Conjugate prior in the Mallows model with Spearman distance
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Introduction

- Ranking and comparing items: crucial for collecting information about preferences in many areas, e.g. marketing, business, politics, genetics.
- Mallows model: powerful model for rankings
- Question: How to include expert opinions into the analysis? How to elicit a meaningful prior on the consensus ranking of a population?

1. The Mallows model with Spearman distance

The Mallows model (Mallows 1957) is a class of non-uniform distributions for $R \in \mathcal{P}_n$, the set of $n$-dimensional permutations, of the form

$$P(R; \rho) = \frac{e^{-s \cdot d_\rho(R)}}{Z(\theta)}$$

where $\rho \in \mathcal{P}_n$, consensus ranking, location parameter

- $s > 0$: precision parameter
- $d_\rho(R)$: right-invariant (Diaconis 1988) distance

Spearman distance: the squared $l_2$ norm on $\mathcal{P}_n$

$$d_\rho(R) = \sum_{i=1}^n (\rho(R)_i - \rho)_i^2$$

Consider $n$ items, ranked by $N$ assessors.

Denote by $R_i = (R_{i1}, R_{i2}, \ldots, R_{in}) \in \mathcal{P}_n$, the ranking of user $i = 1, \ldots, N$.

Under the Mallows model with Spearman distance (MMS), given $\theta$, the likelihood can be written as

$$P(R_1, \ldots, R_N | \rho, \sigma) \propto \exp \left( -\frac{1}{2} \sum_{i=1}^N n \sigma^2 \right)$$

where $\bar{R}_i = \sum_{j=1}^n R_{ij}, i = 1, \ldots, n$, is the sample average of the $i$-th rank.

2. Sufficient statistics and mle

The sufficient statistic for $\rho = (\rho_1, \ldots, \rho_n)$, when $\theta$ is known, is $R = (\bar{R}_1, \ldots, \bar{R}_n)$.

Proposition 1. Let $R_1, \ldots, R_N | \rho, \sigma \sim \text{Mult}(n, \rho, \sigma)$, and define the vector of sample ranks $R$ as above.

Assume $R_i \neq R_j$ for each $i \neq j$, and denote by $Y_i(R) = (Y_{i1}(R), \ldots, Y_{in}(R)) \in \mathcal{P}_n$, the rank vector of $R_i$, i.e. $Y_{ij}(R) = \sum_{k=1}^n 1(R_{ik} \leq R_{ij}), i = 1, \ldots, n$.

Then the unique mle of $\rho$ is

$$\rho_{mle} = \arg \max_{\rho \subset \mathcal{P}_n} \sum_{i=1}^n \bar{R}_i = Y(R).$$

Notice that, in general $R \notin \mathcal{P}_n$. However $R$ lives in the permutahedron of order $n$.

Definition 1. The permutahedron of order $n$, $\mathcal{C}_n$, is an $(n-1)$-dimensional polytope embedded in an $n$-dimensional space, the vertices of which are formed by permuting the coordinates of the vector $(1, 2, 3, \ldots, n)$. Equivalently, it is the convex hull of the points $p \in \mathcal{P}_n$ set of the $n$-dim permutations.

3. The Bayesian Mallows

Vitelli et al. (2018) proposed a framework for performing Bayesian inference on the Mallows model.

They assume that $\rho$ is a priori uniformly distributed, $\rho \sim \text{Unif}(\mathcal{P}_n)$.

Contribution: the objective prior in the sense of Villa & Walker (2015) is the uniform prior density on the space of permutations $\rho \sim \text{Unif}(\mathcal{P}_n)$.

Can we go further? How to put an informative prior on $\rho$?

A: $\theta$ given: conjugate prior for $\rho$

B: $\theta$ not given: conjugate conditioned on $\theta$; clever prior on $\theta$ $\rightarrow$ MH sampling scheme to approximate the posterior

3.A Conjugate prior for $\rho$ (given $\theta$)

Proposition 2. Keeping $\theta$ fixed, the conjugate prior for $\rho \in \mathcal{P}_n$ is

$$\pi(\rho | \theta, N_0) \propto \exp \left( -\theta N_0 \sum_{i=1}^n (\rho_i - \beta_i)^2 \right)$$

where $\rho_i \in [0,1], \beta_i \in (\theta N_0 + N_0, \theta N_0)$.

When $N_0 = 0$, we call $\pi(\rho | \theta, N_0)$ the extended Mallows density: it is a Mallows model where the consensus $\rho_i$ belongs to $\rho_{pp}$.

Posterior consensus: weighted average of prior parameter and observed mean (recall Diaconis et al. 1979):

$$\rho_{N0} = \frac{N_0 \theta + N_0}{N_0 + N_0} = \frac{N_0 \theta}{N_0 + N_0}$$

Then $N_0$ can be interpreted as an equivalent sample size (recall the Gaussian).

Remark 1. When $N_0 = 0$, the conjugate prior reduces to the uniform, for all $\rho_i$. When $\rho_i = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n})$ the conjugate prior reduces to the uniform, for all $\theta_i$.

3.A.bis Toy example 1

Sample $N = 30$ rankings from the MMS with $\rho = (3, 1, 2)$, and $\theta = 0.18$. Conjugate prior with $\rho_{pp} = (1, 2, 3)$, and varying $\theta$.

Ongoing work

- Can we say something about the convergence rate?
- Can we say something about $Z(\cdot)$?
- Applications?
- Interesting data to test our methods on? Please take contact!

References


