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Conjugate prior in the Mallows model with Spearman distance

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Introduction

• Ranking and comparing items: crucial for collecting information about preferences in many areas: e.g. marketing, business, politics, genetics.
• Mallows model: powerful model for rankings
• Question: How to include expert opinions into the analysis? How to elicit a meaningful prior on the consensus ranking of a population?

1. The Mallows model with Spearman distance

The Mallows model (Mallows 1957) is a class of non-uniform distributions for $R \in P_n$, the set of $n$-dimensional permutations, of the form

$$P(R|\rho) = \frac{e^{-s(|\rho - \rho|)}}{Z(\theta)}$$

where $\rho \in P_n$ consensus ranking, location parameter
$\theta \geq 0$: precision parameter
$s(\cdot, \cdot)$: right-invariant (Diaconis 1988) distance

$Z(\theta) = \sum_{\rho \in P_n} e^{-s(|\rho - \rho|)}$: partition function (independent of $\theta$ because of right-invariance of $s(\cdot, \cdot)$)

Spearman distance: the squared $l_2$ norm on $P_n$

$$d_s(\rho, \sigma) = \sum_{i=1}^n (\rho_i - \sigma_i)^2$$

Consider $n$ items, ranked by $N$ assessors. Denote by $R_i = (R_{i1}, R_{i2}, \ldots, R_{in}) \in P_n$, the ranking of user $j$, $i \leq j \leq n$. Under the Mallows model with Spearman distance (MDS), given $\theta$, the likelihood can be written as

$$P(R_1, \ldots, R_N|\rho) \propto \exp \left( -\theta N \sum_{i=1}^n d_s(R_i, \rho) \right)$$

where $R_i = \frac{1}{N} \sum_{j=i}^n R_{ij}$, $i = 1, \ldots, n$, is the sample average of the $i$-th rank.

2. Sufficient statistics and mime

The sufficient statistic for $\rho = (\rho_1, \ldots, \rho_n)$, when $\theta$ is known, is $R = (R_1, \ldots, R_N)$.

Proposition 1. Let $R_1, \ldots, R_N, \theta \sim \text{Mall}(\theta, \rho)$, and define the vector of sample ranks $R$ as above. Assume $R_i \neq R_j$, for each $i \neq j$, and denote by $Y_i(R) = (Y_{i1}(R), \ldots, Y_{in}(R)) \in P_n$, the rank vector of $R_i$, i.e. $Y_{ij}(R) = \sum_{j=1}^n I(R_{ij} \leq R_i)$, $i = 1, \ldots, n$. Then the unique mle of $\rho$ is

$$\rho_{\text{MLE}} = \arg \max_{\rho \in P_n} \left[ -\theta N \sum_{i=1}^n Y_i(R) \right]$$

Notice that, in general $R \notin P_n$. However $R$ lives in the permutohedron of order $n$.

Definition 1. The permutohedron of order $n$, $\text{pp}_{n-1}$, is an $(n-1)$-dimensional polytope embedded in an $n$-dimensional space, the vertices of which are formed by permuting the coordinates of the vector $(1, 2, \ldots, n)$. Equivalently, it is the convex hull of the points $\rho \in P_n$, the set of $n$-dim permutations.

3. The Bayesian Mallows

Vitelli et al. (2018) proposed a framework for performing Bayesian inference on the Mallows model. They assume that $\rho$ is a priori uniformly distributed, $\rho \sim \text{Uni}(P_n)$.

Contribution: the objective prior in the sense of Villa & Walker (2015) is the uniform prior density on the space of permutations $\rho \sim \text{Uni}(P_n)$.

Can we go further? How to put an informative prior on $\rho$?

A. $\theta$ given: conjugate prior for $\rho$
B. $\theta$ not given: conjugate conditioned on $\theta$ + clever prior on $\theta$ → MH sampling scheme to approximate the posterior

3.A Conjugate prior for $\theta$ (given $\theta$)

Proposition 2. Keeping $\theta$ fixed, the conjugate prior for $\rho \in P_n$ is

$$\pi(\rho|\theta) = \frac{1}{Z(\theta)} e^{-\theta\sum_{i=1}^n \rho_i}$$

where $\rho_i \in \mathbb{R}^n$, and $N_i \in \mathbb{N}$. We call $\pi(\rho|\theta)$ the Extended Mallows density: it is a Mallows model where the consensus $\rho$ belongs to $\mathbb{R}^n$.

Posterior consensus: weighted average of prior parameter and observed mean (recall Diaconis et al. 1979):

$$\theta = \frac{\theta N + R}{N + N \rho}$$

Then $N_0 \rho$ can be interpreted as an equivalent sample size (recall the Gaussian).

Remark 1. When $N_0 = 0$, the conjugate prior reduces to the uniform, for all $\rho$. When $\rho_0 = (\frac{1}{n}, \ldots, \frac{1}{n})$ the conjugate prior reduces to the uniform, for all $\theta$.

3.A.bis Toy example 1

Sample $N = 30$ rankings from the MDS with $\rho = (3, 1, 2)$, and $\theta = 0.18$. Conjugate prior with $\rho_0 = (1, 2, 3)$, and varying $\theta_0 = 0.0$.

3.A.bis Toy example 2

Same model. Increasing sample size ($N$). Conjugate prior with $\rho_0 = (1, 2, 3)$ and $\theta_0 = 3$ (i.e. $N_0 = 0$).

3.B Prior when $\theta$ not given

When $\theta$ is unknown, the partition function of $\pi(\rho|\theta, N_0)$ depends on the model parameters and cannot be avoided.

Solution: Let $\pi(\theta) \propto Z(\theta, N_0, \rho_0)$ so that the posterior full conditional is treatable.

3.B Prior when $\theta$ not given

Same model. Increasing sample size ($N$). $\rho_0 = (1, 2, 3)$ and $N_0 = 16$.

3. Ongoing work

• Can we say something about the convergence rate?
• Can we say something about $\mathbb{Z}(\cdot, \cdot)$?
• Applications?

Interesting data to test our methods on? Please take contact!

References