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SIMULATION OF MCKEAN-VLASOV BSDES BY WIENER CHAOS EXPANSION

CÉLINE ACARY-ROBERT, PHILIPPE BRIAND, ABIR GHANNOUM, AND CÉLINE LABART

Abstract. We present an algorithm to solve McKean-Vlasov BSDEs based on Wiener chaos expansion and Picard's iterations and study its convergence. This paper extends the results obtained by Briand and Labart in [BL14] when standard BSDEs were considered. Here we are faced with the problem of the approximation of the law of (Y, Z) in the driver, that we solve by using a particle system. In order to avoid solving a system of BSDEs, which would not be feasible in practice, we use the same particles to approximate the law of (Y, Z) and to compute Monte Carlo approximations. This leads to an algorithm which doesn't cost more than the standard one.

1. Introduction

Backward stochastic differential equations were introduced by Bismut in [Bis73] for the linear case, and by Pardoux and Peng in [PP90] for the general case. These works consisted in finding a pair (Y_t, Z_t) of $\mathcal{F}_t$-adapted processes such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s, \quad 0 \le t \le T, \tag{1.1}$$

where B is a d-dimensional standard Brownian motion, the terminal condition ξ is a real-valued $\mathcal{F}_T$-measurable random variable where $\{\mathcal{F}_t\}_{0\le t\le T}$ stands for the augmented filtration of the Brownian motion B, the generator f is a map from $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}$.

First results on the numerical approximation of (1.1) date from the end of the 90's. The case of a generator f independent of z has been studied in [Che97] and in [CMM99]. The authors introduce a time and space discretization of the BSDE, which is somewhat reminiscent of the dynamic programming equation, introduced a couple of years later. The case of a generator dependent of z has first been done in [Bal97], where the author introduces a random discretization. In [BDM01], the authors generalize the scheme proposed in [Che97] to the case of f depending on z and prove the weak convergence of their scheme. In [BDM02], an approach for the case of path-dependent terminal condition ξ has been presented. The rate of the convergence of this method was left as an open problem. To deal with this question, an approach based on the dynamic programming equation has been introduced by Bouchard and Touzi in [BT04] and Zhang in [Zha04]. Both papers deal with the Markovian case, i.e. $\xi = g(X_T)$ where X is a solution of a stochastic differential equation. To be fully implementable, this algorithm requires to have a good approximation of its associated conditional expectation. Various methods have been developed (see [GLW05], [CMT10], [CT17]). Forward methods have also been introduced to approximate (1.1) : branching diffusion method (see [HLTT14]), multilevel Picard approximation (see [WHJK17]) and Wiener chaos expansion (see [BL14]).

Many extensions of (1.1) have also been considered : high order schemes (see [Cha14], [CC14]), schemes for reflected BSDEs (see [BP03], [CR16]), for fully-coupled BSDEs (see [DM06], [BZ08]),

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for quadratic BSDEs (see [CR15]), for BSDEs with jumps (see [GL16]) and for McKean-Vlasov BSDEs (see [Ala15], [CdRGT15], [CCD17]).

The aim of this paper is to extend the results of [BL14] to the case of McKean-Vlasov BSDEs, i.e. to provide an algorithm based on Wiener chaos expansion to solve BSDEs of the following type

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, [Y_s], [Z_s])ds - \int_t^T Z_s \cdot dB_s, \quad 0 \leq t \leq T, \tag{1.2}$$

where $[\theta]$ is the notation for the law of a random variable θ and f is a map from $[0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d)$ into $\mathbb{R}$. The set $\mathcal{P}_2(\mathbb{R}^d)$ is the set of probability measures with a finite second-order moment, endowed with the Wasserstein distance i.e.

$$\mathcal{W}_2(\mu, \mu') := \inf_\pi \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - x'|^2 d\pi(x, x') \right)^{1/2},$$

for $(\mu, \mu') \in \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$, the infimum being taken over the probability distributions π on $\mathbb{R}^d \times \mathbb{R}^d$ whose marginals on $\mathbb{R}^d$ are respectively μ and μ'. Notice that if X and X' are random variables of order 2 with values in $\mathbb{R}^d$, then by definition we have

$$\mathcal{W}_2([X], [X']) \leq \left[\mathbb{E}|X - X'|^2\right]^{1/2}. \tag{1.3}$$

Such type of BSDEs have been introduced in [BDLP09] and [BLP09] in a more particular framework: in [BDLP09], the authors study the mean field problem in a Markovian setting and prove the existence and the uniqueness of the solution when the terminal condition is of type $\xi = \mathbb{E}\left[g(x, X_T)\right]_{|x=X_T}$ where X is a driving adapted stochastic process, and the generator is defined by $\mathbb{E}\left[f(s, \lambda, \Lambda_s)\right]_{|\lambda=\Lambda_s}$ where $\Lambda_s = (X_s, Y_s, Z_s)$. In [BLP09], the authors extend the result of existence and uniqueness to a more general framework and link the mean-field BSDE to non local partial differential equation.

The study of numerical methods for McKean-Vlasov BSDEs goes back to a few years (see [Ala15], [CdRGT15], [CCD17]). Usually, forward McKean-Vlasov SDEs are solved by using particle algorithms (see [AKH02], [TV03], [Bos05]) in which the McKean term is approximated by the empirical measure of a large number of interacting particles with independent noise. Adapting such algorithms to the backward problem is not obvious as the high dimension of the involved Brownian motion (given by the number of particles) induces, a priori, a high dimension backward problem with bad consequences for the numerical implementation. The above mentioned papers on numerical methods for McKean-Vlasov BSDEs do not use particle systems. In [CdRGT15], the authors present a method based on cubature for decoupled McKean-Vlasov forward backward SDE. In [CCD17], the authors consider the case of strongly coupled forward-backward SDE of McKean-Vlasov type. They propose a scheme whose principle is to implement recursively Picard iterations on small time intervals, since Picard Theorem only applies in small time for fully coupled problems.

In this paper we propose a method based on Wiener chaos expansion and particle system approximation which is neither more complex nor more costly than solving a standard BSDE of type (1.1). The method based on Wiener chaos expansion to solve standard BSDEs has been introduced in [BL14] and consists in writing the Picard scheme of (1.1) in a forward way

$$Y_t^{q+1} = \mathbb{E}\left(\xi + \int_0^T f\left(s, Y_s^q, Z_s^q\right) ds \,\middle|\, \mathcal{F}_t\right) - \int_0^t f\left(s, Y_s^q, Z_s^q\right) ds,$$
$$Z_t^{q+1} = D_t Y_t^{q+1} = D_t \mathbb{E}\left(\xi + \int_0^T f\left(s, Y_s^q, Z_s^q\right) ds \,\middle|\, \mathcal{F}_t\right),$$

(where $D_t X$ stands for the Malliavin derivative of the random variable X) and to use Wiener chaos expansion to easily compute conditional expectations and their Malliavin derivatives. More precisely, all r.v. F in L^2 can be written

$$F = \mathbb{E}\left(F\right) + \sum\nolimits_{k\geq 1} \sum\nolimits_{|n|=k} d_k^n \prod\nolimits_{i\geq 1} K_{n_i}\left(\int_0^T g_i(s) dB_s\right),$$

where K_l denotes the Hermite polynomial of degree l, $(g_i)_{i\geq 1}$ is an orthonormal basis of $\mathrm{L}^2(0,T)$ and, if $n = (n_i)_{i\geq 1}$ is a sequence of integers, $|n| = \sum_{i\geq 1} n_i$. $(d_k^n)_{k\geq 1, |n|=k}$ is the sequence of coefficients ensuing from the decomposition of F. The numerical method consists in working with a finite number of chaos, a finite number of functions $(g_i)_i$ and in using Monte-Carlo approximation to compute the coefficients $(d_k^n)_{k,n}$. In case of McKean-Vlasov BSDE, the generator depends on the laws of the processes. The idea is to use M particles which will serve both to approximate the law of (Y, Z) and to compute the coefficients $(d_k^n)_{k,n}$ by Monte Carlo. By doing this, we manage to get a computational cost which is of the same order as the one obtained in case of standard BSDEs. However, this pooling of particles costs the independance in the Monte Carlo approximation, making the proof of the convergence more difficult and leading to a slower speed of convergence in M.

The outline of this paper is as follows. Section 2 state the notations and recall the main results of [BL14] in order to make the paper as self-contained as possible. In Section 3 we generalize the existence and uniqueness results stated by Pardoux and Peng [PP90] to the case of BSDEs of type (1.2). Section 4 describes precisely the algorithm, Section 5 is devoted to the study of the convergence of the algorithm and finally Section 6 contains some numerical experiments.

2. Preliminaries.

2.1. **Definitions and notations.** Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{R}^d$-valued Brownian motion B, we consider:

- $\{(\mathcal{F}_t); t \in [0,T]\}$, the filtration generated by the Brownian motion B and augmented.
- $L^p(\mathcal{F}_T) := L^p(\Omega, \mathcal{F}_T, \mathbb{P})$, $p \in \mathbb{N}^*$, the space of all $\mathcal{F}_T$-measurable random variables (r.v. in the following) $X : \Omega \longrightarrow \mathbb{R}^d$ satisfying $\|X\|_p^p := \mathbb{E}(|X|^p) < \infty$.
- $\mathbb{E}_t(X) := \mathbb{E}(X|\mathcal{F}_t)$, the conditional expectation of X (in $L^1(\mathcal{F}_T)$) w.r.t. $\mathcal{F}_t$.
- $S^p_{\alpha,T}(\mathbb{R}^d)$, $p \in \mathbb{N}$, $p \geq 2$, $\alpha \geq 0$, the space of all càdlàg predictable processes $\phi : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ such that $\|\phi\|^p_{S^p_{\alpha,T}} = \mathbb{E}(\sup_{t\in[0,T]} e^{\alpha t}|\phi_t|^p) < \infty$. Note that $S^p_T(\mathbb{R}^d) = S^p_{0,T}(\mathbb{R}^d)$.
- $H^p_{\alpha,T}(\mathbb{R}^d)$, $p \in \mathbb{N}$, $p \geq 2$, $\alpha \geq 0$, the space of all predictable processes $\phi : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ such that $\|\phi\|^p_{H^p_{\alpha,T}} = \mathbb{E}\int_0^T e^{\alpha t}|\phi_t|^p dt < \infty$. Note that $H^p_T(\mathbb{R}^d) = H^p_{0,T}(\mathbb{R}^d)$.
- $L^2(0,T)$, the space of all square integrable functions in $[0,T]$.
- $C^{k,l}$, the set of continuously differentiable functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^d$ with continuous derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l).
- $C^{k,l}_b$, the set of continuously differentiable functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^d$ with continuous and uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l). The function ϕ is also bounded.

- $\|\partial^j_{sp} f\|^2_\infty$, the norm of the derivatives of $f([0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R})\times\mathcal{P}(\mathbb{R}^d),\mathbb{R})$ w.r.t. the second and the third component which sum equals j: $\|\partial^j_{sp} f\|^2_\infty := \sum_{|k|=j}\|\partial^{k_0}_y\partial^{k_1}_{z_1}\cdots\partial^{k_d}_{z_d} f\|^2_\infty$, where $|k| = k_0 + \cdots + k_d$.
- C^∞_p, the set of smooth functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with partial derivatives of polynomial growth.
- $\|(.,.)\|^p_{L^p}$, $p \in \mathbb{N}$, $p \geq 2$, the norm on the space $S^p_T(\mathbb{R}) \times H^p_T(\mathbb{R}^d)$ defined by

$$\|(Y,Z)\|^p_{L^p} := \mathbb{E}\Big(\sup_{t\in[0,T]}|Y_t|^p\Big) + \int_0^T \mathbb{E}\big(|Z_t|^p\big)dt. \tag{2.1}$$

  Note that this norm is different from the usual L^p norm for BSDE.

We also recall some useful definitions related to Malliavin calculus. We use the notations of [Nua06].

- $\mathcal{S}$ denotes the class of random variables of the form $F = f(W(h_1),\cdots,W(h_n))$, where $f \in C^\infty_p(\mathbb{R}^{n\times d},\mathbb{R})$, for all $j \leq n$, $h_j = (h^1_j,\cdots,h^d_j) \in L^2([0,T];\mathbb{R}^d)$ and for all $i \leq d$, $W^i(h_j) = \int_0^T h^i_j(t)dW^i_t$.
- $\mathbb{D}^{r,2}$ denotes the closure of $\mathcal{S}$ w.r.t. the following norm on $\mathcal{S}$

$$\|F\|^2_{\mathbb{D}^{r,2}} := \mathbb{E}|F|^2 + \sum_{q=1}^r\sum_{|\alpha|_1=q}\mathbb{E}\Big(\int_0^T\cdots\int_0^T |D^\alpha_{(t_1,\cdots,t_q)}F|^2 dt_1\cdots dt_q\Big),$$

  where α is a multi-index $(\alpha_1,\cdots,\alpha_q) \in \{1,\cdots,d\}^q$, $|\alpha|_1 := \sum_{i=1}^q \alpha_i = q$, and D^α represent the multi-index Malliavin derivative operator. We recall $\mathbb{D}^{\infty,2} = \bigcap_{r=1}^\infty \mathbb{D}^{r,2}$.

Remark 1. *When $d = 1$, $\|F\|^2_{\mathbb{D}^{r,2}} := \mathbb{E}|F|^2 + \sum_{q=1}^r \mathbb{E}(\int_0^T\cdots\int_0^T |D^{(q)}_{(t_1,\cdots,t_q)}F|^2dt_1\cdots dt_q) = \mathbb{E}|F|^2 + \sum_{q=1}^r \|D^{(q)}F\|^2_{L^2(\Omega\times[0,T]^q)}$.*

Let $m \in \mathbb{N}^*$ and $j \in \mathbb{N}$, $j \geq 2$. We also introduce the following notation:

- $\mathcal{D}^{m,j}$ denotes the space of all $\mathcal{F}_T$-measurable r.v. such that

$$\|F\|^j_{m,j} := \sum_{1\leq l\leq m}\sum_{|\alpha|_1=l}\sup_{t_1\leq\cdots\leq t_l}\mathbb{E}(|D^\alpha_{t_1,\cdots,t_l}F|^j) < \infty,$$

  where $\sup_{t_1\leq\cdots\leq t_l}$ means $\sup_{(t_1,\cdots,t_l):t_1\leq\cdots\leq t_l}$.
- $\mathcal{S}^{m,j}$ denotes the space of all couple of processes (Y,Z) belonging to $S^j_T(\mathbb{R}) \times H^j_T(\mathbb{R}^d)$ and such that

$$\|(Y,Z)\|^j_{m,j} := \sum_{1\leq l\leq m}\sum_{|\alpha|_1=l}\sup_{t_1\leq\cdots\leq t_l}\|(D^\alpha_{t_1,\cdots,t_l}Y, D^\alpha_{t_1,\cdots,t_l}Z)\|^j_{L^j} < \infty,$$

  i.e.

$$\|(Y,Z)\|^j_{m,j} = \sum_{1\leq l\leq m}\sum_{|\alpha|_1=l}\sup_{t_1\leq\cdots\leq t_l}\Big\{\mathbb{E}\Big(\sup_{t_1\leq r\leq T}|D^\alpha_{t_1,\cdots,t_l}Y_r|^j\Big) + \int_{t_1}^T \mathbb{E}\big(|D^\alpha_{t_1,\cdots,t_l}Z_r|^j\big)dr\Big\}.$$

  We also denote $\mathcal{S}^{m,\infty} := \bigcap_{j\geq 2}\mathcal{S}^{m,j}$.

2.2. **Chaos decomposition formulas.** We refer to the book [Nua06] for more details on this Section. The notations we use are the ones of [BL14]. Every square integrable random variable F, measurable w.r.t. $\mathcal{F}_T$, admits the following orthogonal decomposition

$$F = d_0 + \sum_{k\geq 1}\sum_{|n|=k} d_k^n \prod_{i\geq 1} K_{n_i}\Big(\int_0^T g_i(s)dB_s\Big), \tag{2.2}$$

where $(g_i)_{i\geq 1}$ is an orthonormal basis of $L^2(0,T)$, K_n is the Hermite polynomial of order n defined by the expansion

$$e^{xt-t^2/2} = \sum_{n\geq 0} K_n(x)t^n$$

with the convention $K_{-1}\equiv 0$, $n=(n_i)_{i\geq 1}$ is a sequence of positive integers and $|n|$ stands for $\sum_{i\geq 1} n_i$. Taking into account the normalization of the Hermite polynomials we use gives

$$d_0 = \mathbb{E}(F), \qquad d_k^n = n!\mathbb{E}\Big(F\times\prod_{i\geq 1} K_{n_i}\Big(\int_0^T g_i(s)dB_s\Big)\Big),$$

where $n! = \prod_{i\geq 1} n_i!$.

To get tractable formulas, we consider a finite number of chaos and a finite number of functions $(g_1,\cdots,g_N)$. The $(g_i)_{1\leq i\leq N}$ functions are chosen such that we can quickly compute $\mathbb{E}(F|\mathcal{F}_t)$ and $D_t\mathbb{E}(F|\mathcal{F}_t)$ (see Section 4.1). We develop in this section the case $d=1$, and we refer to [BL14, Section B.2] when $d>1$.
The first step consists in considering a finite number of chaos. In order to approximate the random variable F, we consider its projection $C_p(F)$ onto the first p chaos, namely

$$C_p(F) = d_0 + \sum_{1\leq k\leq p}\sum_{|n|=k} d_k^n \prod_{i\geq 1} K_{n_i}\Big(\int_0^T g_i(s)dB_s\Big). \tag{2.3}$$

The following two Lemmas give some useful properties of the operator C_p.

Lemma 1. *Let $1\leq m\leq p+1$ and $F\in\mathbb{D}^{m,2}$. We have*

$$\mathbb{E}[|F-C_p(F)|^2] \leq \frac{||D^mF||^2_{L^2(\Omega\times[0,T]^m)}}{(p+2-m)\cdots(p+1)}.$$

We refer to [GL16, Lemma 2.4] for a proof.

Lemma 2.

- *Let F be r.v. in $L^2(\mathcal{F}_T)$. $\forall p\geq 1$, we have $\mathbb{E}(|C_p(F)|^2)\leq\mathbb{E}(|F|^2)$. If F belongs to $L^j(\mathcal{F}_T)$, $\forall j>2$, $\mathbb{E}(|C_p(F)|^j)\leq(1+p(j-1)^{p/2})^j\mathbb{E}(|F|^2)$.*
- *Let H be in $H^2_T(\mathbb{R})$. We have $C_p(\int_0^T H_sds)=\int_0^T C_p(H_s)ds$.*
- *$\forall F\in\mathbb{D}^{1,2}$ and $\forall t\leq r$, $D_t\mathbb{E}_r[C_p(F)]=\mathbb{E}_r[C_{p-1}(D_tF)]$.*

Of course, we still have an infinite number of terms in the sum in (2.3) and the second step consists in working with only the first N functions $g_1,\cdots,g_N$ of an orthonormal basis of $L^2(0,T)$. Let us consider a regular mesh grid of N time steps $\mathcal{T}=\{\tilde t_i = i\frac{T}{N}, i=0,\cdots,N\}$ and the N step functions

$$g_i(t) = \mathbf{1}_{]\tilde t_{i-1},\tilde t_i]}(t)/\sqrt{h}, \quad i=1,\cdots,N, \text{ where } h:=\frac{T}{N}. \tag{2.4}$$

We complete these N functions $g_1, \cdots, g_N$ into an orthonormal basis of $L^2(0,T)$, $(g_i)_{i\geq 1}$. For instance, one can consider the Haar basis on each interval $(\tilde{t}_{i-1}, \tilde{t}_i)$, $i = 1, \cdots, N$. We implicitly assume that $N \geq p$. This leads to the following approximation:

$$C_p^N(F) = d_0 + \sum_{1\leq k\leq p} \sum_{|n|=k} d_k^n \prod_{1\leq i\leq N} K_{n_i}\Big(\int_0^T g_i(s)dB_s\Big). \tag{2.5}$$

Due to the simplicity of the functions g_i, $i = 1, \cdots, N$, we can compute explicitly

$$\int_0^T g_i(s)dB_s = G_i \quad where\ G_i = \frac{B_{\tilde{t}_i} - B_{\tilde{t}_{i-1}}}{\sqrt{h}}.$$

Roughly speaking this means that P_k, the kth chaos, is generated by

$$\{K_{n_1}(G_1)\cdots K_{n_N}(G_N) : n_1 + \cdots + n_N = k\}.$$

Thus the approximation we use for the random variable F is

$$\begin{aligned} C_p^N(F) &= d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n K_{n_1}(G_1)\cdots K_{n_N}(G_N) \\ &= d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n \prod_{1\leq i\leq N} K_{n_i}(G_i), \end{aligned} \tag{2.6}$$

where the coefficients d_0 and d_k^n are given by

$$d_0 = \mathbb{E}(F), \quad d_k^n = n!\mathbb{E}(FK_{n_1}(G_1)\cdots K_{n_N}(G_N)). \tag{2.7}$$

The following Lemma, similar to Lemma 2, gives some useful properties of the operator C_p^N.

Lemma 3. *Let F be r.v. in $L^2(\mathcal{F}_T)$ and H be in $H_T^2(\mathbb{R})$. Then:*

- $\forall (p,N) \in (\mathbb{N}^*)^2$, $\mathbb{E}(|C_p^N(F)|^2) \leq \mathbb{E}(|C_p(F)|^2) \leq \mathbb{E}(|F|^2)$.
- $C_p^N(\int_0^T H_s ds) = \int_0^T C_p^N(H_s)ds$.
- $\forall t \leq r$, $D_t\mathbb{E}_r[C_p^N(F)] = \mathbb{E}_r[C_{p-1}^N(D_tF)]$.

From (2.6), we deduce the expressions of $\mathbb{E}_t(C_p^N F)$ and $D_t\mathbb{E}_t(C_p^N(F))$, useful for the approximation of (Y,Z) by the chaos decomposition (see Section 4.1).

Proposition 1 (Proposition 2.7, [BL14]). *Let F be a real random variable in $L^2(\mathcal{F}_T)$, and let r be an integer in $\{1, \cdots, N\}$. For all $\tilde{t}_{r-1} < t \leq \tilde{t}_r$, we have*

$$\mathbb{E}_t(C_p^N F) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i<r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h}\right)^{(n_r)/2} K_{n_r}\left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}}\right),$$

$$D_t\mathbb{E}_t(C_p^N(F)) = h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n(r)>0}} d_k^n \prod_{i<r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h}\right)^{(n_r-1)/2} K_{n_r-1}\left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}}\right),$$

where, if $r \leq N$ and $n = (n_1, \cdots, n_N)$, $n(r)$ stands for $(n_1, \cdots, n_r)$.

Remark 2 (Remark 1, [BL14]). *For $t = \tilde{t}_r$ and $r \geq 1$, Proposition 1 leads to*

$$\mathbb{E}_{\tilde{t}_r}(C_p^N F) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i\leq r} K_{n_i}(G_i),$$

$$D_{\tilde{t}_r}\mathbb{E}_{\tilde{t}_r}(C_p^N F) = h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n(r)>0}} d_k^n \prod_{i<r} K_{n_i}(G_i) \times K_{n_r-1}(G_r),$$

When $r = 0$, we get $\mathbb{E}_{\tilde{t}_0}(C_p^N F) = d_0$, and we define $D_{\tilde{t}_0}\mathbb{E}_{\tilde{t}_0}(C_p^N F) = \frac{1}{\sqrt{h}} d_1^{e_1}$ (which is the limit of $D_t\mathbb{E}_t(C_p^N F)$ when t tends to 0).

Let us end this subsection by some examples.

Example 1 (Case $p = 2$). *From (2.6)-(2.7), we have*

$$C_2^N(F) = d_0 + \sum_{j=1}^{N} d_1^{e_j} K_1(G_j) + \sum_{j=1}^{N}\sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i)K_1(G_j) + \sum_{j=1}^{N} d_2^{2e_j} K_2(G_j),$$

where e_j denotes the unit vector whose jth component is one, and $e_{ij} = e_i + e_j$. For $j = 1, \cdots, N$ and $i = 1, \cdots, j-1$, it holds

$$d_1^{e_j} = \mathbb{E}\big(FK_1(G_j)\big), \quad d_2^{e_{ij}} = \mathbb{E}\big(FK_1(G_i)K_1(G_j)\big) \quad d_2^{2e_j} = 2\mathbb{E}\big(FK_2(G_j)\big).$$

Remark 2 leads to

$$\mathbb{E}_{\tilde{t}_r}(C_2^N F) = d_0 + \sum_{j=1}^{r} d_1^{e_j} K_1(G_j) + \sum_{j=1}^{r}\sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i)K_1(G_j) + \sum_{j=1}^{r} d_2^{2e_j} K_2(G_j),$$

$$D_{\tilde{t}_r}\mathbb{E}_{\tilde{t}_r}(C_2^N F) = h^{-1/2}\left(d_1^{e_r} + d_2^{2e_r} K_1(G_r) + \sum_{i=1}^{r-1} d_2^{e_{ir}} K_1(G_i)\right).$$

3. Existence, uniqueness and properties of the solution.

Note that the existence and the uniqueness of the solution of (1.2) have been proved in [BLP09] in the case $f(t, Y_t, Z_t, [Y_t], [Z_t]) = \mathbb{E}[g(t, \lambda, Y_t, Z_t)]_{|\lambda=(Y_t,Z_t)}$.

Hypothesis 1. *We assume:*

- *the generator $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$ is Lipschitz continuous: there exists a constant L_f such that for all $t \in \mathbb{R}^+$, $y_1, y_2 \in \mathbb{R}$,$z_1, z_2 \in \mathbb{R}^d$, $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R})$ and $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$*

$$|f(t, y_1, z_1, \mu_1, \nu_1) - f(t, y_2, z_2, \mu_2, \nu_2)| \le L_f\Big(|y_1 - y_2| + |z_1 - z_2| + \mathcal{W}_2(\mu_1, \mu_2) + \mathcal{W}_2(\nu_1, \nu_2)\Big).$$

- $\mathbb{E}(|\xi|^2 + \int_0^T |f(s, 0, 0, [\delta_0], [\delta_0])|^2 ds) < \infty$.

Theorem 1. *Given standard parameters (f, ξ), there exists a unique pair $(Y, Z) \in S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$ which solves (1.2).*

Let us start with a priori estimates that will be useful for our proof.

A Priori Estimates.

Proposition 2. *Let $((f^i, \xi^i); i = 1, 2)$ be two standard parameters of the BSDE and $((Y^i, Z^i); i = 1, 2)$ be two square-integrable solutions. Let L_{f^1} be a Lipschitz constant for f^1, and put $\delta Y_t = Y_t^1 - Y_t^2$ and $\delta_2 f_t = f^1(t, Y_t^2, Z_t^2, [Y_t^2], [Z_t^2]) - f^2(t, Y_t^2, Z_t^2, [Y_t^2], [Z_t^2])$. For any α, $\lambda > 0$ such that $\alpha \ge 8L_{f^1}^2 + 4L_{f^1} + \lambda + \frac{1}{2}$, it follows that*

$$\|\delta Y\|^2_{S^2_{\alpha,T}} + \|\delta Z\|^2_{H^2_{\alpha,T}} \le (8L_{f^1} + 8C^2 + 5)\left[e^{\alpha T}\mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda}\mathbb{E}\left(\int_0^T e^{\alpha s}|\delta_2 f_s|^2 ds\right)\right],$$

where C is a universal constant.

Proof. By applying Itô's formula from $s=t$ to $s=T$ on the semimartingale $e^{\alpha s}|\delta Y_s|^2$, we get

$$\begin{aligned} e^{\alpha t}|\delta Y_t|^2 &+ \alpha\int_t^T e^{\alpha s}|\delta Y_s|^2 ds + \int_t^T e^{\alpha s}|\delta Z_s|^2 ds \\ &= e^{\alpha T}|\delta Y_T|^2 + 2\int_t^T e^{\alpha s}\delta Y_s\Big(f^1(s,Y_s^1,Z_s^1,[Y_s^1],[Z_s^1]) - f^2(s,Y_s^2,Z_s^2,[Y_s^2],[Z_s^2])\Big)ds \\ &- 2\int_t^T e^{\alpha s}\delta Y_s\delta Z_s dB_s. \end{aligned} \tag{3.1}$$

Moreover,

$$\begin{aligned} |f^1(s,&Y_s^1,Z_s^1,[Y_s^1],[Z_s^1]) - f^2(s,Y_s^2,Z_s^2,[Y_s^2],[Z_s^2])| \\ &\le |f^1(s,Y_s^1,Z_s^1,[Y_s^1],[Z_s^1]) - f^1(s,Y_s^2,Z_s^2,[Y_s^2],[Z_s^2])| + |\delta_2 f_s| \\ &\le L_f^1\Big(|\delta Y_s| + |\delta Z_s| + \mathcal{W}_2([Y_s^1],[Y_s^2]) + \mathcal{W}_2([Z_s^1],[Z_s^2])\Big) + |\delta_2 f_s|, \end{aligned}$$

where $L_{f^1}\ge 0$. By using (1.3), we obtain that

$$\begin{aligned} 2|\delta Y_s|\cdot|f^1(s,&Y_s^1,Z_s^1,[Y_s^1],[Z_s^1]) - f^2(s,Y_s^2,Z_s^2,[Y_s^2],[Z_s^2])| \\ &\le 2L_{f^1}\left(|\delta Y_s|^2 + |\delta Y_s||\delta Z_s| + |\delta Y_s|\big(\mathbb{E}(|\delta Y_s|^2)\big)^{1/2} + |\delta Y_s|\big(\mathbb{E}(|\delta Z_s|^2)\big)^{1/2}\right) \\ &+ 2|\delta Y_s||\delta_2 f_s|. \end{aligned} \tag{3.2}$$

Therefore, by Young's inequality with $\lambda>0$, we have

$$\begin{aligned} 2|\delta Y_s|\cdot|f^1(s,&Y_s^1,Z_s^1,[Y_s^1],[Z_s^1]) - f^2(s,Y_s^2,Z_s^2,[Y_s^2],[Z_s^2])| \\ &\le 2L_{f^1}|\delta Y_s|^2 + 4L_{f^1}^2|\delta Y_s|^2 + \frac{1}{4}|\delta Z_s|^2 + 2L_{f^1}|\delta Y_s|\big(\mathbb{E}(|\delta Y_s|^2)\big)^{1/2} \\ &\quad + 4L_{f^1}^2|\delta Y_s|^2 + \frac{1}{4}\mathbb{E}(|\delta Z_s|^2) + \lambda|\delta Y_s|^2 + \frac{1}{\lambda}|\delta_2 f_s|^2 \\ &\le (8L_{f^1}^2 + 2L_{f^1} + \lambda)|\delta Y_s|^2 + 2L_{f^1}|\delta Y_s|\big(\mathbb{E}(|\delta Y_s|^2)\big)^{1/2} \\ &\quad + \frac{1}{4}|\delta Z_s|^2 + \frac{1}{4}\mathbb{E}(|\delta Z_s|^2) + \frac{1}{\lambda}|\delta_2 f_s|^2. \end{aligned} \tag{3.3}$$

On the one hand, it follows from (3.1) and (3.3) that for $t=0$,

$$\begin{aligned} \mathbb{E}(|\delta Y_0|^2) &+ \alpha\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Y_s|^2 ds\Big) + \mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big) \\ &\le e^{\alpha T}\mathbb{E}(|\delta Y_T|^2) + \Big(8L_{f^1}^2 + 4L_{f^1} + \lambda\Big)\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Y_s|^2 ds\Big) + \frac{1}{2}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big) \\ &\quad + \frac{1}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta_2 f_s|^2 ds\Big). \end{aligned} \tag{3.4}$$

Choosing $\alpha \ge 8L_{f^1}^2 + 4L_{f^1} + \lambda + \frac{1}{2}$, this inequality implies

$$\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Y_s|^2 ds\Big) + \mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big) \le 2\Big[e^{\alpha T}\mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta_2 f_s|^2 ds\Big)\Big]. \tag{3.5}$$

On the other hand, by combining equation (3.1) and inequality (3.3), and by using the fact that $\alpha \geq 8L_{f^1}^2 + 2L_{f^1} + \lambda$, we can also obtain

$$e^{\alpha t}|\delta Y_t|^2 \leq e^{\alpha T}|\delta Y_T|^2 + \int_t^T e^{\alpha s}\Big(2L_{f^1}|\delta Y_s|\mathbb{E}(|\delta Y_s|^2) + \frac{1}{4}\mathbb{E}(|\delta Z_s|^2) + \frac{1}{\lambda}|\delta_2 f_s|^2\Big)ds \\ + 2\Big|\int_t^T e^{\alpha s}\delta Y_s \delta Z_s dB_s\Big|,$$

which leads to

$$\mathbb{E}(\sup_{0\leq t\leq T} e^{\alpha t}|\delta Y_t|^2) \leq e^{\alpha T}\mathbb{E}(|\delta Y_T|^2) + 2L_{f^1}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Y_s|^2 ds\Big) + \frac{1}{4}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big) \\ + \frac{1}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta_2 f_s|^2 ds\Big) + 2\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_t^T e^{\alpha s}\delta Y_s\delta Z_s dB_s\Big|\Big). \tag{3.6}$$

By the Burkholder-Davis-Gundy inequality, there exists a universal constant C such that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_t^T e^{\alpha s}\delta Y_s\delta Z_s dB_s\Big|\Big) \leq C\mathbb{E}\Big[\Big(\int_0^T e^{2\alpha s}|\delta Y_s|^2|\delta Z_s|^2 ds\Big)^{1/2}\Big] \\ \leq C\mathbb{E}\Big[\Big(\sup_{0\leq t\leq T} e^{\alpha t}|\delta Y_t|^2\Big)^{1/2}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big)^{1/2}\Big],$$

and since $ab \leq a^2/2 + b^2/2$,

$$2\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big|\int_t^T e^{\alpha s}\delta Y_s\delta Z_s dB_s\Big|\Big) \leq \frac{1}{2}\mathbb{E}\Big(\sup_{0\leq t\leq T} e^{\alpha t}|\delta Y_t|^2\Big) + 2C^2\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big). \tag{3.7}$$

Finally, by combining the inequalities (3.6)-(3.7) and by using (3.5), we derive that

$$\mathbb{E}(\sup_{0\leq t\leq T} e^{\alpha t}|\delta Y_t|^2) \leq 2e^{\alpha T}\mathbb{E}(|\delta Y_T|^2) + 4L_{f^1}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Y_s|^2 ds\Big) + \frac{2}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta_2 f_s|^2 ds\Big) \\ + \frac{(8C^2+1)}{2}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big) \\ \leq (8L_{f^1} + 8C^2 + 3)\Big[e^{\alpha T}\mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta_2 f_s|^2 ds\Big)\Big],$$

then, we can conclude that

$$\mathbb{E}(\sup_{0\leq t\leq T} e^{\alpha t}|\delta Y_t|^2) + \mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta Z_s|^2 ds\Big) \leq (8L_{f^1} + 8C^2 + 5)\Big[e^{\alpha T}\mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha s}|\delta_2 f_s|^2 ds\Big)\Big]. \tag{3.8}$$

□

Proof of Theorem 1. We use a fixed-point theorem for the mapping ϕ from $S^2_{\alpha,T}(\mathbb{R}) \times H^2_{\alpha,T}(\mathbb{R}^d)$ into $S^2_{\alpha,T}(\mathbb{R}) \times H^2_{\alpha,T}(\mathbb{R}^d)$, which maps (y,z) onto the solution (Y,Z) of the BSDE with generator $f(t, y_t, z_t, [y_t], [z_t])$, i.e.,

$$Y_t = \xi + \int_t^T f(s, y_s, z_s, [y_s], [z_s])ds - \int_t^T Z_s \cdot dB_s.$$

Let us remark that the solution $(Y,Z) \in S^2_T(\mathbb{R}) \times H^2_T(\mathbb{R}^d)$ is defined by [PP90], when $(y,z) \in S^2_T(\mathbb{R}) \times H^2_T(\mathbb{R}^d)$.

Let (y^1,z^1), (y^2,z^2) be two elements of $S^2_{\alpha,T}(\mathbb{R})\times H^2_{\alpha,T}(\mathbb{R}^d)$, and let (Y^1,Z^1) and (Y^2,Z^2) be the associated solutions. By applying Proposition 2 with $L_{f^1}=0$ and $\alpha=\lambda+\frac{1}{2}$, we obtain

$$\|\delta Y\|^2_{S^2_{\alpha,T}}+\|\delta Z\|^2_{H^2_{\alpha,T}}\le\frac{(8C^2+5)}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha t}|f(t,y^1_t,z^1_t,[y^1_t],[z^1_t])-f(t,y^2_t,z^2_t,[y^2_t],[z^2_t])|^2dt\Big).$$

Now since f is Lipschitz with constant L_f, we have

$$\begin{aligned}\|\delta Y\|^2_{S^2_{\alpha,T}}+\|\delta Z\|^2_{H^2_{\alpha,T}}&\le\frac{4(8C^2+5)L_f^2}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha t}\Big(|\delta y_t|^2+|\delta z_t|^2+\mathcal{W}_2([y^1_t],[y^2_t])^2+\mathcal{W}_2([z^1_t],[z^2_t])^2\Big)dt\Big)\\&\le\frac{4(8C^2+5)L_f^2}{\lambda}\mathbb{E}\Big(\int_0^T e^{\alpha t}\Big(|\delta y_t|^2+|\delta z_t|^2+\mathbb{E}(|\delta y_t|^2)+\mathbb{E}(|\delta z_t|^2)\Big)dt\Big)\\&\le\frac{8(8C^2+5)L_f^2}{\lambda}\int_0^T\Big(\mathbb{E}(e^{\alpha t}|\delta y_t|^2)+\mathbb{E}(e^{\alpha t}|\delta z_t|^2)\Big)dt\\&\le\frac{8(8C^2+5)(T+1)L_f^2}{\lambda}\Big(\mathbb{E}(\sup_{0\le t\le T}e^{\alpha t}|\delta y_t|^2)+\mathbb{E}\int_0^T e^{\alpha t}|\delta z_t|^2dt\Big)\\&\le\frac{8(8C^2+5)(T+1)L_f^2}{\lambda}\Big(\|\delta y\|^2_{S^2_{\alpha,T}}+\|\delta z\|^2_{H^2_{\alpha,T}}\Big).\end{aligned}\tag{3.9}$$

Choosing $\lambda\ge16(8C^2+5)(T+1)L_f^2$, we see that this mapping ϕ is a contraction from $S^2_{\alpha,T}(\mathbb{R})\times H^2_{\alpha,T}(\mathbb{R}^d)$ onto itself and that there exists a fixed point, which is the unique continuous solution of the BSDE. □

From the proof of Proposition 2 (and more precisely from estimate (3.9)), we derive that the Picard iterative sequence converges almost surely to the solution of the BSDE.

Remark 3. *Let α be such that $\alpha\ge16(8C^2+5)(T+1)L_f^2+\frac{1}{2}$. Let (Y^q,Z^q) be the sequence defined recursively by $(Y^0=0,Z^0=0)$ and*

$$Y^{q+1}_t=\xi+\int_t^T f(s,Y^q_s,Z^q_s,[Y^q_s],[Z^q_s])ds-\int_t^T Z^{q+1}_s\cdot dB_s,\quad 0\le t\le T,\tag{3.10}$$

Then the sequence (Y^q,Z^q) converges to (Y,Z), $d\mathbb{P}\times dt$ a.s. and in $S^2_T(\mathbb{R})\times H^2_T(\mathbb{R}^d)$ as q goes to $+\infty$.

Proof. Let (Y^q,Z^q) be the sequence defined recursively by (3.10). Then, by (3.9),

$$\|Y^{q+1}-Y^q\|^2_{S^2_T}+\|Z^{q+1}-Z^q\|^2_{H^2_T}\le C_T2^{-q},$$

and the result follows easily. □

4. Description of the algorithm.

The algorithm is based on five types of approximations: Picard's iterations, a Wiener chaos expansion up to a finite order, the truncation of an $L^2(0,T)$ basis in order to apply formulas of Proposition 1, a Monte Carlo method to approximate the coefficients d_0 and d^n_k defined in (2.7) and the particle system. We present these five steps of the approximation procedure in Section 4.1. The practical implementation is presented in Section 4.2.

4.1. **Approximation procedure.**

4.1.1. *Picard's iterations.* The first step consists in approximating (Y,Z)—the solution to (1.2)—by Picard's sequence $(Y^q,Z^q)_q$, built as follows: $(Y^0=0,Z^0=0)$ and for all $q\geq 1$

$$Y_t^{q+1}=\xi+\int_t^T f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds-\int_t^T Z_s^{q+1}\cdot dB_s,\quad 0\leq t\leq T. \tag{4.1}$$

From (4.1), under the assumptions that $\xi\in\mathbb{D}^{1,2}$ and $f\in C_b^{0,1,1,0,0}$, we express (Y^{q+1},Z^{q+1}) as a function of the processes (Y^q,Z^q),

$$Y_t^{q+1}=\mathbb{E}_t\Big(\xi+\int_t^T f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds\Big),\quad Z_t^{q+1}=D_tY_t^{q+1}, \tag{4.2}$$

which can also be written

$$\begin{aligned}&Y_t^{q+1}=\mathbb{E}_t\Big(\xi+\int_0^T f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds\Big)-\int_0^t f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds,\\&Z_t^{q+1}=D_tY_t^{q+1},\end{aligned} \tag{4.3}$$

As recalled in the Introduction, the computation of the conditional expectation is the cornerstone in the numerical resolution of BSDEs. Chaos decomposition formulas enable us to circumvent this problem.

4.1.2. *Wiener Chaos expansion.* Computing the chaos decomposition of the r.v. $F=\xi+\int_t^T f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds$ (appearing in (4.2)) in order to compute Y_t^{q+1} is not judicious. F depends on t, and then the computation of Y^{q+1} on the grid $\mathcal{T}=\{\tilde{t}_i=i\frac{T}{N},i=0,\cdots,N\}$ would require $N+1$ calls to the chaos decomposition function. To build an efficient algorithm, we need to call the chaos decomposition function as infrequently as possible, since each call is computationally demanding and brings an approximation error due to the truncation, the Monte Carlo approximation and to the particle approximation (see next sections). Then we look for a r.v. F^q independent of t such that Y_t^{q+1} and Z_t^{q+1} can be expressed as functions of $\mathbb{E}_t(F^q)$, $D_t\mathbb{E}_t(F^q)$ and of Y_q and Z_q. Equation (4.3) gives a more tractable expression of Y^{q+1}. Let F^q be defined by $F^q:=\xi+\int_0^T f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds$. Then

$$Y_t^{q+1}=\mathbb{E}_t(F^q)-\int_0^t f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds,\quad Z_t^{q+1}=D_t\mathbb{E}_t(F^q). \tag{4.4}$$

The second type of approximation consists in computing the chaos decomposition of F^q up to order p. Since F^q does not depend on t, the chaos decomposition function $\mathcal{C}_p$ is called only once per Picard's iteration.

Let $(Y^{q,p},Z^{q,p})$ denote the approximation of (Y^q,Z^q) built at step q using a chaos decomposition with order p: $(Y^{0,p},Z^{0,p})=(0,0)$ and

$$\begin{aligned}&Y_t^{q+1,p}=\mathbb{E}_t\big(C_p(F^{q,p})\big)-\int_0^t f(s,Y_s^{q,p},Z_s^{q,p},[Y_s^{q,p}],[Z_s^{q,p}])ds,\\&Z_t^{q+1,p}=D_t\mathbb{E}_t\big(C_p(F^{q,p})\big),\end{aligned} \tag{4.5}$$

where $F^{q,p}=\xi+\int_0^T f(s,Y_s^{q,p},Z_s^{q,p},[Y_s^{q,p}],[Z_s^{q,p}])ds$. In the sequel, we also use the following equality:

$$Z_t^{q+1,p}=\mathbb{E}_t\big(D_tC_p(F^{q,p})\big). \tag{4.6}$$

4.1.3. *Truncation of the basis.* The third type of approximation comes from the truncation of the orthonormal $L^2(0,T)$ basis used in the definition of C_p (2.3). Instead of considering a basis of $L^2(0,T)$, we only keep the first N functions $(g_1,\cdots,g_N)$ defined by (2.4) to build the chaos decomposition function C_p^N (2.5). Proposition 1 gives us explicit formulas for $\mathbb{E}_t(C_p^N F)$ and $D_t\mathbb{E}_t(C_p^N F)$. From (4.5), we build $(Y^{q,p,N},Z^{q,p,N})_q$ in the following way: $(Y^{0,p,N},Z^{0,p,N})=(0,0)$ and

$$\begin{aligned} Y_t^{q+1,p,N} &= \mathbb{E}_t\big(C_p^N(F^{q,p,N})\big) - \int_0^t f(s,Y_s^{q,p,N},Z_s^{q,p,N},[Y_s^{q,p,N}],[Z_s^{q,p,N}])ds, \\ Z_t^{q+1,p,N} &= D_t\mathbb{E}_t\big(C_p^N(F^{q,p,N})\big), \end{aligned} \tag{4.7}$$

where $F^{q,p,N}=\xi+\int_0^T f(s,Y_s^{q,p,N},Z_s^{q,p,N},[Y_s^{q,p,N}],[Z_s^{q,p,N}])ds$.

Equation (4.7) is tractable as soon as we know closed formulas for the coefficients d_k^n of the chaos decomposition of $\mathbb{E}_t\big(C_p^N(F^{q,p,N})\big)$ and $D_t\mathbb{E}_t\big(C_p^N(F^{q,p,N})\big)$ (see Proposition 1). When it is not the case, we need to use a Monte Carlo method to approximate these coefficients. The next section is devoted to this method.

4.1.4. *Monte Carlo simulations of the chaos decomposition.* Let F denote a r.v. of $L^2(\mathcal{F}_T)$. Practically, when we are not able to compute exactly $d_0=\mathbb{E}(F)$ and/or the coefficients $d_n^k = n!\mathbb{E}(FK_{n_1}(G_1)\cdots K_{n_N}(G_N))$ of the chaos decomposition (2.6)-(2.7) of F, we use Monte Carlo simulations to approximate them. Let $(F^m)_{1\le m\le M}$ be a M i.i.d. sample of F and $(G_1^m,\cdots,G_N^m)_{1\le m\le M}$ be a M i.i.d. sample of $(G_1,\cdots,G_N)$. We propose a method which consists in approximating the expectations $\mathbf{d}:=(d_0,(d_k^n)_{1\le k\le p,|n|=k})$ by empirical means $\widehat{\mathbf{d_M}}:=(\hat d_0,(\hat d_k^n)_{1\le k\le p,|n|=k})$ where

$$\hat d_0 := \frac{1}{M}\sum_{m=1}^M F^m, \qquad \hat d_k^n := \frac{n!}{M}\sum_{m=1}^M F^m K_{n_1}(G_1^m)\cdots K_{n_N}(G_N^m). \tag{4.8}$$

Definition 1. *Let F be a r.v. of $L^2(\mathcal{F}_T)$ and $(F^1,\cdots,F^M)$ be M identically distributed r.v. with the law of F. We denote $C_p^{N,M}((F^m)_{1\le m\le M})$ the following approximation of $C_p^N(F)$*

$$C_p^{N,M}((F^m)_{1\le m\le M}) = \hat d_0 + \sum_{k=1}^p \sum_{|n|=k} \hat d_k^n \prod_{1\le i\le N} K_{n_i}(G_i). \tag{4.9}$$

where $(\hat d_0,(\hat d_k^n)_{1\le k\le p,|n|=k})$ are defined by (4.8).

Remark 4. *When $(F^1,\cdots,F^M)$ is an independent and identically distributed sample of F, we adopt the short notation $C_p^{N,M}(F)$ to refer to $C_p^{N,M}((F^m)_{1\le m\le M})$. The notation $C_p^{N,M}((F^m)_{1\le m\le M})$ will only be used when the r.v. $F^1,\cdots,F^M$ are not independent. This will be the case in the next paragraph, when we introduce the particle system to approximate the law of (Y,Z) (see (4.12)).*

Before introducing the processes $(Y^{q+1,p,N,M},Z^{q+1,p,N,M})$, we define $\mathbb{E}_t(C_p^{N,M}((F^m)_m))$ and $D_t\mathbb{E}_t(C_p^{N,M}((F^m)_m))$, the conditional expectations obtained in Proposition 1 when $(d_0,d_k^n)_{1\le k\le p,|n|=k}$ are replaced by $(\hat d_0,\hat d_k^n)_{1\le k\le p,|n|=k}$, i.e.

$$\mathbb{E}_t(C_p^{N,M}((F^m)_m)) = \hat d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} \hat d_k^n \prod_{i<r} K_{n_i}(G_i)\times\left(\frac{t-\tilde t_{r-1}}{h}\right)^{(n_r)/2} K_{n_r}\left(\frac{B_t-B_{\tilde t_{r-1}}}{\sqrt{t-\tilde t_{r-1}}}\right),$$

$$D_t\mathbb{E}_t(C_p^{N,M}((F^m)_m)) = h^{-1/2}\sum_{k=1}^p \sum_{\substack{|n(r)|=k\\ n(r)>0}} \hat d_k^n \prod_{i<r} K_{n_i}(G_i)\times\left(\frac{t-\tilde t_{r-1}}{h}\right)^{(n_r-1)/2} K_{n_r-1}\left(\frac{B_t-B_{\tilde t_{r-1}}}{\sqrt{t-\tilde t_{r-1}}}\right).$$

Remark 5. *As said in* [BL14, Remark 3.2], *when M samples of $C_p^{N,M}((F^m)_m)$ are needed, we can either use the same samples as the ones used to compute $\widehat{d_0}$ and $\widehat{d_k^n}$: $(C_p^{N,M}((F^m)_m))_m = \widehat{d_0} + \sum_{k=1}^{p} \sum_{|n|=k} \widehat{d_k^n} \prod_{1\leq i\leq N} K_{n_i}(G_i^m)$, or use new ones. In the first case, we only require M samples $(G_1, \cdots, G_N)$. We built $F^1, \cdots, F^M$ from these samples. The coefficients $\widehat{d_k^n}$ and $\widehat{d_0}$ are not independent of $\prod_{1\leq i\leq N} K_{n_i}(G_i^m)$. The notation $\mathbb{E}_t(C_p^{N,M}((F^m)_m))$ introduced above cannot be linked to $\mathbb{E}\left(C_p^{N,M}((F^m)_m)|\mathcal{F}_t\right)$. In the second case, the coefficients $\widehat{d_k^n}$ and $\widehat{d_0}$ are independent of $\prod_{1\leq i\leq N} K_{n_i}(G_i^m)$ and we have $\mathbb{E}_t\left(C_p^{N,M}((F^m)_m)\right) = \mathbb{E}\left(C_p^{N,M}((F^m)_m)|\mathcal{F}_t\right)$. This second approach requires $2M$ samples of F and $(G_1, \cdots, G_N)$ and its variance increases with N. Practically, we use the first technique.*

Let us now introduce the couple of processes $(Y^{q+1,p,N,M}, Z^{q+1,p,N,M})$, which corresponds to the approximation of $(Y^{q+1,p,N}, Z^{q+1,p,N})$ when we use $C_p^{N,M}$ instead of C_p^N, that is, when we use a Monte Carlo procedure to compute the coefficients d_k^n.

$$\begin{aligned} Y_t^{q+1,p,N,M} &= \mathbb{E}_t\big(C_p^{N,M}(F^{q,p,N,M})\big) - \int_0^t f(\theta_s^{q,p,N,M})ds, \\ Z_t^{q+1,p,N,M} &= D_t\mathbb{E}_t\big(C_p^{N,M}(F^{q,p,N,M})\big), \end{aligned} \tag{4.10}$$

where $F^{q,p,N,M} := \xi + \int_0^T f(\theta_s^{q,p,N,M})ds$ and $\theta_s^{q,p,N,M} := (s, Y_s^{q,p,N,M}, Z_s^{q,p,N,M}, [Y_s^{q,p,N,M}], [Z_s^{q,p,N,M}])$.

4.1.5. *Particle system.* In this section, we introduce an interacting particle system associated to (4.10) to approximate the law of $Y_s^{q,p,N,M}$ and $Z_s^{q,p,N,M}$. Indeed we replace a one single stochastic differential equation with unknown processes $Y_s^{q,p,N,M}$ and $Z_s^{q,p,N,M}$, with a system of M ordinary stochastic differential equations, whose solution consists in a system of particles $(Y_s^{q,p,N,M,m}, Z_s^{q,p,N,M,m})_{1\leq m\leq M}$, replacing the law of the processes $Y_s^{q,p,N,M}$ and $Z_s^{q,p,N,M}$ by the empirical mean law

$$\begin{aligned} [Y_s^{q,p,N,M,m}]^M &= \frac{1}{M}\sum_{m=1}^{M} \delta_{Y_s^{q,p,N,M,m}}, \\ [Z_s^{q,p,N,M,m}]^M &= \frac{1}{M}\sum_{m=1}^{M} \delta_{Z_s^{q,p,N,M,m}}. \end{aligned}$$

Our candidates are the particles:

$$\begin{aligned} Y_t^{q+1,p,N,M,m} &= \mathbb{E}_t\big(C_p^{N,M}((F^{q,p,N,M,m})_m)\big) \\ &\quad - \int_0^t f(s, Y_s^{q,p,N,M,m}, Z_s^{q,p,N,M,m}, [Y_s^{q,p,N,M,m}]^M, [Z_s^{q,p,N,M,m}]^M)ds, \\ Z_t^{q+1,p,N,M,m} &= D_t\mathbb{E}_t\big(C_p^{N,M}((F^{q,p,N,M,m})_m)\big), \end{aligned} \tag{4.11}$$

where, for all $m \in \{1, \cdots, M\}$,

$$F^{q,p,N,M,m} = \xi + \int_0^T f(s, Y_s^{q,p,N,M,m}, Z_s^{q,p,N,M,m}, [Y_s^{q,p,N,M,m}]^M, [Z_s^{q,p,N,M,m}]^M)ds. \tag{4.12}$$

Remark 6. *Looking at* (4.11) *and* (4.12), *we notice that we use the M particles to compute $(\hat{d}_0, (\hat{d}_k^n)_{1\leq k\leq p, |n|=k})$. Taking the same M drawings for the Monte Carlo simulations and the particle system has two impacts, one positive and one negative :*

- *the algorithm is not more costly than the one solving standard BSDEs,*
- *the samples $(F^{q,p,N,M,m})_{1\leq m\leq M}$ being not independant, the speed of convergence of the algorithm will be badly impacted (see Remark* (9)*).*

4.2. **Pseudo-code of the algorithm.** We aim at computing M trajectories of an approximation of (Y,Z) on the grid $\mathcal{T}=\{\tilde{t}_i = i\frac{T}{N}, i=0,\cdots,N\}$. Starting from $(Y^{0,p,N,M,m}, Z^{0,p,N,M,m}) = (0,0)$, (4.11) enables to get $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ for each Picard's iteration q on $\mathcal{T}$. Practically, we discretize the integral $\int_0^t f(\theta_s^{q,p,N,M,m})ds$ which leads to approximated values of $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ computed on a grid.

Let us introduce $(\tilde{Y}^{q+1,p,N,M,m}_{\tilde{t}_i}, \tilde{Z}^{q+1,p,N,M,m}_{\tilde{t}_i})_{1\le i\le N}$, defined by $(\tilde{Y}^{0,p,N,M,m}, \tilde{Z}^{0,p,N,M,m}) = (0,0)$ and for all $q\geq 0$

$$\begin{aligned}\tilde{Y}^{q+1,p,N,M,m}_{\tilde{t}_i} &= \mathbb{E}_{\tilde{t}_i}\big(C_p^{N,M}((\tilde{F}^{q,p,N,M,m})_m)\big) - h\sum_{j=1}^{i} f(\tilde{t}_j, \tilde{Y}^{q,p,N,M,m}_{\tilde{t}_j}, \tilde{Z}^{q,p,N,M,m}_{\tilde{t}_j}, [\tilde{Y}^{q,p,N,M,m}_{\tilde{t}_j}]^M,\\ &\qquad [\tilde{Z}^{q,p,N,M,m}_{\tilde{t}_j}]^M),\\ \tilde{Z}^{q+1,p,N,M,m}_{\tilde{t}_i} &= D_{\tilde{t}_i}\mathbb{E}_{\tilde{t}_i}\big(C_p^{N,M}((\tilde{F}^{q,p,N,M,m})_m)\big),\end{aligned} \tag{4.13}$$

where $\tilde{F}^{q,p,N,M,m} := \xi + h\sum_{i=1}^N f(\tilde{t}_i, \tilde{Y}^{q,p,N,M,m}_{\tilde{t}_i}, \tilde{Z}^{q,p,N,M,m}_{\tilde{t}_i}, [\tilde{Y}^{q,p,N,M,m}_{\tilde{t}_i}]^M, [\tilde{Z}^{q,p,N,M,m}_{\tilde{t}_i}]^M)$. Here is the notation we use in the algorithm:

- d: dimension of the Brownian motion;
- q: index of Picard's iteration;
- K_{it} : number of Picard's iterations;
- M: number of Monte Carlo samples;
- N: number of time steps used for the discretization of Y and Z;
- p: order of the chaos decomposition;
- $\mathbf{Y}^q \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $\tilde{Y}^{q,p,N,M,m}$ computed on the grid $\mathcal{T}$;
- for all $l\in\{1,\cdots,d\}$, $(\mathbf{Z}^q)_l \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $(\tilde{Z}^{q,p,N,M,m})_l$ computed on the grid $\mathcal{T}$.

Since ξ in $L^2(\mathcal{F}_T)$, ξ can be written as a measurable function of the Brownian path. Then one gets one sample of ξ from one sample of $(G_1,\cdots,G_N)$ (where G_i represents $\frac{B_{\tilde{t}_i}-B_{\tilde{t}_{i-1}}}{\sqrt{h}}$).
For the sake of clarity, we detail the algorithm for $d=1$.

Algorithm 1 Iterative algorithm

Pick at random $N \times M$ values of standard Gaussian r.v. stored in $\mathbf{G}$.
Using $\mathbf{G}$, compute $(\xi^m)_{0 \le m \le M-1}$.
$\mathbf{Y}^0 \equiv 0$, $\mathbf{Z}^0 \equiv 0$.
for $q = 0 : K_{it} - 1$ **do**
  for $m = 0 : M-1$ **do**
    Compute $F^{q,m} = \xi^m + h \sum_{i=1}^N f(\tilde{t}_i, (\mathbf{Y}^q)_{i,m}, (\mathbf{Z}^q)_{i,m}, [(\mathbf{Y}^q)_i]^M, [(\mathbf{Z}^q)_i]^M)$
  end for
  Compute the vector $\mathbf{d} := (\hat{d}_0, (\hat{d}_k^n)_{1 \le k \le p, |n|=k})$ of the chaos decomposition of $(F^{q,m})_m$
  $\hat{d}_0 := \frac{1}{M} \sum_{m=0}^{M-1} F^{q,m}$, $\hat{d}_k^n := \frac{n!}{M} \sum_{m=0}^{M-1} F^{q,m} K_{n_1}(G_1^m) \cdots K_{n_N}(G_N^m)$
  for $j = 1 : N$ **do**
    for $m = 0 : M-1$ **do**
      Compute $(\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m)))_m$, $(D_{\tilde{t}_j}(\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m))))_m$
      $(\mathbf{Y}^{q+1})_{j,m} = (\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m)))_m - h \sum_{i=1}^j f(\tilde{t}_i, (\mathbf{Y}^q)_{i,m}, (\mathbf{Z}^q)_{i,m}, [(\mathbf{Y}^q)_i]^M, [(\mathbf{Z}^q)_i]^M)$
      $(\mathbf{Z}^{q+1})_{j,m} = (D_{\tilde{t}_j}(\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m))))_m$
    end for
    Compute the empirical distribution of the particles $(\mathbf{Y}^{q+1})_{:,m}$ and $(\mathbf{Z}^{q+1})_{:,m}$
    $[(\mathbf{Y}^{q+1})_j]^M = M^{-1} \sum_{m=1}^M \delta_{(\mathbf{Y}^{q+1})_{j,m}}$, $[(\mathbf{Z}^{q+1})_j]^M = M^{-1} \sum_{m=1}^M \delta_{(\mathbf{Z}^{q+1})_{j,m}}$
  end for
end for
Return $(\mathbf{Y}^{K_{it}})_{0,:} = \hat{d}_0$ and $(\mathbf{Z}^{K_{it}})_{0,:} = \frac{1}{\sqrt{h}} \hat{d}_1^{e_1}$

5. Convergence results

We aim at bounding the error between (Y, Z) — the solution of (1.2) — and $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ defined by (4.11). Before stating the main result of the paper, we recall some hypotheses introduced in [BL14].
In the following, $(t_1, \cdots, t_n)$ and $(s_1, \cdots, s_n)$ denote two vectors such that

$$0 \le t_1 \le \cdots \le t_n \le T, \quad 0 \le s_1 \le \cdots \le s_n \le T \quad \textit{and} \quad \forall i, s_i \le t_i.$$

Hypothesis 2. *(Hypothesis $\mathcal{H}_m$). Let $m \in N^*$. We say that F satisfies Hypothesis $\mathcal{H}_m$ if F satisfies the two following hypotheses:*

- $\mathcal{H}_m^1$*: $\forall j \ge 2$, $F \in \mathcal{D}^{m,j}$, that is $\|F\|_{m,j}^j < \infty$;*
- $\mathcal{H}_m^2$*: $\forall j \ge 2$, $\forall i \in \{1, \cdots, m\}$, $\forall l_0 \le i-1$, $\forall l_1 \le m-i$, $\forall l \in \{1, \cdots, d\}$ and for all multi-indices α_0 and α_1 such that $|\alpha_0| = l_0$ and $|\alpha_1| = l_1 + 1$, there exist two positive constants β_F and k_l^F such that*

$$\sup_{t_1 \le \cdots \le t_{l_0}} \sup_{s_{i+1} \le \cdots \le s_{i+l_1}} \mathbb{E}\big(|D^{\alpha_0}_{t_1, \cdots, t_{l_0}}(D^{\alpha_1}_{t_i, s_{i+1}, \cdots, s_{i+l_1}} F - D^{\alpha_1}_{s_i, \cdots, s_{i+l_1}} F)|^j\big) \le k_l^F(j)(t_i - s_i)^{j\beta_F},$$

where $l = l_0 + l_1 + 1$. In the following, we denote $K_m^F(j) = \sup_{l \le m} k_l^F(j)$.

Remark 7. *If F satisfies $\mathcal{H}_m^2$, for all multi-index α such that $|\alpha| = l$, we have*

$$|\mathbb{E}(D^\alpha_{t_1, \cdots, t_l} F) - \mathbb{E}(D^\alpha_{s_1, \cdots, s_l} F)| \le K_l^F\big((t_1 - s_1)^{\beta_F} + \cdots + (t_l - s_l)^{\beta_F}\big), \tag{5.1}$$

where K_l^F is a constant.

Hypothesis 3. *(Hypothesis $\mathcal{H}_{p,N}^3$). Let $(p, N) \in \mathbb{N}^2$. We say that an r.v. F satisfies $\mathcal{H}_{p,N}^3$ if*

$$V_{p,N}(F) := \mathbb{V}(F) + \sum_{k=1}^p \sum_{|n|=k} n! \mathbb{V}\left(F \prod_{i=1}^N K_{n_i}(G_i)\right) < \infty.$$

Remark 8. *If F is bounded by K, we get $V_{p,N}(F) \leq K^2 \sum_{k=0}^{p} \binom{k}{N}$. Then every bounded r.v. satisfies $\mathcal{H}^3_{p,N}$.*
This remark ensues from $\mathbb{E}(\prod_{i=1}^N K^2_{n_i}(G_i)) = \frac{1}{n!}$.

Theorem 2. *Assume that ξ satisfies $\mathcal{H}_{p+q}$ and there exists a real $r > 4$ s.t. $\xi \in (\mathbb{L}^r \cap \mathcal{D}^{1,r}) \subset \mathcal{H}^3_{p,N}$ and $f \in C_b^{0,p+q,p+q,0,0}$. We have*

$$\|(Y - Y^{q,p,N,M,m}, Z - Z^{q,p,N,M,m})\|^2_{L^2} \leq \frac{A_0}{2^q} + \frac{A_1(q,p)}{(p+1)!} + A_2(q,p)\left(\frac{T}{N}\right)^{2\beta_\xi \wedge 1} + \frac{A_3(q,p,N)}{M} + \frac{A_4(q,p,N)}{M^{(1/2^{(q+2)})}},$$

where A_0 is given in Section 5.1, A_1 is given in Proposition 3, A_2 is given in Proposition 4, A_3 is given in Proposition 5 and A_4 is given in Proposition 6.
If $f \in C_b^{0,\infty,\infty}$ and ξ satisfies $\mathcal{H}_\infty$ and $\mathcal{H}^3_{\infty,\infty}$, we obtain that, for each $m \in 1, \cdots, M$,

$$\lim_{q\to\infty} \lim_{p\to\infty} \lim_{N\to\infty} \lim_{M\to\infty} \|(Y - Y^{q,p,N,M,m}, Z - Z^{q,p,N,M,m})\|^2_{L^2} = 0.$$

Remark 9. *Compared to* [BL14, Theorem 4.6] *we notice that the additional term $\frac{A_4(q,p,N)}{M^{(1/2^{(q+2)})}}$ appears in the error bound. This term corresponds to the error approximation due to the particle system. It is clearly worse that $\frac{A_3(q,p,N)}{M}$ which corresponds to the error due to the Monte Carlo approximation. As we will see in Lemma 7, introducing some dependency between M identically distributed r.v. gives a worse control of the error $C_p^N - C_p^{N,M}$ than the one obtained when considering i.i.d. r.v. (see Lemma 6).*

Proof of Theorem 2. We split the error into 5 terms:

(1) Picard's iterations: $\varepsilon^q = \|(Y - Y^q, Z - Z^q)\|^2_{L^2}$, where (Y_q, Z_q) is defined by (4.1);
(2) the truncation of the chaos decomposition: $\varepsilon^{q,p} = \|(Y^q - Y^{q,p}, Z^q - Z^{q,p})\|^2_{L^2}$, where $(Y_{q,p}, Z_{q,p})$ is defined by (4.5);
(3) the truncation of the $L^2(0,T)$ basis: $\varepsilon^{q,p,N} = \|(Y^{q,p} - Y^{q,p,N}, Z^{q,p} - Z^{q,p,N})\|^2_{L^2}$, where $(Y_{q,p,N}, Z_{q,p,N})$ is defined by (4.7);
(4) the Monte Carlo approximation to compute the expectations: $\varepsilon^{q,p,N,M} = \|(Y^{q,p,N} - Y^{q,p,N,M}, Z^{q,p,N} - Z^{q,p,N,M})\|^2_{L^2}$, where $(Y_{q,p,N,M}, Z_{q,p,N,M})$ is defined by (4.10).
(5) the particle system: $\varepsilon^{q,p,N,M,m} = \|(Y^{q,p,N,M} - Y^{q,p,N,M,m}, Z^{q,p,N,M} - Z^{q,p,N,M,m})\|^2_{L^2}$, where $(Y_{q,p,N,M,m}, Z_{q,p,N,M,m})$ is defined by (4.11).

We have

$$\|(Y - Y^{q,p,N,M,m}, Z - Z^{q,p,N,M,m})\|^2_{L^2} \leq 5(\varepsilon^q + \varepsilon^{q,p} + \varepsilon^{q,p,N} + \varepsilon^{q,p,N,M} + \varepsilon^{q,p,N,M,m}).$$

It remains to combine (5.2), Propositions 3, 4, 5 and 6 to get the first result. The second one is straightforward. □

5.1. **Picard's iterations.** From Remark 3, we know that under Hypothesis 1, the sequence $(Y^q, Z^q)_q$ defined by (4.1) converges to (Y, Z) $d\mathbb{P} \times dt$ a.s. and in $S^2_T(\mathbb{R}) \times H^2_T(\mathbb{R}^d)$. Moreover, we have

$$\varepsilon^q := \|(Y - Y^q, Z - Z^q)\|^2_{L^2} \leq \frac{A_0}{2^q}, \tag{5.2}$$

where A_0 depends on T, $\|\xi\|^2$ and on $\|f(., 0, 0, [0], [0])\|^2_{L^2_{(0,T)}}$.

5.2. **Error due to the truncation of the chaos decomposition.** We assume that the integrals are computed exactly, as well as expectations. The error is only due to the truncation of the chaos decomposition C_p introduced in [Equation(2.3),[BL14]].
For the sequel, we also need the following Lemma. We refer to [BL14, Appendix A.2] for a proof.

Lemma 4. *Assume that ξ satisfies $\mathcal{H}^1_{m+q}$ and $f \in C_b^{0,m+q,m+q,0,0}$. Then $\forall q' \leq q$, $\forall p \in \mathbb{N}$, $(Y^{q'}, Z^{q'})$ and $(Y^{q',p}, Z^{q',p})$ belong to $\mathcal{S}^{m,\infty}$. Moreover*

$$\|(Y^q, Z^q)\|^j_{m,j} + \|(Y^{q,p}, Z^{q,p})\|^j_{m,j} \leq C(\|\xi\|_{m+q,((m+q-1)!/m!)j}, (\|\partial^k_{sp} f\|_\infty)_{k \leq m+q}),$$

where C is a constant depending on $\|\xi\|_{m+q,((m+q-1)!/m!)j}$ and on $(\|\partial^k_{sp} f\|_\infty)_{k\leq m+q}$.

Proposition 3. *Let $m \leq p+1$. Assume that ξ satisfies $\mathcal{H}^1_{m+q}$ and $f \in C_b^{0,m+q,m+q,0,0}$. We recall $\varepsilon^{q,p} = \|(Y^q - Y^{q,p}, Z^q - Z^{q,p})\|^2_{L^2}$. We get*

$$\varepsilon^{q+1,p} \leq C_1 T(T+1) L_f^2 \varepsilon^{q,p} + \frac{K_1(q,m)}{(p+1)\cdots(p+2-m)}, \tag{5.3}$$

where C_1 is a scalar and $K_1(q,m)$ depends on T, m, $\|\xi\|_{m+q,2(m+q-1)!/(m-1)!}$ and on $(\|\partial^k_{sp} f\|_\infty)_{1\leq k\leq m+q}$. Since $\varepsilon^{0,p} = 0$, we deduce from (5.3) that $\varepsilon^{q,p} \leq \frac{A_1(q,m)}{(p+1)\cdots(p+2-m)}$ where $A_1(q,m) := \frac{(C_1 T(T+1)L_f^2)^q - 1}{C_1 T(T+1) L_f^2 - 1} K_1(q,m)$. Then, $(Y^{q,p}, Z^{q,p})$ converges to (Y^q, Z^q) when p tends to ∞ in $\|(.,.)\|_{L^2}$.

Remark 10. *We deduce from Proposition 3 that for all T and L_f, we have $\lim_{q\to\infty} \varepsilon^{q,p} = 0$. When $C_1 T(T+1)L_f^2 < 1$, that is, for T small enough, we also get $\lim_{p\to\infty}\lim_{q\to\infty} \varepsilon^{q,p} = 0$.*

Proof of Proposition 3. For the sake of clearness, we assume $d = 1$. In the following, one notes $\Delta Y_t^{q,p} := Y_t^{q,p} - Y_t^q$, $\Delta Z_t^{q,p} := Z_t^{q,p} - Z_t^q$ and $\Delta f_t^{q,p} := f(t, Y_t^{q,p}, Z_t^{q,p}, [Y_t^{q,p}], [Z_t^{q,p}]) - f(t, Y_t^q, Z_t^q, [Y_t^q], [Z_t^q])$. First, we deal with $\mathbb{E}(\sup_{0\leq t\leq T} |\Delta Y_t^{q+1,p}|^2)$. From (4.4) and (4.5) we get

$$\begin{aligned}
\Delta Y_t^{q+1,p} &= \mathbb{E}_t\big(C_p(F^{q,p}) - F^q\big) - \int_0^t \Delta f_s^{q,p} ds \\
&= \mathbb{E}_t\big(C_p(\xi) - \xi\big) \\
&\quad + \mathbb{E}_t\bigg(C_p\bigg(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}])ds\bigg) - \int_0^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])ds\bigg) \\
&\quad - \int_0^t \Delta f_s^{q,p} ds.
\end{aligned}$$

We introduce $\pm C_p(\int_0^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])ds)$ in the second conditional expectation.
This leads to

$$\begin{aligned}
\Delta Y_t^{q+1,p} &= \mathbb{E}_t\big(C_p(\xi) - \xi\big) + \mathbb{E}_t\bigg(C_p\bigg(\int_0^T \Delta f_s^{q,p} ds\bigg)\bigg) \\
&\quad + \mathbb{E}_t\bigg(\int_0^T C_p(f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])) - f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])ds\bigg) \\
&\quad - \int_0^t \Delta f_s^{q,p} ds.
\end{aligned}$$

where we have used the second property of Lemma 2 to rewrite the third term.
From the previous equation, we bound $\mathbb{E}(\sup_{0\leq t\leq T} |\Delta Y_t^{q+1,p}|^2)$ by using Doob's inequality, the

property of the Wasserstein-2 distance (see (1.3)) and the fact that f is Lipschitz:

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}|\Delta Y_t^{q+1,p}|^2\Big) &\le 16\mathbb{E}\big(|C_p(\xi)-\xi|^2\big)+16\mathbb{E}\Big(\Big|C_p\Big(\int_0^T \Delta f_s^{q,p}ds\Big)\Big|^2\Big|)\\ &\quad+16T\int_0^T \mathbb{E}(|C_p(f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q]))-f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])|^2)ds\\ &\quad+16TL_f^2\int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p}|^2+|\Delta Z_s^{q,p}|^2+\mathcal{W}_2([Y_s^{q,p}],[Y_s^q])^2+\mathcal{W}_2([Z_s^{q,p}],[Z_s^q])^2\big)ds\\ &\le 16\mathbb{E}\big(|C_p(\xi)-\xi|^2\big)+16\mathbb{E}\Big(\Big|C_p\Big(\int_0^T \Delta f_s^{q,p}ds\Big)\Big|^2\Big)\\ &\quad+16T\int_0^T \mathbb{E}(|C_p(f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q]))-f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])|^2)ds\\ &\quad+32TL_f^2\int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p}|^2+|\Delta Z_s^{q,p}|^2\big)ds.\end{aligned}$$

To bound the second expectation of the previous inequality, we use the first property of Lemma 2 and the Lipschitz property of f. Then we bring together this term with the last one to get

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}|\Delta Y_t^{q+1,p}|^2\Big) &\le 16\mathbb{E}\big(|C_p(\xi)-\xi|^2\big)\\ &\quad+16T\int_0^T \mathbb{E}(|C_p(f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q]))-f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])|^2)ds\\ &\quad+160TL_f^2\int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p}|^2+|\Delta Z_s^{q,p}|^2\big)ds.\end{aligned} \tag{5.4}$$

Let us now study the upper bound $\mathbb{E}(\int_0^T|\Delta Z_t^{q+1,p}|^2ds)$. To do so, we use the Itô isometry $\mathbb{E}(\int_0^T|\Delta Z_t^{q+1,p}|^2ds)=\mathbb{E}((\int_0^T \Delta Z_t^{q+1,p}dB_s)^2)$. Using the definitions (4.4)–(4.6) and the Clark–Ocone Theorem leads to

$$\begin{aligned}\int_0^T \Delta Z_s^{q+1,p}dB_s &= F^q-\mathbb{E}\big(F^q\big)-\big(C_p(F^{q,p})-\mathbb{E}\big(C_p(F^{q,p})\big)\big)\\ &= Y_T^{q+1}+\int_0^T f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])ds-Y_0^{q+1}\\ &\quad-\Big(Y_T^{q+1,p}+\int_0^T f(s,Y_s^{q,p},Z_s^{q,p},[Y_s^{q,p}],[Z_s^{q,p}])ds-Y_0^{q+1,p}\Big).\end{aligned}$$

Rearranging this summation makes $\Delta Y_T^{q+1,p}-(\Delta Y_0^{q+1,p})$ appear. We get

$$\begin{aligned}\mathbb{E}\Big(\int_0^T|\Delta Z_s^{q+1,p}|^2ds\Big) &\le 6\mathbb{E}\Big(\sup_{0\le t\le T}|\Delta Y_t^{q+1,p}|^2\Big)\\ &\quad+24TL_f^2\int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p}|^2+|\Delta Z_s^{q,p}|^2\big)ds.\end{aligned} \tag{5.5}$$

Since $\int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p}|^2+|\Delta Z_s^{q,p}|^2\big)ds\le (T+1)\varepsilon^{q,p}$, by computing $7\times$(5.4) + (5.5) we obtain

$$\begin{aligned}\varepsilon^{q+1,p} &\le 112\mathbb{E}\big(|C_p(\xi)-\xi|^2\big)\\ &\quad+112T\int_0^T \mathbb{E}(|C_p(f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q]))-f(s,Y_s^q,Z_s^q,[Y_s^q],[Z_s^q])|^2)ds\\ &\quad+1144T(T+1)L_f^2\varepsilon^{q,p}.\end{aligned}$$

Since ξ and $f(s,Y^q_s,Z^q_s,[Y^q_s],[Z^q_s])$ belong to $\mathbb{D}^{m,2}$ (ξ satisfies $\mathcal{H}^1_{m+q}$, $f\in C_b^{0,m+q,m+q}$ and $(Y^q,Z^q)\in\mathcal{S}^{m,\infty}$ (see Lemma 4)), Lemma 1 gives

$$\begin{aligned}\varepsilon^{q+1,p}\leq&\frac{112}{(p+1)\cdots(p+2-m)}\|D^m\xi\|^2_{L^2(\Omega\times[0,T]^m)}\\&+\frac{112T}{(p+1)\cdots(p+2-m)}\left(\int_0^T\|D^mf(s,Y^q_s,Z^q_s,[Y^q_s],[Z^q_s])\|^2_{L^2(\Omega\times[0,T]^m)}ds\right)\\&+1144T(T+1)L_f^2\varepsilon^{q,p}.\end{aligned}$$

Since $\int_0^T\|D^mf(s,Y^q_s,Z^q_s,[Y^q_s],[Z^q_s])\|^2_{L^2(\Omega\times[0,T]^m)}ds$ is bounded by $C\big(T,m,(\|\partial^k_{sp}f\|_\infty)_{k\leq m},\|(Y^q,Z^q)\|^{2m}_{m,2m}\big)$, Lemma 4 gives the result. □

5.3. **Error due to the truncation of the basis.** We are now interested in bounding the error between $(Y^{q,p},Z^{q,p})$ (defined by (4.5)) and $(Y^{q,p,N},Z^{q,p,N})$ (defined by (4.7)).
Before giving an upper bound for the error, we measure the error between C_p and C_p^N for a r.v. satisfying (5.1) when $m=p$.

Remark 11 (Remark 4.13, [BL14]). *Let ξ satisfy $\mathcal{H}_p$ and $f\in C_b^{0,p,p,0,0}$. Then, for all integer $q\geq 0$, $I_{q,p}:=\int_0^T f(s,Y^{q,p}_s,Z^{q,p}_s,[Y^{q,p}_s],[Z^{q,p}_s])ds$ satisfies (5.1); that is for all multi-index α such that $|\alpha|=r$, we have*

$$|\mathbb{E}(D^\alpha_{t_1,\cdots,t_r}I_{q,p})-\mathbb{E}(D^\alpha_{s_1,\cdots,s_r}I_{q,p})|\leq K_r^{I_{q,p}}\big((t_1-s_1)^{\beta_{I_{q,p}}}+\cdots+(t_r-s_r)^{\beta_{I_{q,p}}}\big),$$

where $\beta_{I_{q,p}}=\frac{1}{2}\wedge\beta_\xi$ and $K_r^{I_{q,p}}$ depends on K_r^ξ, $\|\xi\|_{p,1}$ T and on $(\|\partial^k_{sp}\|_\infty)_{1\leq k\leq p}$.

Lemma 5 (Lemma 4.14, [BL14]). *Let F denote a r.v. in $L^2(\mathcal{F}_T)$ satisfying (5.1) for $m=p$. We have*

$$\mathbb{E}(|(C_p^N-C_p)(F)|^2)\leq(K_p^F)^2\left(\frac{T}{N}\right)^{2\beta_F}\sum_{i=1}^p i^2\frac{T^i}{i!}\leq(K_p^F)^2\left(\frac{T}{N}\right)^{2\beta_F}T(1+T)e^T,$$

where K_p^F and β_F are defined in Hypothesis 2.

Proposition 4. *Assume that ξ satisfies $\mathcal{H}_p$ and $f\in C_b^{0,p,p,0,0}$. We recall $\varepsilon^{q,p,N}:=\|(Y^{q,p}-Y^{q,p,N},Z^{q,p}-Z^{q,p,N})\|^2_{L^2}$. We get*

$$\varepsilon^{q+1,p,N}\leq C_2T(T+1)L_f^2\varepsilon^{q,p,N}+K_2(q,p)T(T+1)e^T\left(\frac{T}{N}\right)^{1\wedge2\beta_\xi},\tag{5.6}$$

where C_2 is a scalar and $K_2(q,p)$ depends on K_p^ξ, T, $\|\xi\|_{p,1}$ and on $(\|\partial^k_{sp}f\|_\infty)_{1\leq k\leq p}$. Since $\varepsilon^{0,p,N}=0$, we deduce from (5.6) that $\varepsilon^{q,p,N}\leq A_2(q,p)(\frac{T}{N})^{1\wedge2\beta_\xi}$ where $A_2(q,p):=K_2(q,p)T(T+1)e^T\frac{(C_2T(T+1)L_f^2)^q-1}{C_2T(T+1)L_f^2-1}$. Then, $(Y^{q,p,N},Z^{q,p,N})$ converges to $(Y^{q,p},Z^{q,p})$ when N tends to ∞ in $\|(.,.)\|_{L^2}$.

Proof of Proposition 4. For the sake of clarity, we assume $d=1$. In the following, one notes $\Delta Y^{q,p,N}_t:=Y^{q,p,N}_t-Y^{q,p}_t$, $\Delta Z^{q,p,N}_t:=Z^{q,p,N}_t-Z^{q,p}_t$ and $\Delta f^{q,p,N}_t:=f(t,Y^{q,p,N}_t,Z^{q,p,N}_t,[Y^{q,p,N}_t],[Z^{q,p,N}_t])-f(t,Y^{q,p}_t,Z^{q,p}_t,[Y^{q,p}_t],[Z^{q,p}_t])$. First, we deal with $\mathbb{E}(\sup_{0\leq t\leq T}|\Delta Y^{q+1,p,N}_t|^2)$. From (4.5) and (4.7) we get

$$\Delta Y^{q+1,p,N}_t=\mathbb{E}_t\big(C_p(F^{q,p,N})-F^{q,p}\big)-\int_0^t\Delta f^{q,p,N}_sds.$$

Following the same steps as in the proof of Proposition 3, we get

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T}|\Delta Y_t^{q+1,p,N}|^2\Big) \le\ & 16\mathbb{E}\big(|C_p^N(\xi)-C_p(\xi)|^2\big)\\
&+16\mathbb{E}\bigg(\bigg|(C_p^N-C_p)\bigg(\int_0^T f(s,Y_s^{q,p},Z_s^{q,p},[Y_s^{q,p}],[Z_s^{q,p}])ds\bigg)\bigg|^2\bigg) \quad (5.7)\\
&+160L_f^2\int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p,N}|^2+|\Delta Z_s^{q,p,N}|^2\big)ds.
\end{aligned}
$$

Let us now study the upper bound $\mathbb{E}(\int_0^T|\Delta Z_s^{q+1,p,N}|^2ds)$. Following the same steps as in the proof of Proposition 3, we get

$$
\begin{aligned}
\mathbb{E}\bigg(\int_0^T|\Delta Z_s^{q+1,p,N}|^2ds\bigg) \le\ & 6\mathbb{E}\Big(\sup_{0\le t\le T}|\Delta Y_t^{q+1,p,N}|^2\Big)\\
&+24TL_f^2\int_0^T\mathbb{E}\big(|\Delta Y_s^{q,p,N}|^2+|\Delta Z_s^{q,p,N}|^2\big)ds.
\end{aligned} \tag{5.8}
$$

Adding $7\times$ (5.7) and (5.8) we obtain

$$
\begin{aligned}
\varepsilon^{q+1,p,N} \le\ & 112\mathbb{E}\big(|(C_p^N-C_p)(\xi)|^2\big)\\
&+112\mathbb{E}\bigg(\bigg|(C_p^N-C_p)\bigg(\int_0^T f(s,Y_s^{q,p},Z_s^{q,p},[Y_s^{q,p}],[Z_s^{q,p}])ds\bigg)\bigg|^2\bigg)\\
&+1144T(T+1)L_f^2\varepsilon^{q,p,N}.
\end{aligned}
$$

Since ξ and $I_{q,p}:=f(s,Y_s^{q,p},Z_s^{q,p},[Y_s^{q,p}],[Z_s^{q,p}])$ satisfy (5.1) (see Remarks 8 and 11), Lemma 5 gives

$$
\begin{aligned}
\varepsilon^{q+1,p,N} \le\ & 112\bigg(\frac{T}{N}\bigg)^{1\wedge 2\beta_\xi}T(T+1)e^T((K_p^\xi)^2+(K_p^{I_{q,p}})^2)\\
&+1144T(T+1)L_f^2\varepsilon^{q,p,N}.
\end{aligned}
$$

and (5.6) follows. □

5.4. **Error due to the Monte Carlo approximation.** We are now interested in bounding the error between $(Y^{q,p,N},Z^{q,p,N})$ (defined by (4.7)) and $(Y^{q,p,N,M},Z^{q,p,N,M})$ (defined by (4.10)). In this section, we assume that the coefficients $\hat{d}_k^n$ are independent of the vector $(G_1,\cdots,G_N)$, which corresponds to the second approach proposed in (Remark 3.2,[BL14]).
Before giving an upper bound for the error, we measure the error between C_p^N and $C_p^{N,M}$ for a r.v. satisfying $\mathcal{H}^3_{p,N}$ (see Hypothesis (3)). The following Lemma deals with $C_p^{N,M}(F)$, i.e. the case of i.i.d. samples of F (see Definition 1). The general case will be stated in Lemma 7.

Lemma 6. *Let F be a r.v. satisfying Hypothesis $\mathcal{H}^3_{p,N}$. We have*

$$\mathbb{E}(|(C_p^N-C_p^{N,M})(F)|^2)=\frac{1}{M}V_{p,N}(F),$$

and

$$\mathbb{E}(|C_p^{N,M}(F)|^2)\le\mathbb{E}(|F|^2)+\frac{1}{M}V_{p,N}(F).$$

We refer to [BL14, Appendix A.5] for the proof of the lemma.

Proposition 5. *Let ξ satisfy Hypothesis $\mathcal{H}^3_{p,N}$ and f is a bounded function. Let $\varepsilon^{q,p,N,M} := \|(Y^{q,p,N} - Y^{q,p,N,M}, Z^{q,p,N} - Z^{q,p,N,M})\|^2_{L^2}$. We get*

$$\varepsilon^{q+1,p,N,M} \leq C_3 T(T+1) L_f^2 \varepsilon^{q,p,N,M} + \frac{K_3(p,N)}{M}, \tag{5.9}$$

where C_3 is a scalar and $K_3(p,N) := 724(V_{p,N}(\xi) + T^2\|f\|^2_\infty \sum_{k=0}^p \binom{k}{N})$.
Since $\varepsilon^{0,p,N,M} = 0$, we deduce from the previous inequality that $\varepsilon^{q,p,N,M} \leq \frac{A_3(q,p,N)}{M}$ where $A_3(q,p,N) := K_3(p,N)\frac{(C_3T(T+1)L_f^2)^q - 1}{C_3T(T+1)L_f^2 - 1}$. Then, $(Y^{q,p,N,M}, Z^{q,p,N,M})$ converges to $(Y^{q,p,N}, Z^{q,p,N})$ when M tends to ∞ in $\|(.,.)\|_{L^2}$.

Proof of Proposition 5. For the sake of clarity, we assume $d = 1$. In the following, note that $\Delta Y_t^{q,p,N,M} := Y_t^{q,p,N,M} - Y_t^{q,p,N}$, $\Delta Z_t^{q,p,N,M} := Z_t^{q,p,N,M} - Z_t^{q,p,N}$ and $\Delta f_t^{q,p,N,M} := f(t, Y_t^{q,p,N,M}, Z_t^{q,p,N,M}, [Y_t^{q,p,N,M}], [Z_t^{q,p,N,M}]) - f(t, Y_t^{q,p,N}, Z_t^{q,p,N}, [Y_t^{q,p,N}], [Z_t^{q,p,N}])$. First, we deal with $\mathbb{E}(\sup_{0\leq t\leq T} |\Delta Y_t^{q+1,p,N,M}|^2)$. From (4.7) and (4.10) we get

$$\Delta Y_t^{q+1,p,N,M} = \mathbb{E}_t\big(C_p^{N,M}(F^{q,p,N,M}) - C_p^N(F^{q,p,N})\big) - \int_0^t \Delta f_s^{q,p,N,M} ds.$$

By introducing $\pm C_p^N(F^{q,p,N,M})$ and by using Lemma 3, Doob's inequality, and the Lipschitz property of f, we obtain

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\leq t\leq T} |\Delta Y_t^{q+1,p,N,M}|^2\Big) &\leq 12\mathbb{E}\Big(|(C_p^{N,M} - C_p^N)(F^{q,p,N,M})|^2\Big) \\
&\quad + 12\mathbb{E}\Big(|C_p^N(F^{q,p,N,M} - F^{q,p,N})|^2\Big) \\
&\quad + 12TL_f^2 \int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2 \\
&\quad + \mathcal{W}_2([Y_s^{q,p,N,M}], [Y_s^{q,p,N}])^2 + \mathcal{W}_2([Z_s^{q,p,N,M}], [Z_s^{q,p,N}])^2\big) ds \\
&\leq 12\mathbb{E}\Big(|(C_p^{N,M} - C_p^N)(F^{q,p,N,M})|^2\Big) \\
&\quad + 12\mathbb{E}\Big(|F^{q,p,N,M} - F^{q,p,N}|^2\Big) \\
&\quad + 24TL_f^2 \int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2\big) ds.
\end{aligned}$$

From Lemma 6, we get $\mathbb{E}(|(C_p^{N,M} - C_p^N)(F^{q,p,N,M})|^2) \leq \frac{2}{M}(V_{p,N}(\xi) + V_{p,N}(\int_0^t f(\theta_s^{q,p,N,M}) ds))$. Then, from Remark 8,

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\leq t\leq T} |\Delta Y_t^{q+1,p,N,M}|^2\Big) &\leq \frac{24}{M}\Big(V_{p,N}(\xi) + T^2\|f\|^2_\infty \sum_{k=0}^p \binom{k}{N}\Big) \\
&\quad + 120TL_f^2 \int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2\big) ds.
\end{aligned} \tag{5.10}$$

Let us now bound $\mathbb{E}\Big(\int_0^T |\Delta Z_s^{q+1,p,N,M}|^2 ds\Big)$. Following the same steps as in the proof of Proposition 4, we get

$$\mathbb{E}\Big(\int_0^T |\Delta Z_s^{q+1,p,N,M}|^2 ds\Big) \leq 6\mathbb{E}\Big(\sup_{0\leq t\leq T} |\Delta Y_t^{q+1,p,N,M}|^2\Big) \\ + 24TL_f^2 \int_0^T \mathbb{E}\big(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2\big) ds. \tag{5.11}$$

Adding $7 \times$ (5.10) and (5.11) gives the result. □

5.5. **Error due to the Particle approximation.** In order to prove the error between $(Y^{q,p,N,M}, Z^{q,p,N,M})$ (defined by (4.10)) and $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ (defined by (4.11)) for all $1 \leq m \leq M$, we introduce the following independent copies of $(Y^{q,p,N,M}, Z^{q,p,N,M})$,

$$\begin{aligned} \bar{Y}_t^{q+1,p,N,M,m} =& \mathbb{E}_t\big(C_p^{N,M}((\bar{F}^{q,p,N,M,m})_m)\big) \\ &- \int_0^t f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}]) ds, \\ \bar{Z}_t^{q+1,p,N,M,m} =& D_t\mathbb{E}_t\big(C_p^{N,M}((\bar{F}^{q,p,N,M,m})_m)\big), \end{aligned} \tag{5.12}$$

where $\bar{F}^{q,p,N,M,m} = \xi + \int_0^T f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}]) ds$.
Note that $[\bar{Y}_s^{q,p,N,M,m}]^M$ (resp. $[\bar{Z}_s^{q,p,N,M,m}]^M$) is the empirical distribution of the particles $\bar{Y}_s^{q,p,N,M,m}$ (resp. $\bar{Z}_s^{q,p,N,M,m}$).
We are now interested in bounding the error between $(\bar{Y}^{q,p,N,M,m}, \bar{Z}^{q,p,N,M,m})$ (defined by (5.12)) and $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ (defined by (4.11)), for all $1 \leq m \leq M$. Before giving an upper bound for the error, we measure the error between C_p^N and $C_p^{N,M}$ for a r.v. F satisfying $\mathcal{H}^3_{p,N}$ (see Hypothesis (3)) when $(F^m)_{1\leq m\leq M}$ are identically distributed r.v., but not necessarily independant.

Lemma 7. *Let F be a r.v. satisfying Hypothesis $\mathcal{H}^3_{p,N}$ and let $(F^m)_{1\leq m\leq M}$ be M identically distributed r.v. with law F. We get*

$$\mathbb{E}(|(C_p^N - C_p^{N,M})((F^m)_m)|^2) \leq \mathbb{E}(|F|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{n!}{2}\mathbb{E}\left(|F|^2 \prod_{i=1}^N K^2_{n_i}(G_i)\right),$$

and so, we obtain

$$\mathbb{E}(|C_p^{N,M}((F^m)_m)|^2) \leq 2\mathbb{E}(|F|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{n!}{2}\mathbb{E}\left(|F|^2 \prod_{i=1}^N K^2_{n_i}(G_i)\right).$$

Proof of Lemma 7. Using definitions (2.6) and (4.9), we have

$$(C_p^N - C_p^{N,M})((F^m)_m) = d_0 - \hat{d}_0 + \sum_{k=1}^p \sum_{|n|=k} (d_k^n - \hat{d}_k^n) \prod_{i=1}^N K_{n_i}(G_i).$$

Since $\hat{d}_k^n$ is independent of $(G_i)_i$, it implies that

$$\mathbb{E}(|(C_p^N - C_p^{N,M})((F^m)_m)|^2) = \mathbb{E}(|d_0 - \hat{d}_0|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{1}{n!}\mathbb{E}(|d_k^n - \hat{d}_k^n|^2).$$

The definition of the coefficients d_0 and d_k^n given in (2.7) leads to

$$\mathbb{E}(|(C_p^N - C_p^{N,M})((F^m)_m)|^2) = \mathbb{V}(\hat{d}_0) + \sum_{k=1}^{p} \sum_{|n|=k} \frac{1}{n!} \mathbb{V}(\hat{d}_k^n)$$
$$\leq \mathbb{E}(|\hat{d}_0|^2) + \sum_{k=1}^{p} \sum_{|n|=k} \frac{1}{n!} \mathbb{E}(|\hat{d}_k^n|^2).$$

From the representation (4.8) of the coefficients $\hat{d}_0$ and $\hat{d}_k^n$, we can prove for the first term of the previous inequality that

$$\mathbb{E}(|\hat{d}_0|^2) = \mathbb{E}\Big(\Big|\frac{1}{M}\sum_{m=1}^{M} F^m\Big|^2\Big) = \frac{1}{M^2}\mathbb{E}\Big(\Big|\sum_{m=1}^{M} F^m\Big|^2\Big) \leq \frac{1}{M}\mathbb{E}\Big(\sum_{m=1}^{M}|F^m|^2\Big) = \mathbb{E}(|F|^2),$$

and for the second term $\mathbb{E}(|\hat{d}_k^n|^2)$, we obtain that

$$\begin{aligned}
\mathbb{E}(|\hat{d}_k^n|^2) &= \mathbb{E}\Big(\Big|\frac{n!}{M}\sum_{m=1}^{M} F^m \prod_{i=1}^{N} K_{n_i}(G_i^m)\Big|^2\Big) \\
&= \frac{n!^2}{M^2}\mathbb{E}\Big(\Big|\sum_{m=1}^{M} F^m \prod_{i=1}^{N} K_{n_i}(G_i^m)\Big|^2\Big) \\
&= \frac{n!^2}{M^2}\Big[\mathbb{E}\Big(\sum_{m=1}^{M}|F^m|^2\prod_{i=1}^{N} K_{n_i}^2(G_i^m)\Big) + \frac{M(M-1)}{2}\mathbb{E}\Big(\Big|F^1\prod_{i=1}^{N} K_{n_i}(G_i^1).F^2\prod_{i=1}^{N} K_{n_i}(G_i^2)\Big|\Big)\Big] \\
&\leq \frac{n!^2}{M}\mathbb{E}\Big(|F|^2\prod_{i=1}^{N} K_{n_i}^2(G_i)\Big) + \frac{M(M-1)n!^2}{2M^2}\mathbb{E}\Big(|F|^2\prod_{i=1}^{N} K_{n_i}^2(G_i)\Big) \\
&\leq n!^2\Big(\frac{1}{2}+\frac{1}{M}\Big)\mathbb{E}\Big(|F|^2\prod_{i=1}^{N} K_{n_i}^2(G_i)\Big).
\end{aligned}$$

Then, the first result follows. To get the second result, we write $C_p^{N,M}((F^m)_m) = (C_p^{N,M} - C_p^N)((F^m)_m) + C_p^N(F)$. Since $\mathbb{E}((C_p^{N,M} - C_p^N)((F^m)_m)C_p^N(F)) = 0$, we get

$$\mathbb{E}(|C_p^{N,M}((F^m)_m)|^2) = \mathbb{E}(|(C_p^{N,M} - C_p^N)((F^m)_m)|^2) + \mathbb{E}(|C_p^N(F)|^2),$$

and Lemma 3 completes the proof.

□

Proposition 6. *Assume that there exists a real $r > 4$ s.t. $\xi \in (\mathbb{L}^r \cap \mathcal{D}^{1,r}) \subset \mathcal{H}_{p,N}^3$ and f is a bounded function. Let $\varepsilon^{q,p,N,M,m} := \|(\bar{Y}^{q,p,N,M,m} - Y^{q,p,N,M,m}, \bar{Z}^{q,p,N,M,m} - Z^{q,p,N,M,m})\|_{L^2}^2$. We get*

$$\varepsilon^{q+1,p,N,M,m} \leq C_4(p,N)\Big(\varepsilon^{q,p,N,M,m}\Big)^{1/2} + \frac{K_4(p,N)}{M^{1/4}}, \tag{5.13}$$

where $C_4(p,N)$ and $K_4(p,N)$ depends on p, N, T and $\|f\|_\infty$. Since $\varepsilon^{0,p,N,M,m} = 0$, we deduce from (5.13) that $\varepsilon^{q,p,N,M,m} \leq \frac{A_4(q,p,N)}{M^{(1/2^{(q+2)})}}$ where $A_4(q,p,N)$ depends on q, p, N, T and $\|f\|_\infty$. Then, $\forall\ 1 \leq m \leq M$, $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ converges to $(\bar{Y}^{q,p,N,M,m}, \bar{Z}^{q,p,N,M,m})$ when M tends to ∞ in $\|(.,.)\|_{L^2}$.

Lemma 8. *Assume that f is a bounded function and $\xi \in \mathbb{L}^r \cap \mathcal{D}^{1,r}$. Then, for all $r \geq 1$ and $1 \leq m \leq M$, there exist a positive constant K_r, depending on f, r and T such that*

$$\mathbb{E}(|\bar{Y}_t^{q,p,N,M,m}|^r) + \mathbb{E}(|\bar{Z}_t^{q,p,N,M,m}|^r) \leq K_r\Big(1 + \mathbb{E}(|\xi|^r) + \mathbb{E}(|D_t\xi|^r)\Big).$$

Proof of Lemma 8. Let $r \geq 1$ and $t \in [0,T]$. From (5.12) and by using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\mathbb{E}(|\bar{Y}_t^{q,p,N,M,m}|^r) &= \mathbb{E}\Bigg(\Bigg|\mathbb{E}_t\big(C_p^N(\bar{F}^{q,p,N,M,m})\big) \\
&\quad - \int_0^t f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}])ds\Bigg|^r\Bigg) \\
&\leq 2^{r-1}\mathbb{E}\Bigg(\Bigg|\mathbb{E}_t\big(C_p^N(\bar{F}^{q,p,N,M,m})\big)\Bigg|^r \\
&\quad + \Bigg|\int_0^t f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}])ds\Bigg|^r\Bigg),
\end{aligned}$$

where $\bar{F}^{q,p,N,M,m} = \xi + \int_0^T f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}])ds$.
Then, since f is a bounded function, there exists a positive constant K_1, depending on f, r and T such that

$$\mathbb{E}(|\bar{Y}_t^{q,p,N,M,m}|^r) \leq K_1(1 + \mathbb{E}(|\xi|^r)).$$

In the same way, we can obtain that

$$\begin{aligned}
\mathbb{E}(|\bar{Z}_t^{q,p,N,M,m}|^r) &= \mathbb{E}\Bigg(\Bigg|D_t\mathbb{E}_t\big(C_p^N(\bar{F}^{q,p,N,M,m})\big)\Bigg|^r\Bigg) \\
&\leq K_2(1 + \mathbb{E}(|D_t\xi|^r)),
\end{aligned}$$

and the result follows. □

Proof of Proposition 6. For the sake of clarity, we assume $d = 1$. In the following, consider that $\Delta Y_t^{q,p,N,M,m} := Y_t^{q,p,N,M,m} - \bar{Y}_t^{q,p,N,M,m}$, $\Delta Z_t^{q,p,N,M,m} := Z_t^{q,p,N,M,m} - \bar{Z}_t^{q,p,N,M,m}$, $\Delta f_t^{q,p,N,M,m} :=$

$f(t, Y_t^{q,p,N,M,m}, Z_t^{q,p,N,M,m}, [Y_t^{q,p,N,M,m}]^M, [Z_t^{q,p,N,M,m}]^M) - f(t, \bar{Y}_t^{q,p,N,M,m}, \bar{Z}_t^{q,p,N,M,m}, [\bar{Y}_t^{q,p,N,M,m}],$ $[\bar{Z}_t^{q,p,N,M,m}])$ and $\Delta F^{q,p,N,M,m} := F^{q,p,N,M,m} - \bar{F}^{q,p,N,M,m}$. First, we deal with $\mathbb{E}(\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M,m}|^2)$. From (5.12) and (4.11) we get

$$\Delta Y_t^{q+1,p,N,M,m} = \mathbb{E}_t\big(C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)\big) - \int_0^t \Delta f_s^{q,p,N,M,m} ds.$$

By using Doob and Cauchy-Schwartz inequalities, we obtain

$$\begin{aligned}
\mathbb{E}\big(\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M,m}|^2\big) &\leq 2\mathbb{E}\Bigg(\sup_{0\leq t\leq T}\Bigg|\mathbb{E}_t\big(C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)\big)\Bigg|^2\Bigg) + 2\mathbb{E}\Bigg(\sup_{0\leq t\leq T}\Bigg|\int_0^t \Delta f_s^{q,p,N,M,m} ds\Bigg|^2\Bigg) \\
&\leq 8\mathbb{E}\Bigg(\Bigg|C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)\Bigg|^2\Bigg) + 2T\mathbb{E}\Bigg(\int_0^T \Bigg|\Delta f_s^{q,p,N,M,m}\Bigg|^2 ds\Bigg).
\end{aligned}$$

From Lemma 7, we have

$$\mathbb{E}(|C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)|^2) \leq 2\mathbb{E}(|\Delta F^{q,p,N,M,m}|^2) + \sum_{k=1}^{p}\sum_{|n|=k}\frac{n!}{2}\mathbb{E}\Bigg(|\Delta F^{q,p,N,M,m}|^2\prod_{i=1}^{N}K_{n_i}^2(G_i)\Bigg),$$

and by using Cauchy-Schwartz inequality and the bounded property of the function f, we get

$$\begin{aligned}\mathbb{E}(|C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)|^2) &\leq 2\mathbb{E}(|\Delta F^{q,p,N,M,m}|^2)\\ &\quad + \sum_{k=1}^{p}\sum_{|n|=k}\frac{n!}{2}\mathbb{E}\Big(|\Delta F^{q,p,N,M,m}|^2\Big)^{1/2}\mathbb{E}\Big(|\Delta F^{q,p,N,M,m}|^2\prod_{i=1}^{N}K_{n_i}^4(G_i)\Big)^{1/2}\\ &\leq 2\mathbb{E}(|\Delta F^{q,p,N,M,m}|^2) + C(p,N)\mathbb{E}\Big(|\Delta F^{q,p,N,M,m}|^2\Big)^{1/2}\\ &\leq \Big(C(p,N) + 4T\|f\|_\infty\Big)\mathbb{E}\Big(|\Delta F^{q,p,N,M,m}|^2\Big)^{1/2},\end{aligned}$$

where $C(p,N) = \sum_{k=1}^{p}\sum_{|n|=k} n!T\|f\|_\infty\mathbb{E}\Big(\prod_{i=1}^{N}K_{n_i}^4(G_i)\Big)^{1/2}$. Then, we can conclude that

$$\begin{aligned}\mathbb{E}\big(\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M,m}|^2\big) &\leq 8\sqrt{T}\Big(C(p,N)+4T\|f\|_\infty\Big)\mathbb{E}\Big(\int_0^T\Big|\Delta f_s^{q,p,N,M,m}\Big|^2 ds\Big)^{1/2}\\ &\quad + 2T\mathbb{E}\Big(\int_0^T\Big|\Delta f_s^{q,p,N,M,m}\Big|^2 ds\Big)\\ &\leq \Big(8\sqrt{T}\big(C(p,N)+4T\|f\|_\infty\big) + 4T\sqrt{T}\|f\|_\infty\Big)\mathbb{E}\Big(\int_0^T\Big|\Delta f_s^{q,p,N,M,m}\Big|^2 ds\Big)^{1/2}\\ &\leq C_1(p,N)\mathbb{E}\Big(\int_0^T\Big|\Delta f_s^{q,p,N,M,m}\Big|^2 ds\Big)^{1/2},\end{aligned}$$

where $C_1(p,N) = 4\sqrt{T}\Big(2C(p,N) + 9T\|f\|_\infty\Big)$.
Moreover, the Lipschitz property of f gives

$$\begin{aligned}\mathbb{E}\big(\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M,m}|^2\big) &\leq 2C_1(p,N)L_f\Big(\int_0^T\mathbb{E}\big(|\Delta Y_s^{q,p,N,M,m}|^2 + |\Delta Z_s^{q,p,N,M,m}|^2\\ &\quad + \mathcal{W}_2([Y_s^{q,p,N,M,m}]^M,[\bar{Y}_s^{q,p,N,M,m}])^2 + \mathcal{W}_2([Z_s^{q,p,N,M,m}]^M,[\bar{Z}_s^{q,p,N,M,m}])^2\big)ds\Big)^{1/2},\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}\big(\mathcal{W}_2([Y_s^{q,p,N,M,m}]^M,[\bar{Y}_s^{q,p,N,M,m}])^2\big) &\leq 2\mathbb{E}\big(\mathcal{W}_2([Y_s^{q,p,N,M,m}]^M,[\bar{Y}_s^{q,p,N,M,m}]^M)^2\big)\\ &\quad + 2\mathbb{E}\big(\mathcal{W}_2([\bar{Y}_s^{q,p,N,M,m}]^M,[\bar{Y}_s^{q,p,N,M,m}])^2\big),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\big(\mathcal{W}_2([Z_s^{q,p,N,M,m}]^M,[\bar{Z}_s^{q,p,N,M,m}])^2\big) &\leq 2\mathbb{E}\big(\mathcal{W}_2([Z_s^{q,p,N,M,m}]^M,[\bar{Z}_s^{q,p,N,M,m}]^M)^2\big)\\ &\quad + 2\mathbb{E}\big(\mathcal{W}_2([\bar{Z}_s^{q,p,N,M,m}]^M,[\bar{Z}_s^{q,p,N,M,m}])^2\big).\end{aligned}$$

On the one hand, by using the property of the Wasserstein distance $\mathcal{W}_2$, we get

$$\begin{aligned}\mathbb{E}\big(\mathcal{W}_2([Y_s^{q,p,N,M,m}]^M,[\bar{Y}_s^{q,p,N,M,m}]^M)^2\big) &\leq \mathbb{E}\Big(\mathcal{W}_2\Big(\frac{1}{M}\sum_{m=1}^{M}\delta_{Y_s^{q,p,N,M,m}},\frac{1}{M}\sum_{m=1}^{M}\delta_{\bar{Y}_s^{q,p,N,M,m}}\Big)^2\Big)\\ &\leq \frac{1}{M}\sum_{m=1}^{M}\mathbb{E}\big(\big|Y_s^{q,p,N,M,m}-\bar{Y}_s^{q,p,N,M,m}\big|^2\big)\\ &= \mathbb{E}\big(\big|\Delta Y_s^{q,p,N,M,m}\big|^2\big).\end{aligned}$$

On the other hand, thanks to Lemma 8 and to the assumption on ξ, we apply [FG15, Theorem 1] to obtain

$$\begin{aligned}\mathbb{E}\big(\mathcal{W}_2([\bar{Y}_s^{q,p,N,M,m}]^M,[\bar{Y}_s^{q,p,N,M,m}])^2\big) &\leq \sup_{0\leq s\leq T}\mathbb{E}\big(\mathcal{W}_2([\bar{Y}_s^{q,p,N,M,m}]^M,[\bar{Y}_s^{q,p,N,M,m}])^2\big)\\ &\leq \frac{C_T}{\sqrt{M}},\end{aligned}$$

where C_T is a positive constant.
In the same way, we have

$$\mathbb{E}\big(\mathcal{W}_2([Z_s^{q,p,N,M,m}]^M,[\bar{Z}_s^{q,p,N,M,m}]^M)^2\big) \leq \mathbb{E}\big(|\Delta Z_s^{q,p,N,M,m}|^2\big),$$

and

$$\mathbb{E}\big(\mathcal{W}_2([\bar{Z}_s^{q,p,N,M,m}]^M,[\bar{Z}_s^{q,p,N,M,m}])^2\big) \leq \frac{C_T'}{\sqrt{M}}.$$

Finally, we can derive the inequality

$$\begin{aligned}&\mathbb{E}\big(\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M,m}|^2\big)\\ &\leq 2C_1(p,N)L_f\left[\int_0^T\mathbb{E}\left(3|\Delta Y_s^{q,p,N,M,m}|^2+3|\Delta Z_s^{q,p,N,M,m}|^2+2\frac{C_T}{\sqrt{M}}+2\frac{C_T'}{\sqrt{M}}\right)ds\right]^{1/2} \qquad (5.14)\\ &\leq \frac{K(p,N)}{M^{1/4}}+C_2(p,N)\left(\int_0^T\mathbb{E}\big(|\Delta Y_s^{q,p,N,M,m}|^2+|\Delta Z_s^{q,p,N,M,m}|^2\big)ds\right)^{1/2},\end{aligned}$$

where $K(p,N)$ and $C_2(p,N)=6C_1(p,N)L_f$ are two constants depending on p, N, T and $\|f\|_\infty$. Let us now study the upper bound $\mathbb{E}(\int_0^T|\Delta Z_s^{q+1,p,N,M,m}|^2ds)$. Following the same steps as in the proof of Proposition 3, we get

$$\mathbb{E}\left(\int_0^T|\Delta Z_s^{q+1,p,N,M,m}|^2ds\right)\leq 6\mathbb{E}\Big(\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M,m}|^2\Big)+3T\mathbb{E}\left(\int_0^T|\Delta f_s^{q,p,N,M,m}|^2\right)ds,$$

and by using the same previous majoration for the second term of the right hand side, we deduce

$$\begin{aligned}\mathbb{E}\left(\int_0^T|\Delta Z_s^{q+1,p,N,M,m}|^2ds\right)\leq{}& 6\mathbb{E}\Big(\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M,m}|^2\Big)+\frac{K_1(p,N)}{M^{1/4}}\\ &+C_3(p,N)\left(\int_0^T\mathbb{E}\big(|\Delta Y_s^{q,p,N,M,m}|^2+|\Delta Z_s^{q,p,N,M,m}|^2\big)ds\right)^{1/2}.\end{aligned} \qquad (5.15)$$

By adding $7\times$ (5.14) and (5.15), the result can follows easily. □

6. Numerical illustrations

In this section, we will illustrate the algorithm by presenting some explicit computations. We consider on $[0,T]$ the following sort of processes

$$Y_t=\xi+\int_t^T\Big(\alpha Y_s+\beta\mathbb{E}(Y_s)+\gamma\mathbb{E}(Z_s)\Big)ds-\int_t^T Z_s\cdot dB_s,\quad 0\leq t\leq T, \qquad (6.1)$$

where α, β and $\gamma\in\mathbb{R}$. First, we study the solution of (6.1) when $\gamma=0$. Then, we study the solution of (6.1) in the general case.

Case (1) $\gamma = 0$. We have

$$Y_t = e^{\alpha(T-t)}\Big(\mathbb{E}_t(\xi) + \mathbb{E}(\xi)(e^{\beta(T-t)} - 1)\Big),$$

and

$$Z_t = e^{\alpha(T-t)}\mathbb{E}_t(D_t\xi).$$

Now, if $\xi = B_T$, we obtain

$$(Y_t, Z_t) = \Big(e^{\alpha(T-t)}B_t, e^{\alpha(T-t)}\Big),$$

and if $\xi = B_T^2$, we get

$$(Y_t, Z_t) = \Big(e^{\alpha(T-t)}\big(B_t^2 - t + Te^{\beta(T-t)}\big), 2e^{\alpha(T-t)}B_t\Big).$$

Case (2) $\gamma \neq 0$. We consider two different values of ξ:
If $\xi = B_T$ and $\beta \neq 0$, we have

$$(Y_t, Z_t) = \Big(e^{\alpha(T-t)}\Big(B_t + \frac{\gamma}{\beta}(e^{\beta(T-t)} - 1)\Big), e^{\alpha(T-t)}\Big).$$

If $\xi = B_T^2$, we have

$$(Y_t, Z_t) = \Big(e^{\alpha(T-t)}\big(B_t^2 - t + Te^{\beta(T-t)}\big), 2e^{\alpha(T-t)}B_t\Big).$$

6.1. **Proofs of the numerical illustrations.**

Proof of case (1). In this case, we have

$$\mathbb{E}(Y_t) = \mathbb{E}(\xi) + \int_t^T (\alpha + \beta)\mathbb{E}(Y_s)ds,$$

from which we derive

$$\mathbb{E}(Y_t) = \mathbb{E}(\xi)e^{(\alpha+\beta)(T-t)},$$

and taking into account (6.1), we get

$$Y_t = \xi + \int_t^T \Big(\alpha Y_s + \beta\mathbb{E}(\xi)e^{(\alpha+\beta)(T-s)}\Big)ds - \int_t^T Z_s dB_s.$$

Applying Itô's formula on $e^{\alpha t}Y_t$, we get

$$\begin{aligned} e^{\alpha t}Y_t &= e^{\alpha T}\xi + \beta\mathbb{E}(\xi)\int_t^T e^{\alpha s}e^{(\alpha+\beta)(T-s)}ds - \int_t^T e^{\alpha s}Z_s dB_s \\ &= e^{\alpha T}\xi + \mathbb{E}(\xi)e^{\alpha T}\Big(e^{\beta(T-t)} - 1\Big) - \int_t^T e^{\alpha s}Z_s dB_s, \end{aligned}$$

which leads

$$Y_t = e^{\alpha(T-t)}\xi + \mathbb{E}(\xi)e^{\alpha(T-t)}\Big(e^{\beta(T-t)} - 1\Big) - e^{-\alpha t}\int_t^T e^{\alpha s}Z_s dB_s.$$

Finally, by taking the conditional expectation of Y_t, we get the explicit form of (Y_t, Z_t) as follows

$$Y_t = e^{\alpha(T-t)}\Big(\mathbb{E}_t(\xi) + \mathbb{E}(\xi)(e^{\beta(T-t)} - 1)\Big),$$

and

$$Z_t = e^{\alpha(T-t)}\mathbb{E}_t(D_t\xi).$$

When the terminal condition is equal to B_T or B_T^2, we easily get the results. □

Proof of case (2). From (6.1), we have

$$Y_t = \mathbb{E}_t(\xi) + \mathbb{E}_t\bigg(\int_t^T \big(\alpha Y_s + \beta\mathbb{E}(Y_s) + \gamma\mathbb{E}(Z_s)\big)ds\bigg).$$

By applying the Malliavin derivative on Y_t, we obtain that for all $s \in [0,T]$,

$$\mathbb{E}(D_s Y_t) = \mathbb{E}(D_s\xi) + \alpha\int_t^T \mathbb{E}(D_s Y_s)ds,$$

and obviously when $s \longrightarrow t$, we get

$$\begin{aligned}\mathbb{E}(Z_t) &= \mathbb{E}(D_t Y_t)\\ &= e^{\alpha(T-t)}\mathbb{E}(D_t\xi).\end{aligned}$$

Therefore,

$$\mathbb{E}(Z_t) = \begin{cases} e^{\alpha(T-t)} & if\ \xi = B_T,\\ 0 & if\ \xi = B_T^2.\end{cases}$$

When $\xi = B_T^2$, we obtain the same equation as in the case (1), and therefore, the same result. Now, when $\xi = B_T$, we get the following form of the process Y_t

$$Y_t = \xi + \int_t^T \Big(\alpha Y_s + \beta\mathbb{E}(Y_s) + \gamma e^{\alpha(T-s)}\Big)ds - \int_t^T Z_s dB_s. \tag{6.2}$$

Since

$$\mathbb{E}(Y_t) = \int_t^T \Big((\alpha+\beta)\mathbb{E}(Y_s) + \gamma e^{\alpha(T-s)}\Big)ds,$$

it implies that

$$\begin{aligned}\mathbb{E}(Y_t) &= \gamma e^{(\alpha+\beta)(T-t)}\int_t^T e^{-(\alpha+\beta)(T-s)}e^{\alpha(T-s)}ds\\ &= \frac{\gamma}{\beta}e^{\alpha(T-t)}\big(e^{\beta(T-t)} - 1\big),\end{aligned}$$

and from (6.2), we deduce that

$$Y_t = \xi + \int_t^T \Big(\alpha Y_s + \gamma e^{(\alpha+\beta)(T-s)}\Big)ds - \int_t^T Z_s dB_s.$$

Finally, by applying Itô's formula and by taking the conditional expectation of $e^{\alpha t}Y_t$, we conclude that

$$(Y_t, Z_t) = \Big(e^{\alpha(T-t)}\Big(B_t + \frac{\gamma}{\beta}(e^{\beta(T-t)} - 1)\Big), e^{\alpha(T-t)}\Big).$$

□

6.2. **Illustrations.** The computations of this section have been done on the following computers

- Dell precision tower 3620 4 cores Intel(R) Xeon(R) CPU E3-1240 v6 @3.7 Ghz with 16 Go of memory for M varying from 10^4 to 10^7
- Dell precision T7920 with 2 Intel Xeon Gold 6128 with 6 cores @3.7 Ghz and 128 Go of memory for $M = 10^8$

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a subdivision of $[0,T]$ of step size $h = T/N$, N being a positive integer, let (Y, Z) be the unique solution of McKean-Vlasov BSDE (1.2) and let, for a given q, p and m, $(\tilde{Y}^{q,p,m}_{t_k}, \tilde{Z}^{q,p,m}_{t_k})_{0\le k\le N}$ be its numerical approximation given by Algorithm 1. For a given integer L, we draw $(\bar{Y}^l, \bar{Z}^l)_{0\le l\le L}$ and $(\tilde{Y}^{q,p,m,l}, \tilde{Z}^{q,p,m,l})_{0\le l\le L}$, L independent copies of (Y, Z). Then we approximate the $\mathbb{L}^2$-error of Theorem 2 by:

$$error = \frac{1}{L}\sum_{l=1}^{L}\left(\max_{0\le k\le N}\left|\bar{Y}^l_{t_k} - \tilde{Y}^{q,p,m,l}_{t_k}\right|^2 + h\sum_{k=0}^{N}\left|\bar{Z}^l_{t_k} - \tilde{Z}^{q,p,m,l}_{t_k}\right|^2\right). \tag{6.3}$$

- **Convergence in** p**.** Table 1 represents the evolution of $\tilde{Y}_0^{q,p,N,M}$ w.r.t q (Picard's iteration index), when $p = 2$ and $p = 3$. We also give the CPU time needed to get $\tilde{Y}_0^{8,p,N,M}$ and $\tilde{Z}_0^{8,p,N,M}$. We fix $M = 10^7$, $N = 20$, $\xi = \sqrt{|B_T|}$ and $f(t, Y, Z) = Y + E(Z)$. The seed of the generator is also fixed. Note that the difference between the values of $\tilde{Y}_0^{8,2,N,M}$ and $\tilde{Y}_0^{8,3,N,M}$ does not exceed 0.12%. This is due to the fast convergence of the algorithm in p. CPU time is 5 times higher when $p = 3$ than when $p = 2$.

Iterations	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Real time**
p=2	0.822301	1.644047	2.033746	2.150164	2.174622	2.178412	2.178843	2.178874	49.596
p=3	0.822236	1.644453	2.035001	2.152181	2.177102	2.181102	2.181609	2.181662	284.947

TABLE 1. Evolution of $\tilde{Y}_0^{q,p,N,M}$ ($p = 2$ and $p = 3$) w.r.t. Picard's iterations, $M = 10^7$, $N = 20$, $\xi = \sqrt{|B_T|}$, $f(t, Y, Z) = Y + E(Z)$ and the real time of calculation.

- **Convergence in** M**.** Figure 1 (resp. Figure 2) represents the evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q when $p = 3$ (resp. when $p = 2$), $N = 20$, $f(t, Y, Z) = Y + E(Z)$ and $\xi = \sqrt{|B_T|}$ (resp. $\xi = B_T^2$) for different values of M. For this set of parameters, the exact solutions are $Y_0 = 2.2352$ (resp. $Y_0 = 2.7183$) and $Z_0 = 0$. The exact solution depicted in Figure 1 is obtained by applying a lot of drawings. Concerning Figure 1, we notice that $\tilde{Y}_0^{q,p,N,M}$ (resp. $\tilde{Z}_0^{q,p,N,M}$) converges to the exact solution when $M \ge 10^5$ (resp. $M \ge 10^6$). Concerning Figure 2, we notice that $\tilde{Y}_0^{q,p,N,M}$ (resp. $\tilde{Z}_0^{q,p,N,M}$) converges to the exact solution when $M \ge 10^5$ (resp. $M \ge 10^7$). In both cases the algorithm stabilizes after very few iterations.

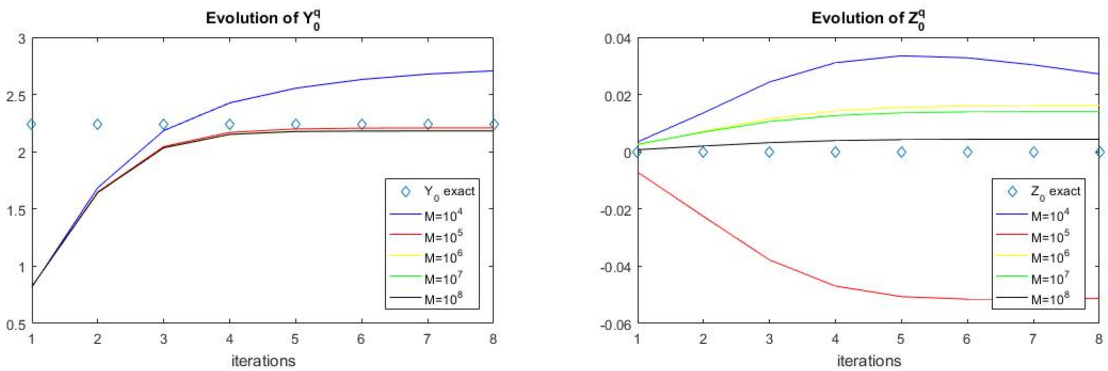

FIGURE 1. Evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q for different values of M when $N = 20$, $p = 3$, $\xi = \sqrt{|B_T|}$, $f(t, Y, Z) = Y + E(Z)$.

- **Convergence in** N**.** Figure 3 represents the evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q when $p = 2$, $M = 10^6$, $f(t, Y, Z) = Y + E(Y) + E(Z)$ and $\xi = B_T$ for different values of N. For this set of parameters, the exact solutions are $Y_0 = 4.6708$ and $Z_0 = 2.7183$. The algorithm converges even when $N = 10$, but $\tilde{Y}_0^{8,p,10,M}$ (resp. $\tilde{Z}_0^{8,p,10,M}$) is quite below

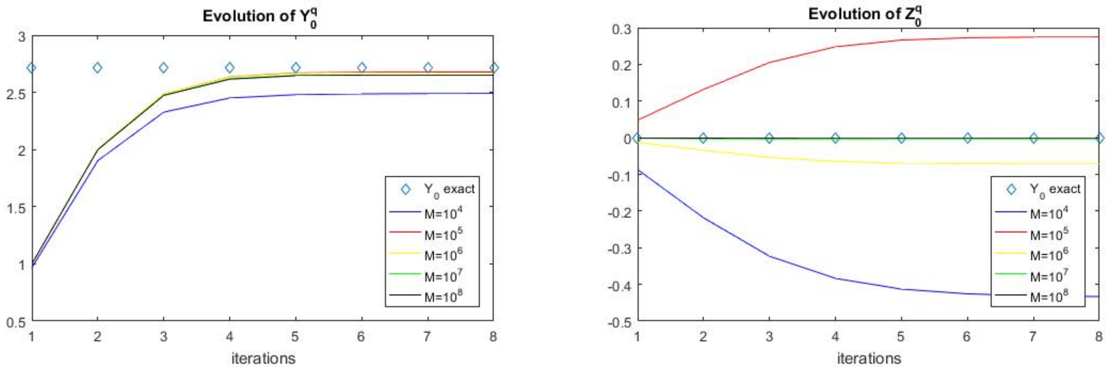

FIGURE 2. Evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q for different values of M when $N = 20$, $p = 2$, $\xi = B_T^2$, $f(t, Y, Z) = Y + E(Z)$.

$\tilde{Y}_0^{8,p,40,M}$ (resp. $\tilde{Z}_0^{8,p,40,M}$). Notice that for $N = 40$ the approximation values are very close to the exact values.

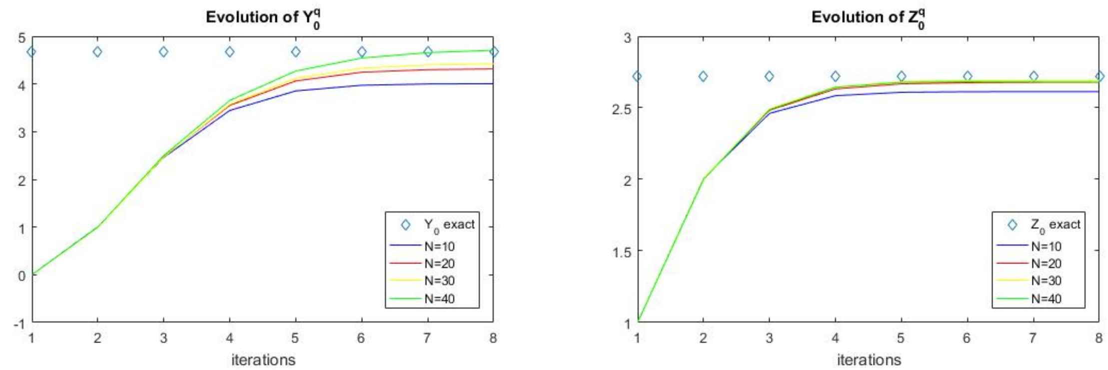

FIGURE 3. Evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q for different values of N when $M = 10^6$, $p = 3$, $\xi = B_T$, $f(t, Y, Z) = Y + E(Y) + E(Z)$.

- **Convergence in M of the error (6.3).** Figure 4 illustrates the error (6.3) (i.e. the error made on the whole path) for the case $p = 2$, $N = 20$, $q = 6$, $\xi = B_T^2$, $f(t, Y, Z) = Y + E(Z)$ for different values of M. We notice that the convergence in M is much faster than the theoretical one, which is $\frac{1}{M^{1/2^{q+2}}}$.

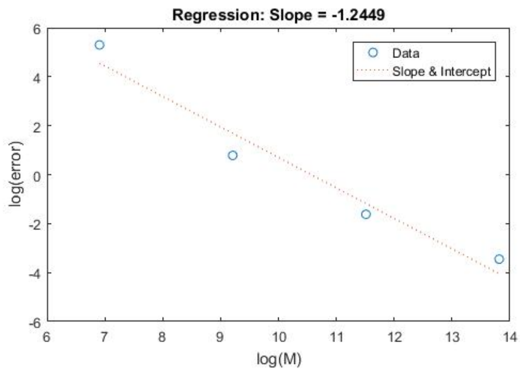

FIGURE 4. Regression of $\log(error)$ w.r.t. $\log(M)$. Data: $error$ when $M = (10^3, 10^4, 10^5, 10^6)$. Parameters: $N = 20$, $p = 2$, $q = 6$, $\xi = B_T^2$, $f(t, Y, Z) = Y + E(Z)$.

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Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France

Email address: Celine.Acary-Robert@univ-smb.fr

Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France

Email address: philippe.briand@univ-smb.fr

Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France

Univ. Libanaise, LaMA-Liban, P.O. Box 37, Tripoli, Liban

Email address: abir.ghannoum@univ-smb.fr

Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France

Email address: celine.labart@univ-smb.fr