato [Ka] : mild solutions are constructed by the Banach contraction principle.

Due to the symmetries of the equations (if $\vec{u}$ is a solution for initial value $\vec{u}_0$, then $\lambda\vec{u}(\lambda^2 t, \lambda(x - x_0))$ is a solution for the initial value $\lambda\vec{u}_0(\lambda(x - x_0)))$, we look for spaces with norms invariant through the transforms $\vec{u}_0(.) \mapsto \lambda\vec{u}(\lambda(. - x_0))$ (for $\lambda > 0$). Moreover, in order to be able to define

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P} \text{ div } (\vec{u} \otimes \vec{v})\, ds$$

at least for $\vec{u} = \vec{v} = e^{t\Delta}\vec{u}_0$ (first step of the Picard iteration to find a fixed-point to $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$), we ask that $\int_{[0,1]\times B(0,1)} |e^{s\Delta}\vec{u}_0(y)|^2\, ds\, dy < +\infty$.

Thus, we are lead to introduce the space $\mathbb{X}$ of distributions v such that

- $\sup_{t>0} \sqrt{t}\|e^{t\Delta}v\|_\infty < +\infty$
- $\sup_{0<t, x_0\in\mathbb{R}^3} t^{-3/2} \int_0^t \int_{B(x_0,\sqrt{t})} |e^{s\Delta}v(y)|^2\, dy\, ds)^{1/2}$

This space $\mathbb{X}$ has been identified by Koch and Tataru [KocT] : this is the Triebel–Lizorkin space $\dot{F}^{-1}_{\infty,2}$, or equivalently the space $\mathrm{BMO}^{-1} = \sqrt{-\Delta}\,\mathrm{BMO}$. Moreover, they proved the following theorem :

Theorem 1 *For $0 < T \le \infty$, define*

$$\|\vec{u}\|_{X_T} =$$

$$\sup_{0<t<T} \sqrt{t}\|\vec{u}(t,.\|_\infty + \sup_{0<t<T, x_0\in\mathbb{R}^3} (t^{-3/2} \int_0^t \int_{B(x_0,\sqrt{t})} |\vec{u}(s,y)|^2\, dy\, ds)^{1/2}$$

There exists C_0 (which does not depend on T) such that if $T \in (0, +\infty]$, if $\vec{u}$ and $\vec{v}$ are defined on $(0,T) \times \mathbb{R}^3$ and if

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\ div\ (\vec{u} \otimes \vec{v})\, ds$$

then

$$\|B(\vec{u},\vec{v})\|_{X_T} \leq C_0 \|\vec{u}\|_{X_T} \|\vec{v}\|_{X_T}.$$

Corollary 1 *If $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$, then the integral Navier–Stokes equations have a solution on $(0,T)$ such that $\|\vec{u}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$.*

This is the unique solution such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$.

A special case of initial data that leads to a solution in some X_T is given by the subspace VMO^{-1} of BMO^{-1}.

Definition 1 VMO^{-1} *is the closure of compactly supported functions in* BMO^{-1}.

If $\vec{u}_0 \in \mathrm{VMO}^{-1}$, then $\lim_{T\to 0} \|e^{t\Delta}\vec{u}_0\|_{X_T} = 0$. Remark that we have the embedding $\dot{B}_{q,p}^{-1+\frac{3}{q}} \subset \mathrm{VMO}^{-1}$ for $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$. As a matter of fact, we may consider VMO^{-1} as a limit case for the scale of spaces $\dot{B}_{q,p}^{-1+\frac{3}{q}}$. Thus, Barker [Ba] raised the following question :

Question 1 *If $\vec{u}_0$ belongs to $L^2 \cap \mathrm{VMO}^{-1}$, does there exists a positive time T such that every weak Leray solution of the Cauchy problem for the Navier–Stokes equations with $\vec{u}_0$ as initial value coincide with the mild solution in X_T ?*

If $\vec{u}_0 \in L^2 \cap \mathrm{VMO}^{-1}$ and if $\|e^{t\Delta}\vec{u}_0\|_{X_T} \leq \frac{1}{4C_0}$, then if $\vec{u}$ is a restricted Leray solution of the Navier–Stokes solutions with initial value $\vec{u}_0$, then $\|\vec{v}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$. In particular, we have uniqueness of *restricted Leray solutions* on $(0,T)$.

As a matter of fact, this proof of local uniqueness of restricted weak Leray solutions holds for a slightly more general class :

Definition 2 BMO_0^{-1} *is the space of distributions u_0 in* BMO^{-1} *such that*

$$\lim_{T\to 0} \|e^{t\Delta}u_0\|_{X_T} = 0.$$

In the following, we will focus on the hypothesis $\vec{u}_0 \in L^2 \cap \mathrm{BMO}_0^{-1}$ and on the issue of uniqueness for Leray solutions.

4 The limiting case

Up to now, we don't know how to prove local uniqueness of the Leray solutions when the initial value $\vec{u}_0$ belongs to $L^2 \cap \mathrm{BMO}_0^{-1}$. What we know for sure is that the mild solution $\vec{u}$ in X_T belongs to $L^\infty((\epsilon, T) \times \mathbb{R}^3)$ for every positive $\epsilon \in (0, T)$. Moreover, $\vec{u}$ is a weak Leray solution and for every other Leray solution $\vec{v}$ and for $\epsilon > 0$, we have

$$\partial_t(\vec{u}.\vec{v}) = \vec{u}.\partial_t\vec{v} + \partial_t\vec{u}.\vec{v}$$

which gives for $0 < \epsilon < t < T$

$$\int \vec{u}(t,x).\vec{v}(t,x)\,dx = \int \vec{u}(\epsilon,x).\vec{v}(\epsilon,x)\,dx - 2\int_\epsilon^t \int \vec{\nabla} \otimes \vec{u}.\vec{\nabla} \otimes \vec{v}\,dx\,ds \\ - \int_\epsilon^t \int \vec{u}.(\vec{v}.\vec{\nabla}\vec{v}) + \vec{v}.(\vec{u}.\vec{\nabla}\vec{u})\,dx\,ds$$

From

$$\int \vec{u}.(\vec{v}.\vec{\nabla}\vec{v})+\vec{v}.(\vec{u}.\vec{\nabla}\vec{u})\,dx = \int \vec{u}.(\vec{v}.\vec{\nabla}(\vec{v}-\vec{u}))+(\vec{v}-\vec{u}).(\vec{u}.\vec{\nabla}\vec{u})\,dx = \int \vec{u}\big((\vec{v}-\vec{u}).\vec{\nabla}(\vec{v}-\vec{u})\big)\,dx$$

and letting ϵ go to 0, we get

$$\int \vec{u}(t,x).\vec{v}(t,x)\,dx = \|\vec{u}_0\|_2^2 - 2\int_0^t \int \vec{\nabla} \otimes \vec{u}.\vec{\nabla} \otimes \vec{v}\,dx\,ds \\ - \lim_{\epsilon \to 0} \int_\epsilon^t \int \vec{u}\big((\vec{v}-\vec{u}).\vec{\nabla}(\vec{v}-\vec{u})\big)\,dx\,ds$$

and (letting $\vec{v} = \vec{u}$)

$$\|\vec{u}(t,.)\|_2^2 = \|\vec{u}_0\|_2^2 - 2\int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2\,ds.$$

Combining those two equalities with the Leray energy inequality for $\vec{v}$

$$\|\vec{v}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2\,ds \le \|\vec{u}_0\|_2^2$$

we get the following inequality for $\vec{w} = \vec{u} - \vec{v}$:

$$\|\vec{w}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2\,ds \le 2\int_0^T |\int \vec{u}.(\vec{w}.\vec{\nabla}\vec{w})\,dx|\,ds. \tag{4}$$

As a matter of fact, the key ingredient in Prodi–Serrin's criterion is the estimation of the integral

$$I(\vec{u},\vec{w}) = \int_0^t \left| \int \vec{u}(.\vec{w}.\vec{\nabla}\vec{w})\, dx \right| ds$$

but, if $\vec{w} = \vec{v} - \vec{u}$ with $\vec{v}$ a Leray solution and $\vec{u}$ the mild solution in X_T, we don't even know whether $I(\vec{u},\vec{w})$ is finite.

In the limiting case of Prodi and Serrin, (for $p = 2$ and $q = +\infty$), we write

$$\|\vec{w}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{w}\|_2^2\, ds \leq 2\int_0^t \int_0^t \left| \int \vec{u}(.\vec{w}.\vec{\nabla}\vec{w})\, dx \right| ds \leq 2\int_0^t \|\vec{u}\otimes\vec{w}\|_2 \|\vec{\nabla}\otimes\vec{w}\|_2\, ds$$

and get

$$\|\vec{w}(t,.)\|_2^2 \leq \int_0^t \|\vec{u}\|_\infty^2 \|\vec{w}\|_2^2\, ds. \tag{5}$$

Of course, we may conclude under the assumption that $\vec{u} \in L^2_t L^\infty_x$. Actually, we shall not be interested in measurabilty issues for functions with values in a non-separable space such as L^∞ (i.e. in Bochner measurability for instance), as we are dealing with locally integrable functions for the Lebesgue measure $dt\, dx$ on $(0,T)\times\mathbb{R}^3$. Thus, for almost every t the quantity $\|\vec{u}(t,.)\|_\infty$ will be wel-defined as a measurable function of t, and $\vec{u} \in L^2_t L^\infty_x$ will simply mean that $\int_0^T \|\vec{u}(t,.)\|^2_{L^\infty(dx)}\, dt < +\infty$. (See [LR 2] for details.)

If $\vec{u} \in L^2_t L^\infty_x$ on $(0,T)\times\mathbb{R}^3$, the Prodi–Serrin criterion proves that every weak Leray solution $\vec{v}$ on $(0,T)$ is equal to the mild solution $\vec{u}$. (This has even be extended to the case $\vec{u} \in L^2_t \mathrm{BMO}_x$ by Kozono and Tanyuchi [KozT]). But it is not easy to translate the condition that $\vec{u} \in L^2_t L^\infty_x$ into an equivalent assumption on $\vec{u}_0$. The problem comes from the fact that the bilinear operator B is not bounded on $L^2_t L^\infty_x$.

On the other hand, if we only assume $\vec{u}_0 \in L^2 \cap \mathrm{BMO}_0^{-1}$, we only know the inequality $\|\vec{u}(t,.)\|_\infty \leq \|\vec{u}\|_{X_T} \frac{o(1)}{\sqrt{t}}$. We find an integrability issue near $t = 0$. To check that this is actually an issue, consider the following example : take $\vec{\omega}$ a divergence-free vector field in the Schwartz class such that the Fourier transform of $\vec{\omega}$ is compactly supported in the annulus $1 < |\xi| < 2$; define

$$\vec{u}_0(x) = \sum_{j=0}^{+\infty} 2^j \frac{1}{\sqrt{1+j}} \vec{\omega}(2^j x);$$

we have $\vec{u}_0 \in L^2$; $\vec{u}_0 \in \dot{B}^{-1+\frac{3}{q}}_{q,p} \subset \dot{B}^{-1}_{\infty,p}$ (with $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$) but $\vec{u}_0 \notin \dot{B}^{-1}_{\infty,2}$; as the bilinear operator B is bounded on $L^p_t L^q_x \cap L^2_t L^\infty_x$, if we assume that the mild solution $\vec{u} \in L^p_t L^q_x$ belongs to $L^2_t L^\infty_x$, we would find that $e^{t\Delta}\vec{u}_0 \in L^2_t L^\infty_x$, and thus $\vec{u}_0 \in \dot{B}^{-1}_{\infty,2}$; thus, we have $\int_0^T \|\vec{u}\|_\infty^2\, dt = +\infty$.

5 Barker's theorem

In this section, we shall sketch the proof of Barker [Ba], as we shall extend it in Section 7 to the case of Besov–Morrey spaces.The main idea in the recent paper of Barker is the following one : if we want to use only the inequality

$$\|\vec{u}(t,.)\|_\infty \le \|\vec{u}\|_{X_T} \frac{o(1)}{\sqrt{t}}$$

to deal with the Gronwall inequality (5), we need to assume more than $\vec{w} \in L^\infty_t L^2_x$. Indeed, we have the easy following lemma :

Lemma 1 *Let $\delta > 0$. Let A and B be locally bounded non-negative measurable functions on $(0,T]$ such that*

$$\lim_{t\to 0} tA(t) = 0 \textit{ and } \sup_{0<t<T} t^{-\delta} B(t) = 0.$$

If we have moreover, for all $t \in (0,T]$,

$$B(t) \le \int_0^t A(s)B(s)\, ds$$

then $B = 0$.

Proof : We have

$$B(t) \le \frac{t^\delta}{\delta} \sup_{0<s<t} sA(s) \sup_{0<\sigma<t} \sigma^{-\delta} B(\sigma)$$

so that $B = 0$ on $(0, T_0]$ as long as $\sup_{0<s<T_0} sA(s) < \delta$. For $t > T_0$, we then write $B(t) \le \sup_{T_0<s<T} A(s) \int_{T_0}^t B(s)\, ds$ and we find $B = 0$. $\diamond$

As $\vec{w} = \vec{v} - \vec{u} = (\vec{v} - e^{t\Delta}\vec{u}_0) - (\vec{u} - e^{t\Delta}\vec{u}_0)$, the extra information on $\|\vec{w}\|_2$ will be provided by the following lemma :

Lemma 2 *Let $\vec{u}_0$ be a divergence-free vector field with $\vec{u}_0 \in L^2$ and let $\vec{v}$ be a weak Leray solution of the Navier–Stokes equations with initial value $\vec{u}_0$. If moreover*

$$\vec{u}_0 \in [L^2, B^{-\gamma}_{\infty,\infty}]_{\theta,\infty} \textit{ for some } -1 < -\gamma < 0 \textit{ and } 0 < \theta < 1$$

then there exists $\delta > 0$ such that

$$\sup_{t>0} t^{-\delta} \|\vec{v}(t,.) - e^{t\Delta}\vec{u}_0\|_2 < +\infty.$$

Proof : Assume $\vec{u}_0 \in [L^2, B^{-\gamma}_{\infty,\infty}]_{\theta,\infty}$. For $0 < \epsilon < 1$, split $\vec{u}_0$ in $\vec{\alpha}_\epsilon + \vec{\beta}_\epsilon$ with

$$\|\vec{\alpha}_\epsilon\|_2 \leq C_0 \epsilon^\theta \text{ and } \|\vec{\beta}_\epsilon\|_{\dot{B}^{-\gamma}_{\infty,\infty}} \leq C_0 \epsilon^{\theta-1}$$

where C_0 does not depend on ϵ (but depends on $\vec{u}_0$).

We have a solution $\vec{U}_\epsilon$ for the Navier-Stokes equations with initial value $\vec{\beta}_\epsilon$ such that $\|\vec{U}_\epsilon(t,.)\|_\infty \leq C_1 t^{-\frac{\gamma}{2}} \epsilon^{\theta-1}$ on an interval $(0, T_\epsilon)$ with $T_\epsilon^{\frac{1-\gamma}{2}} \epsilon^{\theta-1} = C_2$. Moreover $\vec{\beta}_\epsilon = \vec{u}_0 - \vec{\alpha}_\epsilon \in L^2$ and we find that $\sup_{0<t<T_\epsilon} \|\vec{U}_\epsilon\|_2 \leq C_3$ and that $\|\vec{U}_\epsilon - e^{t\Delta}\vec{\beta}_\epsilon\|_2 \leq C_4 t^{(1-\gamma)/2} \epsilon^{\theta-1}$.

Since $\vec{U}_\epsilon$ is in $L^2_t L^\infty_x$ on every bounded interval, we get

$$\int \vec{U}_\epsilon(t,x).\vec{v}(t,x)\,dx = \int \vec{\beta}_\epsilon(x).\vec{u}_0(x)\,dx - 2\int_0^t \int \vec{\nabla} \otimes \vec{U}_\epsilon . \vec{\nabla} \otimes \vec{v}\,dx\,ds$$
$$- \int_0^t \int \vec{U}_\epsilon \big((\vec{v} - \vec{U}_\epsilon).\vec{\nabla}(\vec{v} - \vec{U}_\epsilon) \big)\,dx\,ds$$

and

$$\|\vec{U}_\epsilon(t,.)\|_2^2 = \|\vec{\beta}_\epsilon\|_2^2 - 2\int_0^t \|\vec{\nabla} \otimes \vec{U}_\epsilon\|_2^2\,ds.$$

Combining those two equalities with the Leray energy inequality for $\vec{v}$

$$\|\vec{v}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2\,ds \leq \|\vec{u}_0\|_2^2$$

we get the following inequality for $\vec{W}_\epsilon = \vec{v} - \vec{U}_\epsilon$:

$$\|\vec{W}_\epsilon(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{W}_\epsilon\|_2^2\,ds \leq \|\vec{\alpha}_\epsilon\|_2^2 + 2\int_0^T |\int \vec{U}_\epsilon.(\vec{W}_\epsilon.\vec{\nabla}\vec{W}_\epsilon)\,dx|\,ds.$$

Thus, we get

$$\|\vec{W}_\epsilon(t,.)\|_2^2 \leq C_0^2 \epsilon^{2\theta} + C_1^2 \epsilon^{2(\theta-1)} \int_0^t s^{-\gamma} \|\vec{W}_\epsilon(s,.)\|_2^2\,ds$$

so that

$$\|\vec{W}_\epsilon(t,.)\|_2^2 \leq C_0^2 \epsilon^{2\theta} e^{C_1^2 \epsilon^{2(\theta-1)} \frac{t^{1-\gamma}}{1-\gamma}}.$$

Now, for $\tau < 1$, take $\epsilon = \tau^\mu$ with $\frac{1-\gamma}{2} + \mu(\theta - 1) > 0$. We find that, for $0 < t < T_\epsilon$ with

$$T_\epsilon = C_2^{\frac{2}{1-\gamma}} \epsilon^{\frac{2(1-\theta)}{1-\gamma}} = C_2^{\frac{2}{1-\gamma}} \tau^{\mu \frac{2(1-\theta)}{1-\gamma}} \text{ [where } \mu \frac{2(1-\theta)}{1-\gamma} < 1],$$

we have the inequality

$$\begin{aligned}\|\vec v - e^{t\Delta}\vec u_0\|_2 \leq & \|\alpha_\epsilon\|_2 + \|\vec W_\epsilon\|_2 + \|\vec U_\epsilon - e^{t\Delta}\vec\beta_\epsilon\|_2 \\ \leq & C_0\epsilon^\theta(1 + e^{\frac{C_1^2}{2(1-\gamma)}\frac{t^{1-\gamma}}{T_\epsilon^{1-\gamma}}}) + C_4 t^{\frac{1-\gamma}{2}}\epsilon^{\theta-1}\end{aligned}$$

If τ is small enough, we have $\tau < T_\epsilon$ and we find

$$\|\vec v(\tau,.) - e^{\tau\Delta}\vec u_0\|_2 \leq C_0\tau^{\mu\theta}(1 + e^{\frac{C_1^2}{2(1-\gamma)}}) + C_4\tau^{\frac{1-\gamma}{2}+\mu(\theta-1)}$$

The lemma is proved. $\diamond$ Barker's theorem then reads as :

Theorem 2 *Let $\vec u_0$ be a divergence-free vector field with $\vec u_0 \in L^2$ and let $\vec v$ be a weak Leray solution of the Navier–Stokes equations with initial value $\vec u_0$. If moreover*

$$\vec u_0 \in \mathrm{BMO}_0^{-1} \cap \dot B^{-s}_{q,\infty} \text{ with } 3 < q < +\infty \text{ and } -s > -1 + \frac{2}{q}$$

then there exists $T > 0$ such that if $\vec v$ is a Leray solution and if $\vec u$ is the mild solution with $\|\vec u\|_{X_T} < +\infty$, then $\vec u = \vec v$ on $(0,T)$.

Remark We have the embeddings $L^2 \subset \dot B^{-\frac{3}{2}+\frac{3}{q}}_{q,2} \subset \dot B^{-\frac{3}{2}+\frac{3}{q}}_{q,\infty}$, so that the information conveyed by the hypothesis $\vec u_0 \in \dot B^{-s}_{q,\infty}$ is interesting only for the high frequencies of $\vec u_0$. Moreover the embeddings $L^2 = \dot B^0_{2,2}$ and $\mathrm{BMO}^{-1} \subset \dot B^{-1}_{\infty,\infty}$ gives that

$$L^2 \cap \mathrm{BMO}_0^{-1} \subset \dot B^{-1+\frac{2}{q}}_{q,q} \subset \dot B^{-1+\frac{2}{q}}_{q,\infty}.$$

Thus, the information conveyed by the hypothesis $\vec u_0 \in \dot B^{-s}_{q,\infty}$ is not contained in the assumption $\vec u_0 \in L^2 \cap \mathrm{BMO}_0^{-1}$. Finally, if $-s > -1 + \frac{3}{q} = \frac{2}{p}$, then we have

$$L^2 \cap \mathrm{BMO}_0^{-1} \cap \dot B^{-s}_{q,\infty} \subset \dot B^{-1+\frac{2}{q}}_{q,q} \cap \dot B^{-s}_{q,\infty} \subset \dot B^{-1+\frac{3}{q}}_{q,p}.$$

Thus, the theorem is interesting only in the range $-1 + \frac{2}{q} < -s \leq -1 + \frac{3}{q}$, which corresponds to the gap between $L^2 \cap \mathrm{BMO}_0^{-1}$. (where loca'luniqueness is conjectured to hold) and $\dot B^{-1+\frac{3}{q}}_{q,p}$ (for which the Prodi–Serrin criterion shows that local uniqueness holds).

A further remark is that we have the embedding $\dot B^{-1+\frac{3}{q}}_{q,\infty} \subset \mathrm{BMO}^{-1}$, so that we have a Prodi-Serrin criterion with $\vec u \in L^p_t L^q_x$ (with $\frac{2}{p} + \frac{3}{q} = 1$) replaced with

$$\sup_{0<t<T} t^{\frac{1}{p}}\|\vec u\|_q < +\infty \text{ and } \lim_{t\to 0} t^{\frac{1}{p}}\|\vec u(t,.)\|_q = 0.$$

Proof :

The first step is the use of interpolation inequalities in order to be able to check that $\vec u_0$ fulfills the assuptions of Lemma 2.

- Since $\vec{u}_0 \in L^2 = \dot{B}^0_{2,2}$ and $\vec{u}_0 \in \mathrm{BMO}^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$, we have $\vec{u}_0 \in \dot{B}^{-1+\frac{2}{q}}_{q,q}$.

- Since $\vec{u}_0 \in \dot{B}^{-1+\frac{2}{q}}_{q,q} \cap \dot{B}^{-s}_{q,\infty}$, we have for $-1+\frac{2}{q} < -\sigma < -s$, $\vec{u}_0 \in \dot{B}^{-\sigma}_{q,1}$.

- The Besov space $\dot{B}^{-\sigma}_{q,1}$ is embedded in the Sobolev space $\dot{W}^{-\sigma,q}$.

- For $q < r < \infty$, we have $\dot{W}^{-\sigma,q} = [L^2, \dot{W}^{-\delta,r}]_{[\theta]}$ with $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{r}$ and $-s = -\theta\delta$ (complex inteerpolation)

- Since $\dot{W}^{-\delta,r} \subset \dot{B}^{-1+\frac{3}{r}}_{\infty,\infty}$, we have $[L^2, \dot{W}^{-\delta,r}]_{[\theta]} \subset [L^2, \dot{B}^{-\delta-\frac{3}{r}}_{\infty,\infty}]_{\theta,\infty}$ with
$$-\delta = -\frac{s}{\theta} = -s\frac{\frac{1}{2}-\frac{1}{r}}{\frac{1}{2}-\frac{1}{q}} = -\frac{(-s)}{-1+\frac{2}{q}} + O(\frac{1}{r})$$
so that $-\delta - \frac{3}{r} > -1$ for r large enough

We may now end the proof : recall that if $\vec{v}$ is a Leray solution and if $\vec{u}$ is the mild solution with $\|\vec{u}\|_{X_T} < +\infty$, then the difference $\vec{w} = \vec{u} - \vec{v}$ satisfies a Gronwall estimate :
$$\|\vec{w}(t,.)\|_2^2 \leq \int_0^t \|\vec{u}\|_\infty^2 \|\vec{w}\|_2^2 \, ds.$$

By Lemma 2, we have $\|\vec{v}(t,.) - e^{t\Delta}\vec{u}_0\|_2 = O(t^\delta)$ and $\|\vec{u}(t,.) - e^{t\Delta}\vec{u}_0\|_2 = O(t^\delta)$ for some positive δ. On the other hand, we know that $\|\vec{u}(t,.)\|_\infty = o(\frac{1}{\sqrt{t}})$. Using Lemma 1, we find that $\vec{w} = 0$, and $\vec{v} = \vec{u}$.$\diamond$

6 The Prodi–Serrin criterion for Besov–Morrey spaces

Morrey spaces provide a natural tool for extending the Prodi–Serrin criterion.

Definition 3 *For $1 < r \leq q < +\infty$, we define the Morrey space $\dot{M}^{r,q}$ as the space of Lebesgue measurable functions f on $\mathbb{R}^3$ such that*
$$\sup_{R>0, x_0\in\mathbb{R}^3} R^{\frac{3}{q}-\frac{3}{r}} (\int_{B(x_0,R)} |f(x)|^r \, dx)^{1/r} = \|f\|_{\dot{M}^{r,q}} < +\infty.$$
Similarly, the space $\dot{M}^{1,q}$ is the space of locally finite Borelian (signed) measure μ such that
$$\sup_{R>0, x_0\in\mathbb{R}^3} R^{\frac{3}{q}-3} (\int_{B(x_0,R)} d|\mu|) = \|\mu\|_{\dot{M}^{1,q}} < +\infty.$$

Remark : For absolutely continuous measures $d\mu = f\,dx$ with $f \in L^1_{\rm loc}$, we have

$$\|\mu\|_{\dot M^{1,q}} = \sup_{R>0, x_0\in\mathbb{R}^3} R^{\frac{3}{q}-3}(\int_{B(x_0,R)} |f(x)|\,dx).$$

The key inequality in the proof of the Prodi–Serrin criterion was the inequality (for all $w \in H^1$)

$$\|uw\|_2 \le C\|u\|_q \|w\|_2^{\frac{2}{p}} \|\vec\nabla w\|_2^{\frac{3}{q}}$$

with $\frac{2}{p} + \frac{3}{q} = 1$ and $3 < q < +\infty$. If we want to replace this inequality by a more general inequality

$$\|uw\|_2 \le N(u)\|w\|_2^{\frac{2}{p}} \|\vec\nabla w\|_2^{\frac{3}{q}}$$

(again with $\frac{2}{p} + \frac{3}{q} = 1$ and $3 < q < +\infty$), then we proved in [LR 3] that the existence of a finite $N(u)$ is equivalent to the fact that $u \in \dot M^{2,q}$, and moreover that $N(u) \approx \|u\|_{\dot M^{2,q}}$.

This leads to the following easy extension of the Prodi-Serrin criterion :

Theorem 3 *If $\vec u_0 \in L^2$ and if the Navier-Stokes equations has a solution $\vec u$ such that*

$$\vec u \in L^p_t \dot M^{2,q}_x \text{ with } \frac{2}{p} + \frac{3}{q} = 1 \text{ and } 3 < q < +\infty$$

then if $\vec v$ is a Leray solution we have $\vec u = \vec v$ on $(0,T)$.

If $3 < q < +\infty$, the existence of $T > 0$ and of a solution in $L^p \dot M^{2,q}$ with $\frac{2}{p} + \frac{3}{q}$ is equivalent to the existence of T' such that $e^{t\Delta}\vec u_0 \in L^p \dot M^{2;q}$ on $(0,T')$ (and on $(0,+\infty)$, since $\vec u_0 \in L^2$), thus with

$$\vec u_0 \in \dot B^{-1+\frac{3}{q}}_{\dot M^{2,q},p}.$$

This Besov–Morrey space has been introduced in 1994. by Kozono and Yamazaki [KoY]. It is easy to check that, for $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$, we have the inequality

$$\|e^{t\Delta}u_0\|_{X_T} \le C_q(\int_0^T \|e^{t\Delta}u_0\|^p_{\dot M^{2,q}}\,dt)^{1:p}$$

so that we have the embedding $\dot B^{-1+\frac{3}{q}}_{\dot M^{2,q},p} \subset \mathrm{BMO}_0^{-1}$ for $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$.

7 Barker's theorem and Besov-Morrey spaces

We shall extend Barker's theorem.

Theorem 4 *If*

- $\vec{u}_0 \in L^2 \cap \mathrm{BMO}_0^{-1}$
- $3 < q < +\infty$, $-s > -1 + \frac{2}{q}$ *and* $\vec{u}_0 \in \dot{B}^{-s}_{\dot{M}^{1,q},\infty}$

then there exists $T > 0$ *such that if* $\vec{v}$ *is a suitable Leray solution and if* $\vec{u}$ *is the mild solution with* $\|\vec{u}\|_{X_T} < +\infty$, *then* $\vec{u} = \vec{v}$ *on* $(0, T)$.

Proof : As we shall see, the proof is very similar to Barker's proof for Theorem 2 [Ba]. However, we shall meet some technical issues.

We sketch the proof :

- $\vec{u}_0 \in L^2 = \dot{B}^0_{2,2}$ and $\vec{u}_0 \in \mathrm{BMO}^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$, thus $\vec{u}_0 \in \dot{B}^{-1+\frac{2}{q}}_{q,q}$
- $\vec{u}_0 \in \dot{B}^{-1+\frac{2}{q}}_{q,q} \cap \dot{B}^{-s}_{\dot{M}^{1,q},\infty}$ thus $\vec{u}_0 \in \dot{B}^{-\sigma}_{\dot{M}^{p,q},\infty}$ for $1 < p < q$, $\frac{1}{p} = 1 - \theta + \frac{\theta}{q}$ and $-\sigma = -s(1-\theta) + \theta(-1 + \frac{2}{q}) > -1 + \frac{2}{q}$. We shall take $p > 2$.
- as $p < q$, we have $\dot{B}^{-1+\frac{2}{q}}_{q,q} \subset \dot{B}^{1+\frac{2}{q}}_{q,p}$. Thus, for $-1 + \frac{2}{q} < -\gamma < -\sigma$,

$$\dot{B}^{-1+\frac{2}{q}}_{q,q} \cap \dot{B}^{-\sigma}_{\dot{M}^{p,q},\infty} \subset \dot{B}^{-\gamma}_{\dot{M}^{p,q},1} \subset \dot{W}^{-\gamma,\dot{M}^{p,q}}.$$

We now encounter our first problem. We can no longer write $W^{-\gamma,\dot{M}^{p,q}}$ as a subspace of an interpolate space between L^2 and $\dot{B}^{-1+\delta}_{\infty,\infty}$. More precisely, let us assume $\dot{B}^{-\gamma}_{\dot{M}^{p,q},1} \subset [L^2, \dot{B}^{-1+\delta}_{\infty,\infty}]_{\theta,\infty}$; by homogeneity of the norms, we must have $-\gamma - \frac{3}{q} = -(1-\theta)\frac{3}{2} - \theta(1-\delta)$. We have

$$[L^2, \dot{B}^{-1+\delta}_{\infty,\infty}]_{\theta,\infty} \subset \dot{B}^{-\theta(1-\delta)}_{L^{r,\infty},\infty}$$

with $r = \frac{2}{1-\theta}$. In particular, for $u \in [L^2, \dot{B}^{-1+\delta}_{\infty,\infty}]_{\theta,\infty}$, we have that $e^{\Delta} e^{t\partial_3^2} u$ goes to 0 in $\mathcal{S}'$ when t goes to $+\infty$. But if $3p \leq 2q$, if u depends only on (x_1, x_2) and not on x_3, and if $u \in \dot{B}^{-\gamma}_{\dot{M}^{p,\frac{2q}{3}},1}(\mathbb{R}^2)$, then $u \in \dot{B}^{-\gamma}_{\dot{M}^{p,q},1}(\mathbb{R}^3)$ and $e^{\Delta} e^{t\partial_3^2} u = e^{\Delta} u$. Thus, we have a contradiction.

We better use complex interpolation and write that

$$W^{-\gamma,\dot{M}^{p,q}} = [\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}]^{[\theta]}$$

for $r > p$, $\frac{1-\theta}{2} + \frac{\theta}{r} = \frac{1}{p}$, $\gamma = \theta\rho$. (For interpolation of Morrey spaces, see [LR 4, LR 5])

Then, we remark that $\dot M^{2,\frac{2}{p}q} \subset L^2_{\rm uloc}$ and write that

$$[\dot M^{2,\frac{2}{p}q}, \dot W^{-\rho,\dot M^{r,\frac{r}{p}q}}]^{[\theta]} \subset [\dot M^{2,\frac{2}{p}q}, \dot W^{-\rho,\dot M^{r,\frac{r}{p}q}}]_{\theta,\infty} \subset [L^2_{\rm uloc}, \dot B^{-\rho-\frac{3p}{rq}}_{\infty,\infty}]_{\theta,\infty}$$

In order to finish the proof, we thus need to use the machinery of energy control for suitable local Leray solutions [LR 2, LR 6]. This will be done in the following sections, and we shall finish the proof in Section 10 $\diamond$

8 Weak local Leray solutions

We recall basic results for local weak Leray solutions. We endow $L^2_{\rm uloc}$ with the norm

$$\|u\|_{L^2_{\rm uloc}} = \sup_{k\in\mathbb{Z}^3} \|u\varphi_0(x-k)\|_2,$$

where φ_0 is a non-negative function in $\mathcal{D}$, suppported in a ball $B(0,R_0)$ and such that $\sum_{k\in\mathbb{Z}^3} \varphi_0(x-k) = 1$.

When $\vec u_0 \in L^2_{\rm uloc}$, proof of existence of solutions for the Navier–Stokes equations is based on mollification, energy estimates and compactness arguments (for details, see [LR 6], section 14.1) :

- we solve
$$\partial_t \vec u_\epsilon + (\varphi_\epsilon * \vec u_\epsilon).\vec\nabla \vec u_\epsilon = \Delta \vec u_\epsilon - \vec\nabla p_\epsilon$$
with div $\vec u_\epsilon = 0$ and $\vec u_\epsilon(0,.) = \vec u_0$. Here, $\varphi \in \mathcal{D}$, $\int \varphi\, dx = 1$ and $\varphi_\epsilon(x) = \frac{1}{\epsilon^3}\varphi(\frac{x}{\epsilon})$. Here $\vec\nabla p_\epsilon$ is given by the Leray projection :
$$\vec\nabla p_\epsilon = -(\varphi_\epsilon * \vec u_\epsilon).\vec\nabla u_\epsilon + \mathbb{P}{\rm div}\left((\varphi_\epsilon * \vec u_\epsilon)\otimes \vec u_\epsilon\right).$$

- the solution holds at least on an interval $(0, T_\epsilon)$ where T_ϵ depends on ϵ and on $\|\vec u_0\|_{L^2_{\rm uloc}}$ ($T_\epsilon = \min(1, C_0 \frac{\epsilon^{3/2}}{\|\vec u_0\|^2_{L^2_{\rm uloc}}})$. Moreover, we have the inequalities, for $k \in \mathbb{Z}^3$,
$$\begin{aligned}\int \varphi_0(x-k)|\vec u_\epsilon(t,x)|^2\, dx &+ 2\int_0^t \int \varphi_0(x-k)|\vec\nabla\otimes \vec u_\epsilon(s,x)|^2\, dx\, ds \\ \le& \int \varphi_0(x-k)|\vec u_0(t,x)|^2\, dx + C_1 \int_0^t \int_{|x-k|\le R_0} |\vec u_\epsilon(s,x)|^2\, dx\, ds \\ &+ C_2 \int_0^t \int_{|x-k|>5R_0} \frac{1}{|x-k|^4} |\vec u_\epsilon(s,x)|^3\, dx\, ds \\ &+ C_2 \int_0^t \int_{|x-k|<5R_0} |\vec u_\epsilon(s,x)|^3\, dx\, ds\end{aligned}$$

- defining
$$\alpha_\epsilon(t)=\|\vec u_\epsilon(t,.)\|_{L^2_{\rm uloc}}$$
and
$$\beta_\epsilon(t)=\sup_{k\in\mathbb{Z}^3}\left(\int_0^t\int\varphi_0(x-k)|\vec\nabla\otimes\vec u_\epsilon(s,x)|^2\,dx\,ds\right)^{1/2},$$
we get the inequality
$$\int\varphi_0(x-k)|\vec u_\epsilon(t,x)|^2\,dx+\int_0^t\int\varphi_0(x-k)|\vec\nabla\otimes\vec u_\epsilon(s,x)|^2\,dx\,ds$$
$$\leq\alpha_\epsilon(0)^2+\frac12\beta_\epsilon(t)^2+C_3\int_0^t\alpha_\epsilon(s)^2\,ds+C_3\int_0^t\alpha_\epsilon(s)^6\,ds$$
so that
$$\beta_\epsilon(t)^2\leq 2\alpha_\epsilon(0)^2+2C_3\int_0^t\alpha_\epsilon(s)^2\,ds+2C_3\int_0^t\alpha_\epsilon(s)^6\,ds$$
and finally
$$\alpha_\epsilon(t)^2\leq 2\alpha_\epsilon(0)^2+2C_3\int_0^t\alpha_\epsilon(s)^2\,ds+2C_3\int_0^t\alpha_\epsilon(s)^6\,ds$$
Thus, as long as $8C_3t<1$ and $128\,C_3t\|\vec u_0\|^4_{L^2_{\rm uloc}}<1$, we find that $\alpha_\epsilon(t)\leq 2\|\vec u_0\|_{L^2_{\rm uloc}}$ and $\beta_\epsilon(t)\leq 2\|\vec u_0\|_{L^2_{\rm uloc}}$.
- the solution is then defined on $(0,\min(\frac{1}{8C_3},\frac{1}{128\,C_3\,\|\vec u_0\|^4_{L^2_{\rm loc}}})$ and controlled independently from ϵ. By Rellich theorem, wre find a subsequence that converges strongly in $(L^2_tL^2_x)_{loc}$ to a suitable local Leray solution $\vec u$

An important point is the following one : assume moreover that $\vec u_0\in{\rm BMO}^{-1}_0$ and that $\|e^{t\Delta}\vec u_0\|_{X_T}<\frac{1}{4C_0}$ (where C_0 is the constant of Theorem 1) then $\vec u_\epsilon$ is defined at least on $(0,T)$ and $\|\vec u_\epsilon\|_{X_T}\leq 2\|e^{t\Delta}\vec u_0\|_{X_T}$. As T does not depend on ϵ, we see that the local Leray solution $\vec u$ satisfies $\vec u\in X_S$ with $S=\min(T,\frac{1}{8C_3},\frac{1}{128\,C_3\,\|\vec u_0\|^4_{L^2_{\rm loc}}})$ and $\|\vec u\|_{X_S}\leq 2\|e^{t\Delta}\vec u_0\|_{X_T}$.

Similarly, if we assume that $\vec u_0\in\dot B^{-\gamma}_{\infty,\infty}$ with $-1<-\gamma<0$, then $\vec u_\epsilon$ is defined at least on $(0,T)$ where $T=C\|\vec u_0\|^{\frac{2}{1-\gamma}}_{\dot B^{-\gamma}_{\infty,\infty}}$ and $\sup_{0<t<T}t^{\frac{\gamma}{2}}\|\vec u(t,.)\|_\infty\leq 2\sup_{0<t<T}t^{\frac{\gamma}{2}}\|e^{t\Delta}\vec u_0\|_\infty$. As T does not depend on ϵ, we see that the local Leray solution $\vec u$ satisfies the inequality $\sup_{0<t<S}t^{\frac{\gamma}{2}}\|\vec u(t,.)\|_\infty<+\infty$ where $S=\min(T,\frac{1}{8C_3},\frac{1}{128\,C_3\,\|\vec u_0\|^4_{L^2_{\rm loc}}})$.

9 Comparison of local weak Leray solutions

If $\vec{u}$ and $\vec{v}$ are two local weak Leray solutions, on $(0,T)$ with initial values $\vec{u}_0$ and $\vec{v}_0$, we would like to be able to estimate $\vec{u}(t,.)-\vec{v}(t,.)$ from the estimation of $\vec{u}_0-\vec{v}_0$. This can be done only when at least one of the solutions is regular enough. We shall assume that $\vec{u}\in L^2_t L^\infty_x$. We sketch the computations described in [LR 6], section 14.4.

Define $\vec{w}=\vec{u}-\vec{v}$,

$$\alpha(t)=\|\vec{w}(t,.)\|_{L^2_{\mathrm{uloc}}}$$

and

$$\beta(t)=\sup_{k\in\mathbb{Z}^3}\left(\int_0^t\int\varphi_0(x-k)|\vec{\nabla}\otimes\vec{w}(s,x)|^2\,dx\,ds\right)^{1/2},$$

Using the suitability of $\vec{v}$ and the regularity of $\vec{v}$, we find (for $0<t<\min(1,T)$)

$$\begin{aligned}&\int\varphi_0(x-k)|\vec{w}(t,x)|^2\,dx+\int_0^t\int\varphi_0(x-k)|\vec{\nabla}\otimes\vec{w}(s,x)|^2\,dx\,ds\\ &\leq\alpha(0)^2+\frac{1}{2}\beta(t)^2+C_1\int_0^t\alpha(s)^2\,ds+C_2\int_0^t\alpha(s)^6\,ds+C_3\int_0^t\|\vec{u}(s,.)\|_\infty^2\alpha(s)^2\,ds\end{aligned}$$

where the constants C_i do not depend on T, $\vec{u}$, nor on $\vec{v}$. Finally, we find

$$\begin{aligned}\alpha(t)^2\leq&2\alpha(0)^2+2(C_1+C_3(\|\vec{u}\|_{L^\infty_t L^2_{\mathrm{uloc}}}+\|\vec{v}\|_{L^\infty_t L^2_{\mathrm{uloc}}})^4)\int_0^t\alpha(s)^2\,ds\\ &+2C_3\int_0^t\|\vec{u}(s,.)\|_\infty^2\alpha(s)^2\,ds.\end{aligned}\tag{6}$$

We have the same estimate even if $\vec{u}$ is not intregrable near $t=0$. Let us only assume that $\vec{u}\in L^2_t L^\infty_x$ on every (ϵ,T) with $\epsilon>0$. Considering a time $t_0>0$ which a Lebesgue point for the functions $t\mapsto\int\varphi(x-k)|\vec{u}(t,x)|^2\,dx$ and $t\mapsto\int\varphi(x-k)|\vec{u}(t,x)|^2\,dx$, we have that $\vec{u}$ and $\vec{v}$ are local weak Leray solutions on (t_0,T) with initial values $\vec{u}(t_0,.)$ and $\vec{v}(t_0,.)$. Thus, we shall find that, for $t>t_0$

$$\begin{aligned}\int\varphi_0(x-k)|\vec{w}(t,x)|^2\,dx\leq&2\int\varphi_0(x-k)|\vec{w}(t_0,x)|^2\,dx\\ &+2(C_1+C_3(\|\vec{u}\|_{L^\infty_t L^2_{\mathrm{uloc}}}+\|\vec{v}\|_{L^\infty_t L^2_{\mathrm{uloc}}})^4)\int_{t_0}^t\alpha(s)^2\,ds\\ &+2C_3\int_{t_0}^t\|\vec{u}(s,.)\|_\infty^2\alpha(s)^2\,ds.\end{aligned}\tag{7}$$

It is then enough to let t_0 go to 0 and then take the supremum with respect to k.

10 Proof of Theorem 4

We may now finish the proof. We consider two solutions of the Navier–Stokes equations with initial value $\vec{u}_0 \in L^2 \cap \mathrm{BMO}_0^{-1} \cap \dot{B}^{-s}_{\dot{M}^{1,q},\infty}$ with $-s > -1 + \frac{2}{q}$ (and $3 < q < +\infty$) : we assume that $\vec{v}$ is a suitable Leray solution and $\vec{u}$ is the mild solution in X_T.

As $\vec{v}$ is suitable, $\vec{v}$ is a local Leray solution as well and we may estimate the L^2_{uloc} of $\vec{u} - \vec{w}$: defining $B(t) = \|\vec{u}(t,.) - \vec{v}(t,.)\|^2_{L^2_{\mathrm{uloc}}}$ and

$$A(t) = 2(C_1 + C_3(\|\vec{u}\|_{L^\infty_t L^2_{\mathrm{uloc}}} + \|\vec{v}\|_{L^\infty_t L^2_{\mathrm{uloc}}})^4) + 2C_3\|\vec{u}(t,.)\|^2_\infty,$$

we get

$$B(t) \leq \int_0^t A(s)B(s)\,ds.$$

As $\lim_{t\to 0} tA(t) = 0$, we shall try to apply Lemma 1. Thus, we shall use interpolation estimates to search for a control of $B(t)$ as $O(t^{-\delta})$, in the spirit of Lemma 2:.

Recall that we have introduced the following numbers :

- p such that $2 < p < \frac{2q}{3}$
- $1-\theta$ the barycentric coordinate of $\frac{1}{p}$ in $[\frac{1}{q}, 1]$: $\frac{1}{p} = (1-\theta) + \theta\frac{1}{q}$
- $-\sigma$ the corresponding point in $[-1+\frac{2}{q}, -s]$: $-\sigma = (1-\theta)(-s) + \theta(-1+\frac{2}{q})$
- $-\gamma$ such that $-1 + \frac{2}{q} < -\gamma < -\sigma$
- r such that $p < r < +\infty$
- $1-\eta$ the barycentric coordinate of $\frac{1}{p}$ in the segment $[\frac{1}{r}, \frac{1}{2}]$: $\frac{1}{p} = \frac{1-\eta}{2} + \frac{\eta}{r}$
- $-\rho$ the corresponding point in $[-\gamma, 0]$: $-\rho = \eta(-\gamma)$

We have the following embeddings :

- $B^0_{2,2} \cap \mathrm{BMO}^{-1} \subset \dot{B}^{-1+\frac{2}{q}}_{q,q}$
- $\dot{B}^{-1+\frac{2}{q}}_{q,q} \cap \dot{B}^{-s}_{\dot{M}^{1,q},\infty} \subset \dot{B}^{-\sigma}_{\dot{M}^{p,q},\infty}$

- $\dot{B}^{-1+\frac{2}{q}}_{q,q} \cap \dot{B}^{-\sigma}_{\dot{M}^{p,q},\infty} \subset \dot{B}^{-\gamma}_{\dot{M}^{p,q},1} \subset \dot{W}^{-\gamma,\dot{M}^{p,q}}$.

- $W^{-\gamma,\dot{M}^{p,q}} = [\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}]^{[\eta]}$

- $[\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}]^{[\eta]} \subset [\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}]_{\eta,\infty} \subset [L^2_{\text{uloc}}, \dot{B}^{-\rho-\frac{3p}{rq}}_{\infty,\infty}]_{\eta,\infty}$

If we take r very large, we have $\eta = 1 - \frac{2}{p} + o(1)$ and

$$-\rho - \frac{3p}{rq} = (1 - \frac{2}{p})(-\gamma) + o(1) \in]-1, 0[.$$

Thus far, we have seen that $\vec{u}_0 \in [L^2_{\text{uloc}}, \dot{B}^{-\lambda}_{\infty,\infty}]_{\eta,\infty}$ for some $\eta \in (0,1)$ and some $\lambda \in (0,1)$. We shall now estimate $\vec{v} - e^{t\Delta}\vec{u}_0$ when $\vec{v}$ is a weak local Leray solution. For $0 < \epsilon < 1$, split $\vec{u}_0$ in $\vec{\alpha}_\epsilon + \vec{\beta}_\epsilon$ with

$$\|\vec{\alpha}_\epsilon\|_{L^2_{\text{uloc}}} \leq C_0\epsilon^\eta \text{ and } \|\vec{\beta}_\epsilon\|_{\dot{B}^{-\lambda}_{\infty,\infty}} \leq C_0\epsilon^{\eta-1}$$

where C_0 does not depend on ϵ (but depends on $\vec{u}_0$).

As $\beta_\epsilon = \vec{u}_0 - \alpha_\epsilon$, we have $\|\vec{\beta}_\epsilon\|_{L^2_{\text{uloc}}} \leq \|\vec{u}_0\|_{L^2_{\text{uloc}}} + C_0$ (an estimation which does not depend on ϵ) and we know that we have a (restricted) weak Leray solution of the Navier–Stokes equations $\vec{U}_\epsilon$ with initial value $\vec{\beta}_\epsilon$ such that $\sup_{0<t<T_0} \|\vec{U}_\epsilon(t,.)\|_{L^2_{\text{uloc}}} \leq C_1$, where T_0 and C_1 depends only on $\|\vec{u}_0\|_{L^2_{\text{uloc}}}$ (and not on ϵ).

As $\beta_\epsilon \in \dot{B}^{-\lambda}_{\infty,\infty},$, we have as well that $\|\vec{U}_\epsilon(t,.)\|_\infty \leq C_2 t^{-\frac{\lambda}{2}}\epsilon^{\eta-1}$ on an interval $(0, T_\epsilon)$ with $T_\epsilon^{\frac{1-\lambda}{2}}\epsilon^{\eta-1} = C_3$ It is then easy to check that $\|\vec{U}_\epsilon - e^{t\Delta}\vec{\beta}_\epsilon\|_{L^2_{\text{loc}}} \leq C_4 t^{(1-\lambda)/2}\epsilon^{\eta-1}$.

Using our results on comparison of suitable local Leray solutions, we find that we get the following inequality for $\vec{W}_\epsilon = \vec{v} - \vec{U}_\epsilon$ and $A_\epsilon(t) = \sup_{k\in\mathbb{Z}^3} \int \varphi_0(x-k)|\vec{W}_\epsilon(t,.x)|^2\, dx$:

$$A_\epsilon(t)^2 \leq 2\|\alpha_\epsilon\|^2_{L^2_{\text{uloc}}} + C_5\int_0^t A_\epsilon(s)^2\, ds + C_6\int_0^t \|\vec{u}(s,.)\|^2_\infty A_\epsilon(s)^2\, ds. \quad (8)$$

Thus, we get

$$A_\epsilon(t) \leq C_7\epsilon^{2\eta} + C_8\int_0^t A_\epsilon(s)^2\, ds + C_9\epsilon^{2(\eta-1)}\int_0^t s^{-\lambda}A_\epsilon(s)^2\, ds.$$

so that

$$A_\epsilon(t) \leq C_7\epsilon^{2\eta}e^{C_8 t}e^{C_9\epsilon^{2(\eta-1)}\frac{t^{1-\lambda}}{1-\lambda}}.$$

Now, for $\tau < 1$, take $\epsilon = \tau^\mu$ with $\frac{1-\lambda}{2} + \mu(\eta - 1) > 0$. We find that, for $0 < t < \min(T_\epsilon, T_0)$ with

$$T_\epsilon = C_3^{\frac{2}{1-\lambda}} \epsilon^{\frac{2(1-\eta)}{1-\lambda}} = C_3^{\frac{2}{1-\lambda}} \tau^{\mu \frac{2(1-\eta)}{1-\lambda}} \text{ [where } \mu \frac{2(1-\eta)}{1-\lambda} < 1],$$

we have the inequality

$$\begin{aligned} \|\vec{v} - e^{t\Delta}\vec{u}_0\|_{L^2_{\text{uloc}}} \leq & \|\alpha_\epsilon\|_2 + \|\vec{W}_\epsilon\|_2 + \|\vec{U}_\epsilon - e^{t\Delta}\vec{\beta}_\epsilon\|_2 \\ \leq & C_{10}\epsilon^\eta (1 + e^{C_{11} \frac{t^{1-\lambda}}{T_\epsilon^{1-\lambda}}}) + C_4 t^{\frac{1-\lambda}{2}} \epsilon^{\eta - 1} \end{aligned}$$

If τ is small enough, we have $\tau < T_\epsilon$ and we find

$$\|\vec{v}(\tau, .) - e^{\tau\Delta}\vec{u}_0\|_2 \leq C_0 \tau^{\mu\eta} (1 + e^{\frac{C_1^2}{2(1-\lambda)}}) + C_4 \tau^{\frac{1-\lambda}{2} + \mu(\eta - 1)}$$

The theorem is proved.

References

[Ba] T. Barker, *Uniqueness Results for Weak Leray–Hopf Solutions of the Navier–Stokes System with Initial Values in Critical Spaces*, J. Math. Fluid Mech. 20 (2018), 133–160.

[CKN] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. Pure Appl. Math. 35 (1982), 771–831.

[Ka] T. Kato, *Nonlinear evolution equations in Banach spaces*, in : Proceedings of the Symposium on Applied Mathematics 17 (1965), 50–67.

[KocT] H. Koch and D. Tataru, *Well-posedness for the Navier–Stokes equations*, Adv. Math. 157 (2001), 22–35.

[KozT] H. Kozono and Y. Taniuchi, *Bilinear estimates in* BMO *and Navier–Stokes equations*, Math.Z. 157 (2000), 173–194.

[KoY] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier–Stokes equations with distributions in new function spaces as initial data*, Comm. Partial Differential Equations 19 (1994), 959–1014.

[LR 1] P.G. Lemarié–Rieusset, *Solutions faibles d'énergie infinie pour les équations de Navier–Stokes dans* $\mathbb{R}^3$, C. R. Acad. Sci. Paris, Serie I. 328 (1999), 1133–1138.

[LR 2] P.G. Lemarié-Rieusset, *Recent developments in the Navier–Stokes problem*, CRC Press, 2002.

[LR 3] P.G. Lemarié–Rieusset, *The Navier–Stokes equations in the critical Morrey-Campanato space*, Rev. Mat. Iberoamericana 23 (2007), 897–930.

[LR 4] P.G. Lemarié–Rieusset, *Multipliers and Morrey spaces*, Potential Anal. 38 (2013), 741–752.

[LR 5] P.G. Lemarié–Rieusset, *Erratum to "Multipliers and Morrey spaces"*, Potential Anal. 41 (2014), 13591362.

[LR 6] P.G. Lemarié–Rieusset, *The Navier–Stokes problem in the 21st century*, Chapman & Hall/CRC, (2016).

[Le] J. Leray, *Essai sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math. 63 (1934), 193–248.

[Pr] G. Prodi, *Un teorema di unicitá per le equazioni di Navier–Stokes*, Ann. Mat. Pura Appl. 48 (1959), 173–182.

[Se] J. Serrin, *The initial value problem for the Navier–Stokes equations*, in : Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962), Univ. of Wisconsin Press, Madison, Wis., 1963.