# Robustly Parameterised Higher-Order Probabilistic Models

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## Abstract

We present a method for constructing robustly parameterised families of higher-order probabilistic models. Parameter spaces and models are represented by certain classes of functors in the category of Polish spaces. Maps from parameter spaces to models (parameterisations) are continuous and natural transformations between such functors. Naturality ensures that parameterised models are invariant by change of granularity – ie that parameterisations are intrinsic. Continuity ensures that models are robust with respect to their parameterisation. Our method allows one to build models from a set of basic functors among which the Giry probabilistic functor, spaces of cadlag trajectories (in continuous and discrete time), multisets and compact powersets. These functors can be combined by guarded composition, product and coproduct. Parameter spaces range over the polynomial closure of Giry-like functors. Thus we obtain a class of robust parameterised models which includes the Dirichlet process, point processes and other classical objects of probability theory such as the de Finetti theorem. By extending techniques developed in prior work, we show how to reduce the questions of existence, uniqueness, naturality, and continuity of a parameterised model to combinatorial questions only involving finite spaces.

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## 1 Introduction

In a widely read paper [1], Beylin and Dybjer reinterpret coherence for monoidal categories as a result of normalisation on a linear non-commutative lambda-calculus. They prove that the structural arrows of a monoidal category are characterised by their domain and codomain. In this paper we follow a parallel path for probabilistic models. There are several differences with Dybjer's correspondence, however.

First, we work in **Pol**, the *concrete* category of Polish spaces Polish spaces form a classic and convenient environment to construct a large variety of stochastic models. We exploit this potential to build up a sufficiently expressive stock of structure arrows. Specifically, we build up structure arrows based on a two-sorted polynomial type theory of 'parameter' functors and 'model' ones. Model functors include as primitives the Vietoris functor of compact non-determinism, the Giry functor of probabilities, and most interestingly the

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Skorokhod functor of cadlag functions - i.e. with countably many jumps and values in any Polish space - which captures traces of processes both in discrete and continuous time. Model functors can be combined by products, coproducts, and guarded composition; parameter functors are less malleable (perhaps because we do not understand them well yet). Hence, our type theory allows for iterations of 'compact' non-determinism (as captured by the Vietoris functor) and probabilism (the Giry monad) - which have been a fundamental pursuit in quantitative models of concurrency theory since Segala's early work [18, 10, 19, 14]. In addition, topology (and the possible recourse to metrics) allows one to talk about continuity and to quantify notions of approximate bisimulation. Our arrows come pre-equipped with metric interpretations which several and in particular van Breugel et al. have convincingly argued are fundamental in probabilistic modeling [6, 20].

Thus, our stock of arrows is remarkably expressive. Of course, this would amount to little, without a characterization of arrows equality. Here again, our result is slightly different from Dybjer's. We do not show that natural transformations are uniquely determined by their 'types'. Instead, using the 'machine' built in earlier work [5, 4], we show that structure arrows are completely characterised by their behaviours on finite spaces. In effect, our result makes the structure arrow equality problem (the equivalent of the normalisation of linear lambda-terms if we follow our analogy) purely combinatorial. For some special choices of parameter and model functors (of the Giry type), we can even show rigidity [4], that is to say, types do characterise arrows. Regarding existence, we provide a converse to the above result, and prove that not only are structure arrows wholly determined by the finite case, but that finite data is *enough* to define such arrows.

The standpoint presented here departs from the more common "forward semantics" approach where one knows already what one wants to semanticize. Here we embrace a less trodden path (but usually rewarding, as in the Ehrhard-Regnier invention of differential linear logic via the study of spaces of sequences, or Girard's own carving of linear logic via coherence spaces) of "reverse semantics" where the mathematical tractability of the semantic universe is the primary tool for constructing the universe of computational discourse.

The outline of the paper is as follows: we start with a 'slicing' of the category of Polish spaces in convenient layers, and recall the basic points of our existence and unicity 'Machine', spelling out the conditions on our parameter and model functors. With this behind us, we attack the description of the type theory and develop a string of propositions which justify that choice by building its semantics. We conclude with our "normalisation" theorem and an application of our framework to the celebrated de Finetti theorem, a key result in probability theory and statistics.

## 2 Preliminaries

We work in the category **Pol** of completely metrisable and separable topological spaces and continuous maps. There is the obvious Borel functor $\mathcal{B} : \mathbf{Pol} \to \mathbf{Meas}$ mapping a Polish space to the measurable space with the Borel $\sigma$-algebra and mapping continuous maps to measurable ones, together with the underlying set functor $U : \mathbf{Pol} \to \mathbf{Set}$ with the obvious forgetful action. **Pol** has all countable limits and all countable coproducts [3, IX].

### 2.1 Pol endofunctors

We introduce some **Pol** endofunctors which are going to be used as examples in the rest of the paper and form the primitive bricks for our structure arrows. First, there is the Giry functor $\mathsf{G}$ [9, 17] which maps a space $X$ to the space of Borel probability measures (with the

topology induced by the Kantorovich metric) over $X$. Related to $\mathsf{G}$ are the finite nonzero positive measure functor $\mathsf{M}^+ \cong \mathsf{G} \times \mathbb{R}_{>0}$. A parallel construction is that of the Vietoris functor $\mathsf{V}$ which maps $X$ to the "hyperspace" of its compact subsets topologised with the Hausdorff distance [13, 4.F]. The finite multiset functor $\mathsf{B}$ and the related finite list functor $\mathsf{W}$ are also **Pol** endofunctors [4]. Finally, we have a pair of functional functors. The Skorokhod functor $\mathsf{D}$, which maps $X$ to the space of cadlag (right-continuous with left limits) functions from $[0, \infty)$ to $X$ equipped with the $J_1$ topology [7], which is fundamental to the study of continuous-time stochastic processes. And for any compact Polish set $X$, the functor $\mathsf{C}(X, -)$ which maps any Polish space $Y$ to that of continuous maps from $X$ to $Y$.

Our family of functors covers both probabilistic behaviour through $\mathsf{G}$, compact non-determinism through $\mathsf{V}$ and spaces of trajectories through $\mathsf{D}$ and $\mathsf{C}$. Moreover, working in the category of Polish spaces makes the analogies between these functors all the more striking: $\mathsf{G}$ is metrised by Kantorovich, $\mathsf{V}$ by Hausdorff and the same holds of $\mathsf{D}$ using a metric allowing "time transport". Recent work [16] might shed additional light on these similarities.

## 2.2 The structure of Pol

We slice **Pol** into the following full subcategories: finite Polish spaces $\mathbf{Pol}_f$; compact zero-dimensional spaces $\mathbf{Pol}_{cz}$; zero-dimensional spaces $\mathbf{Pol}_z$; and compact spaces $\mathbf{Pol}_c$:

$$\begin{tikzcd} \mathbf{Pol}_f \arrow[r, hook] & \mathbf{Pol}_{cz} \arrow[r, hook] \arrow[dr, hook] & \mathbf{Pol}_z \arrow[r, hook] & \mathbf{Pol} \\ & & \mathbf{Pol}_c \arrow[ur, hook] & \end{tikzcd} \tag{1}$$

Finite spaces are equipped with the discrete topology. Compact zero-dimensional Polish spaces (such as the Cantor set $2^{\mathbb{N}}$) can be characterised as projective limits of finite Polish spaces. Zero-dimensional spaces (such as the Baire space $\mathbb{N}^{\mathbb{N}}$) are those spaces which admit a *base* of their topology constituted of clopen sets. These subcategories have interesting structures: any zero-dimensional space equipped with a choice of a countable base of clopen sets can be mapped to its compactification, which is compact zero-dimensional [4]. In the other direction, any Polish space equipped with a choice of a countable base can be mapped to a zero-dimensional Polish refinement of its topology. We give some more details on these operations in the next section, together with characterisations of Polish and compact zero-dimensional spaces as respectively colimits and limits of particular diagrams.

### 2.2.1 Characterisations of zero-dimensional spaces

A countable codirected diagram in $\mathbb{A}$ (an $\mathbb{A}$ *ccd* for short) is a functor $D : I^{op} \to \mathbb{A}$ where $I$ is a countable directed partial order and $\mathbb{A}$ is a subcategory of **Pol**. The characterisation of objects of $\mathbf{Pol}_{cz}$ can be formulated as follows:

▶ **Proposition 1** ([11]). *A space $X$ is compact zero-dimensional if and only if there exists a $\mathbf{Pol}_f$ ccd $D$ such that $X \cong \lim D$.*

Any Polish space can be written as the colimit of a diagram in $\mathbf{Pol}_z$:

▶ **Proposition 2** ([5], Proposition 3.2). *Let $X$ be a Polish space and $\mathcal{F}$ a countable base of the topology of $X$. Let $Z_X(\mathcal{F}) \triangleq (U(X), \langle Bool(\mathcal{F})\rangle)$ be the space having the same underlying set as $X$ and the topology generated by the Boolean algebra generated by $\mathcal{F}$. The following holds: (i) $Z_X(\mathcal{F})$ is zero-dimensional Polish, (ii) $\mathcal{B}(Z_X(\mathcal{F})) = \mathcal{B}(X)$. We call $Z_X(\mathcal{F})$ the* zero-dimensionalisation *of $X$ with respect to $\mathcal{F}$.*

The family of zero-dimensionalisations of a space $X$ indexed by all countable bases of $X$ forms a codirected diagram. This diagram is indexed by $Bases(X)$ the set of all countable bases of $X$ partially ordered by inclusion; $Bases(X)$ is directed by closing the union of two bases under finite intersection [4, Def. 2.10]. If $\mathcal{F}$, $\mathcal{G}$ are such bases then, if $\mathcal{F} \subseteq \mathcal{G}$, the identity function is continuous from $Z_X(\mathcal{G})$ to $Z_X(\mathcal{F})$. This defines a codirected diagram from the directed partial order $Bases(X)$ to $\mathbf{Pol}_z$, that we still denote by $Z_X$. The following statement states that any Polish space is the colimit of its diagram of zero-dimensionals:

▶ **Theorem 3** ([5], Th. 3.5; [4], Proposition 2.11)**.** *For every Polish space $X$, $X \cong \operatorname{colim} Z_X$.*

## 2.3 Converging in $\mathsf{G}(X)$

We recall some standard facts about convergence in $\mathsf{G}(X)$ for $X$ Polish. The *boundary* of a set $A \subseteq X$ is the set-theoretic difference between its closure and its interior, and is denoted by $\partial_X A$. By the Portmanteau theorem ([2], Th. 2.1), a sequence $(p_n)_{n\in\mathbb{N}}$ of probability measures converges to $p \in \mathsf{G}(X)$ iff $p_n(A) \to p(A)$ for each Borel set $A$ which is a *$p$-continuity set*, i.e. which verifies $p(\partial_X A) = 0$. Note that for all $p$, $p$-continuity sets form a Boolean algebra that we denote $\mathcal{C}_X(p)$ ([17], Lemma 6.4). We have the following facts:

▶ **Lemma 4.** *Countable Polish spaces are zero-dimensional.*

**Proof.** Let $X$ be Polish, $x \in X$ and $U$ be open in $X$. It is not difficult to use the metric to define a function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$ when $y \in U^c$. If $X$ is countable, then $f[X]$ is countable and thus there exists $p \in [0,1]$ such that $p \notin f[X]$ and it is easy to check that $f^{-1}([0,p)) = f^{-1}([0,p])$ is clopen and included in $U$, and the conclusion follows. ◀

▶ **Lemma 5.** *Let $X, Y$ be Polish, $p \in \mathsf{G}(X)$ and let $f : X \to Y$ be continuous. If $B$ is a $\mathsf{G}(f)(p)$-continuity set then $f^{-1}(B)$ is a $p$-continuity set.*

**Proof.** Direct consequence of the inequality $\partial_X(f^{-1}(B)) \subseteq f^{-1}(\partial_Y B)$. ◀

▶ **Lemma 6.** *Let $X$ be Polish and uncountable and let $\{p_i\}_{i\in I} \subseteq \mathsf{G}(X)$ be a countable family of probability measures. There exists a countable base $\mathcal{F}$ of $X$ such that $\mathcal{F} \subseteq Boole(\mathcal{F}) \subseteq \cap_i \mathcal{C}_X(p_i)$.*

**Proof.** Let $B(x,\epsilon)$ be the open ball of radius $\epsilon > 0$ centered on $x$. Observe that for $0 < \epsilon < \epsilon'$, $\partial_X B(x,\epsilon) \cap \partial_X B(x,\epsilon') = \emptyset$. Therefore, for a given $p$, there can at most be countably many radiuses $\epsilon_k$ such that the $B(x,\epsilon_k)$ are *not* $p$-continuity sets, as otherwise the total mass of $\cup_k \partial_X B(x,\epsilon_k)$ would diverge. Using that a countable union of countable sets is countable, there are at most countably many radiuses $\epsilon_k$ such that $B(x,\epsilon_k) \notin \cap_{i\in I}\mathcal{C}_X(p_i)$. For any dense subset $E \subseteq \mathbb{R}_{>0}$, the open balls $\mathcal{N}(x) = \{B(x,\epsilon)\}_{\epsilon\in E}$ characterise convergence to $x$. For our purposes, it is enough to take $E$ such that $E$ does not intersect the forbidden radii $\{\epsilon_k\}$, which can always be done. The sought base $\mathcal{F}$ is obtained by considering a countable dense subset $D$ of $X$ and taking $\mathcal{F}$ to be the closure under finite intersections of $\{\mathcal{N}(x) \mid x \in D\}$. Since continuity sets form a boolean algebra, we get $Boole(\mathcal{F}) \subseteq \cap_i \mathcal{C}_X(p_i)$. ◀

# 3 The Machine

The parameterised models we are interested in use 'the Machine' [4], a powerful theorem allowing one to extend a class of natural transformations from finite Polish spaces to arbitrary ones. This extension theorem hinges on particular conditions on the domain and

codomain functors of the natural transformation, corresponding respectively to constraints on parameters and models. Accordingly, we will call domain functors '$\mathcal{P}$-functors' and codomain functors '$\mathcal{M}$-functors'. Below, we list these conditions. In Section 4, we will study closure properties of these conditions, and derive a syntax for parameters and models.

## 3.1 Parameter condition

The Machine applies to natural transformations whose $\mathcal{P}$-functor commutes with colimits of diagrams of zero-dimensionals (Section 2.2.1). We call this property *$Z$-cocontinuity*:

▶ **Definition 7** ($Z$-cocontinuity)**.** An endofunctor $F : \mathbf{Pol} \to \mathbf{Pol}$ is $Z$-cocontinuous if for all space $X$, there exists an isomorphism $F(\operatorname{colim} Z_X) \cong \operatorname{colim} FZ_X$.

In order for the Machine to apply, $\mathcal{P}$-functors must also preserve epis. The *parameter condition* is the conjunction of these two conditions.

▶ **Definition 8** (Parameter conditions)**.** An endofunctor $F : \mathbf{Pol} \to \mathbf{Pol}$ satisfies the *parameter condition* (or equivalently, is a $\mathcal{P}$-functor) if (i) $F$ is $Z$-cocontinuous and (ii) $F$ preserves epis.

▶ **Example 9.** The following are $\mathcal{P}$-functors: (i) the identity functor (it is $Z$-cocontinuous by Theorem 3); (ii) the Polish Giry functor $\mathsf{G}$ ($Z$-cocontinuity is proved in Ref. [5, Th. 3.7]), and the related finite positive nonzero measure functor $\mathsf{M}^+$; (iii) the multiset functor $\mathsf{B}$ (see [4]). (iv) for any discrete (and thus at most countable) space $X$, $\mathsf{C}(X, -)$ is trivially $Z$-cocontinuous.

## 3.2 Model condition

The Machine also requires $\mathcal{M}$-functors to verify a list of conditions, corresponding to constraints on models.

### The model condition

Before defining the model condition, we introduce some terminology related to commutation of functors with some limits.

▶ **Definition 10.** Let $\mathbb{A}$ be a subcategory of $\mathbf{Pol}$. An endofunctor $G : \mathbf{Pol} \to \mathbf{Pol}$ is $\mathbb{A}$-continuous if for all *ccd* (Section 2.2) $D : I^{op} \to \mathbb{A}$, $G(\lim D) \cong \lim GD$.

$\mathcal{M}$-functors are endofunctors that satisfies the following.

▶ **Definition 11** (Model conditions)**.** An endofunctor $G : \mathbf{Pol} \to \mathbf{Pol}$ satisfies the *model condition* (or equivalently, is an $\mathcal{M}$-functor) if: (i) $G$ preserves monos, (ii) $G$ preserves embeddings, (iii) $G$ preserves intersections, (iv) $G$ is $\mathbf{Pol}_f$-continuous.

▶ **Example 12.** The following are $\mathcal{M}$-functors: (i) the Giry functor $\mathsf{G}$, and finite measure functors $\mathsf{M}^+$; (ii) the multiset and list functors $\mathsf{B}, \mathsf{W}$; (iii) the Vietoris functor $\mathsf{V}$; (iv) the Skorokhod functor $\mathsf{D}$ and the continuous map functor $\mathsf{C}(X, -)$ from a compact Polish space $X$ (see [4]).

Observe that in Definition 11, all conditions are preserved by composition of endofunctors except the last one. We will come back to this in Section 4.

## 3.3 The Machine

The Machine states that natural transformations (parameterised models) between $\mathcal{M}$-functors and $\mathcal{P}$-functors are entirely characterised by their components on finite spaces.

▶ **Theorem 13** ([4]). *Let $F_1$ be a $\mathcal{P}$-functor and $F_2$ be a $\mathcal{M}$-functor; one has $Nat(F_1, F_2) \cong Nat(F_1|_{\mathbf{Pol}_f}, F_2|_{\mathbf{Pol}_f})$.*

▶ **Example 14.** Let us give some examples of finitely characterised natural transformations.

- The unite of the Giry monad $\eta : \mathbb{1} \Rightarrow \mathsf{G}$ is entirely characterised by its finite components. We conjecture that the parameter condition is closed under composition by $\mathsf{G}$, which would imply that multiplication $\mu : \mathsf{G}^2 \Rightarrow \mathsf{G}$ of the Giry monad is also characterised on the whole of **Pol** by its finite components.
- The normalisation $\nu : \mathsf{M}^+ \Rightarrow \mathsf{G}$ defined by $\nu_X : \mathsf{M}^+X \to \mathsf{G}X, \mu \mapsto \frac{\mu}{\mu(X)}$ is also finitely characterised. Moreover, the Machine allows to prove that it is the unique natural transformation between $\mathsf{M}^+$ and $\mathsf{G}$ [4, Th. 5.2].

Classical objects of statistics can be framed as natural transformations:

- the i.i.d. distribution on sequences of samples $iid : \mathsf{G} \Rightarrow \mathsf{G}(-^{\mathbb{N}})$ (Section 5);
- the Dirichlet process $\mathscr{D} : \mathsf{M}^+ \Rightarrow \mathsf{G}^2$ a cornerstone of Bayesian nonparametrics [8];
- the Poisson process $\mathscr{P} : \mathsf{M}^+ \Rightarrow \mathsf{GB}$ which is the prototypical point process. Using the Machine, it is enough to define the Poisson process on finite sets, this is done via

$$\mathscr{P}_n : \mathsf{M}^+(n) \simeq (\mathbb{R}_+)^n \to \mathsf{GB}(n) \simeq \mathsf{G}(\mathbb{N}^n), (\lambda_1, \ldots, \lambda_n) \mapsto \mathrm{Po}(\lambda_1) \times \ldots \times \mathrm{Po}(\lambda_n)$$

  where $\mathrm{Po}(\lambda)$ is the Poisson measure on $\mathbb{N}$ with parameter $\lambda$.

# 4 A grammar for parameterised models

We turn now to the main question of the paper, which is to find operations on functors under which the parameter and model conditions are closed. For parameters, this result takes the form of a simple grammar over functors. For models, we develop a simple type system over polynomial terms generated from a family of functors, well-typedness implying the model condition. This syntax for parameter and models lifts to natural transformations, giving rise to a language of natural combinators for parameterised models.

As **Pol** has all countable limits and coproducts, the category of **Pol** endofunctors is closed under at most countable coproducts and products (recall that if $F, G : \mathbf{Pol} \to \mathbf{Pol}$ are two endofunctors, their coproduct $F + G$ acts on objects by $(F + G)(X) = F(X) + G(X)$ and on morphisms by $(F + G)(f) = F(f) + G(f)$, and similarly for products). Endofunctors are also trivially closed under composition.

## 4.1 Closure properties of the parameter condition

Let us start with parameter conditions. At the time of writing, we do not know whether these are closed under products and/or functor composition. However, we show that they are closed under coproducts. We also derive specific results for particular functors that altogether yield a sufficiently expressive class of parameterisations.

### Finite coproducts preserve the parameter condition

The following facts are easily verified:

▶ **Proposition 15.** *If $G, H$ preserve epis, then so does $G + H$.*

▶ **Proposition 16.** *The parameter condition is preserved by finite coproducts.*

### Products of Giry-like functors satisfy the parameter condition

Countable products of Giry-like functors (i.e. $\mathsf{G}, \mathsf{M}^+$) satisfy the parameter condition. The case of finite products follow trivially from the same result.

▶ **Proposition 17.** *Let $\{F_k\}_{k\in\mathbb{N}}$ be given with $F_k \in \{\mathsf{G}, \mathsf{M}^+\}$. Then $\prod_k F_k$ satisfies the parameter condition.*

**Proof.** It is enough to treat the case of $\mathsf{G}$ as $\mathsf{M}^+$ is naturally isomorphic to $\mathsf{G} \times \mathbb{R}_{>0}$. Preservation of epis by $\mathsf{G}$ lifts to products of $\mathsf{G}$. Let us prove $Z$-cocontinuity. We reuse the proof method of ([5], Th. 3.7). It is enough to exhibit, for all $X$ and all countable family of converging sequences $\{p_n^{(k)} \to p^{(k)}\}_{k\in\mathbb{N}}$ in $\mathsf{G}(X)^{\mathbb{N}}$, a base of $\mathcal{F} \in \mathit{Bases}(X)$ s.t. the sequence converge in $\mathsf{G}(Z_X(\mathcal{F}))^{\mathbb{N}}$. It follows from Lemma 4 that for countable Polish spaces there is nothing to show, and we can thus assume w.l.o.g. that $X$ is uncountable. Applying Lemma 6, we get a base $\mathcal{F}$ s.t. $\mathit{Boole}(\mathcal{F}) \subseteq \cap_k \mathcal{C}_X(p^{(k)})$. Noting that $\mathit{Boole}(\mathcal{F})$ is a base of $Z_X(\mathcal{F})$, an application of Th. 2.2 in (Billingsley [2]) concludes. ◀

### Giry-like functors over products satisfy the parameter condition

Our grammar for parameterisations admits a way of specifying quantitative relations on points of the underlying space, as shown below in Proposition 18 (which generalises to Giry-like functors $\mathsf{M}^+$). In what follows, we treat countable products. The case of finite products follows easily from the same proof.

▶ **Proposition 18.** $\mathsf{G}(-^{\mathbb{N}})$ *satisfies the parameter condition.*

**Proof.** Preservation of epis still follows from the properties of $\mathsf{G}$. To prove $Z$-cocontinuity, we follow the proof scheme of Proposition 17. We denote $\pi_k : X^{\mathbb{N}} \to X, \pi_{1\ldots k} : X^{\mathbb{N}} \to X^k$ the canonical projections. Let $X$ be Polish and let $(p_n)_{n\in\mathbb{N}} \to p$ be a converging sequence in $\mathsf{G}(X^{\mathbb{N}})$. By Lemma 6, there exists a base $\mathcal{F}$ of $X$ such that for all $k > 0$, $\mathit{Boole}(\mathcal{F}) \subseteq \cap_k \mathcal{C}_X(\mathsf{G}(\pi_k)(p))$. The sets of the form $\pi_k^{-1}(O)$ with $O$ ranging in $\mathit{Boole}(\mathcal{F})$ induce a base $\mathcal{H}$ of $Z_X(\mathcal{F})^{\mathbb{N}}$. Using Lemma 5 plus the fact that continuity sets are closed under finite intersections, we deduce that $\mathcal{H} \subseteq \mathcal{C}_{X^{\mathbb{N}}}(p)$. Therefore, for all $V \in \mathcal{H}$, $p_n(V) \to p(V)$. Using Th. 2.2 in (Billingsley [2]), we get that $p_n \to p$ in $\mathsf{G}(Z_X(\mathcal{F})^{\mathbb{N}})$. We conclude that $\mathsf{G}(-^{\mathbb{N}})$ is $Z$-cocontinuous. ◀

## 4.2 Closure properties of the model condition

We turn now to closure properties of model conditions. As we are going to show, all model conditions (Definition 11) are closed under all polynomial operations with the exception of the last one, namely $\mathbf{Pol}_f$-continuity.

### Finite coproducts preserve the model condition

We prove preservation of the model condition under finite coproducts. We proceed with the other parts of the model condition, namely preservation of embeddings, intersections and $\mathbf{Pol}_f$-continuity. The following proposition is routine.

▶ **Proposition 19.** *(i) If $G, H$ preserve monos, then so does $G + H$. (ii) If $G, H$ preserve embeddings, then so does $G + H$. (iii) If $G, H$ preserve embeddings and intersections, then so does $G + H$.*

The following one is a bit more technical and deserves a proof.

▶ **Proposition 20.** *If $G, H$ are $\mathbb{A}$-continuous, then so is $G + H$.*

**Proof.** Let $(X_i)_{i\in I}$ be a ccd with limit $X$, we're assuming that $GX = \lim_i GX_i$ and $HX = \lim_i HX_i$. The diagrams $(GX_i)_{i\in I}$ and $(HX_i)_{i\in I}$ define a diagram $(GX_i + HX_i)_{i\in I}$ where for each $i < j$ there exists a unique arrow $Gp_{i,j} + Gp_{i,j} : GX_i + HX_i \to GX_j + HX_j$, with $p_{i,j} : X_i \to X_j$ the connecting morphism of the diagram $(X_i)_{i\in I}$. The coproduct $GX + HX$ is a cone over this diagram via the maps $Gp_i + Hp_i : GX + HX \to GX_i + HX_i$ constructed from the canonical projections $p_i : X \to X_i$ by the universal property of coproducts. There must therefore exist a unique continuous map $\phi : GX + HX \to \lim_i(GX_i + HX_i)$ which maps a thread in $GX$ to the obvious thread in $\lim_i(GX_i + HX_i)$ and similarly for threads in $HX$. It is clear that $\phi$ is injective, moreover it is easy to see that threads in $\lim_i(GX_i + HX_i)$ must be the form $(x_i)_{i\in I}$ with every $x_i \in GX_i$ or with every $x_i \in HX_i$, and in particular $\phi$ is surjective. Thus $\phi$ is continuous and bijective, and it only remains to show that it is open. Let $U$ be open in $GX + HX$. By definition of the coproduct topology, $U_G = i_{GX}^{-1}(U)$ and $U_H = i_{HX}^{-1}(U)$ are open, and thus by definition of the topology on the limit,

$$U_G = \bigcup_i Gp_i^{-1}(V_i) \quad \text{and} \quad U_H = \bigcup_j Gq_j^{-1}(W_j)$$

where each $V_i$ (resp. $W_j$) is open in $GX_i$ (resp. $HX_j$). Since $I$ is cofiltered, for every $i, j \in I$ there exists a $k \in I$ and morphisms $p_{k,i} : X_k \to X_i$ and $p_{k,j} : X_k \to X_j$. Let us denote by $i \wedge j$ the choice of such a $k$ for the pair $i, j$ and note that $Gp_i^{-1}(W) = Gp_{i\wedge j}^{-1}(Gp_{i\wedge j,i}^{-1}(W))$. Consider now the set

$$V = \bigcup_{i,j} q_{i\wedge j}^{-1}(Gp_{i\wedge j,i}^{-1}(U_i) \uplus Hp_{i\wedge j,i}^{-1}(U_j))$$

where $q_i$ is the canonical projection $(\lim_i HX_i + GX_i) \to HX_i + GX_i$. By construction it is open in $\lim_i(GX_i + HX_i)$ since it is a union of inverse images of sets which are open in $GX_{i\wedge j} + HX_{i\wedge j}$ by definition of the coproduct topology. We claim that $\phi[U] = V$. For any thread $(x_i) \in U$, if the thread is in the $HX$ component then it must belong to $U_G$ and thus there must exist an $i$ such that $x_i \in V_i$, since we can assume that the connecting morphisms are surjective there exists for each $j$ an element $x_{i\wedge j} \in Gp_{i\wedge j,i}^{-1}(U_i)$ in the thread and it follows that the thread belongs to $V$, and similarly if the thread $(x_i)$ is in the $GX$ component. Similarly, starting from $(x_i)$ in $V$, it is clear that $(x_i)$ belongs to $U$ and it thus $\phi$ is open. ◀

### Finite products preserve the model condition

▶ **Proposition 21.** *(i) If $G, H$ preserve monos, then so does $G \times H$. (ii) If $G, H$ preserve embeddings, then so does $G \times H$. (iii) If $G, H$ preserve intersections, then so does $G \times H$. (iv) If $G, H$ are $\mathbb{A}$-continuous, then so is $G \times H$.*

**Proof.** (i) Straightforward. (ii) Since embeddings are equalizers in **Pol**, this result is simply a case of limits commuting with limits (Mac Lane [15] IX). (iii) Similarly, since intersections are finite limits and they commute with finite limits in **Pol**. (iv) Finally, since ccds are cofiltered limits, they commute with finite products. ◀

### Composition

We now consider the operation of functor composition. The following is trivial:

▶ **Proposition 22.** *The conditions 1. to 3. of Definition 11 are preserved under functor composition.*

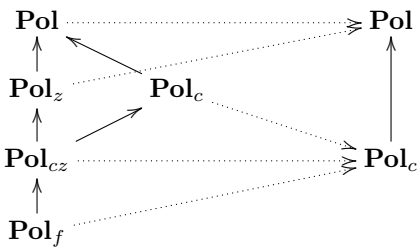



**Figure 1** Signature for G and V.

The condition of $\mathbf{Pol}_f$-continuity (Definition 11, 4.) does not behave as well: if $F, G$ are $\mathbf{Pol}_f$-continuous endofunctors and $F$ maps finite spaces to non-finite spaces, $GF$ has no reason to be $\mathbf{Pol}_f$-continuous. On the other hand, if $F$ maps finite spaces to a subcategory with respect to which $G$ is continuous, then the composition $GF$ will be $\mathbf{Pol}_f$-continuous. In order to make this intuition formal, we need to capture the global behaviour of functors on the subcategories of **Pol**. To do so, we propose to abstract functors as monotonic functions on the poset of subcategories of **Pol**.

▶ **Definition 23** (Partial order on subcategories)**.** We denote by $(\mathbb{P}, \leq)$ the lattice over the subcategories of **Pol** displayed in Equation 1 and generated by the subcategory relation, i.e. $\mathbb{A} \leq \mathbb{B}$ iff $\mathbb{A}$ is a subcategory of $\mathbb{B}$. We will denote by $\wedge$ and $\vee$ the infinimum and the supremum.

Note that $\mathbb{P}$ has as maximal element **Pol** and as minimal element $\mathbf{Pol}_f$. The known behaviour of a endofunctor over **Pol** can be presented as a monotonic function from $\mathbb{P}$ to itself. We call such a function a *signature* assigned to the functor.

▶ **Definition 24** (Signatures and signature assignments)**.** We denote by $Sign(\mathbb{P})$ the set of order-preserving functions from $\mathbb{P}$ to itself. We say that an endofunctor $G$ admits $f \in Sign(\mathbb{P})$ as a signature if for all $\mathbb{A} \in \mathbb{P}$, there exists a functor $G' : \mathbb{A} \to f(\mathbb{A})$ such that the following diagram commutes in **Cat**:

$$\begin{tikzcd} \mathbf{Pol} \arrow[r, "G"] & \mathbf{Pol} \\ \mathbb{A} \arrow[u, "I_{\mathbb{A}\mathbf{Pol}}"] \arrow[r, dashed, "G'"'] & f(\mathbb{A}) \arrow[u, "I_{f(\mathbb{A})\mathbf{Pol}}"'] \end{tikzcd} \tag{2}$$

where $I_{\mathbb{A}\mathbb{B}}$ denotes the obvious inclusion functor. If $G$ admits $f$ as a signature, we call the pair $(G, f)$ a *signature assignment*.

▶ **Example 25.** It is known that the Giry functor G and the Vietoris functor V preserve compactness (see resp. Parthasarathy [17], Th. 6.4 and Kechris [13], Th. 4.26). Therefore, both G and V admit the signature (in dotted arrows) in Figure 1. However, the fact that V maps finite spaces to finite spaces implies that it admits a finer signature (Figure 2). Note also that the functor $\mathsf{M}^+$ does *not* preserve compactness.

The exact signature of a functor might be unknown. However, it is always possible to assign to a functor the signature corresponding to the constant function $\mathbb{A} \in \mathbb{P} \mapsto \mathbf{Pol}$. In fact, the lattice structure on $\mathbb{P}$ lifts to signatures:

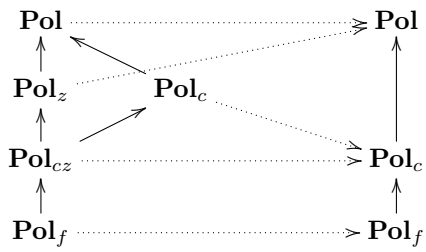



**Figure 2** A finer signature for $\mathsf{V}$.

▶ **Lemma 26.** *Define the relation on $Sign(\mathbb{P})$ $f \leq^* g \Leftrightarrow \forall \mathbb{A} \in \mathbb{P}, f(\mathbb{A}) \leq g(\mathbb{A})$. Set $f \wedge g = \mathbb{A} \mapsto f(\mathbb{A}) \wedge g(\mathbb{A})$ and similarly for $\vee$. Then $(Sign(P), \leq^*)$ is a lattice and the constant function $\mathbb{A} \mapsto \mathbf{Pol}$ is its maximal element.*

**Proof.** Reflexivity and transitivity are trivial. Assume $f \leq^* g$ and $g \leq^* f$, then by antisymmetry of $(P, \leq)$ we have $f = g$. Maximality of the constant **Pol** function is trivial. ◀

Let us now define a criterion for functor composition ensuring preservation of $\mathbf{Pol}_f$-continuity.

▶ **Proposition 27.** *Let $F$ and $G$ be respectively a $\mathbb{A}$-continuous and a $\mathbb{B}$-continuous functor such that $F$ admits signature $f$ and $G$ admits signature $g$. If $f(\mathbf{Pol}_f) \leq \mathbb{B}$, then $GF$ is $\mathbb{C}$-continuous, where $\mathbb{C} = \sup\{\mathbb{C}' \mid \mathbb{C}' \leq \mathbb{A} \wedge f(\mathbb{C}') \leq \mathbb{B}\}$.*

**Proof.** Since $f(\mathbf{Pol}_f) \leq \mathbb{B}$, we know that $\mathbb{C}$ exists and verifies $\mathbf{Pol}_f \leq \mathbb{C}$. Let $D : I^{op} \to \mathbb{C}$ be a $\mathbb{C}$ *ccd*. By assumption of $\mathbb{A}$-continuity and using hat $\mathbb{C} \leq \mathbb{A}$, $F(\lim D) \cong \lim FD$. Since $F$ admits $f$ as a signature, $FD$ is a $f(\mathbb{C})$-*ccd* and since $f(\mathbb{C}) \leq \mathbb{B}$, $G(\lim FD) \cong \lim GFD$. ◀

In the next section, we will leverage Proposition 27 by defining a type system for polynomial composites of endofunctors.

## 4.3 Syntax for parameterisations and models

We capture the results of this section into grammars for parameterisations and models.

### A grammar for parameterisations

In what follows, we let $\mathcal{G} = \{\mathsf{G}, \mathsf{M}^+\}$ be the Giry-like functors (respectively Giry, the finite positive measure and finite nonzero measure functors). We recall that $\Delta$ is the diagonal functor.

▶ **Definition 28** (Parameterisations generated by a family of functors)**.** Parameterisations, denoted by $\mathcal{P}$, are defined by the following grammar:

$$\begin{array}{rcl} \mathcal{P} & ::= & F \mid G\Delta \mid G \times G \mid \mathcal{P} + \mathcal{P} \\ G & ::= & \mathsf{G} \mid \mathsf{M}^+ \end{array}$$

where $F$ ranges over functors satisfying the parameter condition.

We have the following expected result:

▶ **Theorem 29.** *All $P \in \mathcal{P}$ verify the parameter condition.*

**Proof.** By induction, using the results of Section 4.1. ◀

$$\textsc{Axiom}\quad \frac{F \in \mathcal{M}_0 \qquad F \text{ is } \mathbb{A}\text{-continuous} \qquad F \text{ admits signature } f}{F :: (\mathbb{A}, f)}$$

$$\textsc{Sum}\quad \frac{F :: (\mathbb{A}, f) \qquad G :: (\mathbb{B}, g)}{F + G :: (\mathbb{A} \wedge \mathbb{B}, f \vee g)} \qquad \textsc{Product}\quad \frac{F :: (\mathbb{A}, f) \qquad G :: (\mathbb{B}, g)}{F \times G :: (\mathbb{A} \wedge \mathbb{B}, f \vee g)}$$

$$\textsc{Composition}\quad \frac{F :: (\mathbb{A}, f) \qquad G :: (\mathbb{B}, g) \qquad g(\mathbf{Pol}_f) \leq \mathbb{A}}{F \circ G :: (\mathbb{C}, f \circ g) \qquad \text{where } \mathbb{C} = \sup\{\mathbb{C}' \mid \mathbb{C}' \leq \mathbb{A} \wedge f(\mathbb{C}') \leq \mathbb{B}\}}$$

**Figure 3** Inferences rules for the type system on models.

## A grammar for models

Proposition 27 gives a sufficient condition ensuring that composition of functors satisfying the model condition still satisfies the model condition. We integrate this result in a type system for polynomial composites of functors satisfying the model condition.

▶ **Definition 30** (Functor types). A *functor type* is a pair $(\mathbb{A}, f)$ with $\mathbb{A} \in \mathbb{P}$ and $f \in Sign(\mathbb{P})$. The set of types is defined by $Types(\mathbb{P}) \triangleq \mathbb{P} \times Sign(P)$.

Functor types are assigned to elements of the polynomial closure of the set of functors that satisfy the model condition.

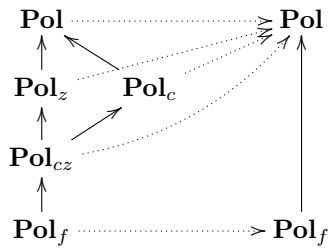
▶ **Definition 31** (Typing judgments). We inductively define a relation between functors satisfying the model condition and functor types through the set of inference rules in Figure 3. The fact that a functor $F$ admits the type $(\mathbb{A}, f)$ will be denoted by $F :: (\mathbb{A}, f)$.

Our type system is sound with respect to the model condition.

▶ **Theorem 32.** *If $M :: (\mathbb{A}, f)$ then $M$ is $\mathbb{A}$-continuous, $m$ admits signature $f$ and $M$ satisfies the model condition.*

**Proof.** The proof is by induction. The properties of preservation of monos, preservation of embeddings and preservation of intersections are treated in Section 4.2. **Sum rule**. Both $F$ and $G$ are $\mathbb{A} \wedge \mathbb{B}$-continuous, therefore by Proposition 20, $F + G$ is $\mathbb{A} \wedge \mathbb{B}$-continuous (and therefore are least $\mathbf{Pol}_f$-continuous). It is clear that a coproduct of finite spaces is finite and similarly for compact zero-dimensional spaces, compact spaces and zero-dimensional spaces. Therefore, $F + G$ admits $f \vee g$ as a signature. The case of the product rule is similar. **Composition**. $\mathbb{C}$-continuity is by Proposition 27. That $FG$ admits $f \circ g$ as a signature is trivial. ◀

▶ **Example 33.** It is instructive to consider the the multiset functor $\mathsf{B}$. It maps finite spaces to finite spaces but we ignore its behaviour on other subcategories, hence the signature:




It is only known to be $\mathbf{Pol}_f$-continuous ([4]). Therefore, $\mathsf{GB}$ is a valid model functor but $\mathsf{BG}$ breaks the third premise of the composition rule in Figure 3: indeed, $\mathsf{G}$ maps finite spaces to compact spaces (Figure 1).

▶ **Definition 34.** The set of models is defined to be that of typeable functors and will be denoted by $\mathcal{M}$.

### Natural parameterised models

Theorem 29 and 32 delineate a class of parameters and models to which the Machine (Theorem 13) applies. These combined results can be reframed concisely as follows:

▶ **Theorem 35.** *For all parameterisation $P \in \mathcal{P}$ and all model $M \in \mathcal{M}$,*

$$Nat(P, M) \cong Nat(P|_{\mathbf{Pol}_f}, M|_{\mathbf{Pol}_f}).$$

## 5 Applications

It is hard to overstate the importance of independently and identically distributed (*i.i.d.*) sequences of random of variables and their generalisation to exchangeable processes to probability and statistics, as witnessed by the wealth of powerful asymptotic results which apply to them – to name a few, the law of large number, the central limit theorem and the de Finetti theorem [12]. We illustrate the usefulness of our framework by recasting i.i.d. processes and the de Finetti theorem as instances of our parameterised models.

### 5.1 The $iid$ natural transformation

Let $X$ be finite Polish. For all integer $n > 0$, we construct an arrow $iid^n_X : \mathsf{G}(X) \to \mathsf{G}(X^n)$ as follows: $iid^n_X(p) = (B_1, B_2, \ldots, B_n) \mapsto p(B_1) \cdot p(B_2) \cdots p(B_n)$. One easily verifies that this map is well-typed and continuous. We have the following result:

▶ **Proposition 36.** *For all all positive integer $n$, the family $iid^n_X$ defines a natural transformation $iid^n : \mathsf{G} \Rightarrow \mathsf{G}(-^n)$.*

**Proof.** Let $X, Y$ be finite spaces and let $f : X \to Y$ be a function. Let $p \in \mathsf{G}(X)$ be given and let $(y_1, \ldots, y_n)$ be a sequence in $Y^n$. We have:

$$\begin{aligned}
(\mathsf{G}(f^n) \circ iid^n_X)(p)(y_1, \ldots, y_n) &= (iid^n_X(p) \circ (f^n)^{-1})(y_1, \ldots, y_n) \\
&= p(f^{-1}(y_1)) \cdot p(f^{-1}(y_2)) \cdots p(f^{-1}(y_n)) \\
&= iid^n_Y(\mathsf{G}(f)(p))(y_1, \ldots, y_n)
\end{aligned}$$

We have proved that the $iid^n$ transformation is well defined on all finite spaces. One easily checks that $\mathsf{G}$ is a $\mathcal{P}$-functor and $\mathsf{G}(-^n)$ is a $\mathcal{M}$-functor. Applying Theorem 35, we conclude that $iid^n$ admits an unique extension to the whole of $\mathbf{Pol}$. ◀

$$\begin{tikzcd} \mathsf{G} \arrow[r, "iid"] \arrow[dr, "iid"'] & \mathsf{G}(-^{\mathbb{N}}) \arrow[d, "G(\hat{\sigma})"] \\ & \mathsf{G}(-^{\mathbb{N}}) \end{tikzcd}$$

**Figure 4** *iid* is exchangeable.

$$\begin{tikzcd} \mathsf{G}^2 \arrow[r, "\mathsf{G}(iid)"] \arrow[dr, "\mathsf{G}(iid)"'] & \mathsf{G}^2(-^{\mathbb{N}}) \arrow[r, "\mu"] \arrow[d, "G^2(\hat{\sigma})"] & \mathsf{G}(-^{\mathbb{N}}) \arrow[d, "G(\hat{\sigma})"] \\ & \mathsf{G}^2(-^{\mathbb{N}}) \arrow[r, "\mu"'] & \mathsf{G}(-^{\mathbb{N}}) \end{tikzcd}$$

**Figure 5** Mixtures of *iid* are exchangeable.

The family of natural transformations $\{iid^n\}_{n>0}$ can in turn be extended to a natural transformation $iid : \mathsf{G} \Rightarrow \mathsf{G}(-^{\mathbb{N}})$. The following result relies on the Bochner extension theorem ([5], Th. 2.5) along with the naturality of the canonical projections $\pi_n : -^{\mathbb{N}} \Rightarrow -^n$ and $\pi_{nm} : -^n \Rightarrow -^m$.

▶ **Proposition 37.** *There exists a unique natural transformation* $iid : \mathsf{G} \Rightarrow \mathsf{G}(-^{\mathbb{N}})$ *such that for all* $n$, $iid^n = \mathsf{G}(\pi_n) \circ iid$.

## 5.2 Exchangeable measures and the de Finetti theorem

For all $n > 0$, we denote by $S_n$ the symmetric group on $\{1, \ldots, n\}$. Each $\sigma \in S_n$ induces a natural transformation $\hat{\sigma} : -^{\mathbb{N}} \Rightarrow -^{\mathbb{N}}$ whose component at $X$ is defined by $\hat{\sigma}_X(x_1, \ldots, x_n, \ldots) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}, \ldots)$. As illustrated by the commutative diagram in Figure 4, the distribution of *iid* is invariant by the action of such permutations. Elements of $\mathsf{G}(X^{\mathbb{N}})$ invariant by the action of $\mathsf{G}(\hat{\sigma})$ for all $n$ and all $\sigma \in S_n$ are called *exchangeable measures*. The diagram in Figure 4 indicates that *iid* is a natural family of exchangeable measures. An easy computation shows that the same property is verified by *mixtures of iids* (see Figure 5). The de Finetti theorem states that *all* exchangeable measures can be represented as such mixtures of *iid*s. We will prove that this representation is *natural*. We introduce a functor mapping any Polish space $X$ to the space of exchangeable measures $\mathsf{G}_{ex}(X^{\mathbb{N}})$. Exchangeable measures form a closed convex subset of $\mathsf{G}(X^{\mathbb{N}})$, therefore they form a Polish space when given the subspace topology. We have the following result:

▶ **Proposition 38.** $\mathsf{G}_{ex}(-^{\mathbb{N}})$ *is a* $\mathcal{P}$*-functor.*

**Proof.** Note that $\mathsf{G}_{ex}(-^{\mathbb{N}})$ is a subfunctor of $\mathsf{G}(-^{\mathbb{N}})$ which is a $\mathcal{P}$-functor (Proposition 18). $Z$-cocontinuity is easily seen to be preserved by subfunctors. Preservation of epis follow from the measurable selection theorem and naturality of $\hat{\sigma}$ for all $\sigma \in S_n$. ◀

Let us introduce another natural transformation: for all $n > 0$, the *empirical measure* $\mathcal{E}_n : -^n \Rightarrow \mathsf{G}$ computes the relative frequencies of elements of a sequence and is defined by $\mathcal{E}_{n,X}(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$. For all $n$, we define the *random empirical measure at time* $n$ $\mathsf{G}(\mathcal{E}_n \circ \pi_n) : \mathsf{G}_{ex}(-^{\mathbb{N}}) \to \mathsf{G}^2$. This defines a sequence of natural transformations indexed by $n$. The de Finetti theorem gives us the following:

▶ **Theorem 39** (de Finetti [12])**.** *Let* $X$ *be Polish.*

- *For all* $P \in \mathsf{G}_{ex}(X^{\mathbb{N}})$*, the limit* $deF_X(P) \triangleq \lim_n \mathsf{G}(\mathcal{E}_n \circ \pi_n)(P)$ *exists in* $\mathsf{G}^2(X)$*;*
- *the associated map* $P \mapsto deF_X(P)$ *is continuous from* $\mathsf{G}_{ex}(X^{\mathbb{N}})$ *to* $\mathsf{G}^2(X)$*;*
- *exchangeable probabilities are mixtures of iids:* $\mu_X \circ \mathsf{G}(iid_X) \circ deF_X = id_{\mathsf{G}_{ex}(X^{\mathbb{N}})}$.

Given this, we can easily prove the following:

▶ **Theorem 40.** *The family of maps deF constructed in Theorem 39 is a natural transformation from* $\mathsf{G}_{ex}(-^{\mathbb{N}})$ *to* $\mathsf{G}^2$.

**Proof.** Let $f : X \to Y$ be a continuous map. We have:

$$\begin{array}{rcll}
(\mathsf{G}^2(f) \circ deF_X)(P) & = & \mathsf{G}^2(f)(\lim_n \mathsf{G}(\mathcal{E}_{n,X} \circ \pi_n)(P)) & \\
& = & \lim_n \mathsf{G}^2(f)(\mathsf{G}(\mathcal{E}_{n,X} \circ \pi_n)(P)) & \text{(Continuity)} \\
& = & \lim_n \mathsf{G}(\mathcal{E}_{n,X}\pi_n \circ f^{\mathbb{N}})(P) & \text{(Naturality)} \\
& = & \lim_n \mathsf{G}(\mathcal{E}_{n,X}\pi_n)(\mathsf{G}_{ex}(f^{\mathbb{N}})(P)) & \\
& = & deF_Y(\mathsf{G}_{ex}(f^{\mathbb{N}})(P)) &
\end{array}$$

◀

This result together with Proposition 38 and Theorem 35 implies that the de Finetti transformation is entirely characterised by its finite components. Concretely, it is enough to prove the de Finetti theorem on finite spaces for our framework to extend it to arbitrary Polish spaces.

## 6 Conclusion

We have proposed a type theory for Polish spaces with the essential functors of the modeling trade: Vietoris, Giry, and Skorokhod. Thus one can re-construct all classical models of mixed probabilistic and non-determinism within our grammar, guaranteeing an adequate level of expressivity for parameterised models. Not only do we subsume classical constructions studied in concurrency theory, but the compass of our type theory also includes probabilities on functions spaces which are hot pursuits in probabilistic modeling - e.g. solutions of stochastic differential equations. For this fledgling type theory, we provide a "normalisation theorem" showing that existence and equality between our structure arrows is completely determined by the finite case.

However, in many ways this type theory is still a draft. An axiomatic or syntactic treatment is not yet in order, as one needs first to decide a certain number of questions which this contribution has left unanswered. For instance, we do not know if parameter functors are closed under products. Further progress might hint at natural such treatments.

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