Quantitative uniqueness for Schrödinger operator with regular potentials

Laurent Bakri, Jean-Baptiste Casteras

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Abstract

We give a sharp upper bound on the vanishing order of solutions to Schrödinger equation with $\mathcal{C}^1$ magnetic potential on a compact smooth manifold. Our method is based on quantitative Carleman type inequalities developed by Donnelly and Fefferman [4]. It also extends the previous work [3] of the first author to the magnetic potential case.

1 Introduction

Let (M, g) be a smooth, compact, connected, n-dimensional Riemannian manifold. The aim of this paper is to obtain quantitative estimate on the vanishing order of solutions to

$$\Delta u + V \cdot \nabla u + Wu = 0. \tag{1.1}$$

We are concerned with H^1, non-trivial, solutions to (1.1) and $\mathcal{C}^1$ potentials (*i.e* W is a $\mathcal{C}^1$-function on M and V is a $\mathcal{C}^1$-vector field). Recall that the vanishing order at a point $x_0 \in M$ of a L^2-function u is

$$\inf \left\{ d > 0 \ ; \ \ \limsup_{r \to 0} \frac{1}{r^d} \left(\frac{1}{r^n} \int_{B_r(x_0)} |u(x)|^2 dv_g(x) \right)^{\frac{1}{2}} > 0 \right\}.$$

With this setting our main result is the following

Theorem 1.1. *The vanishing order of solutions to* (1.1) *is everywhere less than*

$$C(1 + \|W\|_{\mathcal{C}^1}^{\frac{1}{2}} + \|V\|_{\mathcal{C}^1}),$$

where C is a positive constant depending only on (M, g).

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Before proceeding, we want to point out that this bound is sharp with respect to the power of the norms of the potentials V and W. Indeed, consider the function $f_k(x_1, x_2, \cdots, x_{n+1}) = \Re e(x_1 + ix_2)^k$ defined in $\mathbb{R}^{n+1}$. Setting u_k the restriction of f_k to $\mathbb{S}^n$, $(u_k)_k$ is a sequence of spherical harmonics and $-\Delta_{\mathbb{S}^n} u_k = k(k+n-1)u_k$. The vanishing order at the north pole $N = (0, \cdots, 0, 1)$ of u_k is k. Letting $V_k = ((n+k-1)x_1, 0, \ldots, 0)$ and $W_k = k(n+k-1)x_1^2$, one can check that u_k satisfies

$$\Delta u_k = V_k \cdot \nabla u_k + W_k u_k.$$

Since $\|V_k\|_{\mathcal{C}^1} \leq Ck$, $\|W_k\|_{\mathcal{C}^1} \leq Ck^2$, this shows the sharpness.
Now let us discuss briefly our result. We recall that a differential operator P satisfies the strong unique continuation property (SUCP) if the vanishing order of any non-trivial solutions to $Pu = 0$ is finite everywhere. There has been an extensive literature dealing with (SUCP) for solutions to (1.1) with singular potentials. We refer to [11] and the references therein for more details. One of the most useful method to establish (SUCP) is based on Carleman type estimates, some of the principal contributions to (1.1) can be found in ([1, 7, 9, 10, 16, 17, 18]).
As can be seen in Theorem 1.1, our goal is to derive a quantitative version of this unique continuation property. Let us now briefly recall some of the principal results already known in this field.
In the particular case of eigenfunctions of the Laplacian ($W = \lambda$ and $V = 0$), it is a celebrated result of Donnelly and Fefferman [4] that the vanishing order is bounded by $C\sqrt{\lambda}$. In view of this, it seems a natural conjecture (cf [9, 12]) that for solutions to $\Delta u + Wu = 0$, the vanishing order is uniformly bounded by

$$C(1 + \|W\|_\infty^{\frac{1}{2}}).$$

However, this conjecture is not true when one allows complex valued potentials and solutions. In this complex case, it is known that the optimal exponent on $\|W\|_\infty$ is $\frac{2}{3}$ (see [2, 9]). When W is a real bounded function and $V = 0$, Kukavica established in [12] some quantitative results for solutions to (1.1). His method is based on the frequency function (see also [15]) which was introduced by Garofalo and Lin in [5] as an alternative to Carleman estimate for (SUCP). He established that the vanishing order of solutions is every where less than :

$$C(1 + \sqrt{\|W\|_\infty} + (\text{osc}\,(W))^2),$$

where $\text{osc}(W) = \sup W - \inf W$ and C a constant depending only on (M, g). If W is $\mathcal{C}^1$, the first author established in [3] the upper bound

$$C(1 + \|W\|_{\mathcal{C}^1}^{\frac{1}{2}}),$$

with $\|W\|_{\mathcal{C}^1} = \|W\|_\infty + \|\nabla W\|_\infty$ and where the exponent $\frac{1}{2}$ is sharp.In the general case of equation (1.1), it seems that the first algebraic upper bound, depending on $\|V\|_\infty$ and $\|W\|_\infty$, is given in [2] where it is shown that it is everywhere less than

$$C(1 + \|V\|_\infty^2 + \|W\|_\infty^{\frac{2}{3}})$$

For the real case with magnetic potential, I. Kukavica conjectured in [12] that the vanishing order of solutions is less than

$$C(1 + \|V\|_\infty + \|W\|_\infty^{\frac{1}{2}}).$$

Finally in [14] (see also [13]) quantitative uniqueness is shown for singular potentials. This means that vanishing order is everywhere bounded by a constant, which is no longer explicit. Our method is based on L^2-Carleman estimate (Theorem 2.1) in the same spirit as [4] : establish a Carleman estimate on the involved operator (here : $P : u \mapsto \Delta u + V \cdot \nabla u + Wu$) which is only true for great parameter τ, and state explicitly how τ depends on the $\mathcal{C}^1$ norms of the potentials V, W.
Our Carleman estimate will allow us to derive the following doubling inequality

$$\|u\|_{L^2(B_{2r}(x_0))} \leq e^{C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{L^2(B_r(x_0))}. \tag{1.2}$$

This doubling estimate implies Theorem 1.1.

The paper is organized as follows. In section 2 we establish Carleman estimates for the operator $P : u \mapsto \Delta u + V \cdot \nabla u + Wu$. Our method involves repeated integration by parts in the radial and spherical variables. For the sake of clarity, a part of the computation is send to the appendix.

In section 3, we deduce, in a standard manner, a three balls property for solutions to (1.1), then using compactness we derive a doubling inequality which gives immediately Theorem 1.1.

1.1 Notations.

For a fixed point x_0 in M we will use the following standard notations:

- $\Gamma_1(TM)$ will denote the set of $\mathcal{C}^1$ vector fields on M.
- $r := r(x) = d(x, x_0)$ stands for the Riemannian distance from x_0,
- $B_r := B_r(x_0)$ denotes the geodesic ball centered at x_0 of radius r,

- $A_{r_1,r_2} := B_{r_2} \setminus B_{r_1}$.

- ε stands for a fixed number with $0 < \varepsilon < 1$.

- R_0, R_1, c, C, C_1, C_2 will denote positive constants which depend only on (M, g). They may change from a line to another.

- $\| \cdot \|$ stands for the L^2 norm on M and $\| \cdot \|_A$ the L^2 norm on the (measurable) set A. In case T is a vector field (or a tensor), $\|T\|$ has to be understood as $\| \, |T|_g \|$.

2 Carleman estimates

Recall that Carleman estimates are weighted integral inequalities with a weight function $e^{\tau\phi}$, where the function ϕ satisfies some convexity properties. Let us now define the weight function we will use.
For a fixed number ε such that $0 < \varepsilon < 1$ and $T_0 < 0$, we define the function f on $]-\infty, T_0[$ by $f(t) = t - e^{\varepsilon t}$. One can check easily that, for $|T_0|$ great enough, the function f verifies the following properties:

$$\begin{aligned} & 1 - \varepsilon e^{\varepsilon T_0} \leq f'(t) \leq 1 \quad \forall t \in]-\infty, T_0[, \\ & \lim_{t \to -\infty} -e^{-t} f''(t) = +\infty. \end{aligned} \tag{2.1}$$

Finally we define $\phi(x) = -f(\ln r(x))$. Now we can state the main result of this section:

Theorem 2.1. *There exist positive constants R_0, C, C_1, which depend only on M and ε, such that, for any $W \in \mathcal{C}^1(M)$, any $V \in \Gamma_1(TM)$, any $x_0 \in M$, any $u \in C_0^\infty(B_{R_0}(x_0) \setminus \{x_0\})$ and any $\tau \geq C_1(1 + \|W\|_{\mathcal{C}^1}^{\frac{1}{2}} + \|V\|_{\mathcal{C}^1})$, one has*

$$C \left\| r^2 e^{\tau\phi} \left(\Delta u + V \cdot \nabla u + W u \right) \right\| \geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} u \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u \right\|. \tag{2.2}$$

Under the additional assumption that supp(u) is far enough from x_0 we have the following

Corollary 2.2. *Adding to the setting of Theorem 2.1 the supplementary assumption that*

$$\mathrm{supp}(u) \subset \{x \in M; r(x) \geq \delta > 0\},$$

then we have

$$
\begin{aligned}
C\left\|r^2 e^{\tau\phi}\left(\Delta u+V\cdot\nabla u+Wu\right)\right\| &\geq \tau^{\frac{3}{2}}\left\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}u\right\| \\
&+\tau^{\frac{1}{2}}\delta^{\frac{1}{2}}\left\|r^{-\frac{1}{2}}e^{\tau\phi}u\right\|+\tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}}e^{\tau\phi}\nabla u\right\|.
\end{aligned}
\tag{2.3}
$$

Remark 2.3. *One should note that, contrary to the corresponding result of Donnelly and Fefferman [4], we are not able to claim that the constants appearing above depend only on an upper bound of the absolute value of the sectional curvature. This comes from the fact that, working in polar coordinates, we will have to handle terms containing spherical derivative of the metric during the proof of Theorem 2.1. See in particular the computation of* I_3 *in the appendix.*

Remark 2.4. *We will proceed to the proof with the assumption that all functions are real. However it can be easily seen that the same inequality holds with hermitian product for complex valued functions*

Proof of Theorem 2.1. We now introduce the polar geodesic coordinates (r,θ) near x_0. Using Einstein notation, the Laplace operator takes the form

$$
r^2\Delta u=r^2\partial_r^2 u+r^2\left(\partial_r\ln(\sqrt{\gamma})+\frac{n-1}{r}\right)\partial_r u+\frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j u),
$$

where $\partial_i=\dfrac{\partial}{\partial\theta_i}$ and for each fixed r, $\gamma_{ij}(r,\theta)$ is a metric on $\mathbb{S}^{n-1}$, and we write $\gamma=\det(\gamma_{ij})$. Since (M,g) is smooth, we have for r small enough :

$$
\begin{aligned}
\partial_r(\gamma^{ij}) &\leq C(\gamma^{ij}) \quad \text{(in the sense of tensors);}\\
|\partial_r(\gamma)| &\leq C;\\
C^{-1} \leq \gamma &\leq C.
\end{aligned}
\tag{2.4}
$$

Now we set $r=e^t$. In these new variables, we write :

$$
\begin{aligned}
e^{2t}\Delta u &= \partial_t^2 u+(n-2+\partial_t\ln\sqrt{\gamma})\partial_t u+\frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j u),\\
e^{2t}V &= e^{2t}V_t\partial_t+e^{2t}V_i\partial_i.
\end{aligned}
$$

Notice that we will consider the function u to have support in $]-\infty,T_0[\times\mathbb{S}^{n-1}$, where $|T_0|$ will be chosen large enough. The conditions (2.4) become

$$
\begin{aligned}
\partial_t(\gamma^{ij}) &\leq Ce^t(\gamma^{ij}) \quad \text{(in the sense of tensors);}\\
|\partial_t(\gamma)| &\leq Ce^t;\\
C^{-1} \leq \gamma &\leq C.
\end{aligned}
$$

Now we introduce the conjugate operator :

$$L_\tau(u) = e^{2t}e^{\tau\phi}\Delta(e^{-\tau\phi}u) + e^{2t}e^{\tau\phi}g(V,\nabla(e^{-\tau\phi}u)) + e^{2t}Wu, \qquad (2.5)$$

and we compute $L_\tau(u)$:

$$\begin{aligned} L_\tau(u) &= \partial_t^2 u + \left(2\tau f' + e^{2t}V_t + n - 2 + \partial_t \ln\sqrt{\gamma}\right)\partial_t u + e^{2t}V_i\partial_i u \\ &+ \ \left(\tau^2 f'^2 + \tau f' V_t e^{2t} + \tau f'' + (n-2)\tau f' + \tau\partial_t \ln\sqrt{\gamma} f'\right) u \\ &+ \Delta_\theta u + e^{2t}Wu, \end{aligned}$$

with

$$\Delta_\theta u = \frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}\gamma^{ij}\partial_j u\right).$$

It will be useful for us to introduce the following L^2 norm on $]-\infty, T_0[\times\mathbb{S}^{n-1}$:

$$\|v\|_f^2 = \int_{]-\infty,T_0[\times\mathbb{S}^{n-1}} |v|^2\sqrt{\gamma} f'^{-3} dt d\theta,$$

where $d\theta$ is the usual measure on $\mathbb{S}^{n-1}$. The corresponding inner product is denoted by $\langle\cdot,\cdot\rangle_f$, *i.e*

$$\langle u, v\rangle_f = \int uv\sqrt{\gamma} f'^{-3} dt d\theta.$$

We will estimate from below $\|L_\tau u\|_f^2$ by using elementary algebra and integrations by parts. We are concerned, in the computation, by the power of τ and exponential decay when t goes to $-\infty$. We point out that we have

$$e^t(V_t + V_i + \partial_\alpha V_t + \partial_\beta V_i) \le C\|V\|_{\mathcal{C}^1}$$

with the convention that $\partial_\alpha, \partial_\beta = \{\partial_t, \partial_1, \ldots, \partial_{n-1}\}$. First note that by triangular inequality one has

$$\|L_\tau(u)\|_f^2 \ge \frac{1}{2}I - II, \qquad (2.6)$$

with

$$\begin{aligned} I = \Big\| &\partial_t^2 u + \Delta_\theta u + (2\tau f' + e^{2t}V_t)\partial_t u + e^{2t}V_i\partial_i u \\ &+ (\tau^2 f'^2 + \tau f' e^{2t}V_t + (n-2)\tau f' + e^{2t}W)u\Big\|_f^2, \qquad (2.7) \end{aligned}$$

and

$$II = \|\tau f'' u + \tau \partial_t \text{ln}\sqrt{\gamma} f' u + (n-2)\partial_t u + \partial_t \ln \sqrt{\gamma} \partial_t u\|_f^2 . \tag{2.8}$$

We will be able to absorb II later. Now, we want to find a lower bound for I. Therefore, we start by computing it :

$$I = I_1 + I_2 + I_3,$$

with

$$\begin{aligned} I_1 &= \left\| \partial_t^2 u + (\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W) u + \Delta_\theta u \right\|_f^2 , \\ I_2 &= \left\| (2\tau f' + e^{2t} V_t) \partial_t u + e^{2t} V_i \partial_i u \right\|_f^2 , \\ I_3 &= 2 \left\langle \partial_t^2 u + (\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W) u + \Delta_\theta u \right. \\ &\qquad \left. , (2\tau f' + e^{2t} V_t) \partial_t u + e^{2t} V_i \partial_i u \right\rangle_f . \end{aligned} \tag{2.9}$$

We will split the computation into three parts corresponding to the I_i for $i = 1, 2, 3$.

Computation of I_1.
Let $\rho > 0$ be a small number to be chosen later. Since $|f''| \leq 1$ and $\tau \geq 1$, we have :

$$I_1 \geq \frac{\rho}{\tau} I_1', \tag{2.10}$$

where I_1' is defined by :

$$I_1' = \left\| \sqrt{|f''|} \left[\partial_t^2 u + (\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W) u + \Delta_\theta u \right] \right\|_f^2 . \tag{2.11}$$

Now, we decompose I_1' into three parts

$$I_1' = K_1 + K_2 + K_3, \tag{2.12}$$

with

$$K_1 = \left\| \sqrt{|f''|} \left(\partial_t^2 u + \Delta_\theta u \right) \right\|_f^2 , \tag{2.13}$$

$$K_2 = \left\| \sqrt{|f''|} \left(\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) u \right\|_f^2 , \tag{2.14}$$

$$K_3 = 2 \left\langle \left(\partial_t^2 u + \Delta_\theta u \right) |f''| , \left(\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) u \right\rangle_f . \tag{2.15}$$

We just ignore K_1 since it is positive. To estimate K_2, we first note that

$$K_2 \geq \frac{\tau^4}{2} \left\| \sqrt{|f''|} f'^2 u \right\|_f^2 - \left\| \sqrt{|f''|} \left(\tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) u \right\|_f^2 .$$

On the other hand, we have

$$\begin{aligned} &\left\| \sqrt{|f''|} \left(\tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) u \right\|_f^2 \\ &\qquad\qquad\qquad \leq c\tau^4 \left\| \sqrt{|f''|} e^t u \right\|_f^2 + \tau^2 \left\| \sqrt{|f''|} u \right\|_f^2 . \end{aligned}$$

Therefore using the assumptions on τ, and the exponential decay at $-\infty$, we have for T_0 large enough, that every other term in K_2 can be absorbed in $\tau^4 \| \sqrt{|f''|} u \|$. That is :

$$K_2 \geq c\tau^4 \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \tag{2.16}$$

Now, we derive a suitable lower bound for K_3. Integrating by parts gives :

$$\begin{aligned} K_3 = & 2 \int f'' \left(\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ & + 2 \int \partial_t \left[f'' \left(\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) \right] \partial_t u u \sqrt{\gamma} f'^{-3} dt d\theta \\ & - 6 \int \left(f''^2 f'^{-1} \left(\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) \right) \partial_t u u \sqrt{\gamma} f'^{-3} dt d\theta \\ & + 2 \int f'' \left(\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) \partial_t \ln \sqrt{\gamma} \partial_t u u f'^{-3} \sqrt{\gamma} dt d\theta \\ & + 2 \int f'' \left(\tau^2 f'^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W \right) |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ & + 2 \int f'' e^{2t} \partial_i W \cdot \gamma^{ij} \partial_j u u f'^{-3} \sqrt{\gamma} dt d\theta \\ & + 2\tau \int f'' f' e^{2t} \partial_i V_t \cdot \gamma^{ij} \partial_j u u f'^{-3} \sqrt{\gamma} dt d\theta, \end{aligned} \tag{2.17}$$

where $|D_\theta u|^2$ stands for

$$|D_\theta u|^2 = \partial_i u \gamma^{ij} \partial_j u.$$

The condition $\tau \geq C(1 + \|V\|_{\mathcal{C}^1} + \|W\|_{\mathcal{C}^1}^{\frac{1}{2}})$, the Young's inequality and the fact that f' is close to 1 imply

$$\int |f''| e^{2t} |(\partial_i W + f'\partial_i V_t)\gamma^{ij}\partial_j u u| f'^{-3}\sqrt{\gamma}dtd\theta \leq c\tau^2 \int |f''|(|D_\theta u|^2 + |u|^2) f'^{-3}\sqrt{\gamma}dtd\theta.$$

Now since $2\partial_t u u \leq u^2 + |\partial_t u|^2$, we can use conditions (2.1) and (2.5) to get

$$K_3 \geq -c\tau^2 \int |f''| \left(|\partial_t u|^2 + |D_\theta u|^2 + |u|^2\right) f'^{-3}\sqrt{\gamma}dtd\theta. \tag{2.18}$$

Therefore, inserting (2.16), (2.18) in (2.12) (recall that $K_1 \geq 0$), we have

$$I_1' \geq -c\tau^2 \int |f''| \left(|\partial_t u|^2 + |D_\theta u|^2 + |u|^2\right) f'^{-3}\sqrt{\gamma}dtd\theta + c\tau^4 \int |f''||u|^2 f'^{-3}\sqrt{\gamma}dtd\theta. \tag{2.19}$$

From the definition of I_1' (see (2.10)), we get

$$I_1 \geq -\rho c\tau \int |f''| \left(|\partial_t u|^2 + |D_\theta u|^2\right) f'^{-3}\sqrt{\gamma}dtd\theta + C\tau^3\rho \int |f''||u|^2 f'^{-3}\sqrt{\gamma}dtd\theta. \tag{2.20}$$

Computation of I_2.
We begin by recalling that

$$I_2 = \left\|(2\tau f' + e^{2t}V_t)\partial_t u + e^{2t}V_i\partial_i u\right\|_f^2.$$

In the same way as for I_1, using that $\tau \geq 1$, we have

$$I_2 \geq \frac{1}{\tau} I_2.$$

Using the triangular inequality, one has

$$I_2 \geq \frac{1}{2}\left\|2\tau f'\partial_t u\right\|_f^2 - \left\|e^{2t}V_t\partial_t u + e^{2t}V_i\partial_i u\right\|_f^2.$$

Now, using the assumptions on τ, we note that

$$\left\|e^{2t}V_t\partial_t u + e^{2t}V_i\partial_i u\right\|_f^2 \leq 2\tau^2\|e^t\partial_t u\|_f^2 + 2\tau^2\|e^t D_\theta u\|_f^2.$$

From the last three previous inequalities and since e^t is small for T_0 largely negative, we see that the following estimate holds

$$I_2 \geq c\tau \|\partial_t u\|_f^2 - c\tau \|e^t D_\theta u\|_f^2. \tag{2.21}$$

Computation of I_3.
Since this computation is quite lengthy, we send it to the Appendix. There, we show that

$$\begin{aligned} I_3 \geq 3\tau \int |f''|\,|D_\theta u|^2\, {f'}^{-3}\sqrt{\gamma}dtd\theta - c\tau^3 \int e^t |u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \\ - c\tau \int |f''|\,|\partial_t u|^2\, {f'}^{-3}\sqrt{\gamma}dtd\theta - c\tau^2 \int |f''|\,|u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta. \end{aligned} \tag{2.22}$$

Lower bound for $L_\tau u$.

Now recalling that $I = I_1 + I_2 + I_3$ and using (2.20), (2.21) and (2.22), we obtain

$$\begin{aligned} I \geq c\tau \int |\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta + 3\tau \int |f''|\,|D_\theta u|^2\, {f'}^{-3}\sqrt{\gamma}dtd\theta \\ + C\tau^3\rho \int |f''||u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta - c\rho\tau \int |f''||D_\theta u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \\ - c\tau \int e^{2t}|D_\theta u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta - c\tau^3 \int e^t|u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \\ - c\tau \int |f''||\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta - c\tau^2 \int |f''||u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta. \end{aligned}$$

Now we want to derive a lower bound for I. Then one needs to check that every non-positive term in the right hand side of (2.23) can be absorbed. We first fix ρ small enough (*i.e.* $\rho \leq \frac{2}{c}$) such that

$$\rho c\tau \int |f''| \cdot |D_\theta u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \leq 2\tau \int |f''| \cdot |D_\theta u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta$$

where c is the constant appearing in (2.23). Now the other negative terms of (2.23) can then be absorbed by comparing powers of τ and decay rate at $-\infty$. Indeed conditions (2.1) imply that e^t is small compared to $|f''|$.
Thus we obtain :

$$\begin{aligned} CI \geq \tau \int |\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta + \tau \int |f''||D_\theta u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \\ + \tau^3 \int |f''||u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta. \end{aligned} \tag{2.23}$$

Now we can check that II can be absorbed in I for $|T_0|$ and τ large enough. Indeed from (2.8), using (2.1) and (2.5) ($|\partial_t \ln\sqrt{\gamma}| \le Ce^t$), one gets

$$
\begin{aligned}
II &= \|\tau f'' u + \tau\partial_t \ln\sqrt{\gamma} f' u + (n-2)\partial_t u + \partial_t \ln\sqrt{\gamma}\partial_t u\|_f^2 \\
&\le \tau^2 \int |f''|^2 |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + \tau^2 \int e^{2t} |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad + C \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta.
\end{aligned} \tag{2.24}
$$

And each term in the right hand side can easily be absorbed in (2.23). Then we obtain

$$
\|L_\tau u\|_f^2 \ge C\tau^3 \|\sqrt{|f''|} u\|_f^2 + C\tau \|\partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} D_\theta u\|_f^2. \tag{2.25}
$$

Note that, since $\sqrt{|f''|} \le 1$, one has

$$
\|L_\tau u\|_f^2 \ge C\tau^3 \|\sqrt{|f''|} u\|_f^2 + c\tau \|\sqrt{|f''|}\partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} D_\theta u\|_f^2, \tag{2.26}
$$

and the constant c can be chosen arbitrary smaller than C.

End of the proof.
If we set $v = e^{-\tau\phi} u$ and use the triangular inequality on the second right-sided term of (2.26), then we have

$$
\begin{aligned}
\left\|e^{2t} e^{\tau\phi}(\Delta v + V \cdot \nabla v + W v)\right\|_f^2 &\ge C\tau^3 \left\|\sqrt{|f''|} e^{\tau\phi} v\right\|_f^2 - c\tau^3 \left\|\sqrt{|f''|} f' e^{\tau\phi} v\right\|_f^2 \\
&\quad + \frac{c}{2}\tau \left\|\sqrt{|f''|} e^{\tau\phi}\partial_t v\right\|_f^2 + C\tau \left\|\sqrt{|f''|} e^{\tau\phi} D_\theta v\right\|_f^2
\end{aligned} \tag{2.27}
$$

Finally since f' is close to 1 one can absorb the negative term to obtain

$$
\begin{aligned}
\left\|e^{2t} e^{\tau\phi}(\Delta v + V \cdot \nabla v + W v)\right\|_f^2 &\ge C\tau^3 \left\|\sqrt{|f''|} e^{\tau\phi} v\right\|_f^2 \\
&\quad + C\tau \left\|\sqrt{|f''|} e^{\tau\phi}\partial_t v\right\|_f^2 + C\tau \left\|\sqrt{|f''|} e^{\tau\phi} D_\theta v\right|_f^2
\end{aligned} \tag{2.28}
$$

It remains to get back to the usual L^2 norm. First note that since f' is close to 1, we can get the same estimate without the term $(f')^{-3}$ in the integrals. Recall that in polar coordinates (r, θ) the volume element is $r^{n-1}\sqrt{\gamma} dr d\theta$, we can deduce from (2.23) that :

$$
\begin{aligned}
\|r^2 e^{\tau\phi}(\Delta v + V \cdot \nabla v + W v) r^{-\frac{n}{2}}\|^2 &\ge C\tau^3 \|r^{\frac{\varepsilon}{2}} e^{\tau\phi} v r^{-\frac{n}{2}}\|^2 \\
&\quad + C\tau \|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla v r^{-\frac{n}{2}}\|^2.
\end{aligned} \tag{2.29}
$$

Finally one can get rid of the term $r^{-\frac{n}{2}}$ by replacing τ with $\tau + \frac{n}{2}$. Indeed, from $e^{\tau\phi}r^{-\frac{n}{2}} = e^{(\tau+\frac{n}{2})\phi}e^{-\frac{n}{2}r^{\varepsilon}}$, one can easily check that, for r small enough

$$\frac{1}{2}e^{(\tau+\frac{n}{2})\phi} \leq e^{\tau\phi}r^{-\frac{n}{2}} \leq e^{(\tau+\frac{n}{2})\phi}.$$

This achieves the proof of Theorem 2.1.

$\square$

Next, we demonstrate the Corollary 2.2.

Proof of Corollary 2.2. Now suppose that $\mathrm{supp}(u) \subset \{x \in M; r(x) \geq \delta > 0\}$ and define $T_1 = \ln\delta$.

Cauchy-Schwarz inequality applied to

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = 2\int u\partial_t u e^{-t}\sqrt{\gamma}dtd\theta$$

gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \leq 2\left(\int (\partial_t u)^2 e^{-t}\sqrt{\gamma}dtd\theta\right)^{\frac{1}{2}}\left(\int u^2e^{-t}\sqrt{\gamma}dtd\theta\right)^{\frac{1}{2}}. \tag{2.30}$$

On the other hand, integrating by parts gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = \int u^2e^{-t}\sqrt{\gamma}dtd\theta - \int u^2e^{-t}\partial_t(\ln(\sqrt{\gamma}))\sqrt{\gamma}dtd\theta. \tag{2.31}$$

Now since $|\partial_t \ln\sqrt{\gamma}| \leq Ce^t$ for $|T_0|$ large enough we can deduce :

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \geq c\int u^2e^{-t}\sqrt{\gamma}dtd\theta. \tag{2.32}$$

Combining (2.30), (2.32) and by the assumption on supp(u), we find

$$\begin{aligned} c^2\int u^2e^{-t}\sqrt{\gamma}dtd\theta &\leq 4\int (\partial_t u)^2 e^{-t}\sqrt{\gamma}dtd\theta \\ &\leq 4e^{-T_1}\int (\partial_t u)^2\sqrt{\gamma}dtd\theta. \end{aligned}$$

Finally, dropping all terms except $\tau\int|\partial_t u|^2 f'^{-3}\sqrt{\gamma}dtd\theta$ in (2.23) gives :

$$CI \geq \tau\delta\int e^{-t}|u|^2 f'^{-3}\sqrt{\gamma}dtd\theta.$$

Inequality (2.23) can then be replaced by :

$$\begin{aligned} I \geq C\tau \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ + C\tau^3 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau\delta \int e^{-t} |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \tag{2.33}$$

The rest of the proof follows in a way similar to the last part of the proof of Theorem 2.1. □

3 Vanishing order

We now proceed to establish an upper bound on the vanishing order of solutions to (1.1), from our Carleman estimate. This is inspired by [4]. We choose to establish a doubling inequality. We recall that doubling inequality implies vanishing order estimate. Before proceeding, we would like to emphasize that if $u \in H^1(B_r(x_0))$, by standard elliptic regularity theory, one has that $u \in H^2_{\text{loc}}(B_r(x_0))$ (see by example [6] Theorem 8.8). Therefore, by density, we see that we can apply inequality (2.3) of Corollary 2.2 to χu for χ a cut-off function null in a neighborhood of x_0

3.1 Three balls inequality

We first want to derive from (2.3), a control on the local behavior of solutions in the form of an Hadamard three circles type theorem. To obtain such result the basic idea is to apply Carleman estimate to χu where χ is an appropriate cut-off function and u a solution of (1.1). This is standard [3, 8] and the proof adapted to our weight function is given for the sake of completeness.

Proposition 3.1 (Three balls inequality)**.** *There exist positive constants R_1, C_1, C_2 and $0 < \alpha < 1$ which depend only on (M, g) such that, if u is a solution to (1.1) with $W \in \mathcal{C}^1(M)$ and $V \in \Gamma_1(TM)$, then for any $R < R_1$, and any $x_0 \in M$, one has*

$$\|u\|_{B_R(x_0)} \leq e^{C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})} \|u\|^{\alpha}_{B_{\frac{R}{2}}(x_0)} \|u\|^{1-\alpha}_{B_{2R}(x_0)}. \tag{3.1}$$

Proof. Let x_0 be a point in M. Let u be a solution to (1.1) and R such that $0 < R < \frac{R_0}{2}$ with R_0 as in Corollary 2.1. Let $\psi \in \mathcal{C}_0^\infty(B_{2R})$, $0 \leq \psi \leq 1$, a function with the following properties:

- $\psi(x) = 0$ if $r(x) < \frac{R}{4}$ or $r(x) > \frac{5R}{3}$,

- $\psi(x) = 1$ if $\frac{R}{3} < r(x) < \frac{3R}{2}$,
- $|\nabla \psi(x)| \leq \frac{C}{R}$,
- $|\nabla^2 \psi(x)| \leq \frac{C}{R^2}$.

First since the function ψu is supported in the annulus $A_{\frac{R}{3},\frac{5R}{3}}$, we can apply estimate (2.3) of theorem 2.1. In particular we have, since the quotient between $\frac{R}{3}$ and $\frac{5R}{3}$ doesn't depend on R. :

$$C \left\| r^2 e^{\tau\phi} \left(\Delta\psi u + 2\nabla u \cdot \nabla\psi + V \cdot u\nabla\psi\right)\right\| \geq \tau^{\frac{1}{2}} \left\| e^{\tau\phi}\psi u \right\|. \tag{3.2}$$

Notice that

$$\|r^2 e^{\tau\phi} V \cdot u\nabla\psi\| \leq \|V\|_\infty \|r^2 e^{\tau\phi} u\nabla\psi\|.$$

Then, from the properties of ψ and since $\tau \geq \|V\|_\infty$ we get

$$\tau^{\frac{1}{2}} \|e^{\tau\phi}\psi u\| \leq C\left(\|e^{\tau\phi}u\|_{\frac{R}{4},\frac{R}{3}} + \|e^{\tau\phi}u\|_{\frac{3R}{2},\frac{5R}{3}}\right) \tag{3.3}$$

$$+ C\left(R\|e^{\tau\phi}\nabla u\|_{\frac{R}{4},\frac{R}{3}} + R\|e^{\tau\phi}\nabla u\|_{\frac{3R}{2},\frac{5R}{3}}\right) \tag{3.4}$$

$$+ \ C\tau\|re^{\tau\phi}u\|_{\frac{R}{4},\frac{R}{3}} + C\tau\|re^{\tau\phi}u\|_{\frac{3R}{2},\frac{5R}{3}} \tag{3.5}$$

Now since r is small, we bound (3.5) and the right hand side of (3.3) from above by $\tau\|e^{\tau\phi}u\|_{\frac{R}{4},\frac{R}{3}} + \tau\|e^{\tau\phi}u\|_{\frac{3R}{2},\frac{5R}{3}}$. Then, dividing both sides of the previous inequality by τ and noticing that $\tau^{-\frac{1}{2}} \leq 1$, one has :

$$\|e^{\tau\phi}u\|_{\frac{R}{3},\frac{3R}{2}} \leq C\tau^{\frac{1}{2}}\left(\|e^{\tau\phi}u\|_{\frac{R}{4},\frac{R}{3}} + \|e^{\tau\phi}u\|_{\frac{3R}{2},\frac{5R}{3}}\right)$$

$$+ C\left(R\|e^{\tau\phi}\nabla u\|_{\frac{R}{4},\frac{R}{3}} + R\|e^{\tau\phi}\nabla u\|_{\frac{3R}{2},\frac{5R}{3}}\right). \tag{3.6}$$

Recall that $\phi(x) = -\ln r(x) + r(x)^\varepsilon$. In particular ϕ is radial and decreasing (for small r). Then one has,

$$\|e^{\tau\phi}u\|_{\frac{R}{3},\frac{3R}{2}} \leq C\tau^{\frac{1}{2}}\left(e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{4},\frac{R}{3}} + e^{\tau\phi(\frac{3R}{2})}\|u\|_{\frac{3R}{2},\frac{5R}{3}}\right)$$

$$+ C\left(Re^{\tau\phi(\frac{R}{4})}\|\nabla u\|_{\frac{R}{4},\frac{R}{3}} + Re^{\tau\phi(\frac{3R}{2})}\|\nabla u\|_{\frac{3R}{2},\frac{5R}{3}}\right). \tag{3.7}$$

Now we recall the following elliptic estimates : since u satisfies (1.1) then :

$$\|\nabla u\|_{aR} \leq C\left(\frac{1}{(1-a)R} + \|W\|_\infty^{1/2} + \|V\|_\infty\right)\|u\|_R, \quad \text{for } 0 < a < 1. \tag{3.8}$$

Moreover since $A_{R_1,R_2} \subset B_{R_2}$ and $\tau \geq 1$, multiplying formula (3.8) by $e^{\tau\phi(\frac{3R}{2})}$, we find

$$e^{\tau\phi(\frac{3R}{2})}\|\nabla u\|_{\frac{3R}{2},\frac{5R}{3}} \leq C\tau^{\frac{1}{2}}\left(\frac{1}{R} + \|W\|_\infty^{1/2} + \|V\|_\infty\right) e^{\tau\phi(\frac{3R}{2})}\|u\|_{2R}.$$

Using (3.7) and noting that $\|e^{\tau\phi}u\|_{\frac{R}{3},\frac{3R}{2}} \geq e^{\tau\phi(R)}\|u\|_{\frac{R}{3},R}$, one has :

$$\|u\|_{\frac{R}{3},R} \leq C\tau^{\frac{1}{2}}(1 + \|W\|_\infty^{1/2} + \|V\|_\infty)\left(e^{\tau A_R}\|u\|_{\frac{R}{2}} + e^{-\tau B_R}\|u\|_{2R}\right),$$

with $A_R = \phi(\frac{R}{4}) - \phi(R)$ and $B_R = -(\phi(\frac{3R}{2}) - \phi(R))$. From the properties of ϕ we may assume that, we have $0 < A^{-1} \leq A_R \leq A$ and $0 < B \leq B_R \leq B^{-1}$ where A and B don't depend on R. We may assume that $C\tau^{\frac{1}{2}}(1 + \|W\|_\infty^{1/2} + \|V\|_\infty) \geq 2$. Then we can add $\|u\|_{\frac{R}{3}}$ to each side and bound it in the right hand side by $C\tau^{\frac{1}{2}}(1 + \|W\|_\infty^{1/2} + \|V\|_\infty)e^{\tau A}\|u\|_{\frac{R}{2}}$. We get :

$$\|u\|_R \leq C\tau^{\frac{1}{2}}(1 + \|W\|_\infty^{1/2} + \|V\|_\infty)\left(e^{\tau A}\|u\|_{\frac{R}{2}} + e^{-\tau B}\|u\|_{2R}\right). \tag{3.9}$$

Now we want to find τ such that

$$C\tau^{\frac{1}{2}}(1 + \|W\|_\infty^{1/2} + \|V\|_\infty)e^{-\tau B}\|u\|_{2R} \leq \frac{1}{2}\|u\|_R$$

which is true for $\tau \geq -\frac{2}{B}\ln\left(\frac{1}{2C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)}\frac{\|u\|_R}{\|u\|_{2R}}\right)$. Since τ must also satisfy

$$\tau \geq C_1(1 + \|W\|_{\mathcal{C}^1}^{\frac{1}{2}} + \|V\|_{\mathcal{C}^1}),$$

we choose

$$\tau = -\frac{2}{B}\ln\left(\frac{1}{2C(1 + \|W\|_\infty^{1/2} + \|V\|_\infty)}\frac{\|u\|_R}{\|u\|_{2R}}\right) + C_1(1 + \|W\|_{\mathcal{C}^1}^{\frac{1}{2}} + \|V\|_{\mathcal{C}^1}).$$

Since, of course, $\|U\|_{\mathcal{C}^1} \geq \|U\|_\infty$, one has :

$$\|u\|_R^{\frac{B+2(A+1)}{B}} \leq e^{C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{2R}^{\frac{2(A+1)}{B}}\|u\|_{\frac{R}{2}}, \tag{3.10}$$

Finally, defining $\alpha = \frac{2(A+1)}{2(A+1)+B}$, we see that (3.10) gives the result.

□

3.2 Doubling estimates

Now we intend to show that the vanishing order of solutions to (1.1) is everywhere bounded by $C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})$. This is an immediate consequence of the following :

Theorem 3.2 (doubling estimate)**.** *There exists a positive constant C, depending only on (M,g) such that : if u is a solution to (1.1) on M then for any x_0 in M and any $r>0$, one has*

$$\|u\|_{B_{2r}(x_0)} \leq e^{C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{B_r(x_0)}. \tag{3.11}$$

To prove Theorem 3.2, we need to use the standard overlapping chains of balls argument ([4, 8, 12]) to show :

Proposition 3.3. *For any $R>0$ there exists $C_R>0$ such that for any $x_0\in M$, any $W\in\mathcal{C}^1(M)$, any $V\in\Gamma_1(TM)$, and any solutions u to (1.1) :*

$$\|u\|_{B_R(x_0)} \geq e^{-C_R(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{L^2(M)}.$$

Proof. We may assume without loss of generality that $R<R_1$, with R_1 as in the three balls inequality (Proposition 3.1). Up to multiplication by a constant, we can assume that $\|u\|_{L^2(M)}=1$. We denote by $\bar{x}$ a point in M such that $\|u\|_{B_R(\bar{x})}=\sup_{x\in M}\|u\|_{B_R(x)}$. This implies that one has $\|u\|_{B_R(\bar{x})}\geq D_R$, where D_R depends only on M and R. One has (from Proposition 3.1) at an arbitrary point x of M :

$$\|u\|_{B_{R/2}(x)} \geq e^{-c(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{B_R(x)}^{\frac{1}{\alpha}}. \tag{3.12}$$

Let γ be a geodesic curve between x_0 and $\bar{x}$ and define $x_1,\cdots,x_m=\bar{x}$ such that $x_i\in\gamma$ and $B_{\frac{R}{2}}(x_{i+1})\subset B_R(x_i)$, for any i from 0 to $m-1$. The number m depends only on $\mathrm{diam}(M)$ and R. Then the properties of $(x_i)_{1\leq i\leq m}$ and inequality (3.12) give for all i, $1\leq i\leq m$:

$$\|u\|_{B_{R/2}(x_i)} \geq e^{-c(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{B_{R/2}(x_{i+1})}^{\frac{1}{\alpha}}. \tag{3.13}$$

The result follows by iteration and the fact that $\|u\|_{B_R(\bar{x})}\geq D_R$.

□

Corollary 3.4. *For all $R>0$, there exists a positive constant C_R depending only on M and R such that at any point x_0 in M one has*

$$\|u\|_{R,2R} \geq e^{-C_R(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{L^2(M)}.$$

Proof. Recall that $\|u\|_{R,2R} = \|u\|_{L^2(A_{R,2R})}$ with $A_{R,2R} := \{x; R \leq d(x,x_0) \leq 2R)\}$. Let $R < R_1$ where R_1 is from Proposition 3.1, note that $R_1 \leq \mathrm{diam}(M)$. Since M is geodesically complete, there exists a point x_1 in $A_{R,2R}$ such that $B_{x_1}(\frac{R}{4}) \subset A_{R,2R}$. From Proposition 3.3 one has

$$\|u\|_{B_{\frac{R}{4}}(x_1)} \geq e^{-C_R(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{L^2(M)}$$

which gives the result. □

Proof of Theorem 3.2. We proceed as in the proof of three balls inequality (Proposition 3.1) except for the fact that now we want the first ball to become arbitrary small in front of the others. Let $R = \frac{R_1}{4}$ with R_1 as in the three balls inequality, let δ such that $0 < 3\delta < \frac{R}{8}$, and define a smooth function ψ, with $0 \leq \psi \leq 1$ as follows:

- $\psi(x) = 0$ if $r(x) < \delta$ or if $r(x) > R$,
- $\psi(x) = 1$ if $r(x) \in [\frac{5\delta}{4}, \frac{R}{2}]$,
- $|\nabla\psi(x)| \leq \frac{C}{\delta}$ and $|\nabla^2\psi(x)| \leq \frac{C}{\delta^2}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$,
- $|\nabla\psi(x)| \leq C$ and $|\nabla^2\psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$.

Keeping appropriate terms in (2.3) applied to ψu gives :

$$\begin{aligned}\tau^{\frac{3}{2}}\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}\psi u\| &+ \tau^{\frac{1}{2}}\delta^{\frac{1}{2}}\|r^{-\frac{1}{2}}e^{\tau\phi}\psi u\| \\ &\leq C\left(\|r^2e^{\tau\phi}\nabla u\cdot\nabla\psi\| + \|r^2e^{\tau\phi}\Delta\psi u\| + \|r^2e^{\tau\phi}Vu\nabla\psi\|\right). \quad (3.14)\end{aligned}$$

Using properties of ψ and since $\tau \geq \|V\|_\infty$, one finds

$$\begin{aligned}\tau^{\frac{3}{2}}\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}u\|_{\frac{R}{8},\frac{R}{4}} + \tau^{\frac{1}{2}}\|e^{\tau\phi}u\|_{\frac{5\delta}{4},3\delta} &\leq C(\delta\|e^{\tau\phi}\nabla u\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}\nabla u\|_{\frac{R}{2},R}) \\ &+ C(\|e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}u\|_{\frac{R}{2},R}) \\ &+ C\frac{\tau}{\delta}\|r^2e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}} + C\tau\|r^2e^{\tau\phi}u\|_{\frac{R}{2},R}.\end{aligned}$$

Now, we bound from above the two last terms of the previous inequality by $C\tau\left(\|e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}u\|_{\frac{R}{2},R}\right)$. Then we divide both sides of (3.15) by $\tau^{\frac{1}{2}}$. Noticing that $\tau \geq 1$, this yields to

$$\begin{aligned}\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}u\|_{\frac{R}{8},\frac{R}{4}} + \|e^{\tau\phi}u\|_{\frac{5\delta}{4},3\delta} &\leq C\tau^{\frac{1}{2}}\left(\delta\|e^{\tau\phi}\nabla u\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}\nabla u\|_{\frac{R}{2},R}\right) \\ &+ C\tau^{\frac{1}{2}}\left(\|e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}u\|_{\frac{R}{2},R}\right).\end{aligned} \quad (3.15)$$

From the elliptic estimate (3.8) and the decreasing of ϕ, we get

$$e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8},\frac{R}{4}} + e^{\tau\phi(3\delta)}\|u\|_{\frac{5\delta}{4},3\delta} \leq C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{1/2}+\|V\|_\infty)\left(e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}}\right).$$

Adding $e^{\tau\phi(3\delta)}\|u\|_{\frac{5\delta}{4}}$ to each sides and noting that we can bound it from above by $C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{1/2}+\|V\|_\infty)e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}}$, we find that

$$e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8},\frac{R}{4}} + e^{\tau\phi(3\delta)}\|u\|_{3\delta} \leq C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{1/2}+\|V\|_\infty)\left(e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}}\right).$$

Now we want to choose τ such that

$$C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{1/2}+\|V\|_\infty)e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}} \leq \frac{1}{2}e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8},\frac{R}{4}}.$$

For the same reasons as before we choose

$$\tau = \frac{2}{\phi(\frac{R}{3})-\phi(\frac{R}{4})}\ln\left(\frac{1}{2C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)}\frac{\|u\|_{\frac{R}{8},\frac{R}{4}}}{\|u\|_{\frac{5R}{3}}}\right) + C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1}).$$

Define $D_R = -\left(\phi(\frac{R}{3})-\phi(\frac{R}{4})\right)^{-1}$; like before one has $0 < E^{-1} \leq D_R \leq E$, with E a fixed real number. Dropping the first term in the left hand side and noting that $0 < \phi(\delta)-\phi(3\delta) \leq C$, one has

$$\|u\|_{3\delta} \leq e^{C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\left(\frac{\|u\|_{\frac{R}{8},\frac{R}{4}}}{\|u\|_{\frac{5R}{3}}}\right)^{-E}\|u\|_{\frac{3\delta}{2}}$$

Finally, from Corollary 3.4, we define $r = \frac{3\delta}{2}$ to have :

$$\|u\|_{2r} \leq e^{C(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_r.$$

Thus, the theorem is proved for all $r \leq \frac{R_1}{16}$. Using Proposition 3.3 we have for $r \geq \frac{R_1}{16}$:

$$\|u\|_{B_{x_0}(r)} \geq \|u\|_{B_{x_0}(\frac{R_0}{16})} \geq e^{-C_0(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{L^2(M)} \geq e^{-C_1(1+\|W\|_{\mathcal{C}^1}^{\frac{1}{2}}+\|V\|_{\mathcal{C}^1})}\|u\|_{B_{x_0}(2r)}.$$

$\square$

Finally Theorem 1.1 is an easy and direct consequence of this doubling estimate.

4 Appendix.

The aim of this appendix is to prove the claim (2.22) we used in the proof of Theorem 2.1. More precisely, we show the following lemma.

Lemma 4.1. *We have*

$$\begin{aligned} I_3 \geq 3\tau \int |f''| \, |D_\theta u|^2 \, {f'}^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta \\ - c\tau \int |f''| \, |\partial_t u|^2 \, {f'}^{-3} \sqrt{\gamma} dt d\theta - c\tau^2 \int |f''| \, |u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \tag{4.1}$$

Proof. We begin by recalling the definition of I_3 :

$$\begin{aligned} I_3 = 2 \Big\langle \partial_t^2 u + (\tau^2 {f'}^2 + \tau f' e^{2t} V_t + (n-2)\tau f' + e^{2t} W) u + \Delta_\theta u \\ , (2\tau f' + e^{2t} V_t) \partial_t u + e^{2t} V_i \partial_i u \Big\rangle_f . \end{aligned}$$

We also recall the following estimates on the weight and the metric :

$$\begin{aligned} 1 - \varepsilon e^{\varepsilon T_0} \leq f'(t) \leq 1 \qquad \forall t \in]-\infty, T_0[, \\ \lim_{t \to -\infty} -e^{-t} f''(t) = +\infty, \end{aligned} \tag{4.2}$$

and, $\forall \; i, j, k \in \{1, \ldots n-1\}$,

$$\begin{aligned} \partial_t(\gamma^{ij}) &\leq C e^t (\gamma^{ij}) \;\; \text{(in the sense of tensors);} \\ \partial_k(\gamma^{ij}) &\leq C(\gamma^{ij}) \qquad \text{(in the sense of tensors);} \\ |\partial_t(\gamma)| &\leq C e^t; \\ C^{-1} &\leq \gamma \leq C. \end{aligned} \tag{4.3}$$

We will also use the key assumption on τ :

$$\tau \geq C_1 (1 + \sqrt{\|W\|_{C^1}} + \|V\|_{C^1}). \tag{4.4}$$

In order to compute I_3 we write it in a convenient way:

$$I_3 = \sum_{i=1}^{16} J_i, \tag{4.5}$$

where the integrals J_i are defined by :

$$
\begin{aligned}
J_1 &= 2\tau \int f' \partial_t(|\partial_t u|^2) {f'}^{-3} \sqrt{\gamma} dt d\theta \\
J_2 &= 4\tau \int f' \partial_t u \partial_i \left(\sqrt{\gamma} \gamma^{ij} \partial_j u \right) {f'}^{-3} dt d\theta \\
J_3 &= \int \left(2\tau^3 + 2(n-2)\tau^2 {f'}^{-1} + 2\tau {f'}^{-2} e^{2t} W \right) \partial_t |u|^2 \sqrt{\gamma} dt d\theta \\
J_4 &= 2\tau^2 \int e^{2t} V_t \partial_t |u|^2 {f'}^{-1} \sqrt{\gamma} dt d\theta \\
J_5 &= \int e^{2t} V_t \partial_t (|\partial_t u|^2) {f'}^{-3} \sqrt{\gamma} dt d\theta \\
J_6 &= \tau^2 \int e^{2t} V_t \partial_t |u|^2 {f'}^{-1} \sqrt{\gamma} dt d\theta \\
J_7 &= \tau \int e^{4t} V_t^2 \partial_t |u|^2 {f'}^{-2} \sqrt{\gamma} dt d\theta \\
J_8 &= (n-2)\tau \int e^{2t} V_t \partial_t |u|^2 {f'}^{-2} \sqrt{\gamma} dt d\theta \\
J_9 &= \int e^{4t} W V_t \partial_t |u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta \\
J_{10} &= 2 \int e^{2t} V_t \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u) \partial_t u {f'}^{-3} dt d\theta \\
J_{11} &= 2 \int e^{2t} V_i \partial_i u \partial_t^2 u {f'}^{-3} \sqrt{\gamma} dt d\theta \\
J_{12} &= \tau^2 \int e^{2t} V_i \partial_i |u|^2 {f'}^{-1} \sqrt{\gamma} dt d\theta \\
J_{13} &= \tau \int e^{4t} V_i V_t \partial_i |u|^2 f'^{-2} \sqrt{\gamma} dt d\theta \\
J_{14} &= (n-2)\tau \int e^{2t} V_i \partial_i |u|^2 f'^{-2} \sqrt{\gamma} dt d\theta \\
J_{15} &= \int e^{4t} V_i W \partial_i |u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta \\
J_{16} &= 2 \int e^{2t} V_k \partial_k u \partial_i \left(\sqrt{\gamma} \gamma^{ij} \partial_j u \right) f'^{-3} dt d\theta.
\end{aligned}
$$

Here we noticed that $2\partial_t u \partial_t^2 u = \partial_t(|\partial_t u|^2)$ and $2u\partial_t u = \partial_t |u|^2$. Before we start the computation, we want to point out that the only positive term of (4.5) comes from J_2. Now we will use integration by parts to estimate each J_i. Note that f is radial.

We begin with J_1. We find that :

$$J_1 = \int (4\tau f'')\, |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - \int 2\tau f' \partial_t \ln\sqrt{\gamma} |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta.$$

The conditions (4.3) imply that $|\partial_t \ln \sqrt{\gamma}| \leq Ce^t$. Then properties (4.2) on f give for large $|T_0|$ that $|\partial_t \ln \sqrt{\gamma}|$ is small compared to $|f''|$. Then one has

$$J_1 \geq -c\tau \int |f''| \cdot |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \tag{4.6}$$

In order to estimate J_2 we first integrate by parts with respect to ∂_i :

$$J_2 \;=\; -2 \textstyle\int 2\tau f' \partial_t \partial_i u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta.$$

Then we integrate by parts with respect to ∂_t. We get :

$$\begin{aligned} J_2 = -4\tau \int f'' \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ + \int 2\tau f' \partial_t \ln\sqrt{\gamma} \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ + \int 2\tau f' \partial_t (\gamma^{ij}) \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

Recall that $|D_\theta u|^2$ denotes $|D_\theta u|^2 = \partial_i u \gamma^{ij} \partial_j u$. Now using that $-f''$ is non-negative and τ is large, the conditions (4.2) and (4.3) give for $|T_0|$ large enough:

$$J_2 \geq \frac{7}{2}\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \tag{4.7}$$

Similarly computation of J_3 gives :

$$\begin{aligned} J_3 = -2 \int (\tau^3 + (n-2)\tau^2 f'^{-1}) \partial_t \ln(\sqrt{\gamma}) u^2 \sqrt{\gamma} dt d\theta \\ - \int (4f' - 4f'' + 2f' \partial_t \ln \sqrt{\gamma}) \tau e^{2t} W u^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ + 2 \int (n-2)\tau^2 f'' f' |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ - \int 2\tau f' e^{2t} \partial_t W |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

From (4.2) and (4.3) one can see that if C_1 and $|T_0|$ are large enough, then

$$J_3 \geq -c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^2 \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \tag{4.8}$$

We now compute the terms involving only radial derivatives, that is to say J_i for $i = 4, \ldots, 9$. We have

$$J_4 = 2\tau^2 \int e^{2t} V_t \partial_t |u|^2 f'^{-1} \sqrt{\gamma} dt d\theta$$
$$= -2\tau^2 \int u^2 e^{2t} \left(2V_t f'^2 - V_t f'' f' + f'^2 \partial_t V_r + f'^2 V_t \partial_t (\ln \sqrt{\gamma}) \right) f'^{-3} \sqrt{\gamma} dt d\theta$$

$$J_4 \geq -c\tau^3 \int e^t u^2 f'^{-3} \sqrt{\gamma} dt d\theta, \tag{4.9}$$

and

$$J_5 = \int e^{2t} V_t \partial_t (|\partial_t u|^2) f'^{-3} \sqrt{\gamma} dt d\theta$$
$$= - \int e^{2t} |\partial_t u|^2 \left(2V_t + \partial_t V_t - 3V_t f'' f'^{-1} + V_t \partial_t (\ln \sqrt{\gamma}) \right) f'^{-3} \sqrt{\gamma} dt d\theta$$

$$J_5 \geq -c\tau \int |f''| |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \tag{4.10}$$

In the last inequality, we use that e^t is small compared to $|f''|$. Let's resume our computation. We obtain

$$J_6 = \tau^2 \int e^{2t} V_t \partial_t |u|^2 f'^{-1} \sqrt{\gamma} dt d\theta$$
$$= -\tau^2 \int e^{2t} u^2 \left(2V_t f'^2 + \partial_t V_t f'^2 - f'' f' V_t + V_t f'^2 \partial_t (\ln \sqrt{\gamma}) \right) f'^{-3} \sqrt{\gamma} dt d\theta$$

$$J_6 \geq -c\tau^3 \int e^t u^2 f'^{-3} \sqrt{\gamma} dt d\theta, \tag{4.11}$$

$$J_7 = \tau \int e^{4t} |V_t|^2 \partial_t |u|^2 f'^{-2} \sqrt{\gamma} dt d\theta$$
$$= -\tau \int u^2 e^{4t} \left(4|V_t|^2 f' + 2V_t \partial_t V_t f' \right) f'^{-3} \sqrt{\gamma} dt d\theta$$
$$+ \tau \int u^2 e^{4t} \left(2|V_t|^2 f'' - |V_t|^2 f' \partial_t (\ln \sqrt{\gamma}) \right) f'^{-3} \sqrt{\gamma} dt d\theta$$

$$J_7 \geq -c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta, \tag{4.12}$$

and

$$J_8 = (n-2)\tau \int e^{2t} V_t \partial_t |u|^2 f'^{-2} \sqrt{\gamma} dt d\theta$$
$$= -(n-2)\tau \int e^{2t} \left(2V_t f' + \partial_t V_t f' - 2f'' V_t + V_t \partial_t \ln \sqrt{\gamma} \right) |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta$$

$$J_8 \geq -c\tau^2 \int |f''||u|^2 \sqrt{\gamma} {f'}^{-3} dtd\theta, \tag{4.13}$$

where we use once more that e^t is small compared to $|f''|$. Finally, for J_9, we get

$$\begin{aligned} J_9 &= \int e^{4t} V_t W \partial_t |u|^2 f'^{-3} \sqrt{\gamma} dtd\theta \\ &= -\int u^2 e^{4t} \left(4V_t W + \partial_t V_t W + V_t \partial_t W\right) f'^{-3} \sqrt{\gamma} dtd\theta \\ &\quad + \int u^2 e^{4t} \left(3f'' f'^{-1} V_t W - V_t W \partial_t (\ln \sqrt{\gamma})\right) f'^{-3} \sqrt{\gamma} dtd\theta \end{aligned}$$

$$J_9 \geq -c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dtd\theta. \tag{4.14}$$

Now, we deal with the terms involving spherical derivative. We recall that

$$J_{10} = 2 \int e^{2t} V_t \partial_t u \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u) f'^{-3} dtd\theta.$$

Integrating by parts in the spherical variables gives

$$\begin{aligned} J_{10} = -2 \int & e^{2t} V_t \partial_i \partial_t u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dtd\theta \\ & - 2 \int e^{2t} \partial_i V_t \partial_t u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dtd\theta. \end{aligned}$$

Now, we use the identity $\partial_t |D_\theta u|^2 = 2\gamma^{ij} \partial_t \partial_i u \partial_j u + \partial_t \gamma^{ij} \partial_i u \partial_j u$ to find

$$\begin{aligned} J_{10} = -\int & e^{2t} V_t (\partial_t |D_\theta u|^2 - \partial_t \gamma^{ij} \partial_i u \partial_j u) f'^{-3} \sqrt{\gamma} dtd\theta \\ & - 2 \int e^{2t} \partial_i V_t \partial_t u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dtd\theta. \end{aligned}$$

Finally, integrating by parts with respect to the radial variable,

$$\begin{aligned} J_{10} = \int & e^{2t} |D_\theta u|^2 (2V_t + \partial_t V_t - 3V_t f'' f'^{-1} + V_t \partial_t (\ln \sqrt{\gamma})) f'^{-3} \sqrt{\gamma} dtd\theta \\ & + \int e^{2t} V_t \partial_t \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dtd\theta \\ & - 2 \int e^{2t} \partial_i V_t \partial_t u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dtd\theta, \end{aligned}$$

we obtain

$$J_{10} \geq -c\tau \int |f''||\partial_t u|^2 f'^{-3} \sqrt{\gamma} dtd\theta - c\tau \int e^t |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dtd\theta. \tag{4.15}$$

Integrating by parts the following

$$J_{11} = 2\int e^{2t} V_i \partial_i u \partial_t^2 u f'^{-3} \sqrt{\gamma} dt d\theta,$$

gives

$$J_{11} = -2\int e^{2t} \partial_i u \partial_t u \left(2V_i + \partial_t V_i - 3f'' f'^{-1} V_i + V_i \partial_t (\ln \sqrt{\gamma})\right) f'^{-3} \sqrt{\gamma} dt d\theta \\ - 2\int e^{2t} V_i \partial_t \partial_i u \partial_t u f'^{-3} \sqrt{\gamma} dt d\theta.$$

Noticing that $2\partial_t \partial_i u \partial_t u = \partial_i |\partial_t u|^2$, we have

$$J_{11} = -2\int e^{2t} \partial_i u \partial_t u \left(2V_i + \partial_t V_i - 3f'' f'^{-1} V_i + V_i \partial_t (\ln \sqrt{\gamma})\right) f'^{-3} \sqrt{\gamma} dt d\theta \\ - \int e^{2t} V_i \partial_i (|\partial_t u|^2) f'^{-3} \sqrt{\gamma} dt d\theta,$$

then integrating by parts the last integral of the right hand side gives

$$J_{11} = -2\int e^{2t} \partial_i u \partial_t u \left(2V_i + \partial_t V_i - 3f'' f'^{-1} V_i + V_i \partial_t (\ln \sqrt{\gamma})\right) f'^{-3} \sqrt{\gamma} dt d\theta \\ + \int e^{2t} |\partial_t u|^2 \left(\partial_i V_i + V_i \partial_i (\ln \sqrt{\gamma})\right) f'^{-3} \sqrt{\gamma} dt d\theta.$$

Therefore we can state that

$$J_{11} \geq -c\tau \int |f''| |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau \int e^t |D_\theta u|^2) f'^{-3} \sqrt{\gamma} dt d\theta. \tag{4.16}$$

From the definition of J_{12}

$$J_{12} = \tau^2 \int e^{2t} V_i \partial_i |u|^2 f'^2 f'^{-3} \sqrt{\gamma} dt d\theta,$$

integrating by parts with respect to the spherical variables gives

$$J_{12} = -\tau^2 \int u^2 e^{2t} f'^2 \left(\partial_i V_i + V_i \partial_i (\ln \sqrt{\gamma})\right) f'^{-3} \sqrt{\gamma} dt d\theta, \tag{4.17}$$

therefore we can derive the estimate

$$J_{12} \geq -c\tau^3 \int |u|^2 e^t f'^{-3} \sqrt{\gamma} dt d\theta. \tag{4.18}$$

In the same way, we have

$$
\begin{aligned}
J_{13} &= \tau \int e^{4t} V_i V_t \partial_i |u|^2 f'^{-2} \sqrt{\gamma} dt d\theta \\
&= -\tau \int u^2 e^{4t} f' \left(\partial_i V_i V_t + V_i \partial_i V_t + V_i V_t \partial_i (\ln \sqrt{\gamma}) \right) f'^{-3} \sqrt{\gamma} dt d\theta
\end{aligned}
$$

$$
J_{13} \geq -c\tau^3 \int |u|^2 e^t f'^{-3} \sqrt{\gamma} dt d\theta, \tag{4.19}
$$

$$
\begin{aligned}
J_{14} &= (n-2)\tau \int e^{2t} V_i \partial_i |u|^2 f'^{-2} \sqrt{\gamma} dt d\theta \\
&= -(n-2)\tau \int e^{2t} (\partial_i V_i + \partial_i \ln \sqrt{\gamma}) |u|^2 f'^{-2} \sqrt{\gamma} dt d\theta
\end{aligned}
$$

$$
J_{14} \geq -c\tau^2 \int |f''||u|^2 f'^{-3} \sqrt{\gamma} dt d\theta, \tag{4.20}
$$

and

$$
\begin{aligned}
J_{15} &= \int e^{4t} V_i W \partial_i |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&= -\int e^{4t} (\partial_i V_i W + V_i \partial_i W + V_i W \partial_i \ln \sqrt{\gamma}) |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta
\end{aligned}
$$

$$
J_{15} \geq -c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \tag{4.21}
$$

We now turn to J_{16}

$$
J_{16} = 2 \int e^{2t} V_k \partial_k u \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u) f'^{-3} dt d\theta.
$$

We first integrate by parts with respect to ∂_t

$$
J_{16} = -2 \int e^{2t} (\partial_i V_k \partial_k u + V_k \partial_i \partial_k u) \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta,
$$

and use the identity $\partial_i \partial_k u \gamma^{ij} \partial_j u = 2(\partial_k |D_\theta u|^2 - \partial_k \gamma^{ij} \partial_i u \partial_j u)$ to find

$$
\begin{aligned}
J_{16} = &-2 \int e^{2t} \partial_i V_k \partial_k u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\
&- \int e^{2t} V_k (\partial_k |D_\theta u|^2 - \partial_k \gamma^{ij} \partial_i u \partial_j u) f'^{-3} \sqrt{\gamma} dt d\theta.
\end{aligned}
$$

Then an integration by parts with respect to ∂_k gives

$$\begin{aligned} J_{16} = &-2\int e^{2t}\partial_i V_k \partial_k u \gamma^{ij}\partial_j u f'^{-3}\sqrt{\gamma}dtd\theta \\ &+\int e^{2t}(\partial_k V_k + V_k\partial_k(\ln\sqrt{\gamma}))|D_\theta u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \\ &+\int e^{2t}V_k\partial_k\gamma^{ij}\partial_i u\partial_j u f'^{-3}\sqrt{\gamma}dtd\theta. \end{aligned}$$

This yields to :

$$J_{16} \geq -c\tau\int |D_\theta u|^2 e^t f'^{-3}\sqrt{\gamma}dtd\theta. \tag{4.22}$$

Therefore, combining all the previous estimates on the J_i ($i.e$ (4.6) to (4.22)) and noticing that

$$c\tau\int |D_\theta u|^2 e^t f'^{-3}\sqrt{\gamma}dtd\theta \leq \frac{1}{2}\tau\int |f''|\,|D_\theta u|^2\, f'^{-3}\sqrt{\gamma}dtd\theta,$$

we have established that

$$\begin{aligned} I_3 \geq\ & 3\tau\int |f''|\,|D_\theta u|^2\, f'^{-3}\sqrt{\gamma}dtd\theta - c\tau^3\int e^t|u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \\ &- c\tau\int |f''|\,|\partial_t u|^2\, f'^{-3}\sqrt{\gamma}dtd\theta - c\tau^2\int |f''|\,|u|^2 f'^{-3}\sqrt{\gamma}dtd\theta. \end{aligned}$$

□

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