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Abstract
In this paper we present an extension of existing results on limit cycles for Liénard systems and formulate sufficient conditions for existence and uniqueness of limit cycles for Liénard systems with non-differentiable vector fields. As an application we consider the example of a linear systems with the saturation nonlinearity.

Keywords: Liénard systems, limit cycles, saturation.

1. INTRODUCTION

In this paper, we consider the Liénard system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1}$$

$x \in \mathbb{R}$, $y \in \mathbb{R}$, $F, g : \mathbb{R} \to \mathbb{R}$. For $F \in \mathcal{C}^1$ define $f(x) := \frac{dF(x)}{dx}$. We are interested in obtaining sufficient conditions on functions F and g for the existence and uniqueness of a limit cycle. In particular, we aim to relax standard regularity assumptions on F and g for this class of systems. Indeed, in (Perko, 2013, Chapter 3.8,Theorem 1) F ($F \in \mathcal{C}^1$) and g are required to be continuously differentiable, in Villari (1987), function F is continuously differentiable and g is locally Lipschitz continuous.

Clearly, in case F is not differentiable, the above properties are not verified for f. As an example, take $F(x) = a_1(x) + b_1 sat(x)$ and $g = a_2(x) + b_2 sat(x)$ where $sat(x) = sign(x)min(|x|, 1)$, which corresponds to a linear saturating system. This class of system appears in control systems, where the control action is limited and feedback laws have are constrained. The saturation function is then used to model these constrained systems. There is therefore a practical interest on studying such a class of systems and hence it is of interest to extend the results of Liénard systems to non-differentiable functions F and g.

This short note presents a generalization of Liénards theorem for non-differentiable functions F and g in (1) in Section 2 and illustrates its application to the class of systems with input saturation in Section 3.

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2. MAIN RESULT

The theorem below is the main result of this paper, relaxing assumptions on the regularity of the functions defining the Liénard system

Theorem 2.1. Under the assumptions

(1) $F, g \in \mathcal{C}^0(\mathbb{R})$ are odd and globally Lipschitz,
(2) $xg(x) > 0$, for $x \neq 0$,
(3) $xF(x) < 0$, for $|x| < \epsilon$,
(4) F has a single positive zero at $x = a$,
(5) F increases monotonically to infinity for $x \geq a$ as $x \to +\infty$,

the Liénard system (1) has a unique stable limit cycle.

Before presenting the proof of this theorem, we start with a few observations. Under Assumption 1, Cauchy problem for system (1) have unique solutions. Under Assumption 2, the origin is the only critical point. Due to the central symmetry of the vector field (Assumption 1), we restrict our study to the plan $x \geq 0$. By continuity of F, under Assumptions 3, 4 and 5 we have

$$\begin{cases} F(x) < 0 & 0 < x < a, \\ F(x) = 0 & x \in \{0, a\}, \\ F(x) > 0 & a < x. \end{cases}$$

It follows that

$$\exists c > 0,\ F(x) > -c.$$

We denote

$$\begin{cases} P(x, y) = y - F(x) \\ Q(x, y) = -g(x). \end{cases}$$

Proof. Let $P_0 = (x_0, y_0) \in \{(x, y) \in \mathbb{R}^2 | x \geq 0\}$ and Γ be the trajectory of the Liénard system (1) starting at P_0, that is $(x_0, y_0) = (x(t_0), y(t_0))$. We first prove that any trajectory that starts from P_0 intersects the curve $y = F(x)$ Consider two cases regarding the sign of $F(x_0) - y_0$

Case 1 $F(x_0) < y_0$

We construct the proof by contradiction. Assume that if $\forall (x(t), y(t)) \in \Gamma$ then $(x(t), y(t)) \cap \{(x, y) \in \mathbb{R}^2 | F(x) = y\} = \emptyset$ that is Γ never intersects the curve $y = F(x)$. Thus, $\forall\, t > t_0$ we have

$$dx = (y - F(x))dt > 0$$

since the trajectory does not interest the curve $y = F(x)$ x remains positive and from Assumption 2

$$dy = -g(x)dt < 0$$

Thus, $x(t)$ is increasing, $y(t)$ is decreasing and we have

$$\forall\, t > t_0, \begin{cases} x(t) > 0, \\ y(t) < y_0. \end{cases}$$

For both $x(t)$ and $y(t)$, we consider the following cases

$$\lim_{t \to +\infty} x(t) = \begin{cases} +\infty \\ x^* \end{cases} \qquad \lim_{t \to +\infty} y(t) = \begin{cases} -\infty \\ y^*. \end{cases}$$

If $x(t) \to +\infty$, as $y(t) < y_0$ and $F(+\infty) = +\infty$, $y - F(x)$ cannot remain positive. Thus,

$$\lim_{t \to +\infty} x(t) = x^*. \tag{2}$$

If $y(t) \to -\infty$, as $F(x) > -c$, $y - F(x)$ can not remain positive. Thus,

$$\lim_{t \to +\infty} y(t) = y^*.$$

Let's prove that (x^*, y^*) is a critical point. We suppose that $P(x^*, y^*) \neq 0$ (the proof is the same for the case $Q(x^*, y^*) \neq 0$). Thus,

$$\exists\, T > 0,\ \forall\, t \geq T, \quad |P(x(t), y(t))| \geq \left| \frac{P(x^*, y^*)}{2} \right|$$

and we have $\forall\, t \geq T$

$$|x(t) - x(T)| = \left| \int_T^t P(x(s), y(s)) ds \right|$$

As $P(x(t), y(t))$ preserves its sign for $t > T$

$$\left| \int_T^t P(x(s), y(s)) ds \right| = \int_T^t |P(x(s), y(s))| ds \geq \left| \frac{P(x^*, y^*)}{2} \right| (t - T).$$

It follows that $\lim_{t \to +\infty} x(t) = +\infty$, therefore contradicting (2). Thus $P(x^*, y^*) = 0$ ($Q(x*, y*) = 0$) is a critical point. Since by assumption $(0, 0)$ is the only critical point of system (1), and $x^* \neq 0$, we arrive at a contradiction. Thus, any trajectory passing by a point above $y = F(x)$ intersect this curve.

Case 2 $F(x_0) \geq y_0$

As in the previous case we proceed by contradiction. Let us show that Γ intersects $\{(x, y) \in \mathbb{R}^2 | x = 0, y < 0\}$. Suppose

$$dy = -g(x)dt < 0\ \forall t > t_0.$$

Due to the nature of the flow on the curve $y = F(x)$ for $x > 0$, Γ remains below the curve for all $t > t_0$ and we have

$$\forall\, t > t_0,\ dx = (y - F(x))dt \leq 0.$$

Thus, both $x(t)$ and $y(t)$ are decreasing. If we suppose that

$$\lim_{t \to +\infty} x(t) = x_{min} > 0,$$

since the origin is the only critical point, with the same reasoning used above, we have

$$\lim_{t \to +\infty} y(t) = -\infty,$$

and

$$dx = y(t) - F(x(t)) < y(t) + c \to -\infty.$$

which is a contradiction, thus we have $\Gamma \cap \{(x, y) \in \mathbb{R}^2 | x = 0, y < 0\} \neq \emptyset$. Assumption 3 implies that in the neighborhood of the origin, $F(x) < 0$. As $y(t)$ is decreasing and $F(x) \geq y$, Γ must intersect the $y - axis$ for $y < 0$. Thus, any trajectory passing through a point below $y = F(x)$ intersects $\{(x, y) \in \mathbb{R}^2 | x = 0, y < 0\}$.

It follows that any curve Γ starting from $P_0 = (0, y_0)$ $y_0 > 0$ intersects the curve $y = F(x)$ for $x > 0$ followed by an intersection with the $y - axis$ for $y < 0$. Next part of the proof makes use of the Figure 1 and especially, of points $P_j = (x_j, y_j)$, $j = 0, \ldots, 4$. Note that points P_1 and P_3 exist only if $x_2 > a$. By central symmetry,

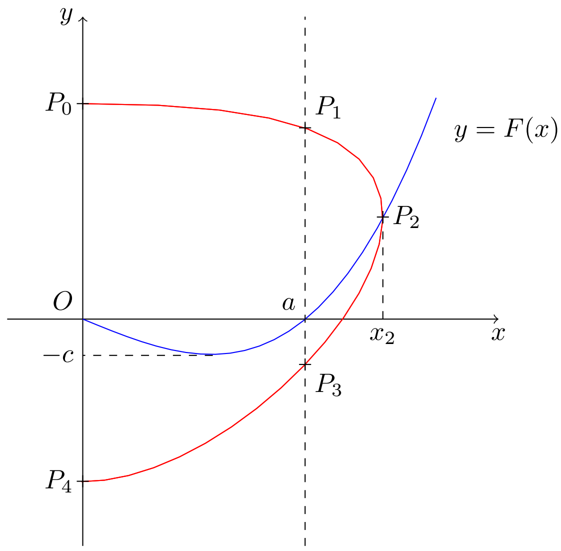

Figure 1. Typical behaviour of function $F(x)$ and a trajectory Γ of the Liénard system (1).

Γ is a closed trajectory of (1) if and only if $y_4 = -y_0$; and for $u(x, y) := \frac{y^2}{2} + \int_0^x g(s)ds$, this is equivalent to $u(0, y_0) = u(0, y_4)$.

Let A be the arc $\widehat{P_0 P_4}$. We proved that A intersects $\{(x, y) | F(x) = y\}$ in P_2. Thus, for any arc A, there exists only one $\alpha = x_2$, the abscissa of P_2. By uniqueness of solution of (1), for any $\alpha = x_2$ the arc A is unique, thus, the line integral

$$\Phi(\alpha) = \int_A du = u(0, y_4) - u(0, y_0)$$

is a bijection and Γ is a closed trajectory of (1) if and only if $\Phi(\alpha) = 0$. To prove that the Liénard system (1) has a single limit cycle, let us show that the function $\Phi(\alpha)$ has exactly one zero.

Define, along the trajectory Γ

$$du = g(x)dx + ydy.$$

Since for system (1)

$$\begin{cases} dx = (y - F(x))dt \\ dy = -g(x)dt \end{cases}$$

we obtain the following expression for du

$$du = \frac{-F(x)g(x)}{y - F(x)}dx = F(x)dy. \qquad (3)$$

From this expression it follows that if $0 < \alpha \leq a$, we have $F(x) < 0$ and $dy < 0$, thus, $du > 0$ and $\Phi(\alpha) > 0$, and Γ must intersect $\{(x, y)|y = F(x)\}$ at a point $P_2 = (x_2, y_2)$ with $0 < x_2 \leq a$. Hence, if Γ is a closed trajectory, $x_2 = \alpha > a$.

To prove that there exists only one limit cycle, let show that for $\alpha > a$, the line integral $\Phi(\alpha)$ decreases monotonically from $\Phi(a) > 0$ to $-\infty$. We recall that for $\alpha > a$, P_1 and P_3 exist. We split the arc A into three parts $A_1 = \widehat{P_0P_1}$, $A_2 = \widehat{P_1P_3}$ and $A_3 = \widehat{P_3P_4}$. It follows that

$$\Phi(\alpha) = \Phi_1(\alpha) + \Phi_2(\alpha) + \Phi_3(\alpha)$$

where

$$\Phi_1(\alpha) = \int_{A_1=\widehat{P_0P_1}} du,$$
$$\Phi_2(\alpha) = \int_{A_2=\widehat{P_1P_3}} du,$$
$$\Phi_3(\alpha) = \int_{A_3=\widehat{P_3P_4}} du.$$

With (3), we have

$$du = F(x)dy$$

Along the arc A_1 and A_3, $F(x) < 0$ and $dy < 0$. Therefore, $\Phi_1(\alpha) > 0$ and $\Phi_3(\alpha) > 0$. Similarly, along the arc A_2, $F(x) > 0$, $dy < 0$ and $\Phi_2(\alpha) < 0$.

The arc A_1 always connects a point in $\{(x, y) \in \mathbb{R}|x = 0\}$ to a point of the line $\{(x, y) \in \mathbb{R}|x = a\}$. Thus, since $dx > 0$ along arc A_1, we can define the function

$$\gamma_{1,\alpha} \colon [0, a] \to \mathbb{R}$$
$$x \mapsto y$$

such that

$$\Phi_1(\alpha) = \int_0^a \frac{-F(x)g(x)}{\gamma_{1,\alpha}(x) - F(x)}dx.$$

Since trajectories of system (1) can not intersect we have that, for $\alpha_1 < \alpha_2$, the corresponding arc A_{α_1} stays in the interior of A_{α_2} defined on $\mathbb{R}_{\geq 0}$. Thus,

$$\forall x \in [0, a],\ \gamma_{1,\alpha_1}(x) < \gamma_{1,\alpha_2}(x).$$

From the above we can deduce that

$$0 < \gamma_{1,\alpha_1}(x) - F(x) < \gamma_{1,\alpha_2}(x) - F(x)$$
$$\frac{1}{\gamma_{1,\alpha_1}(x) - F(x)} > \frac{1}{\gamma_{1,\alpha_2}(x) - F(x)}$$
$$\frac{-F(x)g(x)}{\gamma_{1,\alpha_1}(x) - F(x)} > \frac{-F(x)g(x)}{\gamma_{1,\alpha_2}(x) - F(x)}$$
$$\int_0^a \frac{-F(x)g(x)}{\gamma_{1,\alpha_1}(x) - F(x)}dx > \int_0^a \frac{-F(x)g(x)}{\gamma_{1,\alpha_2}(x) - F(x)}dx$$
$$\Phi_1(\alpha_1) > \Phi_1(\alpha_2)$$

thus we obtain that $\Phi_1(\alpha)$ is strictly decreasing.

With the same reasoning, the arc A_3 is joining a point of the line $x = a$ to a point of the line $x = 0$ and $dx < 0$. Thus, we can define the function

$$\gamma_{3,\alpha} \colon [0, a] \to \mathbb{R}$$
$$x \mapsto y$$

such that

$$\Phi_3(\alpha) = \int_a^0 \frac{-F(x)g(x)}{\gamma_{3,\alpha}(x) - F(x)}dx.$$

For $\alpha_1 < \alpha_2$ we have

$$\forall x \in [0, a],\ \gamma_{3,\alpha_1}(x) > \gamma_{3,\alpha_2}(x).$$

It follows that

$$0 > \gamma_{1,\alpha_1}(x) - F(x) > \gamma_{1,\alpha_2}(x) - F(x)$$
$$\frac{1}{\gamma_{1,\alpha_1}(x) - F(x)} < \frac{1}{\gamma_{1,\alpha_2}(x) - F(x)}$$
$$\frac{-F(x)g(x)}{\gamma_{1,\alpha_1}(x) - F(x)} < \frac{-F(x)g(x)}{\gamma_{1,\alpha_2}(x) - F(x)}$$
$$\int_a^0 \frac{-F(x)g(x)}{\gamma_{1,\alpha_1}(x) - F(x)}dx > \int_a^0 \frac{-F(x)g(x)}{\gamma_{1,\alpha_2}(x) - F(x)}dx$$
$$\Phi_1(\alpha_1) > \Phi_1(\alpha_2)$$

and $\Phi_3(\alpha)$ is strictly decreasing.

For the arc A_2, we have

$$\Phi_2(\alpha) = \int_{A_2=\widehat{P_1P_3}} du$$
$$= \int_{y_{1,\alpha}}^{y_{3,\alpha}} F(x)dy.$$

As $dy < 0$, we can define the function

$$\beta_{2,\alpha} \colon [y_{1,\alpha}, y_{3,\alpha}] \to \mathbb{R}$$
$$y \mapsto x$$

such that

$$\Phi_2(\alpha) = \int_{y_{1,\alpha}}^{y_{3,\alpha}} F(\beta_{2,\alpha}(y))dy.$$

Since system trajectories from different initial conditions do not cross, for $\alpha_1 < \alpha_2$, the corresponding arc A_{α_1} stays in the interior of A_{α_2} defined on $\mathbb{R}_{\geq 0}$. Thus,

$$\forall y \in [y_{1,\alpha_1}, y_{3,\alpha_1}],\ \beta_{2,\alpha_1}(y) < \beta_{2,\alpha_2}(y).$$

Thus, from Assumption 5 we have

$$0 < F(\beta_{2,\alpha_1}(y)) < F(\beta_{2,\alpha_2}(y))$$
$$\int_{y_{1,\alpha_1}}^{y_{3,\alpha_1}} F(\beta_{2,\alpha_1}(y))dy > \int_{y_{1,\alpha_1}}^{y_{3,\alpha_1}} F(\beta_{2,\alpha_2}(y))dy$$
$$\Phi_2(\alpha_1) > \int_{y_{1,\alpha_1}}^{y_{3,\alpha_1}} F(\beta_{2,\alpha_2}(y))dy.$$

Since $F(x) > 0$, $y_{1,\alpha_1} < y_{1,\alpha_2}$ and $y_{3,\alpha_1} > y_{3,\alpha_2}$, we have

$$\int_{y_{1,\alpha_2}}^{y_{1,\alpha_1}} F(\beta_{2,\alpha_2}(y))dy < 0, \quad \int_{y_{3,\alpha_1}}^{y_{3,\alpha_2}} F(\beta_{2,\alpha_2}(y))dy < 0,$$

and

$$\Phi_2(\alpha_1) > \int_{y_{1,\alpha_2}}^{y_{3,\alpha_2}} F(\beta_{2,\alpha_2}(y))dy = \Phi_2(\alpha_2).$$

We conclude that $\Phi_2(\alpha)$ is strictly decreasing.

As a result we obtain that, for $\alpha > a$

$$\Phi(\alpha) = \Phi_1(\alpha) + \Phi_2(\alpha) + \Phi_3(\alpha)$$

is strictly decreasing too. Since $\Phi(\alpha) > 0$ for $\alpha \leq a$, it suffices to show that $\Phi_2(\alpha) \to -\infty$ as $\alpha \to \infty$ to prove

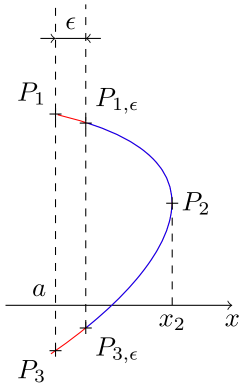

Figure 2. The reduced arc A_2

that $\Phi(\alpha)$ has only one zero α_0.
Let $\epsilon > 0$, $\epsilon < \alpha$ and define $y_{1,\epsilon}$, $y_{3,\epsilon}$ such that

$$P_{1,\epsilon} := (a+\epsilon, y_{1,\epsilon}) \in A_2$$
$$P_{3,\epsilon} := (a+\epsilon, y_{3,\epsilon}) \in A_2,$$

as it is depicted in the Figure 2.
Next we use this partitioning of the arc to obtain $\Phi(\alpha)$ and write

$$\begin{aligned}\Phi_2(\alpha) &= \int_{\widehat{P_1P_3}} F(x)dy \\ &= \int_{\widehat{P_1P_{1,\epsilon}}} F(x)dy + \int_{\widehat{P_{1,\epsilon}P_{3,\epsilon}}} F(x)dy + \int_{\widehat{P_{3,\epsilon}P_3}} F(x)dy.\end{aligned}$$

On the other hand, since $dy < 0$ and $F(x) \geq 0$ we have that the following three inequalities hold

$$\int_{\widehat{P_1P_{1,\epsilon}}} F(x)dy < 0, \qquad \int_{\widehat{P_{3,\epsilon}P_3}} F(x)dy < 0$$

and

$$\Phi_2(\alpha) < \int_{\widehat{P_{1,\epsilon}P_{3,\epsilon}}} F(x)dy.$$

Finally, using Assumption 5, we have that $F(x)$ is increasing and therefore

$$\begin{aligned}\int_{\widehat{P_{1,\epsilon}P_{3,\epsilon}}} F(x)dy &< F(a+\epsilon)\int_{\widehat{P_{1,\epsilon}P_{3,\epsilon}}} dy \\ &< F(a+\epsilon)\int_{y_{1,\epsilon}}^{y_{3,\epsilon}} dy \\ &< F(a+\epsilon)(y_{3,\epsilon}-y_{1,\epsilon}) \\ &< F(a+\epsilon)(y_{3,\epsilon}-y_{1,\epsilon}).\end{aligned}$$

Since $y_{3,\epsilon} < 0 < y_2 < y_{1,\epsilon}$ for ϵ small enough, we have

$$\begin{aligned}\Phi_2(\alpha) &< F(a+\epsilon)(y_{3,\epsilon}-y_{1,\epsilon}) \\ &< -F(a+\epsilon)y_{2,\epsilon}.\end{aligned}$$

According to Assumption 5, $y_2 = F(x_2 = \alpha) \to \infty$ when $\alpha \to \infty$ and $\Phi_2(\alpha) \to -\infty$ when $\alpha \to \infty$.

With the above we complete the proof of existence and uniqueness of the limit cycle. To prove the stability of the limit cycle, let us define the sequence of intersection points between Γ and the positive $y-axis$ that we denote $(y_0)_n$. If $\alpha < \alpha_0$, we have $\Phi(\alpha) > 0$ and thus $y_0 < -y_4$. By central symmetry, we conclude that $(y_0)_n$ is an increasing sequence. As trajectories do not cross, $(y_0)_n$ is bounded and thus, converge to the limit cycle. Since the limit cycle is the only possible limit for Γ, it follows that the limit cycle is stable. ∎

3. APPLICATION TO SINGLE INPUT PLANAR SATURATING SYSTEMS

We consider the class of single input planar systems with saturation

$$\dot{\xi} = A\xi + B\,\text{sat}(K\xi) \tag{4}$$

where $\xi \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times2}$, $B \in \mathbb{R}^{2\times1}$ and $K \in \mathbb{R}^{1\times2}$. We assume that the pair (A, K) in system (4) is observable. This implies that the system observability matrix, defined as,

$$\mathcal{O} := \begin{bmatrix} K \\ KA \end{bmatrix}$$

satisfies the rank condition $rank(\mathcal{O}) = 2$. We introduce next the following notations $\tilde{q} \in \mathbb{R}^{2\times1}$ the second column of $\mathcal{O}^{-1}$ and the matrix $T := [A\tilde{q}\ \tilde{q}]$. Thus, using the change of coordinates $\xi = T\bar{\xi}$ the system can be written as

$$\dot{\bar{\xi}} = \bar{A}\bar{\xi} + \bar{B}\,\text{sat}(\bar{K}\bar{\xi})$$

where

$$\bar{A} = T^{-1}AT = \begin{bmatrix} a_1 & 1 \\ a_0 & 0 \end{bmatrix}$$
$$\bar{B} = T^{-1}B = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}$$
$$\bar{K} = KT = [1\ \ 0]$$

with a_1, a_0, b_1, $b_0 \in \mathbb{R}$. With $\bar{\xi} = \begin{bmatrix} x \\ y \end{bmatrix}$, we have

$$\begin{cases} \dot{x} = a_1x + y + b_1\,\text{sat}(x) \\ \dot{y} = a_0x + b_0\,\text{sat}(x) \end{cases}$$

which is a Liénard system (1) with

$$\begin{cases} F(x) = -a_1x - b_1\,\text{sat}(x) \\ g(x) = -a_0x - b_0\,\text{sat}(x). \end{cases} \tag{5}$$

Next we show that all the assumptions of Theorem 2.1 are satisfied for this system. We use the fact that the function sat is continuous on $\mathbb{R}$, differentiable on $\mathbb{R}\backslash(-1, 1)$ and

$$\text{sat}(x) = \int_0^x s(\theta)d\theta, \quad s(\theta) = \begin{cases} 1 & \text{if } |\theta| \leq 1 \\ 0 & \text{if } |\theta| > 1 \end{cases}$$

3.1 Functions F and g are odd and globally Lipschitz

Property 3.1. Let x_a and x_b be two vectors of $\mathbb{R}^2$. Thus

$$\|\text{sat}(Kx_a) - \text{sat}(Kx_b)\| \leq \|K\|\|x_a - x_b\|$$

and sat is a 1-Lipschitz function.

Proof. With the notation introduced above, we have

$$\begin{aligned}\|\text{sat}(Kx_a) - \text{sat}(Kx_b)\| &= \|\int_0^{Kx_a} s(\theta)d\theta - \int_0^{Kx_b} s(\theta)d\theta\| \\ &= \|\int_{Kx_b}^{Kx_a} s(\theta)d\theta\| \\ &\leq \int_{\min(Kx_a,Kx_b)}^{\max(Kx_a,Kx_b)} |s(\theta)|d\theta \\ &\leq \int_{\min(Kx_a,Kx_b)}^{\max(Kx_a,Kx_b)} 1d\theta \\ &\leq \|Kx_a - Kx_b\| \\ &\leq \|K\|\|x_a - x_b\|.\end{aligned}$$

∎

Property 3.2. $Ax + B\,\mathrm{sat}(Kx)$ is globally Lipschitz.

Proof. Let x_a and x_b be two vectors of $\mathbb{R}^2$. Thus

$$\begin{aligned}\|\mathcal{F}(x_a) - \mathcal{F}(x_b)\| &= \|A(x_a - x_b) \\ &\quad + B(\mathrm{sat}(Kx_a) - \mathrm{sat}(Kx_b))\| \\ &\leq \|A(x_a - x_b)\| \\ &\quad + \|B(\mathrm{sat}(Kx_a) - \mathrm{sat}(Kx_b))\| \\ &\leq \|A\|\|x_a - x_b\| + \|B\|\|K\|\|x_a - x_b\| \\ &\leq (\|A\| + \|B\|\|K\|)\|x_a - x_b\|.\end{aligned}$$

■ Thus, function F and g defined in (5) are odd and globally Lipschitz.

3.2 The function $g(\cdot)$ satisfies the inequality $xg(x) > 0$, for $x \neq 0$

Let show the following property based on the definition of g in (5).

Property 3.3. $xg(x) = -a_0x^2 - b_0x\,\mathrm{sat}(x) > 0$ for $x \neq 0$ if and only if $a_0 < 0$ and $a_0 + b_0 < 0$

Proof. We suppose that $xg(x) = -a_0x^2 - b_0x\,\mathrm{sat}(x) > 0$, for $x \neq 0$. Thus, $a_0 < 0$ and $a_0 + b_0 < 0$.

We suppose that $a_0 < 0$ and $a_0 + b_0 < 0$. Let us prove the result for $x > 0$, the parity of $x \to xg(x)$ will give us the result for $x \neq 0$. By definition

$$xg(x) = \begin{cases} -(a_0 + b_0)x^2 & \text{if } 0 < x < 1 \\ -a_0x^2 - b_0x & \text{if } 1 < x. \end{cases}$$

Since $a_0 + b_0 < 0$, $xg(x) > 0$ for $0 < x < 1$. And as $a_0 < 0$, if $1 < x$ we have

$$\begin{aligned} a_0 &> a_0x \\ 0 > a_0 + b_0 &> a_0x + b_0 \\ 0 &< -a_0x^2 - b_0x \end{aligned}$$

which complete the proof. ■

3.3 The function $F(\cdot)$ satisfies the inequality $xF(x) < 0$, for $|x| < \epsilon$

Let show the following property based on the definition of F in (5).

Property 3.4. $xF(x) = -a_1x^2 - b_1x\,\mathrm{sat}(x) < 0$ for $|x| < \epsilon$ if and only if $a_1 + b_1 > 0$.

Proof. We suppose that $xF(x) = -a_1x^2 - b_1x\,\mathrm{sat}(x) < 0$ for $|x| < \epsilon$. Since in the neighborhood of the origin $xF(x) = -(a_1 + b_1)x^2$, the inequality $a_1 + b_1 > 0$ must hold.

We suppose that $a_1 + b_1 > 0$. For $|x| < 1$, $xF(x) = -(a_1 + b_1)x^2 < 0$ which complete the proof. ■

3.4 F has a single positive zero denoted a

Let show the following property based on the definition of F in (5).

Property 3.5. F has a single positive zero if and only if a_1 and $a_1 + b_1$ have different signs.

Proof. We suppose that F has a single positive zero at $x = a > 0$. Thus, $F(a) = -a_1a - b_1\,\mathrm{sat}(a) = 0$. From the definition of F, we must have $a > 1$. Hence, $-a_1a - b_1 = 0$ and $a = -\frac{b_1}{a_1} > 1$. If a_1 and $a_1 + b_1$ have the same sign, $\frac{a_1+b_1}{a_1} = 1 - a > 0$ which leads to a contradiction. Thus, a_1 and $a_1 + b_1$ have different sign.

We suppose that a_1 and $a_1 + b_1$ have different sign. Thus, $-\frac{b_1}{a_1} > 1$ and $F(-\frac{b_1}{a_1}) = a_1\frac{b_1}{a_1} - b_1 = 0$ which complete the proof. ■ Let us note that to satisfy both assumptions 3.3 and 3.4, we must have $a_1 + b_1 > 0$ and $a_1 < 0$.

3.5 F increases monotonically to infinity for $x \geq a$ as $x \to +\infty$

Based on the definition of F in (5), this property is immediately verified.

3.6 Corollary

We can now formulate the following corollary which is a direct application of Theorem 2.1.

Corollary 3.1. Any system

$$\dot{x} = \begin{bmatrix} a_1 & 1 \\ a_0 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \mathrm{sat}([1\ \ 0]\,x).$$

which satisfy

$$\begin{cases} a_1 < 0 \\ a_0 < 0 \end{cases} \qquad \begin{cases} a_1 + b_1 > 0 \\ a_0 + b_0 < 0 \end{cases}$$

has exactly one limit cycle and it is stable.

4. CONCLUSION

We have presented an extension of the Liénard Theorem, establishing conditions for the existence and stability of a limit cycle for planar systems with non-differentiable right hand sides. We have then applied the result to show the existence of limit cycles to planar systems with a saturation nonlinearity.

We currently investigate conditions on the vector field for estimating the amplitude and frequency of the limit cycle of for planar systems with saturations. We are also interested to obtain conditions for the design of feedback laws guaranteeing the existence of limit cycles in closed loop.

Also, we are currently studying the extension of the results presented in this paper for the case where the right hand side is not defined by odd functions, namely for the case where F and g are not odd functions.

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